Jacobi's four square theorem

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The Four Square Theorem

Theorem

Let $n \in \mathbb{N}$, then,

$$r_4(n) = \#\{(a, b, c, d) \in \mathbb{Z}^4 \mid a^2 + b^2 + c^2 + d^2 = n\} = \sum_{d|n, d \notin 4\mathbb{Z}} 8d$$

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Theta functions

Define $\theta(z) = \sum_{m \in \mathbb{Z}} e^{2\pi i z m^2}$ on $\mathbb{H}$.

$$\theta(z)^k = \sum_{n \in \mathbb{Z}} \left(\sum_{(a_1, \ldots a_k) | \sum a_i^2 = n} 1 \right) e^{2\pi i z n} = \sum_{n \in \mathbb{Z}} r_k(n) e^{2\pi i z n}$$

So, $\quad \theta(z)^4 = \sum_{n \in \mathbb{Z}} r_4(n) e^{2\pi i z n}$

Modular forms on SL(2, $\mathbb{Z}$)

Definition

Let k be any integer. A holomorphic function $f : \mathbb{H} \to \mathbb{C}$ is a *modular form* of weight k if

- $f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z)$ for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z})$.
- f is holomorphic at ∞.

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- f is holomorphic at ∞.

Holomorphic at infinity

We have $f(z+1) = (1)^k f(z) = f(z)$.
Hence $f(z) = g(e^{2\pi iz})$ for some holomorphic $g : D \setminus \{0\} \to \mathbb{C}$.
Thus $g(q)$ has a laurent series $g(q) = \sum_{n=-\infty}^{\infty} a_n q^n$ where $q = e^{2\pi iz}$.

We say f is holomorphic at ∞ if $a_n = 0$ for all $n < 0$.

Additionally, f is a cusp form if $a_0 = 0$

Modular forms on SL(2, $\mathbb{Z}$)

Example

Consider the functions $G_k(z) = \sum_{(m,n)\in\mathbb{Z}^2\setminus(0,0)} \frac{1}{(mz+n)^k}$ for $k \geq 3$.

$SL(2,\mathbb{Z})$ is generated by matrices $S = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $T = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

Hence it is sufficient to check condition (1) for these matrices.

Clearly $G_k(z+1) = G_k(z)$. Also,

$$G_k\left(\frac{-1}{z}\right) = \sum_{(m,n)\in\mathbb{Z}^2\setminus(0,0)} \frac{1}{(m\frac{-1}{z}+n)^k} = z^k \sum_{(m,n)\in\mathbb{Z}^2\setminus(0,0)} \frac{1}{(nz+m)^k} = z^k G_k(z)$$

It can be shown that $G_k(z)$ is holomorphic at infinity by showing it is bounded by the value at $\omega = e^{2\pi i/3}$ and furthermore

$$G_k(\infty) = \lim_{z\to\infty} \sum_{(m,n)\in\mathbb{Z}^2\setminus(0,0)} \frac{1}{(mz+n)^k} = \sum_{n\in\mathbb{Z}\setminus 0} \frac{1}{n^k} = 2\zeta(k)$$

The space of modular forms

The vector space of modular forms of weight k is denoted by $\mathcal{M}_k$ and the space of cusp forms by S_k

Theorem

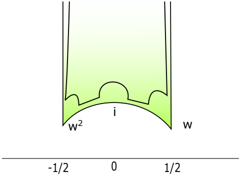
The space $\mathcal{M} = \oplus_{k \in \mathbb{Z}} \mathcal{M}_k$ *of all modular forms on* $SL(2, \mathbb{Z})$ *is isomorphic to* $\mathbb{C}[G_4, G_6]$

Proof Sketch

We obtain a bound on dimension of $\mathcal{M}_k$ by computing a contour integral around the fundamental domain of the action of $SL(2, \mathbb{Z})$ on $\mathbb{H}$

Proof Sketch

f be a non-zero modular form of weight k.
Let $v_p(f)$ = degree of zero at p.
Then,
$v_\infty(f) + \frac{1}{2} v_i(f) + \frac{1}{3} v_\omega(f) + \sum_{p \in \mathbb{H}/\Gamma} v_p(f) = \frac{k}{12}$



This gives us that $\mathcal{M}_k = 0$ for $k \leq 2$ and odd k
Furthermore, we can establish the following isomorphisms:

$$\mathcal{M}_k \cong S_k \oplus \mathbb{C} G_k \text{ and } M_{k-12} \cong S_k$$

The second comes from recognizing
$\triangle = 60^3 (G_4(z))^3 - 27 \cdot 140^2 (G_6(z))^2$ is a cusp form of weight 12.

Using the isomorphisms, one can show that $\mathcal{M}_k$ has dimension 1 for $k = 4, 6, 8, 10$ and explicitly producing basis elements: $G_4, G_6, (G_4)^2, G_4 G_6$ respectively.

For higher k, the dimension of the space is $\lfloor k/12 \rfloor$ from the second isomorphism and a basis for the space is $\{ G_4^a G_6^b \mid 4a + 6b = k \}$ proving the theorem.

Definition

The principle congruence subgroup of level N is denoted:

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}) \middle| \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mod N \right\}$$

A congruence subgroup Γ of level N is any subgroup of $SL(2,\mathbb{Z})$ such that $\Gamma(N) \subset \Gamma \subset SL(2,\mathbb{Z})$.

We will focus on the congruence subgroup

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}) \middle| \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \mod N \right\}$$

Modular forms on subgroups

Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z})$ and $f : \mathbb{H} \to \mathbb{C}$ holomorphic. Define

$f[\gamma]_k = j(\gamma, z)^{-k} f(\gamma(z))$ where $j(\gamma, z) = cz+d$ and $\gamma(z) = \dfrac{az+b}{cz+d}$

Definition

Let k be any integer and Γ be a congruence subgroup. A holomorphic function $f : \mathbb{H} \to \mathbb{C}$ is a *modular form* of weight k with respect to Γ if

- $f[\gamma]_k = f(z)$ for all $\gamma \in \Gamma$.
- $f[\gamma]_k$ is holomorphic at ∞ for all $\gamma \in SL(2, \mathbb{Z})$.

The condition $f[\gamma]_k = f(z)$ for all $\gamma \in \Gamma$ is consistent with condition 1 for modular forms of $SL(2,\mathbb{Z})$.

If $f[\gamma]_k(z) = f(z)$, then $(cz+d)^{-k} f(\frac{az+b}{cz+d}) = f(z)$ giving us $f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z)$

The vector space of modular forms of weight k over Γ is denoted by $\mathcal{M}_k(\Gamma)$

Note

$j(\gamma\gamma', z) = j(\gamma, \gamma'(z)) j(\gamma', z)$ and $f[\gamma\gamma']_k = (f[\gamma']_k)[\gamma]_k$.
Thus checking the first condition is equivalent to checking it for a generating set of Γ

Theorem

$\theta(z)^4$ is an element of $\mathcal{M}_2(\Gamma_0(4))$

Proof

First we note that $\Gamma_0(4)$ is generated by $\pm\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}, \pm\begin{pmatrix}1 & 0\\ 4 & 1\end{pmatrix}$

Clearly $\theta(z) = \sum\limits_{m\in\mathbb{Z}} e^{2\pi izm^2} = \theta(z+1)$

We use the Poisson summation formula: $\sum_{n\in\mathbb{Z}} f(n) = \sum_{n\in\mathbb{Z}} \hat{f}(n)$ (where $\hat{f}$ is the fourier transform of f) to show $\theta(\frac{-1}{4z}) = \sqrt{-2iz}\theta(z)$
Defining $f(x) = e^{-\pi tx^2}$ gives us

$$\hat{f}(n) = \int_{-\infty}^{\infty} e^{-\pi tx^2 - 2\pi ixn} dx = e^{\frac{\pi n^2}{t}} \int_{-\infty}^{\infty} e^{-\pi t(x-\frac{ni}{t})^2} dx$$
$$= e^{\frac{\pi n^2}{t}} \int_{-\infty}^{\infty} e^{-\pi tx} dx = \frac{1}{\sqrt{t}} e^{\frac{\pi n^2}{t}}$$

Proof (contd.)

... $\hat{f}(n) = \frac{1}{\sqrt{t}} e^{\frac{\pi n^2}{t}}$ Substituting $z = \frac{-t}{2i}$ gives us

$$\theta\Big(\frac{-1}{4z}\Big) = \sum e^{\frac{\pi}{t} n^2} = \sqrt{t} \sum e^{-\pi t n^2} = \sqrt{-2iz}\theta(z)$$

This implies

$$\begin{aligned} \theta\Big(\frac{z}{4z+1}\Big) &= \theta\Big(\frac{-1}{4(\frac{1}{4z}-1)}\Big) \\ &= \sqrt{2i(\tfrac{1}{4z}+1)}\theta(\tfrac{-1}{4z}-1) \\ &= \sqrt{2i(\tfrac{1}{4z}+1)}\theta(\tfrac{-1}{4z}) \\ &= \sqrt{2i(\tfrac{1}{4z}+1)}\sqrt{-2iz}\ \theta(z) \\ &= \sqrt{4z+1}\ \theta(z) \end{aligned}$$

Raising to the fourth power proves the modularity of $\theta(z)^4$.

Modular forms on subgroups

Modular forms (of weight $2k$) can be as k forms invariant of the Γ action: $\gamma : z \mapsto \frac{az+b}{cz+d}$

Definition

Let k be any integer and Γ be a congruence subgroup. A holomorphic function $f : \mathbb{H} \to \mathbb{C}$ is a *modular form* of weight $2k$ with respect to Γ if

- $f(z)(dz)^k$ is a k form defined on $\mathbb{H}/\Gamma$
- $f[\gamma]_k$ is holomorphic at ∞ for all $\gamma \in SL(2,\mathbb{Z})$.

This is equivalent to the previous definition because $d(\gamma(z)) = d(\frac{az+b}{cz+d}) = \frac{ad-bc}{(cz+d)^2}dz = \frac{dz}{(cz+d)^2}$ Hence,

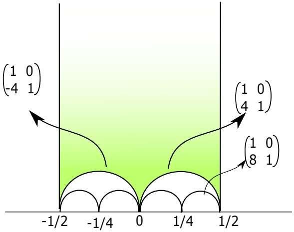
$$f(\gamma(z))(d\gamma(z))^k = f(\gamma(z))\frac{(dz)^k}{(cz+d)^{2k}} = f(z)(dz)^k$$
$$\iff (cz+d)^{-2k}f(\gamma(z)) = f(z)$$

Dimension of $\mathcal{M}_2(\Gamma_0(4))$

Modular forms of weight 2 can be thought of as 1-forms on the Riemann surface $\mathbb{H}/\Gamma_0(4)$. We find a fundamental domain of $\mathbb{H}$ for this action by $\Gamma_0(4)$.

Since the translation matrix: $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ is in the subgroup, the fundamental domain is contained in $\{|\mathrm{Im}(z)| \leq \frac{1}{2}\}$.

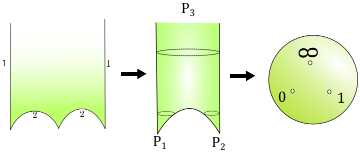
For each matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4)$, draw semicircles centered at $\frac{a}{c}$ with radius $\frac{1}{|c|}$.



The fundamental domain for the action by $\Gamma_0(4)$ is the region outside the largest semicircles.

Dimension of $\mathcal{M}_2(\Gamma_0(4))$

The fundamental domain under appropriate identification is a sphere with 3 punctures.



Any 1-forms on this thrice punctured sphere can have simple poles at each of the punctures. Thus the space of 1-forms is generated by 2 elements: $\frac{dz}{z}$ and $\frac{dz}{z-1}$. The 1-form $a\frac{dz}{z} + b\frac{dz}{z-1}$ has simple poles at 0 and 1. It also has a simple pole at infinity:

Set $\xi = \frac{1}{z}$. $d\xi = \frac{-1}{z^2}dz$. So the 1-form becomes: $-a\frac{d\xi}{\xi} - b\frac{d\xi}{\xi(1-\xi)}$ which has a simple pole at $\xi = 0$

Thus $\dim \mathcal{M}_2(\Gamma_0(4)) = 2$

Dimension of $\mathcal{M}_k(\Gamma)$

There are general formulas for the dimension of the space of modular forms of weight k over a subgroup Γ.

$$\dim \mathcal{M}_k(\Gamma) = (k-1)(g-1) + \left\lfloor \frac{k}{4} \right\rfloor e_2 + \left\lfloor \frac{k}{3} \right\rfloor e_3 + \frac{k}{2} e_\infty$$

where g is the genus of the Riemann surface $\overline{\mathbb{H}/\Gamma}$

e_2 is the number of orbits of elliptic points of order 2

e_3 is the number of orbits of elliptic points of order 3

e_∞ is the number of orbits of cusps

Almost invariance of G_2

The function $G_2(z) = \sum\limits_{(m,n)\in\mathbb{Z}^2\setminus 0} \frac{1}{(mz+n)^2}$ isn't absolutely convergent.
However it is conditionally convergent written as

$\sum\limits_{m\in\mathbb{Z}} \sum\limits_{n\in\mathbb{Z}_m} \frac{1}{(mz+n)^2}$ where $\mathbb{Z}_m = \mathbb{Z}\setminus 0$ if $m=0$ and $\mathbb{Z}$ otherwise.

Consider $\pi cot\pi z = \frac{1}{z} + \sum\limits_{d=1}^{\infty} \frac{1}{z-d} + \frac{1}{z+d}$.

Also, $\pi cot\pi z = \pi\frac{cos\pi z}{sin\pi z} = \pi i\frac{e^{i\pi z}+e^{-i\pi z}}{e^{i\pi z}-e^{-i\pi z}} = -\pi i - 2\pi i\sum\limits_{n=0}^{\infty} e^{2\pi inz}$

Taking the derivative of $\pi cot\pi z$, $\sum\limits_{d=-\infty}^{\infty} \frac{1}{(z+d)^2} = -4\pi^2 \sum\limits_{n=0}^{\infty} ne^{2\pi inz}$

Almost invariance of G_2

$$
\begin{aligned}
G_2(z) &= \sum_{m\in\mathbb{Z}} \sum_{n\in\mathbb{Z}_m} \frac{1}{(mz+n)^2} \\
&= \sum_{n\neq 0} \frac{1}{n^2} + \sum_{m>0} \sum_{n\in\mathbb{Z}} \frac{1}{(mz+n)^2} + \sum_{m<0} \sum_{n\in\mathbb{Z}} \frac{1}{(mz+n)^2} \\
&= 2\zeta(2) + 2 \sum_{m>0} \sum_{n\in\mathbb{Z}} \frac{1}{(mz+n)^2} \\
&= 2\zeta(2) + 2 \sum_{m>0} -4\pi^2 \sum_{n=0}^{\infty} n e^{2\pi i n m z} = 2\zeta(2) - 8\pi^2 \sum_{n=1}^{\infty} \sigma(n) e^{2\pi i n z}
\end{aligned}
$$

We have $\left| \sum_{m>0} \sum_{n=0}^{\infty} n e^{2\pi i n m z} \right| \leq C \sum_{m>0}^{\infty} \frac{1}{|(1-e^{2\pi i z m})^2|} \leq$
$C' \sum_{m>0}^{\infty} |e^{-4\pi i m z}| = \frac{C''}{1-|e^{-4\pi i z}|}$

Almost invariance of $G_2(z)$

It is clear that $G_2(z) = 2\zeta(2) - 8\pi^2 \sum_{m>0} \sum_{n=0}^{\infty} n e^{2\pi i n m z} = G_2(z+1)$

With more computation, one can show that

$$G_2\left[\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right]_2 (z) = G_2(z) - \frac{2\pi i}{z}$$

Inductively one shows $G_2[\gamma]_2(z) = G_2(z) - \frac{2\pi i c}{cz+d}$ where

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$$

If we define $G_2'(z) = G_2(z) - \frac{\pi}{Im(z)}$, the above statement gives $G_2'[\gamma]_2(z) = G_2'(z)$ but G_2' is not holomorphic.

Weak modular forms with respect to $\Gamma_0(4)$

Consider the functions $G_{2,N}(z) = G_2(z) - NG_2(Nz)$.

By straightforward computation $G_{2,N}[\gamma]_2 = G_{2,N}(z)$ for all $\gamma \in \Gamma_0(N)$.

If we can prove these are holomorphic at ∞, we have $G_{2,N} \in \mathcal{M}_2(\Gamma_0(N))$. So, $G_{2,4} \in \mathcal{M}_2(\Gamma_0(4))$ and $G_{2,2} \in \mathcal{M}_2(\Gamma_0(2)) \subseteq \mathcal{M}_2(\Gamma_0(4))$

A bounding theorem

Theorem

Let $f : \mathbb{H} \to \mathbb{C}$ be weakly modular on $\mathbb{H}$, with respect to Γ, a congruence subgroup of level N. If there exist positive constants C, r, such that the Fourier expansion of f satisfies $f(z) = \sum_{n \geq 0} a_n e^{2\pi inz/N}$ with $a_n \leq Cn^r$ for all $n > 0$, then

$$|f(z)| \leq C_0 + C\left(\int_0^\infty t^r e^{-2\pi ty/N} dt\right) + \frac{C_1}{y^r}$$

Proof

We have $|f(z)| \leq |a_0| + \sum_{n>0} Cn^r e^{-2\pi iny/N}$.

Consider the function $g(t) = t^r e^{-2\pi ty/N}$.

$g'(t) > 0$ when $t \in (0, \frac{rN}{2\pi y})$ and $g'(t) < 0$ when $t > \frac{rN}{2\pi y}$

Proof (contd.)

... Calling $k = \left\lfloor \frac{rN}{2\pi y} \right\rfloor$, we get $\sum\limits_{1}^{k-1} n^r e^{-2\pi i n y} < \int\limits_0^k t^r e^{-2\pi t y/N} dt$ and $\sum\limits_{k+2}^{\infty} n^r e^{-2\pi i n y} < \int\limits_k^{\infty} t^r e^{-2\pi t y/N} dt$

Thus

$$
\begin{aligned}
|f(z)| &\leq |a_0| + C\Bigg(k^r e^{-2\pi k y/N} + (k+1)^r e^{-2\pi (k+1) y/N} \\
&\quad + \sum_{1}^{k-1} n^r e^{-2\pi i n y} + \sum_{k+2}^{\infty} n^r e^{-2\pi i n y} \Bigg) \\
&\leq C_0 + C\left(\int_0^{\infty} t^r e^{-2\pi t y/N} dt \right) + \frac{C_1}{y^r}
\end{aligned}
$$

Theorem

If a weakly modular function satisfies the above condition on fourier coefficients, it is a modular form with respect to Γ.

Proof

For all $\gamma \in SL(2, \mathbb{Z})$, we need $f[\gamma]_k$ to be holomorphic at ∞. $f[\gamma]_k$ is invariant under $\gamma^{-1}\Gamma\gamma$ and hence has a Laurent expansion: $f[\gamma]_k = \sum_{n \in \mathbb{Z}} b_n e^{2\pi i n z/N}$.

As $z \to \infty$,
$|f[\gamma]_k| = (cz + d)^{-k} f(\gamma(z)) = O(y^{-k})O((Im(\gamma(z))^{-r}) = O(y^{r-k})$

Thus, $lim_{z\to\infty}|f[\gamma]_k e^{2\pi i z/N}| = lim_{z\to\infty} O(y^{r-k})e^{-2\pi y/N} \to 0$

Elements of $\mathcal{M}_2(\Gamma_0(4))$

The functions $G_{2,2}$ and $G_{2,4}$ have fourier expansions as follows:

$$\begin{aligned} G_{2,2}(z) &= G_2 - 2G_2(2z) \\ &= 2\zeta(2) - 8\pi^2 \sum_{n=1}^{\infty} \sigma(n) e^{2\pi izn} - 2\Big(2\zeta(2) - 8\pi^2 \sum_{n=1}^{\infty} \sigma(n) e^{4\pi izn}\Big) \\ &= -\frac{\pi^2}{3} - 8\pi^2 \sum_{n=1}^{\infty} \Bigg(\sum_{d|n, d\notin 2\mathbb{Z}} d \Bigg) e^{2\pi izn} \end{aligned}$$

$$\text{Similarly, } G_{2,4}(z) = -\pi^2 - 8\pi^2 \sum_{n=1}^{\infty} \Bigg(\sum_{d|n, d\notin 4\mathbb{Z}} d \Bigg) e^{2\pi izn}$$

The fourier coefficients are bounded by $8\pi^2\sigma(n) \leq 8\pi^2 n^2$ hence, $G_{2,2}, G_{2,4} \in \mathcal{M}_2(\Gamma_0(4))$

Proving Jacobi's four square theorem

We have the following:

- $\theta^4(z) \in \mathcal{M}_2(\Gamma_0(4))$
- $\dim \mathcal{M}_2(\Gamma_0(4)) = 2$
- $G_{2,2}$ and $G_{2,4}$ are a basis for $\mathcal{M}_2(\Gamma_0(4))$

Thus, $\theta^4(z) = aG_{2,2} + bG_{2,4}$.

$$1 + 8e^{2\pi iz} + \ldots = \frac{-a\pi^2}{3}(1 + 24e^{2\pi iz} + \ldots) - b\pi^2(1 + 8e^{2\pi iz} + \ldots)$$

Comparing coefficients gives us

$$\theta^4(z) = \tfrac{-1}{\pi^2} G_{2,4} = \sum_{n \in \mathbb{Z}} \Big(8 \sum_{d|n, d \notin 4\mathbb{Z}} d \Big) e^{2\pi izn}$$

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