OPEN PROBLEMS FOR THE SECOND 2022 BARBADOS WORKSHOP

MAINTAINED BY TUNG H. NGUYEN

1. Bogdan Alecu

Let $k, n \in \mathbb{N}$. Call a graph a *perfectly matched k-partite graph* if it is a k-partite graph with stable sets $A_1, \ldots, A_k$ such that $|A_i| = n$ for all i, and the graph induced by $A_i \cup A_j$ for all $i \neq j$ is a perfect matching. We note that perfectly matched k-partite graphs are not a hereditary class (and in fact, any graph is induced in some large perfectly matched k-partite graph).

Problem 1. *Let us suppose that we are asked to construct a perfectly matched k-partite graph for some large k, and we are given control of n (that is, we can make the size of the bags very large if we so wish) and of the matchings we put between bags. Subject to this, what kinds of constraints can we hope to satisfy when constructing our graph?*

For instance, with a bit of work, we can always ensure that for any fixed g, our perfectly matched k-partite graph has girth at least g (the proof I have uses really large bags though – something like $(g!)^{k^g}$; is it possible to do better?). But what about other constraints? When making k large, can we for instance always arrange to have "small" treewidth? What about other parameters? What about avoiding various induced subgraphs other than small cycles?

Background. I am afraid the background for this problem isn't particularly glamorous – I didn't find it anywhere, I just came up with it and looked at it for a few weeks during my thesis. But perfectly matched k-partite graphs seem very natural, and I was surprised not to find any literature on them (maybe I was googling the wrong things). If you have seen them anywhere else, please let me know! (Also, any questions about them other than the ones I've suggested would be interesting.)

An earlier version of the problem asked for bounded treewidth; but this is not possible since the degeneracy is not always bounded.

Progress.

- (Illingworth, Steiner) The bags can be chosen to be of size $O((kg\log(kg))^g)$.

2. Marthe Bonamy

Problem 2. *Is it true that for all large enough 5-connected planar graphs G, $\alpha(G) > |G|/4$?*

Background. "Large enough" because there is a graph G on 24 vertices with $\alpha(G) = |G|/4$, and there is another graph on 28 vertices. Another conjecture: There are only a finite number of minimal planar graph G with $\alpha(G) = |G|/4$. 11 such graphs have been found and these could be all. It could be true that $\alpha \geq (1/4 + \varepsilon)|G|$ for some ε.

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3. Édouard Bonnet

There is a simple proof that triangle-free graphs of twin-width at most d are $d+2$-colorable (see [5, Figures 3 and 4], and the same paper for the definition of twin-width). But how large can the chromatic number of a triangle-free graph of twin-width d be?

The $d+2$ bound is tight when $d = 0$, since K_2 is a triangle-free 2-chromatic graph of twin-width 0. This bound is however not tight when $d = 1$, since every triangle-free graph of twin-width 1 is bipartite. (Indeed, it can be observed that any odd cycle that is not a triangle has twin-width exactly 2.)

For every non-negative integer d, let

$$f(d) = \max\{\chi(G) : G \text{ is a triangle-free graph of twin-width } d\}$$

(or if you prefer "of twin-width at most d"), where $\chi(G)$ denotes the chromatic number of G.

Problem 3. *Is $f(d) = \Omega(d)$? More generally, the question is to obtain bounds on $f(d)$ refining $f(d) = \Omega(\log d)$ (given, for instance, by the Mycielski graphs) and $f(d) \leqslant d + 2$.*

4. Matija Bucić

Problem 4. *Is it true that for every $a > 0$ and every G with $\alpha(G) \geq a|G| = an$, there is a vertex subset of size $o(n)$ hitting all maximum stable sets?*

This is an old problem of Bollobas, Erdos and Tuza from the 90's. It is known that $\sqrt{n}$ is needed.

5. Linda Cook

Let (T, β) be a rooted tree-decomposition of G. (T, β) is *squirrel-friendly* if there is a function $\alpha\colon V(T) \to V(G)$ such that $\alpha(t) \in \beta(t) \setminus \beta(\text{parent}(t))$ for all $t \in T$ (here the parent of the root is void) and $\alpha(t)\alpha(\text{parent}(t)) \in E(G)$ for all $t \neq r$.

Problem 5. *Does there exist a function $f : \mathbb{N} \to \mathbb{N}$ such that for every graph G, there is a squirrel friendly tree-decomposition with width bounded from above by $f(\text{tw}(G))$?*

Progress.

- (McCarty et al.) False... Similar counterexample to another conjecture of Dvořák.

Let $\mathcal{F} = \{C_3, C_4, \ldots\}$ be the set of all cycles. Eun Jung Kim and Ojoung Kwon proved that $\{C_4, \ldots\}$ has the *induced* version of the Erdős-Posá property, but any $\mathcal{H} \subseteq \mathcal{F}$ with $C_3, C_4 \notin \mathcal{H}$ doesn't have this property. Kwon and Huynh proved that $\{C_4\} \cup \{C_6, \ldots\}$ has this property.

Problem 6. *Are there any other subclasses of $\mathcal{F}$ with this property?*

6. James Davies

Problem 7. *Is it true that for every set S of algebraically independent positive real numbers, the graph on vertex set $\mathbb{R}^2$ with edge set consisting of the pairs of distance in S has bounded chromatic number?*

Background. This problem was first posed by Bukh in 2008. The problem is known to be true if $S \subseteq [a, b]$ for any $0 < a < b < \infty$. False if we require each color class to be measurable. Another problem posed by Soifer in 2010 is

Problem 8. *Is it true if $S = \{2^n : n \in \mathbb{N}_0\}$?*

7. Matt DeVos

Problem 9. *G vertex-transitive, connected, $|G| = n$, what is the size of the longest (not necessarily induced) cycle in G?*

Background. This is an old question. Note that there are several vertex-transitive graphs (Petersen for example) which are not Hamiltonian. Babai in 1979 proved that such a graph G has a cycle of length at least $\sqrt{3n}$ for all $n \geq 3$; and he conjectured that one cannot do better than εn for some $\varepsilon > 0$. It is known that $cn^{3/5}$ is doable. Can one do better than this?

8. Jim Geelen

Given a simple graph G we define a Cayley graph C_G whose vertex set consists of all subsets of $V(G)$ and two vertices are adjacent if the symmetric difference of the sets is an edge of G.

Problem 10. *Are the chromatic numbers of G and C_G qualitatively related?*

Background. This question was implicitly posed by François Jaeger in the 1980s. This much we know:

(i) G is bipartite if and only if C_G is bipartite;
(ii) $\chi(C_G) \leq 2^{\lceil \log_2(\chi(G)) \rceil} < 2\chi(G)$ (proved by Jaeger, but not published); and
(iii) $\chi(C_G) \neq 3$ (proved by Payan [14]).

This is a special case of a question about the critical number of binary matroids. Both (i) and (ii) are easy in the context of matroids. The proof of (iii) is pretty and relies upon a nice generalization of the Mycielski construction.

9. Sepehr Hajebi

A conjecture of Aravind and several other people in 2021 that there is some $c > 0$ such that for all triangle-free G, there is an induced cycle (or path) of length $c\chi(G) \log \chi(G)$. A more doable question is

Problem 11. *What is the length of a longest induced path in the kth Mycielskian?*

It's not hard to show that one can get $\Omega(k)$. What about a superlinear bound in k?

Another conjecture of Aravind in the 2010's says that every properly colored triangle-free graph G contains a *rainbow* induced path on $\chi(G)$ vertices. Scott and Seymour in 2017 [16] proved that there is a function f such that for all t and for all G with $\chi(G) > f(t, \omega(G))$, every proper coloring of G contains a rainbow induced path on t vertices.

Yet another conjecture of Aravind et al. in 2021 [2] says that every properly colored graph with chromatic number χ and clique number ω contains a rainbow induced path on $\chi^{\frac{1}{\omega-1}}$; and they proved in the same paper that every such graph contains a rainbow stable set of size at least $\lceil \chi^{\frac{1}{\omega-1}}/2 \rceil$. It is not known that one could get a rainbow induced matching of size at least $\lfloor (\chi^{\frac{1}{\omega-1}} + 1)/3 \rfloor$.

Problem 12. *Is it true that for every $t \geq 1$, every properly colored graph G contains a rainbow induced copy of rP_t with $r = \lfloor (\chi^{\frac{1}{\omega-1}} + 1)/(t+1) \rfloor$?*

Problem 13. *For all $t \geq 1$ there exists $w = w(t)$ such that every graph G with treewidth at least w contains either K_t or $K_{t,t}$ as an induced subgraph or a 2-degenerate induced subgraph.*

10. Kevin Hendrey

A *linear cycle* in a hypergraph is a sequence $v_1, h_1, v_2, h_2, \ldots, v_t, h_t$ of distinct vertices v_i and hyperedges h_i with $t \geq 3$ such that each hyperedge h_i intersects the union of the other hyperedges of the linear cycle in exactly the set $\{v_i, v_{i+1}\}$ (or $\{v_t, v_1\}$ if $i = t$).

Problem 14. *Given a 3-uniform hypergraph H, can we determine in polynomial time whether H contains a linear cycle?*

Background. For $k \geq 4$, the k-uniform version of this problem is NP-hard (there is a straightforward reduction to 3-SAT). For graphs, every cycle is a linear cycle, and cycles can be detected in linear time.

11. Claire Hilaire

A graph U is *minor-universal* for a family of graphs $\mathcal{F}$ if every graph of $\mathcal{F}$ is a minor of U.

Problem 15. *What is the number of vertices of a smallest planar graph that is minor-universal for the family of all planar graphs on at most n vertices?*

Background. Bodini in 2002 [3] showed that the smallest size of a minor-universal tree for the family of n-vertex trees is at least $\Omega(n \log n)$ and at most $O(n^{1.984\ldots})$. A celebrated result of Robertson, Seymour, and Thomas in 1994 [15] shows that the $2n \times 2n$ grid is minor-universal for the family of n-vertex planar graphs; is there a better example with $o(n^2)$ vertices?

12. Freddie Illingworth

A hypergraph is *linear* if $|e \cap f| \leq 1$ for every pair of distinct edges e, f.

Problem 16. *Is there a linear 3-uniform hypergraph H such that $R(H, R_t^{(3)})$ is superpolynomial in t? Even the case when H is a Fano plane is not known.*

Background. In 2010, Bohman, Frieze, and Mubayi [4, Conjecture 9] conjectured that there are linear 3-uniform H for which $R(H, K_t^{(3)})$ is superpolynomial. In fact they made the equivalent conjecture that there is a linear 3-uniform H and H-free hypergraphs with chromatic number greater than k and $k^{3+o(1)}$ edges. At a 2015 AIM workshop Conlon also asked the same problem.

What is known. For higher uniformities there do exist linear H with $R(H, R_t^{(3)})$ exponential in t. Here's a 4-uniform example. Start with a single vertex G_0 and iterate the following procedure: G_{i+1} is obtained by take two copies of G_i and adding all edges that have exaftly two vertices in each of the copies. Then G_n has 2^n vertices and independence number $n+1$. Let H be a Steiner quadruple system on an odd number of vertices (there exist for any integer that is 1 mod 12). Then G_n is H-free for all n. Indeed, let n be minimal with G_n containing H. By minimality, the copy of H muse have a vertex in both copies of G_{n-1}. Fix a vertex v in the left copy of G_{n-1} and consider its link: every edge has an even number of vertices in the right copy of G_{n-1}. But its link covers all vertices of $V(H) \setminus \{v\}$ exactly once so H has an even number of vertices in the right copy of G_{n-1}. Similarly for the left copy and so H has an even number of vertices, a contradiction. Hence $R(H, K_{n+2}^{(4)}) \geq 2^n$.

13. Peter Nelson

Let $R(s,t)$ be the minimum n so that for every red/blue coloring of $\mathbb{F}_2^n \setminus \{\mathbf{0}\}$ contains a red s-dimensional subspace or a blue t-dimensional subspace. Thus, $R(2,t)$ is the smallest number of colors such that either there is a red triple $\{x, y, x+y\}$ red or t-dimensional subspace in blue. Is $R(2,t)$ poly in t?

Problem 17. *Does there exist some $C > 0$ such that $R(2,t) \leq t^C$ for all t?*

Background. The best known bound is currently $R(2,t) \leq e^{Ct}$. Evidence from the triangle-free process shows that $R(2,t)$ could be linear in t. Another formulation: $A \subseteq \mathbb{F}_2^n \setminus \{\mathbf{0}\}$ implies $\omega(A^c) \geq n^\varepsilon$ for some $\varepsilon > 0$. Observe that if A is maximal such that $0 \notin A+A+A$ then $A+A = A^c$; so yet another formulation: is it true that for all $A \subseteq \mathbb{F}_2^n \setminus \{\mathbf{0}\}$, either A^c or $A + A$ contains a subspace of dimension at least n^c?

14. Tung Nguyen

Problem 18. *Is it true that for every integer $k \geq 1$, every graph of chromatic number at least $3k$ contains a k-connected subgraph of chromatic number at least k?*

Background. This problem has a fairly long history. Thomassen in 1983 [18] claimed to have a proof that this is true; but the proof has an error and remains unfixed. Alon–Kleitman–Saks–Seymour–Thomassen in 1987 [1] proved that chromatic number $O(k^3)$ suffices, and Chudnovsky–Penev–Scott–Trotignon in 2013 [6] pushed this down to $O(k^2)$. Girão and Narayanan in 2022 [8] showed that $7k$ works, a linear bound which is crucial in recent progress on linear Hadwiger's conjecture. Nguyen [13] very recently achieved the bound $(3 + \frac{1}{16})k$ which still falls short of the desired $3k$. At the other extreme, a construction in [1] gives a lower bound of $2k$.

15. Sergey Norin

Problem 19. *Optimize the edge density of a minor on t vertices in any graph with chromatic number at least t.*

Hadwiger's conjecture implies that one could get 100%. It is not hard to obtain 25%. What about average/minimum degree instead of chromatic number? Random graph examples show that one cannot do better than 62% in this case.

Progress.

- (Hendrey, Norin, Steiner, Turcotte) 37% is obtainable.

16. Paul Seymour

Let G be a connected graph. A *cut* of G means the set of edges between X and $V(G) \setminus X$, for some $X \subseteq V(G)$ with $X \neq \emptyset, V(G)$. For an integer $t \geq 1$, ν_t denotes the largest k such that there is a list $D_1, \ldots, D_k$ (possibly with repetition) with the property that for every edge e, $e \in D_i$ for at most t values of $i \in \{1, \ldots, k\}$.

Problem 20. *Is it always true that ν_4 is even? That $2\nu_2 = \nu_4$? That $2\nu_2 = \lim_{t\to\infty} \nu_t/t$?*

Progress.

- (Linda Cook and Sophie Spirkl) This is false; consider the following counterexample. Let $D_1, ..., D_k$ be the cuts of G. Then Paul's question is (equivalently) asking about the following LP: $\max \sum_{i=1}^k x_i$ subject to $\sum_{D_i \ni e} x_i \leq 1$ for all $e \in E(G)$ and $x_i \geq 0$ for all i.

  Its dual is: $\min \sum_{e\in E(G)} y_e$ subject to $\sum_{e \in D_i} y_e \geq 1$ for all i, and $y_e \geq 0$ for all e.

  Consider the following graph: A pyramid, with seven vertices (apex a, base b_1, b_2, b_3, and each path with one internal vertex, say c_1, c_2, c_3).

  There is a primal solution given by:
  - For all i, the cut induced by $\{c_i\}$ has weight $3/4$;
  - For all i, the cut induced by $\{b_i\}$ has weight $1/4$;
  - For all i, the cut induced by $\{c_i, b_i\}$ has weight $1/4$.

A matching dual solution gives weight 1/4 to all edges of the form b_ib_j, and weight 1/2 to all other edges.

This shows: $\nu_2/2 \neq \nu_4/4$, since $\nu_2/2$ is half-integral and $\nu_4/4$ is 3.75 as shown by the above solutions. Also, $\nu_4 = 15$. It's still possible that $\nu_4/4$ is always equal to the LP value.

17. Vaidy Sivaraman

Problem 21. *Let $\mathcal{F}$ be a class of graphs that is closed under induced subgraphs and disjoint unions, such that every forbidden induced subgraph for $\mathcal{F}$ has at most c vertices. Let $\mathcal{F}'$ be the class of graphs containing a vertex whose removal results in a graph in $\mathcal{F}$. Then every forbidden induced subgraph for $\mathcal{F}'$ has at most $2c$ vertices.*

Progress.

- (Hendrey) $\frac{1}{4}c^2 + O(c)$ is actually the correct answer.

Problem 22. *For every tree T, the class of $\{T, \overline{P_5}\}$-free graphs is polynomially χ-bounded.*

Background. This implies the well-known Gyárfás–Sumner conjecture for triangle-free graphs, which is still open. Scott, Seymour, and Spirkl proved that the class of $\{T, C_4\}$-free graphs is polynomially χ-bounded.

18. Maya Stein

An *oriented graph* is a graph all whose edges have been given a direction. The *semidegree* of a vertex v is $d^0(v) := \min\{d^-(v), d^+(v)\}$, and the *minimum semidegree* of D is $\delta^0(D) := \min\{d^0(v) : v \in V(D)\}$.

Considering for a moment undirected graphs, we know that a greedy embedding argument shows that a graph of minimum degree $\geq k$ contains each path of length k a a subgraph. Dirac's theorem shows that a minimum degree of $n/2$ is enough to find a Hamilton cycle in an n-vertex graph. With a similar proof, one can show that every connected graph on at least k vertices and of minimum degree at least $k/2$ contains a path of length k.

Do these results extend to oriented graphs, if we use the semidegree notion? Again, using a greedy embedding argument, it is clear that any oriented graph with $\delta^0(D) \geq k$ must contain each oriented k-edge path. Thomassen asked in the 1970's for minimum semidegree conditions that ensure a directed Hamilton cycle, and more recently, other orientations of Hamilton cycles have been considered. In particular, for oriented graphs, a minimum semidegree of $3n/8 + o(n)$ is enough to ensure any orientation of a Hamilton cycle, and in fact, any orientation of any cycle of length between 3 and n [10].

But how about finding shorter paths P_k of any orientation with a semidegree condition that is close to the order of the path (but lower than the trivial bound $\delta^0(D) \geq k$)? More precisely, in analogy to the situation for undirected graphs, we ask:

Problem 23. *Does every oriented graph D with $\delta^0(D) > \frac{k}{2}$ contain each oriented k-edge path?*

This would be best possible as one can see by considering a regular tournament with k vertices (if k is even). For antidirected paths one couls also consider the blow-up of a directed triangle (Antidirected paths are oriented paths where every vertex has either in-degree 0 or out-degree 0.)

For directed paths, the answer to Problem 23 is yes: this was shown by Jackson in 1981 [9]. For antidirected paths the assertion holds approximately in large dense graphs [17]: For all $\eta \in (0, 1)$ there is n_0 such that for all $n \geq n_0$ and $k \geq \eta n$ every oriented graph D on n vertices with $\delta^0(D) > (1 + \eta)\frac{k}{2}$ contains every antidirected path with k edges. Also for antidirected paths, a short proof [11] shows that if we replace the semidegree condition with $\delta^0(D) \geq 3k/4$, then we can guarantee an antipath of length k (this result could perhaps be improved with some more careful analysis).

19. Raphael Steiner

Problem 24. *Is it true that there is a constant $c > 0$ such that for every integer $t \geq 1$ there is a (not necessarily induced) C_5-free graph of treewidth at most t and chromatic number at least ct? Same question for $C_7, C_9, C_{11}, \ldots$*

Background. In a 2022 Oberwolfach workshop, Matija Bucić presented the following problem and attributed it to Bucić–Fox–Sudakov:

> *For which graphs H does there exist a constant $\varepsilon > 0$ (depending solely on H) such that all (not necessarily induced) H-free graphs with no K_t-minor have chromatic number $O(t^{1-\varepsilon})$?*

Kühn and Osthus [12] proved that this holds for all bipartite graphs H, and it follows from a result of Dvořák and Kawarabayashi [7] that the above does not hold when H contains a triangle, even when strengthening the condition "K_t-minor-free" to "treewidth at most $t-2$." The posed problem is thus an attempt to understand whether their lower bound construction (based on online colorings) may extend from triangle-free graphs to graphs that exclude a fixed odd cycle.

The following was asked by Erdős in 1987: Is it true that for every $k \geq 4$ and every integer $r \geq 1$, there is a k-vertex-critical graph G with $A \subseteq E(G)$ such that $|A| \leq r$ and $\chi(G \setminus A) = k$?

Problem 25. *Is it true for $r = 2$?*

Dirac conjectured this for $r = 1$; and Jensen in 2004 proved this for all $k \geq 5$.

20. Jérémie Turcotte

Problem 26. *Does there exist a linklessly embeddable graph with girth at least five and minimum degree at least four?*

Problem 27. *Does every connected induced P_5-free graph contains $a \neq b, c$ such that $N[a] \subseteq N[b] \cup N[c]$?*

Background. These questions are motivated by various similar results in recent years and by some questions on the cop number of graphs.

21. Alexandra Wesolek

Problem 28. *For which H, is the crossing number problem NP-hard on H-minor-free graphs?*

Background. For $H = K_t$ with $t \geq 6$, Mohar and Isabella proved this is true; and this is false for $H = K_4$. What about $H = K_5$? It is also known that this is true for $H = K_{4,t}$ for $t \geq 4$, but false for $H = K_{2,3}$. Would be interesting to decide the $H = K_{2,4}$ case.

22. Liana Yepremyan

The Turán number $\mathrm{ex}(n, H)$ is the maximum number of edges in an n-vertex graph with no H subgraph. Keevash–Mubayi–Sudakov in 2008 introduced the rainbow Turán number $\mathrm{ex}^*(n, H)$, which is the maximum number of edges in an n-vertex graph G with edges properly colored (any number of colors) such that G has no rainbow (not necessarily induced) copy of H. Obviously $\mathrm{ex}^*(n, H) \geq \mathrm{ex}(n, H)$. It is known that if H is non-bipartite then $\mathrm{ex}^*(n, H) = (1 + o(1))\,\mathrm{ex}(n, H)$.

Problem 29. *Is it true that $\mathrm{ex}^*(n, H) = O(\mathrm{ex}(n, H))$ for all bipartite H?*

Yanzer in 2020 proved that $\mathrm{ex}^*(n, C_{2k}) = O(n^{1+1/k})$ for all k. Also $\mathrm{ex}^*(n, \text{rainbow cycles}) = O(n(\log n)^4)$ by Yanzer, which was improved by Tomon to $O(n(\log n)^{2+o(1)})$, and then very recently $O(n \log n \log \log n)$ by Alon, Sauermann,.... It is also known that $\mathrm{ex}^*(n, \text{rainbow cycles}) = \Omega(n \log n)$ which comes from a special coloring of the hypercube.

Bukh and Conlon proved in 2015 that for every rational $r \in (1, 2)$ there is a family (of graphs $\mathcal{F}$ such that $\mathrm{ex}(n, \mathcal{F}) = \Theta(n^r)$.

Problem 30. *Is it true that for every rational $r \in (1, 2)$ there is a family of graphs $\mathcal{F}$ such that $\mathrm{ex}^*(n, \mathcal{F}) = \Theta(n^r)$?*

Problem 31. *Is it true that for every bipartite H, if $\mathrm{ex}^*(n, H) = O(n^{1+\alpha})$ then $\mathrm{ex}^*(n, \mathcal{F}) = O(n^{1+\alpha/2})$ where $\mathcal{F}$ is the class of all 1-subdivisions of H? Easy for $\mathrm{ex}(n, \mathcal{F})$.*

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