Lie Algebra Structure on Hochschild Cohomology

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This talk is organized in the following way

- MOTIVATION
- HOCHSCHILD COHOMOLOGY
- QUIVER & KOSZUL ALGEBRAS
- HOMOTOPY LIFTING MAPS
- EXAMPLES & APPLICATIONS

Motivation

Let k be a field of characteristic 0.

Defnition: A differential graded Lie algebra (DGLA) over k is a graded vector space $L = \bigoplus_{i \in I} L^i$ with a bilinear map $[\cdot, \cdot] : L^i \otimes L^j \to L^{i+j}$ and a differential $d : L^i \to L^{i+1}$ such that

- bracket is anticommutative i.e. $[x, y] = -(-1)^{|x||y|}[y, x]$
- bracket satisfies the Jacobi identity i.e.

$$(-1)^{|x||z|}[x,[y,z]]+(-1)^{|y||x|}[y,[z,x]]+(-1)^{|z||y|}[z,[x,y]] = 0$$

- bracket satisfies the Liebniz rule i.e. $d[x, y] = [d(x), y] + (-1)^{|x|}[x, d(y)]$

Examples

1 Every Lie algebra is a DGLA concentrated in degree 0.

2 Let $A = \bigoplus_i A^i$ be an associative graded-commutative k-algebra i.e. $ab = (-1)^{|a||b|}ba$ for a, b homogeneous and $L = \bigoplus_i L^i$ a DGLA. Then $L \otimes_k A$ has a natural structure of DGLA by setting:

$$(L \otimes_k A)^n = \bigoplus_i (L^i \otimes_k A^{n-i}),\ d(x \otimes a) = d(x) \otimes a,$$

$[x \otimes a, y \otimes b] = (-1)^{|a||y|}[x, y] \otimes ab.$

3 Space of Hochschild cochains $C^*(\Lambda, M)$ of an algebra Λ is a DGLA where $[\cdot, \cdot]$ is the Gerstenhaber bracket, and M a Λ-bimodule.

Deformation philosophy

Over a field of characteristic 0,
it is well known that every deformation problem is governed by a differential graded Lie algebra (DGLA) via solutions of the **Maurer-Cartan equation** modulo gauge action.[6]

$$\{\textit{Deformation problem}\} \rightsquigarrow \{\textit{DGLA}\} \rightsquigarrow \{\textit{Deformation functor}\}$$

The first arrow is saying that the DGLA you obtain depends on the data from the deformation problem and the second arrow is saying for DGLAs that are quasi-isomorphic, we obtain an isomorphism of deformation functor.

Definition: An element x of a DGLA is said to satisfy the Maurer-Cartan equation if

$$d(x) + \frac{1}{2}[x, x] = 0.$$

Hochschild cohomology

Hochschild cohomology

Let $\mathbb{B} = \mathbb{B}_\bullet(\Lambda)$ denote the bar resolution of Λ.

$\Lambda^e = \Lambda \otimes \Lambda^{op}$ the enveloping algebra of Λ.

$$\mathbb{B} : \cdots \rightarrow \Lambda^{\otimes(n+2)} \xrightarrow{\delta_n} \Lambda^{\otimes(n+1)} \rightarrow \cdots \xrightarrow{\delta_2} \Lambda^{\otimes 3} \xrightarrow{\delta_1} \Lambda^{\otimes 2} (\xrightarrow{\pi} \Lambda)$$

The differentials δ_n's are given by

$$\delta_n(a_0 \otimes a_1 \otimes \cdots \otimes a_{n+1}) = \sum_{i=0}^{n} (-1)^i a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{n+1}$$

for each elements $a_i \in \Lambda$ $(0 \leq i \leq n+1)$ and π, the multplication map.

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for each elements $a_i \in \Lambda$ $(0 \leq i \leq n+1)$ and π, the multplication map.Let M be a left Λ^e-module. The Hochschild cohomology of Λ with coefficients in M is defined as

$$HH^*(\Lambda, M) = C^*(\Lambda, M) = \bigoplus_{n \geq 0} H^n(Hom_{\Lambda^e}(\mathbb{B}_\bullet(\Lambda), M))$$

If $M = \Lambda$, we write $HH^*(\Lambda)$.

Multiplicative structures on $HH^*(\Lambda)$

- Cup product

$$\smile : HH^m(\Lambda) \times HH^n(\Lambda) \to HH^{m+n}(\Lambda)$$

$$\alpha \smile \beta(a_1 \otimes \cdots \otimes a_{m+n}) = (-1)^{mn} \alpha(a_1 \otimes \cdots \otimes a_m) \beta(a_{m+1} \otimes \cdots \otimes a_{m+n})$$

- Gerstenhaber bracket of degree -1.

$$[\cdot, \cdot] : HH^m(\Lambda) \times HH^n(\Lambda) \to HH^{m+n-1}(\Lambda)$$

defined originally on the bar resolution by $[\alpha, \beta] = \alpha \circ \beta - (-1)^{(m-1)(n-1)} \beta \circ \alpha$ where

where $\alpha \circ \beta = \sum_{j=1}^{m} (-1)^{(n-1)(j-1)} \alpha \circ_j \beta$ with

$$(\alpha \circ_j \beta)(a_1 \otimes \cdots \otimes a_{m+n-1}) = \alpha(a_1 \otimes \cdots \otimes a_{j-1} \otimes \beta(a_j \otimes \cdots \otimes a_{j+n-1}) \otimes a_{j+n} \otimes \cdots \otimes a_{m+n-1}). \quad (1)$$

Make sense of Equation (1) without using $\mathbb{B}$

- Hochschild cohomology as the Lie algebra of the derived Picard group (B. Keller) - 2004
- Brackets via contracting homotopy using certain resolutions (C. Negron and S. Witherspoon) - 2014
  $[\alpha, \beta] = \alpha \circ_\phi \beta - (-1)^{(m-1)(n-1)} \beta \circ_\phi \alpha$
- Completely determine $[HH^1(A), HH^m(A)]$ using derivation operators on any resolution $\mathbb{P}$. (M. Suárez-Álvarez) - 2016
  $[\alpha^1, \beta] = \alpha^1 \beta - \beta \tilde{\alpha_m}$ where $\tilde{\alpha_m} : \mathbb{P}_m \to \mathbb{P}_m$.
- Completely determine $[HH^*(A), HH^*(A)]$ using homotopy lifting on any resolution. (Y. Volkov) - 2016
  $[\alpha, \beta] = \alpha \psi_\beta - (-1)^{(m-1)(n-1)} \beta \psi_\alpha$

Quiver algebras and Koszul algebras

Quiver algebras

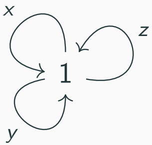
A quiver is a directed graph where loops and multiple arrows between vertices are allowed. It is often denoted by $Q = (Q_0, Q_1, o, t)$, where Q_0 is the set of vertices, Q_1 set of arrows and $o, t : Q_1 \to Q_0$ taking every path $a \in Q$ to its origin vertex $o(a)$ and terminal vertex $t(a)$.

Define kQ to be the vector k-vector space having the set of all paths as its basis. If p and q are two paths, we say pq is possible if $t(p) = o(q)$ otherwise, $pq = 0$. By this, kQ becomes an associative algebra. Let kQ_i be a vector subspace spanned by all paths of length i, then kQ is graded.

$$kQ = \bigoplus_{n \geq 0} kQ_n$$

Examples of quiver algebras

- Let Q be the quiver with a vertex 1 (with a trivial path e_1 of length 0). Then $kQ \cong k$.
- Let Q be the quiver with two vertices and a path: $1 \xrightarrow{\alpha} 2$. There are two trivial paths e_1 and e_2 associated with the vertices $1, 2$. There is a relation $e_1\alpha = e_1\alpha e_2 = \alpha e_2$. Define a map $kQ \to \mathbb{M}_2(k)$, by $e_1 \mapsto \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, $e_2 \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $\alpha \mapsto \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. Then $kQ \cong \{A \in \mathbb{M}_2(k) : A_{12} = 0\}$.
- Let Q be the quiver with a vertex and 3 paths x, y, z.



Then $kQ \cong k\langle x, y, z\rangle$.

Koszul algebras

A relation on Q is a k-linear combination of paths of length $n \geq 2$ having same origin and terminal vertex. Let I be the subspace spanned by some relations, we denote by (Q, I) a quiver with relations and kQ/I the quiver algebra associated to (Q, I).
We are interested in quiver algebras that are Koszul. Let $\Lambda = kQ/I$ be Koszul:

- Λ is quadratic. This means that I is a homogenous admissible ideal of kQ_2
- Λ admits a grading $\Lambda = \bigoplus_{i \geq 0} \Lambda_i$, Λ_0 is isomorphic to k or copies of k and has a minimal graded free resolution.

A canonical construction of a projective resolution for Koszul quiver algebras

Let $\mathbb{L} \to \Lambda_0$ be a minimal projective resolution of Λ_0 as a right Λ-module, $\mathbb{L}$

- contains all the necessary information needed to construct a minimal projective resolution of Λ_0 as a left Λ-module
- contains all the necessary information to construct a minimal projective resolution of Λ over the enveloping algebra Λ^e.

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- contains all the necessary information to construct a minimal projective resolution of Λ over the enveloping algebra Λ^e.
- There exist integers $\{t_n\}_{n\geq 0}$ and elements $\{f_i^n\}_{i=0}^{t_n}$ in $R = kQ$ such that $\mathbb{L}$ can be given in terms of a filtration of right ideals

$$\cdots \subseteq \bigoplus_{i=0}^{t_n} f_i^n R \subseteq \bigoplus_{i=0}^{t_{n-1}} f_i^{n-1} R \subseteq \cdots \subseteq \bigoplus_{i=0}^{t_0} f_i^0 R = R$$

- The f_i^n can be choosen so that they satisfy a comultiplicative structure.

A result of E.L. Green, G. Hartman, E. Marcos, Ø. Solberg [2]

Theorem 1

Let $\Lambda = kQ/I$ be a Koszul algebra. Then for each r, with $0 \leq r \leq n$, and i, with $0 \leq i \leq t_n$, there exist elements $c_{pq}(n, i, r)$ in k such that for all $n \geq 1$,

$$f_i^n = \sum_{p=0}^{t_r} \sum_{q=0}^{t_{n-r}} c_{pq}(n, i, r) f_p^r f_q^{n-r} \quad (\textbf{comultiplicative structure})$$

Theorem 2

Let $\Lambda = kQ/I$ be a Koszul algebra. The resolution $(\mathbb{K}, d)$ is a minimal projective resolution of Λ with Λ^e-modules

$$\mathbb{K}_n = \bigoplus_{i=0}^{t_n} \Lambda o(f_i^n) \otimes_k t(f_i^n) \Lambda$$

with each $\mathbb{K}_n$ having free basis elements $\{\varepsilon_i^n\}_{i=0}^{t_n}$ and they are given explicitly by $\varepsilon_i^n = (0, \ldots, 0, o(f_i^n) \otimes_k t(f_i^n), 0, \ldots, 0)$.

Homotopy lifting maps

Making sense of Equation (1) using homotopy lifting

Definition

Let $\mathbb{K} \xrightarrow{\mu} \Lambda$ be a projective resolution of Λ as Λ^e-module. Let $\Delta : \mathbb{K} \to \mathbb{K} \otimes_\Lambda \mathbb{K}$ be a chain map lifting the identity map on Λ and $\eta \in \mathrm{Hom}_{\Lambda^e}(\mathbb{K}_n, \Lambda)$ a cocycle. A module homomorphism $\psi_\eta : \mathbb{K} \to \mathbb{K}[1-n]$ is called a **homotopy lifting** map of η with respect to Δ if

$$d\psi_\eta - (-1)^{n-1}\psi_\eta d = (\eta \otimes 1 - 1 \otimes \eta)\Delta \qquad \text{and} \tag{2}$$

$$\mu\psi_\eta \quad \text{is cohomologous to} \quad (-1)^{n-1}\eta\psi \tag{3}$$

for some $\psi : \mathbb{K} \to \mathbb{K}[1]$ for which $d\psi - \psi d = (\mu \otimes 1 - 1 \otimes \mu)\Delta$.

Remark.

For Koszul algebras Equation (3) holds.

Theorem [5, a slight variation presented by Y. Volkov]
Let $\mathbb{K} \to \Lambda$ be a projective resolution of Λ^e-modules. Suppose that $\alpha \in \mathrm{Hom}_{\Lambda^e}(\mathbb{K}_n, \Lambda)$ and $\beta \in \mathrm{Hom}_{\Lambda^e}(\mathbb{K}_m, \Lambda)$ are cocycles representing elements in $\mathrm{HH}^*(\Lambda)$, ψ_α and ψ_β are homotopy liftings of α and β respectively, then the bracket $[\alpha, \beta] \in \mathrm{Hom}_{\Lambda^e}(\mathbb{K}_{n+m-1}, \Lambda)$ on Hochschild cohomology can be expressed as

$$[\alpha, \beta] = \alpha\psi_\beta - (-1)^{(m-1)(n-1)}\beta\psi_\alpha$$

at the chain level.
Remark: The above formula is given as reformulated by S. Witherspoon in her new book [3].

Homotopy lifting, comultiplicative structure, and $\mathbb{K}$

Notation

If $\theta : \mathbb{K}_n \to \Lambda$ is defined by $\varepsilon_0^n \mapsto \lambda_0, \varepsilon_1^n \mapsto \lambda_1$ and so on until $\varepsilon_{t_n}^n \mapsto \lambda_{t_n}$, we write $\theta = \sum_i \theta^i$

$$\theta = \begin{pmatrix} \lambda_0^{(0)} & \cdots & \lambda_i^{(i)} & \cdots & \lambda_{t_n}^{(t_n)} \end{pmatrix}, \quad \theta^i = \begin{pmatrix} 0 & \cdots & \lambda_i^{(i)} & \cdots & 0 \end{pmatrix}$$

Theorem 3 [7, T.Oke]

Let $\Lambda = kQ/I$ and $\mathbb{K}$ be the projective resolution of Theorem 2. Let $\eta : \mathbb{K}_n \to \Lambda$ be a cocycle such that $\eta = \begin{pmatrix} 0 & \cdots & 0 & (f_w^1)^{(i)} & 0 & \cdots & 0 \end{pmatrix}$, for some f_w^1 path of length 1. There are scalars $b_{m,r}(m-n+1, s)$ in k for which the map $\psi_\eta : \mathbb{K}_m \to \mathbb{K}_{m-n+1}$, defined by

$$\psi_\eta(\varepsilon_r^m) = b_{m,r}(m-n+1, s)\varepsilon_s^{m-n+1}$$

is a homotopy lifting map for η, with the scalars satisfying some equations.

Theorem 4 [7, T.Oke]
Let $\Lambda = kQ/I$ and $\mathbb{K}$ be the projective resolution of Theorem 2.
Let $\eta : \mathbb{K}_n \to \Lambda$ be a cocycle such that
$\eta = \begin{pmatrix} 0 & \cdots & 0 & (f^2_w)^{(i)} & 0 & \cdots & 0 \end{pmatrix}$, for some $f^2_w = f^1_{w_1} f^1_{w_2}$ path of length 2. There are scalars $b_{m,r}(m-n+1, s)$ in k for which the map $\psi_\eta : \mathbb{K}_m \to \mathbb{K}_{m-n+1}$, defined by

$$\psi_\eta(\varepsilon^m_r) = b_{m,r}(m-n+1, s+1) f^1_{w_1} \varepsilon^{m-n+1}_{s+1} + b_{m,r}(m-n+1, s) \varepsilon^{m-n+1}_s f^1_{w_2}$$

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In Theorem 3 for instance, the scalars $b_{*,*}(*,*)$ satisfy

For all α,

(i). $B = \begin{cases} c_{i,\alpha}(m,r,1) & \text{when } p = w \\ 0 & \text{when } p \neq w \end{cases}$, and

(ii). $B' = \begin{cases} (-1)^{n(m-n)} c_{p,i}(m,r,m-n) & \text{when } p = w \\ 0 & \text{when } p \neq w \end{cases}$,

where

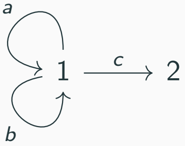
$$B = b_{m,r}(m-n+1,s)c_{p\alpha}(m-n+1,s,1) + (-1)^n b_{m-1,j}(m-n,\alpha)c_{p\alpha}(m-n+1,r,1),$$

$$B' = (-1)^{m+1}(-1)^n[b_{m,r}(m-n+1,s)c_{\alpha q}(m-n+1,s,m-n) + b_{m-1,j}(m-n,\alpha)c_{\alpha q}(m-n+1,r,m-n)].$$

Examples

Let Q be the quiver with two vertices and 3 paths a, b, c of length 1. Let $I_q = \langle a^2, b^2, ab - qba, ac \rangle$ be a family of ideal and take

$$\{\Lambda_q = kQ/I_q\}_{q \in k} \qquad Q :=$$



to be a family of quiver algebras.

- Let $\eta : \mathbb{K}_1 \to \Lambda_q$ defined by $\eta = \begin{pmatrix} a & 0 & 0 \end{pmatrix}$ be a degree 1 cocycle. Then for each n and r, $(\psi_\eta)_n(\varepsilon_r^n) = \begin{cases} (n-r)\varepsilon_r^n & \text{when } r = 0, 1, 2, \ldots, n \\ (n+1)\varepsilon_r^n & \text{when } r = n+1, \end{cases}$ are homotopy lifting maps associated to η.

- Let $\chi : \mathbb{K}_2 \to \Lambda_q$ defined by $\chi = \begin{pmatrix} 0 & 0 & ab & 0 \end{pmatrix}$ be a degree 2 cocycle $(\psi_\chi)_1(\varepsilon_i^1) = 0, \quad (\psi_\chi)_2(\varepsilon_i^2) = \begin{cases} 0 & \text{if } i = 0 \\ 0, & \text{if } i = 1 \\ a\varepsilon_1^1 + \varepsilon_0^1 b & \text{if } i = 2 \\ 0 & \text{if } i = 3 \end{cases}$,

$$(\psi_\chi)_3(\varepsilon_i^3) = \begin{cases} 0, & \text{if } i = 0 \\ 0, & \text{if } i = 1 \\ -a\varepsilon_1^2, & \text{if } i = 2 \\ \varepsilon_1^2 b, & \text{if } i = 3 \\ 0, & \text{if } i = 4 \end{cases}$$

are the first, second and third homotopy lifting maps associated to χ.

Applications

(1) Cup product and bracket structure

Theorem [R.O. Buchweitz, E. L. Green, N. Snashall, Ø. Solberg]
Let $\Lambda = kQ/I$ be a Koszul algebra. Suppose that $\eta : \mathbb{K}_n \to \Lambda$ and $\theta : \mathbb{K}_m \to \Lambda$ represent elements in $HH^*(\Lambda)$ and are given by $\eta(\varepsilon_i^n) = \lambda_i$ for $i = 0, 1, \ldots, t_n$ and $\theta(\varepsilon_i^m) = \lambda'_i$ for $i = 0, 1, \ldots, t_m$.
Then $\eta \smile \theta : \mathbb{K}_{n+m} \to \Lambda$ can be expressed as

$$(\eta \smile \theta)(\varepsilon_j^{n+m}) = \sum_{p=0}^{t_n} \sum_{q=0}^{t_m} c_{pq}(n+m, i, n)\lambda_p \lambda'_q,$$

Theorem [7, T. Oke]
Under the same hypothesis with each $\lambda_i, \lambda'_i = \beta_i$ paths of length 1, the r-th component of the bracket on the r-th basis element is

$$[\eta, \theta]^r(\varepsilon_r^{m+n-1}) = \sum_{i=0}^{t_n} \sum_{j=0}^{t_m} b_{m-n+1,r}(n, i)\lambda_i - (-1)^{(m-1)(n-1)}(b_{m-n+1,r}(m, j)\beta_j.$$

(2) Specify solutions to the Maurer-Cartan equation

The space of Hochschild cochains $C^*(\Lambda, \Lambda)$ is a DGLA with $\bar{d}[\alpha, \beta] = [\bar{d}(\alpha), \beta] + (-1)^{m-1}[\alpha, \bar{d}(\beta)]$ for all $\alpha \in \mathrm{HH}^m(\Lambda), \beta \in \mathrm{HH}^n(\Lambda)$ and $\bar{d}(\alpha) = (-1)^{m-1}\alpha\delta$.

Using these results, the Maurer-Cartan equation for an Hochschild 2-cocycle η is the following

$$(-1)^{2-1}\eta d = -\frac{1}{2}[\eta, \eta] = -\frac{1}{2}(\eta\psi_\eta + \eta\psi_\eta)$$

$$\eta d(\varepsilon_r^3) = \eta\psi_\eta(\varepsilon_r^3)$$

$$\eta\{\text{a k-linear combination of } f_p^1\varepsilon_s^2, \varepsilon_s^2 f_q^1\}_{p,q,s} = \eta\psi_\eta(\varepsilon_r^3)$$

If $\eta(\varepsilon_s^2) = f_w^1$, the left hand side is a linear combination of paths of length 2 but the right hand is a linear combination of paths of length 1. **This is a contradiction!**. There are solutions however if $\eta(\varepsilon_s^2) = f_w^2$ for some w.

Thanks for listening!

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