Specialization of Integral Closure of Ideals by General Elements

Based on joint work with Rachel Lynn

Lindsey Hill

Purdue University

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Basic Definitions

Definition

Let $I$ be an ideal of a ring $R$. An element $x \in R$ is integral over $I$ if it satisfies an equation of integral dependence of the form

$$x^n + a_1 x^{n-1} + \ldots + a_n = 0$$

with $a_i \in I^i$. The collection of all elements integral over $I$ is the integral closure of $I$, denoted $\overline{I}$.
Example of Integral Closure

Example

Let $R = k[x, y]$ and $I = (x^3, x^2y, y^3)$. Then $\overline{I} = (x^3, x^2y, xy^2, y^3)$.

- Fact: The integral closure of a monomial ideal is a monomial ideal.
- Notice that $xy^2$ satisfies $z^2 - (x^2y)(y^3) = 0$, so $(x, y)^3 \subset \overline{I}$.
- Any monomial integral over $I$ has degree at least 3, hence $\overline{I} \subset (x, y)^3$. 
Question

Given an integrally closed ideal, can we reduce the height and maintain integral closedness?
Let $R = k[x, y]$ and let $m = (x, y)$. Notice $m^2 = (x^2, xy, y^2)$ is integrally closed ideal of height two. Is $\frac{m^2}{(x^2)}$ an integrally closed ideal of $\frac{R}{(x^2)}$? The answer: No. Notice that $x$ satisfies an equation of integral dependence $z^2 = 0$ in $R/(x^2)$ and therefore, $x \in \frac{m^2}{(x^2)} \setminus \frac{m^2}{(x^2)}$. 
The generic element approach

Let $R$ be a Noetherian (local) ring and $I = (a_1, \ldots, a_n)$ an $R$-ideal. Let $T_1, \ldots, T_n$ be variables over $R$. Recall that $R[T_1, \ldots, T_n]$ and $R(T) = R[T_1, \ldots, T_n]_{m_R} R[T]$ are faithfully flat extensions of $R$. Then

- $\text{ht } I = \text{ht } IR[T] = \text{ht } IR(T)$
- $\bar{I}R[T] = \bar{IR}[T]$
- $\bar{I}R(T) = \bar{IR}(T)$

and $\alpha = a_1 T_1 + a_2 T_2 + \ldots + a_n T_n$ is a generic element of $IR[T]$ or $IR(T)$. 
A theorem of Itoh (1989)

Let \((R, m)\) be an analytically unramified, Cohen-Macaulay local ring of dimension \(d \geq 2\). Let \(I\) be a parameter ideal for \(R\). Assume that \(R/m\) is infinite. Then there exists a system of generators \(x_1, \ldots, x_d\) for \(I\) such that if we put \(x = \sum_i x_i T_i\) and \(I' = IR(T)\), where \(R(T) = R[T]_{m[T]}\) with \(T = (T_1, \ldots, T_d)\) \(d\) indeterminates, then

\[
\overline{I'/(x)} = \overline{I'/(x)}.
\]
A generalization by Hong-Ulrich (2014)

Let $R$ be a Noetherian, locally equidimensional, universally catenary ring such $R_{red}$ is locally analytically unramified. Let $I = (a_1, \ldots, a_n)$ be an $R$-ideal of height at least 2. Let $R' = R[T_1, \ldots, T_n]$ be a polynomial ring in the variables $T_1, \ldots, T_n$, $I' = IR'$, and $x = \sum_{i=1}^{n} T_i a_i$. Then

$$\frac{I'}{(x)} = \frac{I'}{(x)}.$$
Applications of Hong-Ulrich

1. Enables proofs by induction on the height of an integrally closed ideal.
2. Gives a quick proof of a result proved independently by Huneke and Itoh: Let $R$ be a Noetherian, locally equidimensional, universally catenary ring such that $R_{\text{red}}$ is locally analytically unramified. Let $I$ be a complete intersection $R$-ideal. Then $I^{n+1} \cap I^n = \overline{I}I^n$ for all $n \geq 0$. 
Specialization by general elements (–, Lynn)

Let \((R, m)\) be a local equidimensional excellent \(k\)-algebra, where \(k\) is a field of characteristic 0. Let \(I\) be an \(R\)-ideal of height at least 2 and let \(x\) be a general element of \(I\). Then \(\overline{I}/(x) = \overline{I}/(x)\).
Main Ingredients of the Proof

1. (Extended) Rees Algebras and Their Integral Closures
2. General Elements and Bertini’s Theorems
Let \( R \) be a ring, \( I \) an ideal of \( R \) and \( t \) a variable over \( R \). The Rees algebra of \( I \) is a subring of \( R[t] \) defined by

\[
R[I t] = \bigoplus_{n \geq 0} I^n t^n.
\]

The extended Rees algebra of \( I \) is the subring of \( R[t, t^{-1}] \) defined as

\[
R[I t, t^{-1}] = \bigoplus_{n \in \mathbb{Z}} I^n t^n
\]

with \( I^n = R \) for \( n \leq 0 \).
Let $R$ be a ring, $t$ a variable over $R$ and $I$ an ideal of $R$. Then

$$\overline{R[It]}^{R[t]} = R \oplus \overline{I}t \oplus \overline{I}^2t^2 \oplus \overline{I}^3t^3 \oplus \ldots$$

and

$$\overline{R[It, t^{-1}]}^{R[t, t^{-1}]} = \ldots \oplus Rt^{-2} \oplus Rt^{-1} \oplus R \oplus \overline{I}t \oplus \overline{I}^2t^2 \oplus \ldots$$
Bertini’s Theorems

Let \( I = (x_1, \ldots, x_n) \). Then a general element \( x_\alpha \) of \( I \) is
\[
x_\alpha = \sum_{i=1}^{n} \alpha_i x_i
\]
where \( \alpha = (\alpha_1, \ldots, \alpha_n) \) is in a Zariski open subset of \( k^n \).

**A theorem of Bertini**

Let \( A \) be a local excellent \( k \)-algebra over the field \( k \) of characteristic 0 and let \( x_1, \ldots, x_n \in m_A \). Let \( U \subseteq D(x_1, \ldots, x_n) \) be open, so that for \( p \in U \) the ring \( A_p \) satisfies Serre’s Conditions \((S_r)\) or \((R_s)\) respectively. For general \( \alpha \in k^n \) and \( p \in U \cap V(x_\alpha) \) the ring \((A/x_\alpha A)_p\) also satisfies the conditions \((S_r)\) or \((R_s)\).
Sketch of the proof

1. Reduce to the case where $R$ is a local normal domain.
2. Define:

\[ A = R[lt, t^{-1}] \]
\[ B = \frac{R}{(x)} \left[ \frac{I}{(x)} t, t^{-1} \right] \]
\[ \overline{A} = \frac{R[lt, t^{-1}]}{R[t, t^{-1}]} \]
\[ \overline{B} = \frac{R}{(x)} \left[ \frac{I}{(x)} t, t^{-1} \right] \]
\[ \overline{B} = \frac{R[(x)]}{(x)} \left[ \frac{I}{(x)} t, t^{-1} \right] \]
Sketch of the proof

3. Consider the natural map

\[ \varphi : \frac{\mathcal{A}}{xt\mathcal{A}} \to \mathcal{B}. \]

Notice that \( \left[ \frac{\mathcal{A}}{xt\mathcal{A}} \right]_1 = I/(x) \) and \( \left[ \mathcal{B} \right]_1 = I/(x) \). For this reason, it suffices to show that the \( C = \text{coker}(\varphi) \) vanishes in degree 1.

4. Define \( J = (lt, t^{-1})\mathcal{A} \). Show that for \( p \in \text{Spec}(\mathcal{A}) \setminus V(J\mathcal{A}) \), \( \varphi_p \) is an isomorphism. In the case where \( lt \not\in p \), we apply Bertini’s Theorem to \( \mathcal{A} \) to say \( (\mathcal{A}/xt\mathcal{A})_p \) is normal, and since the extension \( (\mathcal{A}/xt\mathcal{A})_p \to \mathcal{B}_p \) is integral, \( \varphi_p \) is an isomorphism.

5. Step 4 implies that \( C = H^0_j(C) \). From this, we have an embedding \( [C]_n \hookrightarrow [H^2_j(\mathcal{A})]_{n-1} \). We use a local cohomology vanishing theorem proved by Hong and Ulrich to say \( [C]_1 = 0 \).