# OPEN PROBLEMS FOR THE SECOND 2022 BARBADOS GRAPH THEORY WORKSHOP 

MAINTAINED BY TUNG H. NGUYEN

## 1. Bogdan Alecu

Let $k, n \in \mathbb{N}$. Call a graph a perfectly matched $k$-partite graph if it is a $k$-partite graph with stable sets $A_{1}, \ldots, A_{k}$ such that $\left|A_{i}\right|=n$ for all $i$, and the graph induced by $A_{i} \cup A_{j}$ for all $i \neq j$ is a perfect matching. We note that perfectly matched $k$-partite graphs are not a hereditary class (and in fact, any graph is induced in some large perfectly matched $k$-partite graph).

Problem 1. Let us suppose that we are asked to construct a perfectly matched $k$-partite graph for some large $k$, and we are given control of (that is, we can make the size of the bags very large if we so wish) and of the matchings we put between bags. Subject to this, what kinds of constraints can we hope to satisfy when constructing our graph?

For instance, with a bit of work, we can always ensure that for any fixed $g$, our perfectly matched $k$-partite graph has girth at least $g$ (the proof I have uses really large bags though - something like $(g!)^{k^{g}}$; is it possible to do better?). But what about other constraints? When making $k$ large, can we for instance always arrange to have "small" treewidth? What about other parameters? What about avoiding various induced subgraphs other than small cycles?

I am afraid the background for this problem isn't particularly glamorous - I didn't find it anywhere, I just came up with it and looked at it for a few weeks during my thesis. But perfectly matched $k$-partite graphs seem very natural, and I was surprised not to find any literature on them (maybe I was googling the wrong things). If you have seen them anywhere else, please let me know! (Also, any questions about them other than the ones I've suggested would be interesting.)

An earlier version of the problem asked for bounded treewidth; but this is not possible since the degeneracy is not always bounded.

## Progress.

- (Freddie Illingworth and Raphael Steiner) The bags can be chosen to be of size $O\left((k g \log (\mathrm{~kg}))^{g}\right)$.


## 2. Marthe Bonamy

Problem 2. Is it true that for all large enough 5-connected planar graphs $G, \alpha(G)>|G| / 4$ ?
Problem 3. Does there exist $\varepsilon>0$ such that $\alpha(G) \geq(1 / 4+\varepsilon)|G|$ for all large enough 5 -connected planar graphs $G$ ?

These two problems ask for "large enough" because there is a graph $G$ on 24 vertices with $\alpha(G)=|G| / 4$, and there is another graph on 28 vertices with the same property. Another problem is:
Problem 4. Are there only a finite number of minimal planar graphs $G$ with $\alpha(G)=|G| / 4$ ?
Eleven such graphs have been found and these could be all.

## 3. Édouard Bonnet

There is a simple proof that triangle-free graphs of twin-width at most $d$ are $(d+2)$-colourable. But how large can the chromatic number of a triangle-free graph of twin-width $d$ be?

The $d+2$ bound is tight when $d=0$, since $K_{2}$ is a triangle-free 2 -chromatic graph of twin-width 0 . This bound is however not tight when $d=1$, since every triangle-free graph of twin-width 1 is bipartite. (Indeed, it can be observed that any odd cycle that is not a triangle has twin-width exactly 2 .)

For every non-negative integer $d$, let

$$
f(d)=\max \{\chi(G): G \text { is a triangle-free graph of twin-width } d\}
$$

[^0](or if you prefer "of twin-width at most $d$ "), where $\chi(G)$ denotes the chromatic number of $G$.
Problem 5. Is $f(d)=\Omega(d)$ ? More generally, the question is to obtain bounds on $f(d)$ refining $f(d)=$ $\Omega(\log d)$ (given, for instance, by the Mycielski graphs) and $f(d) \leqslant d+2$.

## Progress.

- (Édouard Bonnet, Romain Bourneuf, Julien Duron, Colin Geniet, Stéphan Thomassé, Nicolas Trotignon) It turns out that $f(d) \geq d+1$.


## 4. Matija Bucić

Problem 6. Is it true that for every $a>0$ and every $G$ with $\alpha(G) \geq a|G|=a n$, there is a vertex subset of size o(n) hitting all maximum stable sets?

This is an old problem of Bollobás, Erdős and Tuza from the 90 's. Alon constructed a sequence $\left(G_{n}\right)_{n \geq 1}$ of graphs such that for each $n \geq 1,\left|G_{n}\right|=n, \alpha\left(G_{n}\right) \geq n / 4$, but $G_{n}$ has no vertex subset of size less than $\sqrt{n} / 2$ that hits all maximum stable sets.

## 5. Linda Сook

Let $(T, \beta)$ be a rooted tree-decomposition of $G . \quad(T, \beta)$ is squirrel-friendly if there is a function $\alpha: V(T) \rightarrow V(G)$ such that $\alpha(t) \in \beta(t) \backslash \beta(\operatorname{parent}(t))$ for all $t \in T$ (here the parent of the root is void) and $\alpha(t) \alpha(\operatorname{parent}(t)) \in E(G)$ for all $t \neq r$.

Problem 7. Does there exist a function $f: \mathbb{N} \rightarrow \mathbb{N}$ such that for every graph $G$, there is a squirrel friendly tree-decomposition with width bounded from above by $f(\operatorname{tw}(G))$ ?

## Progress.

- (Pablo Blanco, Linda Cook, Meike Hatzel, Claire Hilaire, Freddie Illingworth, Rose McCarty) The answer is negative. The counterexamples also answer another question of Dvořák in the negative.
Let $\mathcal{F}=\left\{C_{3}, C_{4}, \ldots\right\}$ be the class of all cycles. Kim and Kwon proved that $\left\{C_{4}, \ldots\right\}$ has the induced version of the Erdős-Posá property, but any $\mathcal{H} \subseteq \mathcal{F}$ with $C_{3}, C_{4} \notin \mathcal{H}$ doesn't have this property. Kwon and Huynh proved that $\left\{C_{4}\right\} \cup\left\{C_{6}, \ldots\right\}$ has this property.

Problem 8. Are there any other subclasses of $\mathcal{F}$ with the induced Erdös-Posá property?

## 6. James Davies

Problem 9. Is it true that for every set $S$ of algebraically independent positive real numbers, the graph on vertex set $\mathbb{R}^{2}$ with edge set consisting of the pairs of distance in $S$ has bounded chromatic number?

This problem was first posed by Bukh in 2008. The problem is known to be true if $S \subseteq[a, b]$ for any $0<a<b<\infty$, and false if we require each colour class to be measurable. Another problem posed by Soifer in 2010 is

Problem 10. Is Problem 9 true if $S=\left\{2^{n}: n \in \mathbb{N}_{0}\right\}$ ?

## 7. Matt DeVos

Problem 11. For a vertex-transitive and connected graph $G$ on $n$ vertices, what is the size of the longest (not necessarily induced) cycle in $G$ ?

This is an old question. Note that there are several vertex-transitive graphs (Petersen for example) which are not Hamiltonian. Babai in 1979 proved that such a graph $G$ has a cycle of length at least $\sqrt{3 n}$ for all $n \geq 3$; and he conjectured that one cannot do better than $\varepsilon n$ for some $\varepsilon>0$. It is known that $c n^{3 / 5}$ is doable. Can one do better than this?

## 8. Jim Geelen

Given a simple graph $G$ we define a Cayley graph $C_{G}$ whose vertex set consists of all subsets of $V(G)$ and two vertices are adjacent if the symmetric difference of the sets is an edge of $G$.
Problem 12. Are the chromatic numbers of $G$ and $C_{G}$ qualitatively related?
This question was implicitly posed by François Jaeger in the 1980s. This much we know:
(i) $G$ is bipartite if and only if $C_{G}$ is bipartite;
(ii) $\chi\left(C_{G}\right) \leq 2^{\left[\log _{2}(\chi(G))\right\rceil}<2 \chi(G)$ (proved by Jaeger, but not published); and
(iii) $\chi\left(C_{G}\right) \neq 3$ (proved by Payan).

This is a special case of a question about the critical number of binary matroids. Both (i) and (ii) are easy in the context of matroids. The proof of (iii) is pretty and relies upon a nice generalization of the Mycielski construction.

## 9. Sepehr Hajebi

Aravind et al. in 2021 conjectured that there is some $c>0$ such that for all triangle-free $G$, there is an induced cycle (or path) of length at least $c \chi(G) \log \chi(G)$. A more doable question seems to be:

Problem 13. What is the length of a longest induced path in the kth Mycielskian?
It's not hard to show that one can get $\Omega(k)$. What about a superlinear bound in $k$ ?
Another conjecture of Aravind in the 2010's says that every properly coloured triangle-free graph $G$ contains a rainbow induced path on $\chi(G)$ vertices. Scott and Seymour in 2017 proved that there is a function $f$ such that for all $t$ and for all $G$ with $\chi(G)>f(t, \omega(G))$, every proper colouring of $G$ contains a rainbow induced path on $t$ vertices.

Yet another conjecture of Aravind et al. says that every properly coloured graph with chromatic number $\chi$ and clique number $\omega$ contains a rainbow induced path on $\chi^{\frac{1}{\omega-1}}$; and they proved in the same paper that every such graph contains a rainbow stable set of size at least $\left\lceil\chi^{\frac{1}{\omega-1}} / 2\right\rceil$. It is not known that one could get a rainbow induced matching of size at least $\left\lfloor\left(\chi^{\frac{1}{\omega-1}}+1\right) / 3\right\rfloor$.

Problem 14. Is it true that for every $t \geq 1$, every properly coloured graph $G$ contains a rainbow induced copy of $r P_{t}$ with $r=\left\lfloor\left(\chi^{\frac{1}{\omega-1}}+1\right) /(t+1)\right\rfloor$ ?
Problem 15. For all $t \geq 1$ there exists $w=w(t)$ such that every graph $G$ with treewidth at least $w$ contains either $K_{t}$ or $K_{t, t}$ as an induced subgraph or a 2-degenerate induced subgraph of treewidth $t$.

## 10. Kevin Hendrey

A linear cycle in a hypergraph is a sequence $v_{1}, h_{1}, v_{2}, h_{2}, \ldots, v_{t}, h_{t}$ of distinct vertices $v_{i}$ and hyperedges $h_{i}$ with $t \geq 3$ such that each hyperedge $h_{i}$ intersects the union of the other hyperedges of the linear cycle in exactly the set $\left\{v_{i}, v_{i+1}\right\}$ (or $\left\{v_{t}, v_{1}\right\}$ if $i=t$ ).
Problem 16. Given a 3 -uniform hypergraph $H$, can we determine in polynomial time whether $H$ contains a linear cycle?

For $k \geq 4$, the $k$-uniform version of this problem is NP-hard (there is a straightforward reduction to 3 -SAT). For graphs, every cycle is a linear cycle, and cycles can be detected in linear time.

## 11. Claire Hilaire

A graph $U$ is minor-universal for a family of graphs $\mathcal{F}$ if every graph of $\mathcal{F}$ is a minor of $U$.
Problem 17. What is the number of vertices of a smallest planar graph that is minor-universal for the family of all planar graphs on at most $n$ vertices?

Bodini in 2002 showed that the smallest size of a minor-universal tree for the family of $n$-vertex trees is at least $\Omega(n \log n)$ and at most $O\left(n^{1.984 \ldots}\right)$. A celebrated result of Robertson, Seymour, and Thomas in 1984 shows that the $2 n \times 2 n$ grid is minor-universal for the family of $n$-vertex planar graphs; is there a better example with $o\left(n^{2}\right)$ vertices?

## 12. Freddie Illingworth

A hypergraph is linear if $|e \cap f| \leq 1$ for every pair of distinct edges $e, f$.
Problem 18. Is there a linear 3-uniform hypergraph $H$ such that $R\left(H, R_{t}^{(3)}\right)$ is superpolynomial in $t$ ? Even the case when $H$ is a Fano plane is not known.

In 2010, Bohman, Frieze, and Mubayi conjectured that there are linear 3-uniform $H$ for which $R(H$, $\left.K_{t}^{(3)}\right)$ is superpolynomial. In fact they made the equivalent conjecture that there is a linear 3 -uniform $H$ and $H$-free hypergraphs with chromatic number greater than $k$ and $k^{3+o(1)}$ edges. At a 2015 AIM workshop, Conlon also asked the same problem.

However, for higher uniformities, there do exist linear $H$ with $R\left(H, R_{t}^{(3)}\right)$ exponential in $t$. Here's a 4-uniform example. Start with a single vertex $G_{0}$ and iterate the following procedure: $G_{i+1}$ is obtained by take two copies of $G_{i}$ and adding all edges that have exactly two vertices in each of the copies. Then $G_{n}$ has $2^{n}$ vertices and independence number $n+1$. Let $H$ be a Steiner quadruple system on an odd number of vertices (there exist for any integer that is $1 \bmod 12$ ). Then $G_{n}$ is $H$-free for all $n$. Indeed, let $n$ be minimal with $G_{n}$ containing $H$. By minimality, the copy of $H$ muse have a vertex in both copies of $G_{n-1}$. Fix a vertex $v$ in the left copy of $G_{n-1}$ and consider its link: every edge has an even number of vertices in the right copy of $G_{n-1}$. But its link covers all vertices of $V(H) \backslash\{v\}$ exactly once so $H$ has an even number of vertices in the right copy of $G_{n-1}$. Similarly for the left copy and so $H$ has an even number of vertices, a contradiction. Hence $R\left(H, K_{n+2}^{(4)}\right) \geq 2^{n}$.

## 13. Peter Nelson

Let $R(s, t)$ be the minimum $n$ so that for every red/blue colouring of $\mathbb{F}_{2}^{n} \backslash\{\mathbf{0}\}$ contains a red $s$ dimensional subspace or a blue $t$-dimensional subspace. Thus, $R(2, t)$ is the smallest number of colours such that either there is a red triple $\{x, y, x+y\}$ red or $t$-dimensional subspace in blue.

Problem 19. Does there exist some $C>0$ such that $R(2, t) \leq t^{C}$ for all $t$ ?
The best known bound is currently $R(2, t) \leq e^{C t}$. Evidence from the triangle-free process shows that $R(2, t)$ could be linear in $t$. Another formulation: $A \subseteq \mathbb{F}_{2}^{n} \backslash\{\mathbf{0}\}$ implies $\omega\left(A^{c}\right) \geq n^{\varepsilon}$ for some $\varepsilon>0$. Observe that if $A$ is maximal such that $0 \notin A+A+A$ then $A+A=A^{c}$; so yet another formulation: is it true that for all $A \subseteq \mathbb{F}_{2}^{n} \backslash\{\mathbf{0}\}$, either $A^{c}$ or $A+A$ contains a subspace of dimension at least $n^{c} ?$

## Progress.

- (Cosmin Pohoata and Zach Hunter) It turns out that $R(2, t) \leq C t^{7}$ for some universal constant $C>0$; the proof builds on recent work of Kelley and Meka on the 3-AP problem.


## 14. Tung Nguyen

Problem 20. Is it true that for every integer $k \geq 1$, every graph of chromatic number at least $3 k$ contains a $k$-connected subgraph of chromatic number at least $k$ ?

This problem has a fairly long history. Thomassen in 1983 claimed to have a proof that this is true; but the proof has an error and remains unfixed. Alon, Kleitman, Saks, Seymour, and Thomassen in 1987 proved that chromatic number $O\left(k^{3}\right)$ suffices, and Chudnovsky, Penev, Scott, and Trotignon in 2013 pushed this down to $O\left(k^{2}\right)$. Girão and Narayanan in 2022 showed that $7 k$ works, a linear bound which is crucial in recent progress on linear Hadwiger's conjecture. The best known upper bound is $\left(3+\frac{1}{16}\right) k$. At the other extreme, a construction of Alon et al. gives a lower bound of $2 k$.

## 15. SERGEY NORIN

Problem 21. Optimize the edge density of a minor on $t$ vertices in any graph with chromatic number at least $t$.

Hadwiger's conjecture implies that one could get $100 \%$. It is not hard to obtain $25 \%$. What about average/minimum degree instead of chromatic number?

## Progress.

- (Kevin Hendrey, Sergey Norin, Raphael Steiner, Jeremie Turcotte) For average degree, $41 \%$ is obtainable and one cannot do better than $75 \%$.


## 16. Paul Seymour

Let $G$ be a connected graph. A cut of $G$ means the set of edges between $X$ and $V(G) \backslash X$, for some $X \subseteq V(G)$ with $X \neq \emptyset, V(G)$. For an integer $t \geq 1, \nu_{t}$ denotes the largest $k$ such that there is a list $D_{1}, \ldots, D_{k}$ (possibly with repetition) with the property that for every edge $e, e \in D_{i}$ for at most $t$ values of $i \in\{1, \ldots, k\}$.
Problem 22. Is it always true that $\nu_{4}$ is even? That $2 \nu_{2}=\nu_{4}$ ? That $2 \nu_{2}=\lim _{t \rightarrow \infty} \nu_{t} / t$ ?

## Progress.

- (Linda Cook and Sophie Spirkl) This is false; consider the following counterexample. Let $D_{1}, \ldots, D_{k}$
be the cuts of $G$. Then Paul's question is (equivalently) asking about the following LP: $\max \sum_{i=1}^{k} x_{i}$ subject to $\sum_{D_{i} \ni e} x_{i} \leq 1$ for all $e \in E(G)$ and $x_{i} \geq 0$ for all $i$.

Its dual is: $\min \sum_{e \in E(G)} y_{e}$ subject to $\sum_{e \in D_{i}} y_{e} \geq 1$ for all $i$, and $y_{e} \geq 0$ for all $e$.
Consider the following graph: A pyramid, with seven vertices (apex $a$, base $b_{1}, b_{2}, b_{3}$, and each path with one internal vertex, say $\left.c_{1}, c_{2}, c_{3}\right)$.

There is a primal solution given by:

- For all $i$, the cut induced by $\left\{c_{i}\right\}$ has weight $3 / 4$;
- For all $i$, the cut induced by $\left\{b_{i}\right\}$ has weight $1 / 4$;
- For all $i$, the cut induced by $\left\{c_{i}, b_{i}\right\}$ has weight $1 / 4$.

A matching dual solution gives weight $1 / 4$ to all edges of the form $b_{i} b_{j}$, and weight $1 / 2$ to all other edges.

This shows: $\nu_{2} / 2 \neq \nu_{4} / 4$, since $\nu_{2} / 2$ is half-integral and $\nu_{4} / 4$ is 3.75 as shown by the above solutions. Also, $\nu_{4}=15$. It's still possible that $\nu_{4} / 4$ is always equal to the LP value.

## 17. Vaidy Sivaraman

Problem 23. Let $\mathcal{F}$ be a class of graphs that is closed under induced subgraphs and disjoint unions, such that every forbidden induced subgraph for $\mathcal{F}$ has at most $c$ vertices. Let $\mathcal{F}^{\prime}$ be the class of graphs containing a vertex whose removal results in a graph in $\mathcal{F}$. Then every forbidden induced subgraph for $\mathcal{F}^{\prime}$ has at most $2 c$ vertices.

## Progress.

- (Kevin Hendrey) $\frac{1}{4} c^{2}+O(c)$ is actually the correct answer.

Problem 24. For every tree $T$, the class of $\left\{T, \overline{P_{5}}\right\}$-free graphs is polynomially $\chi$-bounded.
This problem implies the Gyárfás-Sumner conjecture for triangle-free graphs, which is still open. Scott, Seymour, and Spirkl proved a weakening that the class of $\left\{T, C_{4}\right\}$-free graphs is polynomially $\chi$-bounded.

## 18. Maya Stein

An oriented graph is a graph all whose edges have been given a direction. The semidegree of a vertex $v$ is $d^{0}(v):=\min \left\{d^{-}(v), d^{+}(v)\right\}$, and the minimum semidegree of $D$ is $\delta^{0}(D):=\min \left\{d^{0}(v): v \in V(D)\right\}$.

Considering for a moment undirected graphs, we know that a greedy embedding argument shows that a graph of minimum degree $\geq k$ contains each path of length $k$ a a subgraph. Dirac's theorem shows that a minimum degree of $n / 2$ is enough to find a Hamilton cycle in an $n$-vertex graph. With a similar proof, one can show that every connected graph on at least $k$ vertices and of minimum degree at least $k / 2$ contains a path of length $k$.

Do these results extend to oriented graphs, if we use the semidegree notion? Again, using a greedy embedding argument, it is clear that any oriented graph with $\delta^{0}(D) \geq k$ must contain each oriented $k$-edge path. Thomassen asked in the 1970's for minimum semidegree conditions that ensure a directed Hamilton cycle, and more recently, other orientations of Hamilton cycles have been considered. In particular, for oriented graphs, Kelly, Kühn, and Osthus proved that a minimum semidegree of $3 n / 8+o(n)$ is enough to ensure any orientation of a Hamilton cycle, and in fact, any orientation of any cycle of length between 3 and $n$.

But how about finding shorter paths $P_{k}$ of any orientation with a semidegree condition that is close to the order of the path (but lower than the trivial bound $\left.\delta^{0}(D) \geq k\right)$ ? More precisely, in analogy to the situation for undirected graphs, we ask:

Problem 25. Does every oriented graph $D$ with $\delta^{0}(D)>\frac{k}{2}$ contain each oriented $k$-edge path?

This would be best possible as one can see by considering a regular tournament with $k$ vertices (if $k$ is even). For antidirected paths one couls also consider the blow-up of a directed triangle (Antidirected paths are oriented paths where every vertex has either in-degree 0 or out-degree 0 .)

For directed paths, the answer to Problem 25 is yes: this was shown by Jackson in 1981. For antidirected paths, Stein and Zárate-Guerén showed that the assertion holds approximately in large dense graphs: for all $\eta \in(0,1)$ there is $n_{0}$ such that for all $n \geq n_{0}$ and $k \geq \eta n$ every oriented graph $D$ on $n$ vertices with $\delta^{0}(D)>(1+\eta) \frac{k}{2}$ contains every antidirected path with $k$ edges. Also for antidirected paths, a short proof by Klimošová and Stein shows that if we replace the semidegree condition with $\delta^{0}(D) \geq 3 k / 4$, then we can guarantee an antipath of length $k$ (this result could perhaps be improved with some more careful analysis).

## 19. Raphael Steiner

In a 2022 Oberwolfach workshop, Matija Bucić presented the following problem and attributed it to Bucić, Fox, and Sudakov:

For which graphs $H$ does there exist a constant $\varepsilon>0$ (depending solely on $H$ ) such that all (not necessarily induced) $H$-free graphs with no $K_{t}$-minor have chromatic number $O\left(t^{1-\varepsilon}\right)$ ?
Kühn and Osthus proved that this holds for all bipartite graphs $H$, and it follows from a result of Dvořák and Kawarabayashi that the above does not hold when $H$ contains a triangle, even when strengthening the condition " $K_{t}$-minor-free" to "treewidth at most $t-2$." The following problem is an attempt to understand whether their lower bound construction (based on online colourings) may extend from triangle-free graphs to graphs that exclude a fixed odd cycle.

Problem 26. Is it true that there is a constant $c>0$ such that for every integer $t \geq 1$ there is a (not necessarily induced) $C_{5}$-free graph of treewidth at most $t$ and chromatic number at least ct? Same question for $C_{7}, C_{9}, C_{11}, \ldots$

The following was asked by Erdős in 1989:

> Is it true that for every $k \geq 4$ and every integer $r \geq 1$, there is a $k$-vertex-critical graph $G$ with $A \subseteq E(G)$ such that $|A| \leq r$ and $\chi(G \backslash A)=k$ ?

Problem 27. Is this problem true for $r=2$ ?
Dirac conjectured this for $r=1$; and Jensen in 2002 proved this for all $k \geq 5$.

## 20. Jérémie Turcotte

Problem 28. Does there exist a linklessly embeddable graph with girth at least five and minimum degree at least four?

Problem 29. Does every connected induced $P_{5}$-free graph contain vertices $a \neq b, c$ with $N[a] \subseteq N[b] \cup$ $N[c]$ ?

These questions are motivated by various similar results in recent years and by some questions on the cop number of graphs.

## Progress.

- (Maria Chudnovsky, Sergey Norin, Jeremie Turcotte, Paul Seymour) Positive answer to Problem 29.


## 21. Alexandra Wesolek

Problem 30. For which $H$ is the crossing number problem NP-hard on H-minor-free graphs?
The answer is (trivially) negative when $H \in\left\{K_{4}, K_{2,3}\right\}$; and Cabello and Mohar showed that the answer is positive when $H=K_{t}$ with $t \geq 6$ and when $H=K_{3, s}$ with $s \geq 4$. What if $H \in\left\{K_{5}, K_{2,4}\right\}$ ?

## 22. Liana Yepremyan

The Turán number ex $(n, H)$ is the maximum number of edges in an $n$-vertex graph with no $H$ subgraph. Keevash, Mubayi, Sudakov, and Verstraëte in 2008 introduced the rainbow Turán number ex* $(n, H)$, which is the maximum number of edges in an $n$-vertex graph $G$ with edges properly coloured (any number of colours) such that $G$ has no rainbow (not necessarily induced) copy of $H$. Obviously ex* $n$, $H) \geq \operatorname{ex}(n, H)$. It is known that if $H$ is non-bipartite then $\operatorname{ex}^{*}(n, H)=(1+o(1)) \operatorname{ex}(n, H)$.
Problem 31. Is it true that $\operatorname{ex}^{*}(n, H)=O(\operatorname{ex}(n, H))$ for all bipartite $H$ ?
Janzer in 2020 proved that $\operatorname{ex}^{*}\left(n, C_{2 k}\right)=O\left(n^{1+1 / k}\right)$ for all $k$. He also proved ex* $(n$, rainbow cycles $)=$ $O\left(n(\log n)^{4}\right)$, which was improved by Tomon to $O\left(n(\log n)^{2+o(1)}\right)$, and then to $O(n \log n \log \log n)$ by Alon, Bucić, Sauermann, Zakharov, and Zamir. It is also known that ex* $(n$, rainbow cycles $)=\Omega(n \log n)$ which comes from a special colouring of the hypercube.

Bukh and Conlon proved in 2015 that for every rational $r \in(1,2)$ there is a family (of graphs $\mathcal{F}$ such that $\operatorname{ex}(n, \mathcal{F})=\Theta\left(n^{r}\right)$.

Problem 32. Is it true that for every rational $r \in(1,2)$ there is a family of graphs $\mathcal{F}$ such that $\operatorname{ex}^{*}(n$, $\mathcal{F})=\Theta\left(n^{r}\right)$ ?
Problem 33. Is it true that for every bipartite $H$, if $\operatorname{ex}^{*}(n, H)=O\left(n^{1+\alpha}\right)$ then $\operatorname{ex}^{*}(n, \mathcal{F})=O\left(n^{1+\alpha / 2}\right)$ where $\mathcal{F}$ is the class of all 1-subdivisions of $H$ ? Easy for $\operatorname{ex}(n, \mathcal{F})$.


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