We prove that the Hilbert-Kunz function of the ideal \((I,J)/R(I)\), where \(I\) is an \(m\)-primary ideal and \(J\) is a parameter ideal of a 1-dimensional local ring \((R,m)\), is a quasi-polynomial for large values.

For \(s \in \mathbb{N}\), we compute the Hilbert-Samuel function of the \(R\)-module \(I^{(s)}\) and obtain an explicit description of the generalized Hilbert-Kunz function of the ideal \((I,J)/R(I)\) when \(I\) is a parameter ideal in a Cohen-Macaulay local ring of dimension \(d \geq 2\), proving that the generalized Hilbert-Kunz function is a piecewise polynomial in this case.

**Preliminaries**

Let \((R,m)\) be a \(d\)-dimensional Noetherian local ring of prime characteristic \(p > 0\) and let \(I\) be an \(m\)-primary ideal.

- \(\ell(R/I^{[p^n]}) = \ell_{HK}(I,R)\) is a parameter ideal in a Cohen-Macaulay local ring of dimension \(d \geq 2\), proving that the generalized Hilbert-Kunz function is a piecewise polynomial in this case.

**Joint work with Mitra Koley and Jugul K. Verma**

**STANLEY-REISNER RINGS**

Let \(S = k[x_1, \ldots, x_n]\) be a polynomial ring in \(r\) variables over a field \(k\) and let \(m\) be the maximal homogeneous ideal of \(S\). Let \(P_i, \ldots, P_m\), for \(\alpha \geq 2\), be distinct \(S\)-ideals generated by subsets of \(\{x_1, \ldots, x_n\}\). Let \(I = \cap_{i=1}^m P_i\) and \(R = S/I\). Suppose \(n = m/\ell\) denotes the maximal homogeneous ideal of \(R\) and \(\text{dim}(R) = d\).

**Theorem 7.** Set \(\delta = \max\{\alpha_i(R) : \alpha_i(R) \neq 0\} - \infty\). Then for \(s > \delta\),

\[
\ell\left(\frac{R(I)}{(I,J)^{[s]}}\right) = (n,m)\left[\frac{n}{n,m}\right]\text{ is a polynomial in } s \text{ of degree } d + 1.
\]

**Examples**

**Example 8.** Let \(\Delta\) be the simplicial complex

\[
\begin{align*}
\{x_1, x_2, x_3, x_4\} \cap \{x_1, x_3, x_4\} \\
\{x_1, x_2, x_4\} \cap \{x_1, x_4\} \\
\{x_1, x_3\} \cap \{x_1, x_3, x_4\} \\
\{x_2, x_3\} \cap \{x_2, x_3, x_4\} \\
\{x_2, x_4\} \cap \{x_2, x_4\} \\
\{x_3, x_4\} \cap \{x_1, x_3, x_4\}
\end{align*}
\]

Then \(R = k[x_1, x_2, x_3, x_4]/((x_1x_2) \cap (x_2x_4))\), the Stanley-Reisner ring of \(\Delta\) with \(f\)-vector \(f(\Delta) = (1, 6, 15, 10)\) and \(b\)-vector of \(R\) is \(h(\Delta) = (1, 3, 6, 0)\). For \(s > 1\),

\[
\ell\left(\frac{R(I)}{(I,J)^{[s]}}\right) = 300 + \frac{3s^3 + 13s^2}{2} + 372 \frac{s^2 + 1}{2} - 41s.
\]

**Example 9.** (Triangulation of real projective plane) Let \(\Delta\) be the triangulation of the real projective plane. Let \(R\) be the corresponding Stanley-Reisner ring of \(\Delta\). The \(f\)-vector of \(R\) is \(f(\Delta) = (1, 6, 15, 10)\) and \(b\)-vector of \(R\) is \(h(\Delta) = (1, 3, 6, 0)\).

**Theorem 5.** Let \(R\) be a Cohen-Macaulay local ring of dimension \(d \geq 2\) and let \(I\) be a parameter ideal of \(R\). For a fixed \(s \in \mathbb{N}\), \(\ell(R/I^{[s]}) = \ell_{HK}(I,R)\) is a polynomial in \(s\), whenever the limit exists.

**Theorem 6.** Let \(R\) be a Cohen-Macaulay local ring of dimension \(d \geq 2\) and let \(I\) be a parameter ideal of \(R\). For a fixed \(s \in \mathbb{N}\), \(\ell(R/I^{[s]}) = \ell_{HK}(I,R)\) is a polynomial in \(s\), whenever the limit exists.

**Theorem 5.** Let \(R\) be a Cohen-Macaulay local ring of dimension \(d \geq 2\) and let \(I\) be a parameter ideal of \(R\). For a fixed \(s \in \mathbb{N}\), \(\ell(R/I^{[s]}) = \ell_{HK}(I,R)\) is a polynomial in \(s\), whenever the limit exists.

The edge ideal of \(K_{3,4}\) is the ideal \(I = (x, y) | 1 \leq x, y \leq 5, 1 \leq y \leq 4\). Let \(R = S/I\). For all \(s > 4\),

\[
\ell\left(\frac{R(I)}{(I,J)^{[s]}}\right) = 61 \frac{s^3 + 19s^2 + 1}{72} + \frac{1}{12} \frac{s^2 - 7}{24} + \frac{9}{25} s - 1.
\]

**References**