Singularities of generic hypersurface projections

**Problem 1.** Let $X \subseteq \mathbb{P}^r$ be a smooth surface.

(a) Adapt the proof that curves are birational to a nodal plane curve to show that there is a generic linear projection $\pi: \mathbb{P}^r \dashrightarrow \mathbb{P}^5$ such that $\pi|_X$ is an isomorphism onto its image.

(b) If one projects to $\mathbb{P}^4$ or $\mathbb{P}^3$, what singularities do you think the image of $X$ could have?

*Remark.* A theorem of Severi says that the only non-degenerate smooth surface in $\mathbb{P}^5$ that projects to a smooth surface in $\mathbb{P}^4$ is the Veronese surface $\nu_2(\mathbb{P}^2) \subseteq \mathbb{P}^5$. 
For the next problem, we use the following:

**Fedder’s Criterion.** Let $k$ be a field of characteristic $p > 0$, and consider $f \in R = k[x_1, x_2, \ldots, x_n]$. Then, $R/(f)$ is $F$-pure if and only if

$$f^{p-1} \not\in (x_1^p, x_2^p, \ldots, x_n^p).$$

In particular, if $f^{p-1} \not\in (x_1^p, x_2^p, \ldots, x_n^p)$, then $R/(f)$ is weakly normal.

**Problem 2.** Let $k$ be a field of characteristic $p > 0$. Using Fedder’s criterion, prove the following rings are $F$-pure, and hence weakly normal:

(a) $k[x, y, z]/(xyz)$.

(b) $k[x, y, z]/(x^2 - yz^2)$, if char $k \neq 2$.

(c) $k[s, t, u, x, y, z]/(stux^2 - stuzy^2)$, if char $k \neq 2$. 