Initial Ideals of Closed Determinantal Facet Ideals

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Betti Numbers of Initial Ideals

For any closed determinantal facet ideal, the initial ideal is of degree $n$ and squarefree, so that Stanley-Reisner theory may be employed to compute the $\mathbb{Z}^n$-graded Betti numbers.

**Notation.** Let $\text{in}(J_\Delta)$ denote the initial ideal with respect to $< \in \Delta$.

**Theorem (AV20)**

Let $\Delta$ be a pure $(n-1)$-dimensional simplicial complex which is closed. Then the $\mathbb{Z}^n$-graded Betti numbers of $\text{in}(J_\Delta)$ are either 0 or 1.

**Idea of Proof.** We study the Stanley-Reisner complex $\Gamma$ of $\text{in}(J_\Delta)$. We observe that the restriction of $\Gamma$ to monomials of certain forms is homotopy equivalent to a sphere, and that the restriction of $\Gamma$ to any other monomial is contractible. By Hochster’s formula, the result follows.

**Linear Strand of $\text{in}(J_\Delta)$**

**Definition.** Let $\alpha = (a_1, \ldots, a_n)$ with $|a_i| = \ell$ and $I = (i_1 < \cdots < i_\ell)$. Define the indexing set

$$I_{\Delta}(\alpha, I) := \{i \in \{1, \ldots, n\} : |a_i| > 0, |a_{\ell+1}| \leq j \leq |a_{\ell+1}|$$

**Definition.** $C_{\sigma}(\Delta, M) = \mathbb{N}^G$. For $i \geq 1$, let $C_{\sigma}^i(\Delta, M) \subseteq D_{\sigma^i}(G^\sigma) \otimes \mathbb{N}^{i+1}$ denote the free submodule generated by all elements of the form $g^{(\alpha)} \otimes f_{\alpha^i}$, where $\sigma = \Delta^{\alpha}_{\sigma^i}$ with $|\sigma| = n + i - 1$ and $\alpha = (a_1, \ldots, a_n)$ with $|\alpha| = i - 1$. Let $C_{\sigma}^i(\Delta, M)$ denote the complex induced by the differentials

$${d_i(g^{(\alpha)} \otimes f_{\alpha^i})} = -(-1)^{j+1} x_{\alpha^i} g^{(\alpha{}_{\ell+1})} \otimes f_{\alpha^i}$$

on the submodules defined above.

**Theorem (AV20)**

Assume that $\Delta$ is an $(n-1)$-pure closed simplicial complex. Let $F_{\text{lin}}$ denote the minimal graded free resolution of $\text{in}(J_\Delta)$ and let $F_{\text{lin}}$ denote its linear strand; then

$F_{\text{lin}} \cong C_{\sigma}(\Delta, M)$.

**Example 1.** Let $G$ be the closed graph below, so $J_G = \{[1, 2], [1, 3], [2, 3], [2, 4], [3, 4]\}.

The clique complex of $G$ has facets $[1, 2, 3]$ and $[2, 3, 4]$. Consider the basis element $g^{(\alpha)} \otimes f_{\alpha}$ where $|\alpha| = 0, 1$ and $\alpha = \{2, 3, 4\}$. Then $I_{\Delta}(\alpha, \sigma) = \{[2, 3], [2, 4]\}$ so

$$d_2(g^{(\alpha)} \otimes f_{\alpha}) = x_{2} x_{4} + x_{2} x_{3} x_{4}.$$