When does a $p^1$-linear map exist?

Land acknowledgment: I work on the traditional territories of The Three Fires People:
The Ojibwe (keepers of faith)
The Odawa (keepers of trade)
The Bodéwadmi (keepers of the fire)
I am giving this talk on the land of the Kiikaapoi.

Based on joint work with Takumi Murayama '20, Karen Smith '18, 
& Takumi Murayama & Karen Smith in preparation.
Throughout $R$ is a noetherian domain of prime char. $p > 0$ and

$$K := \text{Frac}(R).$$

**Frobenius**: $F^e : R \to F^e_* R$ \[\text{[Detects singularities of } R]\]

Here $F^e_* R$ is $R$ as a ring but with $R$-mod structure given by

$$r \in R, \ x \in F^e_* R \implies r \cdot x = r^p x \ (\text{restriction of scalars})$$

A $p^e$-linear map is an $R$-linear map

$$\varphi : F^e_* R \to R.$$

**Example**: $R = \mathbb{F}_p[x, y]$, then $F_* R$ is a free $R$-mod with basis

$$x^i y^j, \quad 0 \leq i, j \leq p - 1.$$

$$\varphi : F_* R \to R$$

on the basis given by $\varphi(x^i y^j) = \begin{cases} 1 & \text{if } i = 0 = j \\ 0 & \text{otherwise} \end{cases}$.

This is a Frobenius splitting.
Why do we care about existence of nonzero $p^{-e}$-linear maps?

- Global variants of such maps on a variety $X$, especially splittings, imply $X$ satisfies Kodaira vanishing \[\text{[Mehta-Ramanathan]}\]
  
  Kodaira vanishing fails in general in char. $p$ (Raynaud)

- Used extensively in the theory of test ideals, a prime char. analogue of multiplier ideals
  
  \[\text{[Hochster, Huneke, Smith, Hara, Yoshida, Takagi, Watanabe, Lyubeznik, Aberbach, Enescu, Schwede, Blickle, Tucker, Sharp among others]}\]

- Used in the study of $F$-signature, and more recently, its non-local variant.
  
  \[\text{[Smith, Van den Bergh, Huneke, Leuschke, Tucker, Aberbach, Enescu, Yao, Singh, De Stefani, Polska among others]}\]

- Existence of "sufficiently many" such maps implies $R$ is Cohen-Macaulay \[\text{[Hochster-Huneke]}\]
  
  Strongly $F$-regular rings.
• If $K = \text{Frac}(R)$ satisfies $[K : K^p] < \infty$, then existence of a nonzero $p^{-1}$-linear map implies $R$ is excellent [Smith - D]

▷ Large class of rings that behave well under integral closures, completions, openness of regular and other loci.

▷ Deep thems such as Resolution of Singularities conjectured to hold for this class.

Question: When does $R$ have nonzero $p^{-e}$-linear maps?

Example/Exercise: If $F : R \to F_*R$ is finite, then nonzero $p^{-e}$-linear maps exist!

If $[K : K^p] < \infty$, then existence of a nonzero $p^{-e}$-linear maps $\Rightarrow$ Frobenius is finite. [Smith - D]
Above example and its converse give many examples of non-excellent rings with $\neq$ nonzero $p^e$-linear maps.

**Folklore**: If $R$ is “nice”, for example, if $R$ is excellent, then does $R$ admit nonzero $p^e$-linear maps?

**Theorem A [Murayama-D]**: For each integer $n > 0$, $\exists$

- excellent
- regular local
- Henselian

ring $R$ of Krull dim $n$ that does not admit any nonzero $p^e$-linear map.

Thus, $\exists$ excellent $F$-pure rings that are NOT $F$-split.

Answers a long-standing question of Hochster, also raised by others like Smith, Zhang, Schwede, Blickle etc.

Folklore question has positive answer for large class of excellent, but non-$F$-finite rings.

**Theorem B [Murayama-D]**: If $R$ is essentially of finite type over a complete local ring, then $R$ has nonzero $p^1$-linear maps.

Furthermore, for such $R$, $F$-pure $\Rightarrow$ $F$-split.
Open Question: Are there non-excellent local \( R \) that admit non-trivial \( \Phi^c \)-linear maps?

If we drop local hypothesis then can construct such examples (forthcoming work Murayama-D)

Thm A proof sketch Krull dim 1 : We use a construction from rigid analytic geometry.

A NA field \((k, \|\cdot\|)\) is a field equipped with

\[ \|\cdot\|: k \to \mathbb{R}_{>0} \]

satisfying

1. \( \|x\| = 0 \iff x = 0 \)
2. \( \|xy\| = \|x\| \|y\| \)
3. \( \|x+y\| \leq \max\{\|x\|, \|y\|\} \) (ultrametric \( \Delta \)-inequality)

\((k, \|\cdot\|)\) becomes a metric space via \( \|x-y\| \) and we assume \( k \) is complete with this metric.

For such \( k \) have the Tate algebra

\[ T_1(k) := \left\{ \sum_{i=0}^{\infty} a_i x^i \in k[[x]] : \|a_i\| \to 0 \text{ as } i \to \infty \right\}. \]

\( T_1(k) \) is regular (not local)
- excellent (Kiehl)
- Euclidean domain.
Murayama-D: For $(k, 11)$ of char $p > 0$, $T_i(k)$ has a nonzero $p^e$-linear map $\iff k$ has a nonzero continuous $p^e$-linear map.

Gabber/Blaszczyk (now Rzepka)-Kuhlmann: $\exists$ NA fields $k$ that do not admit continuous $p^1$-linear maps.

This uses non-Archimedean functional analysis.

To get local, Henselian counterexample you localize $T_i(k)$ at the max ideal $(x)$ and then Henselize, for a NA field $k$ given by Gabber/Rzepka-Kuhlmann.

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