Quantum bumpless pipe dreams

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Contents

Schubert Calculus

Geometric problem

Count the number of points of intersection of some subvarieties of some varieties X.

Algebra

Cohomology of X.

Polynomial representative

Family of polynomial that encodes the geometric question we care about.

Introduction

Cohomology	Polynomials
$H_T^*(\mathrm{Fl}_n)$	Double Schubert polynomials
$QH_T^*(\mathrm{Fl}_n)$	Quantum double Schubert polynomials

- Double Schubert polynomials has a combinatorial formula called bumpless pipe dreams.
- Quantum bumpless pipe dreams: combinatorial formula for quantum Schubert polynomials.

Divided difference operators

Definition

Consider the action of S_n on $\mathbb{Z}[x,y] := \mathbb{Z}[x_1,\dots,x_n,y_1,\dots,y_n]$ by permuting the y variables. In particular, s_i acts by swapping y_i and y_{i+1}:

$$s_i f(x, y_1, \dots, y_i, y_{i+1}, \dots, y_n) = f(x, y_1, \dots, y_{i+1}, y_i, \dots, y_n)$$

Definition

We define the *divided difference operator* ∂_i^y on $\mathbb{Z}[x,y]$:

$$\partial_i^y f = \frac{f - s_i f}{y_i - y_{i+1}}$$

Divided difference operators

Definition

A reduced word for $w \in S_n$ is an expression $w = s_{a_1} \cdots s_{a_k}$ using exactly $\ell(w)$ s_i's.

Definition

Let $w \in S_n$, and let $w = s_{a_1} s_{a_2} \cdots s_{a_k}$ be a reduced word for w. The divided difference operator ∂_w^y is defined as follows.

$$\partial_w^y = \partial_{a_1}^y \partial_{a_2}^y \cdots \partial_{a_k}^y$$

Remark

The definition above does not depends on the choice of reduced word since the ∂_i^y operator satifies the relations $\partial_i^y \partial_j^y = \partial_j^y \partial_i^y$ for $|i - j| > 1$ and the braid relations $\partial_i^y \partial_{i+1}^y \partial_i^y = \partial_{i+1}^y \partial_i^y \partial_{i+1}^y$.

Double Schubert polynomials

Definition (Double Schubert polynomials)

For the longest permutation $w_0 = n, n-1, \ldots, 1$,

$$\mathfrak{S}_{w_0}(x, y) = \prod_{i+j \leq n} (x_i - y_j)$$

and for other permutations

$$\mathfrak{S}_w(x, y) = (-1)^{\ell(ww_0)} \partial^y_{ww_0} \mathfrak{S}_{w_0}(x, y)$$

Example

$$\mathfrak{S}_{213}(x, y) = \partial^y_2 \partial^y_1 \mathfrak{S}_{321}(x, y)$$

Monk's rule for double Schubert polynomial

Theorem

For any k and any permutation w,

$$\mathfrak{S}_{s_k}(x,y)\mathfrak{S}_w(x,y) = \sum_{\substack{a \le k < b, \\ \ell(wt_{ab}) = \ell(w)+1}} \mathfrak{S}_{wt_{ab}}(x,y) + \sum_{i=1}^{k} (y_{w(i)} - y_i)\mathfrak{S}_w(x,y)$$

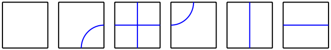
Example

$$\mathfrak{S}_{s_2}(x,y)\mathfrak{S}_{213}(x,y) = \mathfrak{S}_{312}(x,y) + \mathfrak{S}_{231}(x,y)$$

Bumpless pipe dreams

Definition

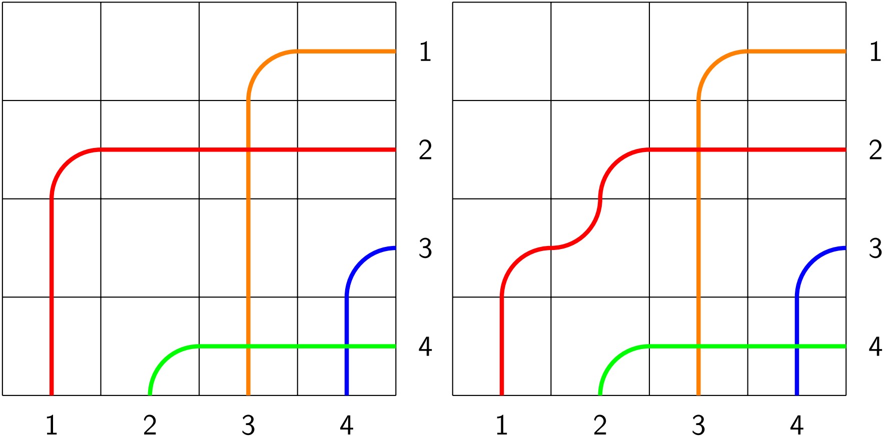
A bumpless pipe dream (BPD) is a tiling of $n \times n$ grid filled with tiles



so that

- The tiling forms n pipes;
- Each pipe starts horizontally at the right edge of the grid and ends vertically at the bottom edge of the grid;
- No two pipes cross more than once (reduced.)

Bumpless pipe dreams



Bumpless pipe dreams for 3142

Bumpless pipe dreams formula

Definition

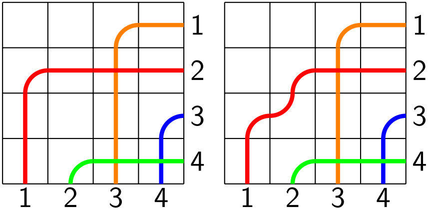
The binomial weight of a bumpless pipe dream P, denoted $\text{bwt}(P)$ is

$$\text{bwt}(P) := \prod_{P(i,j)=\square} (x_i - y_j)$$

Theorem (Lam, Lee, Shimozono '21)

The double Schubert polynomial indexed by $w \in S_n$ is the sum of binomial weights of all BPDs associated to w:

$$\mathfrak{S}_w(x, y) = \sum_{P \in \text{BPD}(w)} \text{bwt}(P)\,.$$



Bumpless pipe dreams for 3142

Example

$$\mathfrak{S}_{3142}(x,y) = (x_1 - y_1)(x_1 - y_2)(x_3 - y_2) + (x_1 - y_1)(x_1 - y_2)(x_2 - y_1)$$

Quantum double Schubert polynomials

- Quantum Schubert polynomials were introduced by Fomin, Gelfand, and Postnikov.
- Polynomial representative for $QH_T^*(\mathrm{Fl}_n)$
- Lies in $\mathbb{Z}[x, y, q]$, where x, y variables are of degree 1 and q variables are of degree 2.
- Not monomially positive.

Quantum double Schubert polynomials

Definition (Fomin, Gelfand, Postnikov '97)

Define $E_i^k(x_1, \ldots, x_k)$ to be the coefficient of λ^i in the characteristic polynomial $\det(I + \lambda G_k)$ where

$$G_k = \begin{bmatrix} x_1 & q_1 & 0 & \ldots & 0 \\ -1 & x_2 & q_2 & \ldots & 0 \\ 0 & -1 & x_3 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & x_k \end{bmatrix}.$$

Quantum double Schubert polynomials

Definition (Ciocan-Fontanine–Fulton–Pragacz '98)

The quantum double Schubert polynomials $\mathfrak{S}^q_w$ are defined as follows. For the longest permutation w_0, we have

$$\mathfrak{S}^q_{w_0}(x, y) := \prod_{k=1}^{n-1} E^k_k(x_1 - y_{n-k}, \ldots, x_k - y_{n-k}),$$

and, for any permutation w, we have

$$\mathfrak{S}^q_w(x, y) = (-1)^{\ell(ww_0)} \partial^y_{ww_0} \mathfrak{S}^q_{w_0}(x, y).$$

Remark

Specializing the q variables to 0 yields the double Schubert polynomials.

Monk's rule for quantum double Schubert polynomials

Theorem

For any k and any permutation w,

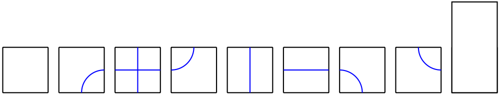
$$\mathfrak{S}^q_{s_k}(x,y)\mathfrak{S}^q_w(x,y) = \sum_{\substack{a \le k < b,\\ \ell(wt_{ab}) = \ell(w)+1}} \mathfrak{S}^q_{wt_{ab}}(x,y) + \sum_{\substack{c \le k < d,\\ \ell(wt_{cd}) = \ell(w) - \ell(t_{cd})}} q_{cd}\mathfrak{S}^q_{wt_{cd}}(x,y) + \sum_{i=1}^{k}(y_{w(i)} - y_i)\mathfrak{S}^q_w(x,y),$$

where $q_{cd} := q_c q_{c+1} \cdots q_{d-1}$.

Quantum bumpless pipe dreams

Definition (L., Ouyang, Restivo, Tao, Zhang '25)

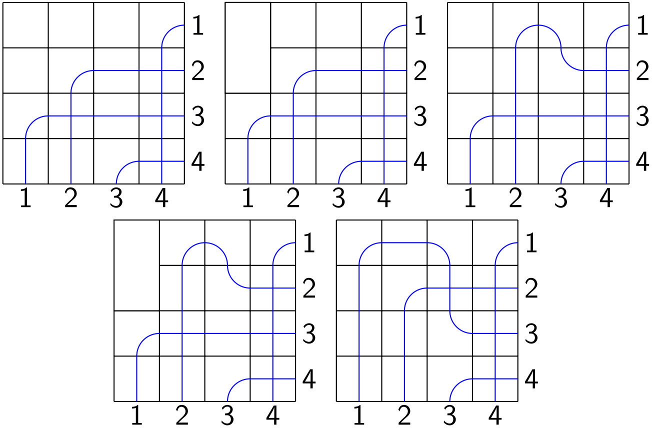
A *quantum bumpless pipe dream* (QBPD) is a tiling of an $n \times n$ grid filled with tiles



so that

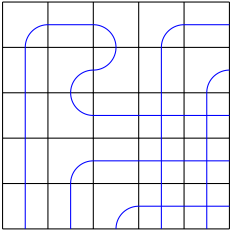
- The tiling forms n pipes;
- Each pipe starts horizontally at the right edge of the grid and ends vertically at the bottom edge of the grid;
- The pipes only move upward, downward, or leftward (but not rightward) when moving from the right edge to the bottom edge;
- No two pipes cross more than once (reduced.)

Examples



QPBDs for 4213

Non-example



A non-example of a QBPD

Binomial weights of QBPDs

Definition

The *binomial weight* for a QBPD P, denoted $\mathrm{bwt}(P)$, is the product of factors contributed by the following rules:

- An empty tile on row i, column j contributes $x_i - y_j$;
- A domino tile whose upper cell is on row i contributes q_i;
- A cross tile on row i where the vertical pipe moves upwards contributes q_i;
- A southwest elbow on row i contributes $-q_i$;
- A vertical tile on row i where the pipe moves upward contributes $-q_i$.

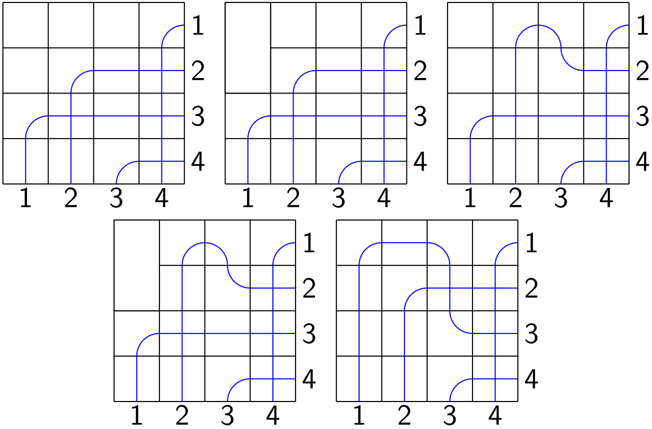
QBPDs formula

Theorem (L., Ouyang, Restivo, Tao, Zhang '25)

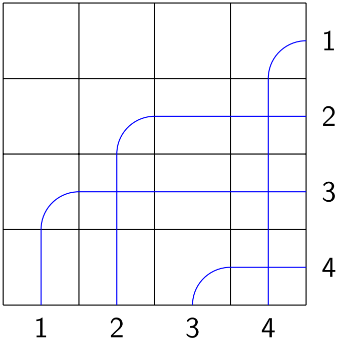
The quantum double Schubert polynomial indexed by $w \in S_n$ is the sum of binomial weights of all QBPDs associated to w:

$$\mathfrak{S}_w^q(x,y) = \sum_{P \in \mathrm{QBPD}(w)} \mathrm{bwt}(P)\,.$$

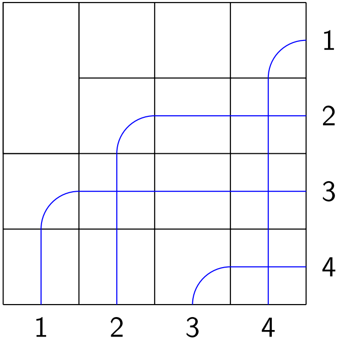
Examples



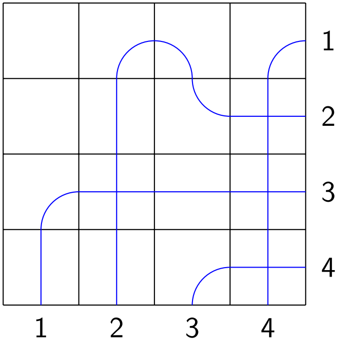
QPBDs for 4213



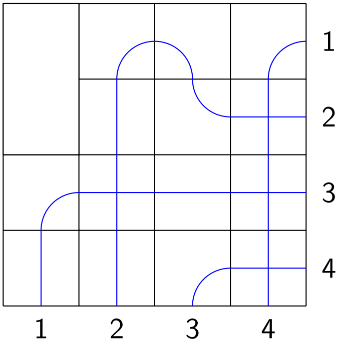
$$(x_1 - y_1)(x_1 - y_2)(x_1 - y_3)(x_2 - y_1)$$



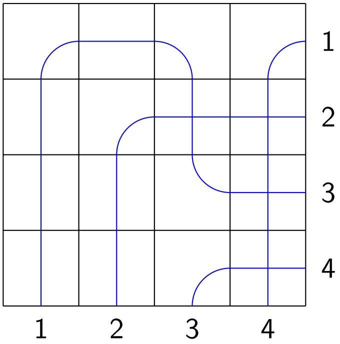
$$q_1(x_1 - y_2)(x_1 - y_3)$$



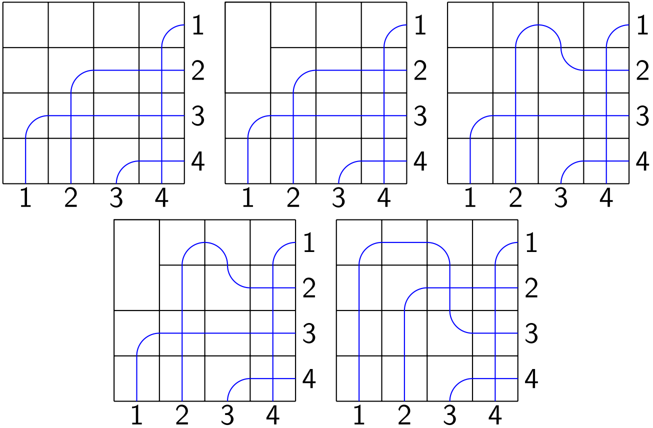
$$(x_1 - y_1)(x_2 - y_1)(-q_1)$$



$$q_1(-q_1)$$



$$(-q_1)q_2$$



Example

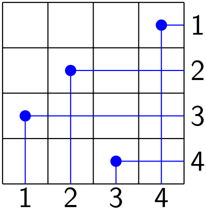
$$\mathfrak{S}^q_{4213}(x,y) = (x_1-y_1)(x_1-y_2)(x_1-y_3)(x_2-y_1) + q_1(x_1-y_2)(x_1-y_3) + (x_1-y_1)(x_2-y_1)(-q_1) + q_1(-q_1) + (-q_1)q_2\,.$$

Proving the QBPDs formula

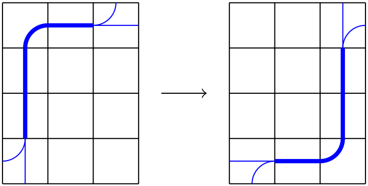
Preliminaries

- Rothe diagram and droop moves
- Lift moves
- Stability: quantum double Schubert polynomial are stable under the inclusion $S_n \to S_{n+1}$.
- A defining transition equation

Rothe diagram and droop moves



Rothe diagram for 4213



A droop move

Proposition (Lam, Lee, Shimozono '21)

All bumpless pipe dreams for a given permutation can be generated from the Rothe diagram using a sequence of droop moves.

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What about QBPDs?

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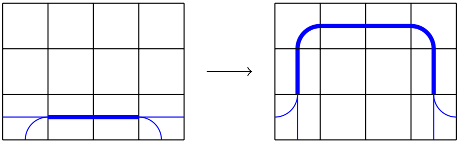
What about QBPDs?



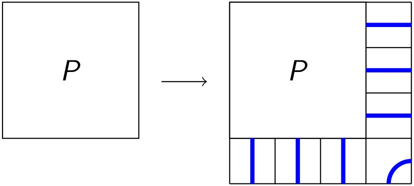
A lift move

Proposition (L., Ouyang, Restivo, Tao, Zhang '25)

All domino-free quantum bumpless pipe dreams for a given permutation can be generated from the Rothe diagram using a sequence of droop and lift moves.

Stability

Given $w \in S_n \subseteq S_{n+1}$ and a QBPD P of w:



Extending a $n \times n$ QBPD to a $(n+1) \times (n+1)$ QBPD

Reverse is possible for $w \in S_{n+1}$ with $w(n+1) = n+1$.

A defining transition equation

Proposition

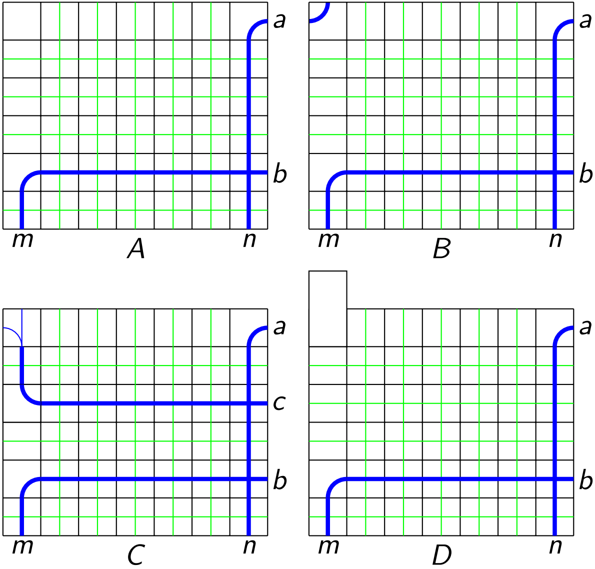
Let $\pi \in S_\infty, \pi \neq \mathrm{id}$, and let n be the largest number such that $\pi(n) \neq n$. Let $a = \pi^{-1}(n)$ and $b > a$ be such that

$$\pi(b) = \max_{n \geq i > a} \pi(i) =: m,$$

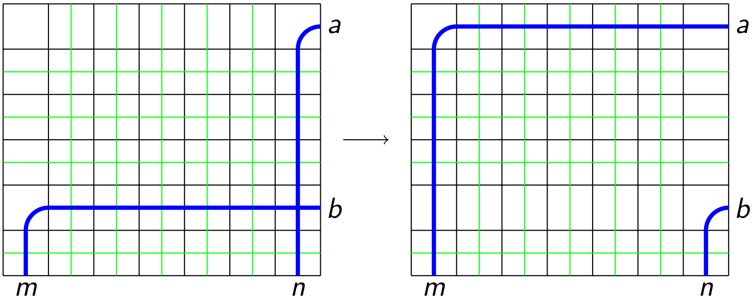
Let $\sigma = \pi t_{ab}$. Then

$$\mathfrak{S}^q_\pi(x,y) = (x_a - y_{\sigma(a)})\mathfrak{S}^q_\sigma(x,y) + \sum_{\substack{c<a,\\ \ell(\sigma t_{ca}) = \ell(\sigma)+1}} \mathfrak{S}^q_{\sigma t_{ca}}(x,y)$$

$$- \sum_{\substack{a<c,\\ \ell(\sigma t_{ac}) = \ell(\sigma) - \ell(t_{ca})}} q_{ac}\mathfrak{S}^q_{\sigma t_{ac}}(x,y) + \sum_{\substack{c<a,\\ \ell(\sigma t_{ca}) = \ell(\sigma) - \ell(t_{ca})}} q_{ca}\mathfrak{S}^q_{\sigma t_{ca}}(x,y).$$

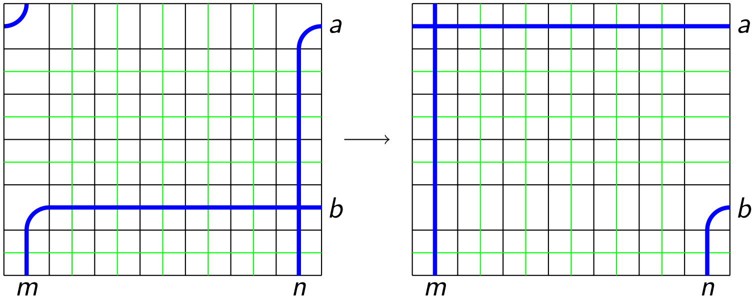


Configurations for QBPDs of π. Recall: $\pi(a) = n, \pi(b) = m$.



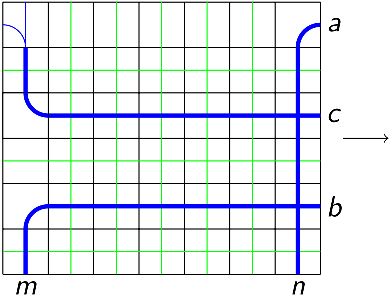
The bijection $\phi_A \colon \mathrm{QBPD}(\pi, A) \to \mathrm{QBPD}(\sigma)$

Recall: $\sigma = \pi t_{ab}$, $n = \pi(a) = \sigma(b), m = \pi(b) = \sigma(a)$.



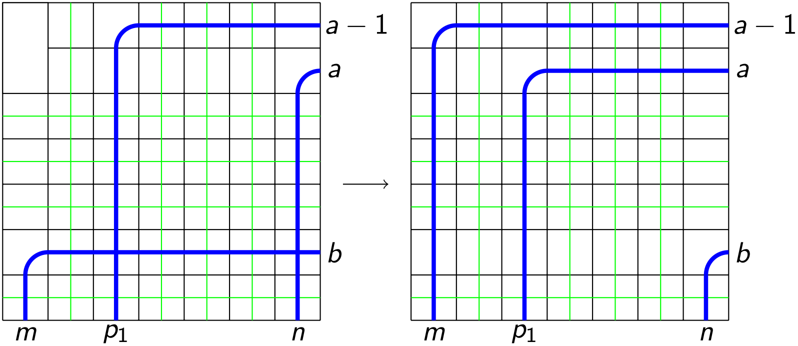
The bijection $\phi_B\colon \operatorname{QBPD}(\pi, B) \to \bigcup_{\substack{c<a \\ \ell(\sigma t_{ca})=\ell(\sigma)+1}} \operatorname{QBPD}(\sigma t_{ca})$

Recall: $\sigma = \pi t_{ab}$, $n = \pi(a) = \sigma(b), m = \pi(b) = \sigma(a)$.



The bijection $\phi_C\colon \ \mathrm{QBPD}(\pi, C) \to \bigcup_{\substack{c>a \\ \ell(\sigma t_{ac})=\ell(\sigma)-\ell(t_{ac})}} \mathrm{QBPD}(\sigma t_{ac})$

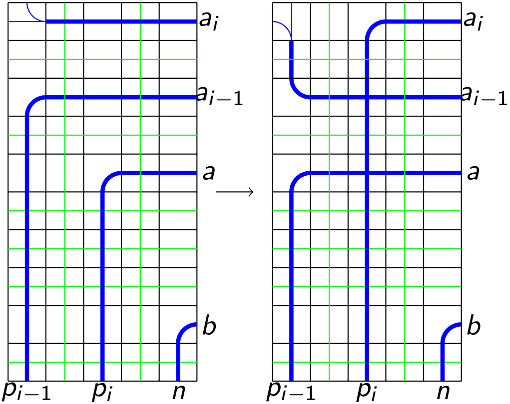
Recall: $\sigma = \pi t_{ab},\ n = \pi(a) = \sigma(b), m = \pi(b) = \sigma(a).$



The bijection $\phi_D\colon \text{QBPD}(\pi, D) \to \text{QBPD}(\sigma t_{a-1,a}, X_1)$

Recall: Last term is sum over $c < a, \ell(\sigma t_{ca}) = \ell(\sigma) - \ell(t_{ca})$.

Remaining terms are shown to cancel outs via another series of bijections.



The bijection $\phi_i\colon \mathrm{QBPD}(\sigma t_{a_i a}, X_i) \to \mathrm{QBPD}(\sigma t_{a_{i-1} a}, Y_{i-1})$

Thank you for listening!