MAT 540 : Problem Set 9

Due Thursday, November 21

1 Abelian subcatgeories of triangulated categories

Let $\mathscr{D}$ be a triangulated category. We denote the shift functors by $X \longmapsto X[1]$, and we write triangles as $X \to Y \to Z \to X[1]$ or $X \to Y \to Z \xrightarrow{+1}$. For every $X, Y \in \mathrm{Ob}(\mathscr{D})$ and every $n \in \mathbb{Z}$, we write $\mathrm{Hom}^n_{\mathscr{D}}(X,Y) = \mathrm{Hom}(X, Y[n])$.

(a). Let $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$ and $X' \xrightarrow{u'} Y' \xrightarrow{v'} Z' \xrightarrow{w'} X'[1]$ be two distinguished triangles of $\mathscr{D}$, and let $g : Y \to Y'$ be a morphism.

  (i) (2 points) Show that the following conditions are equivalent:

    (1) $v' \circ g \circ u = 0$;

    (2) there exists $f : X \to X'$ such that $u' \circ f = g \circ u$;

    (3) there exists $h : Z \to Z'$ such that $h \circ v = v' \circ g$;

    (4) there exist $f : X \to X'$ and $h : Z \to Z'$ such that (f, g, h) is a morphism of triangles.

  (ii) (1 point) Suppose that the conditions (i) hold and that $\mathrm{Hom}^{-1}_{\mathscr{D}}(X, Z') = 0$. Show that the morphisms f and h of (i)(2) and (i)(3) are unique.

(b). Let $\mathscr{C}$ be a full subcategory of $\mathscr{D}$, and suppose that $\mathrm{Hom}^n(X,Y) = 0$ if $X, Y \in \mathrm{Ob}(\mathscr{C})$ and $n < 0$.

  (i) (2 points) Let $f : X \to Y$ be a morphism of $\mathscr{C}$. Take a distinguished triangle $X \xrightarrow{f} Y \to S \xrightarrow{+1}$ in $\mathscr{D}$, and suppose that we have a distinguished triangle $N[1] \to S \to C \xrightarrow{+1}$ with $N, C \in \mathrm{Ob}(\mathscr{C})$. In particular, we get morphisms $\alpha : N[1] \to S \to X[1]$ and $\beta : Y \to S \to X$.

  Show that $\alpha[-1] : N \to X$ is a kernel of f and that $\beta : Y \to C$ is a cokernel of f.

  We say that a morphism f of $\mathscr{C}$ is *admissible* if there exist distinguished triangles satisfying the conditions of (i). We say that a sequence $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ of morphisms of $\mathscr{C}$ is an *admissible short exact sequence* if there exists a distinguished triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{+1}$ in $\mathscr{D}$.

  (ii) (2 points) Suppose that $\mathscr{C}$ as a zero object. If $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{+1}$ is a distinguished triangle in $\mathscr{D}$ with $X, Y, Z \in \mathrm{Ob}(\mathscr{C})$, show that f and g are admissible, that f is a kernel of g and that g is a cokernel of f.

  (iii) (2 points) If $f : X \to Y$ is an admissible monomorphism (resp. epimorphism) in $\mathscr{C}$

and $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{+1}$ is a distinguished triangle in $\mathscr{D}$, show that f has a cokernel (resp. a kernel) in $\mathscr{C}$ and that $Z \simeq \mathrm{Coker}(f)$ (resp. $Z[-1] \simeq \mathrm{Ker}(f)$).

(iv) (4 points) Suppose that every morphism of $\mathscr{C}$ is admissible and $\mathscr{C}$ is an additive subcategory of $\mathscr{D}$. Show that $\mathscr{C}$ is an abelian category and that every short exact sequence in $\mathscr{C}$ is admissible.

(v) (3 points) Suppose that $\mathscr{C}$ is an abelian category and that every short exact sequence in $\mathscr{C}$ is admissible. Show that every morphism of $\mathscr{C}$ is admissible.

Solution.

(a). (i) Obviously, point (4) implies (2) and (3). Also, as $v' \circ u' = 0$ and $v \circ u = 0$ by Proposition V.1.1.11(i) of the notes, points (2) and (3) each imply (1). Also, by axiom (TR4), we have that (2) implies (4). So it remains to show that (1) implies (2). Applying the cohomological functor $\mathrm{Hom}_{\mathscr{D}}(X, \cdot)$ to the distinguished triangle $X' \to Y' \to Z' \xrightarrow{+1}$, we get an exact sequence

$$\mathrm{Hom}_{\mathscr{D}}(X, X') \xrightarrow{u'\circ(\cdot)} \mathrm{Hom}_{\mathscr{D}}(X, Y') \xrightarrow{v'\circ(\cdot)} \mathrm{Hom}_{\mathscr{D}}(X, Z').$$

So, if $v' \circ (g \circ u) = 0$ (that is, if (1) holds), then there exists $f \in \mathrm{Hom}_{\mathscr{D}}(X, X')$ such that $u' \circ f = g \circ u$ (that is, (2) holds).

(ii) In the exact sequence of (i), the kernel of $u' \circ (\cdot) : \mathrm{Hom}_{\mathscr{D}}(X, X') \to \mathrm{Hom}_{\mathscr{D}}(X, Y')$ is the image of the morphism $w'[-1] \circ (\cdot) : \mathrm{Hom}_{\mathscr{D}}(X, Z'[-1]) \to \mathrm{Hom}_{\mathscr{D}}(X, X')$. This gives the uniqueness of f in (2) (if it exists). To show the uniqueness of h, suppose that we have two morphisms $h, h' : Z \to Z'$ such that $h \circ v = v' \circ g = h' \circ v$, so that $(h - h') \circ v = 0$. Applying the cohomological functor $\mathrm{Hom}_{\mathscr{D}}(\cdot, Z')$ to the distinguished triangle $X \to Y \to Z \xrightarrow{+1}$, we get an exact sequence

$$\mathrm{Hom}_{\mathscr{D}}(X[1], Z') = \mathrm{Hom}_{\mathscr{D}}(X, Z'[-1]) = 0 \to \mathrm{Hom}_{\mathscr{D}}(Z, Z') \to \mathrm{Hom}_{\mathscr{D}}(Z, Y').$$

So the morphism $(\cdot) \circ v : \mathrm{Hom}_{\mathscr{D}}(Z, Z') \to \mathrm{Hom}_{\mathscr{D}}(Y, Z')$ is injective, which shows that $h = h'$.

(b). (i) We show that β is a cokernel of f. Let $g : Y \to Z$ be a morphism of $\mathscr{C}$ such that $g \circ f = 0$. We want to show that there exists a unique morphism $g' : C \to Z$ such that $g' \circ \beta = g$. By (TR1) and (TR3), we have a distinguished triangle $0 \to Z \xrightarrow{\mathrm{id}_Z} Z \to 0[1] = 0$. Applying question (a) to the diagram

$$\begin{tikzcd}
X \arrow[r, "f"] & Y \arrow[r] \arrow[d, "g"] & S \arrow[r] \arrow[d, dashed, "h"] & X[1] \\
0 \arrow[r] & Z \arrow[r, "\mathrm{id}_Z"'] & Z \arrow[r] & 0
\end{tikzcd}$$

and using the fact that $\mathrm{Hom}_{\mathscr{D}}^{-1}(X, Z) = 0$ (because $X, Z \in \mathrm{Ob}(\mathscr{C})$), we see that there exists a unique morphism $h : S \to Z$ making the diagram commute. This already implies the uniqueness of g' (if it exists). To show the existence of g', we apply question (a) again to the diagram

$$\begin{tikzcd}
N[1] \arrow[r] & S \arrow[r] \arrow[d, "h"] & C \arrow[r] \arrow[d, dashed, "g'"] & N[2] \\
0 \arrow[r] & Z \arrow[r, "\mathrm{id}_Z"'] & Z \arrow[r] & 0
\end{tikzcd}$$

The hypothesis of (a) is satisfied, because the composition of h and of $N[1] \to S$ is an element of $\mathrm{Hom}_{\mathscr{D}}(N[1], Z) = \mathrm{Hom}^{-1}_{\mathscr{D}}(N, Z) = 0$.

We show that $\alpha[-1]$ is a kernel of f. The proof is similar. Let $g : Z \to X$ be a morphism of $\mathscr{C}$ such that $f \circ g = 0$. We want to show that there exists a unique morphism $g' : Z \to N$ such that $\alpha[-1] \circ g' = g$. First, we apply question (a) to the diagram

$$\begin{tikzcd} Z \arrow[r, "\mathrm{id}_Z"] \arrow[d, dashed, "h"'] & Z \arrow[r] \arrow[d, "g"] & 0 \arrow[r] & 0 \\ S[-1] \arrow[r] & X \arrow[r, "f"'] & Y \arrow[r] & S \end{tikzcd}$$

Using the fact that $\mathrm{Hom}^{-1}_{\mathscr{D}}(Z, Y) = 0$ (because $Y, Z \in \mathrm{Ob}(\mathscr{C})$), we see that there is a unique morphism $h : Z \to S[-1]$ making the diagram commute. This impluies the uniqueness of g'. To show the existence of g', we apply question (a) to the diagram

$$\begin{tikzcd} Z \arrow[r, "\mathrm{id}_Z"] \arrow[d, dashed, "g'"'] & Z \arrow[r] \arrow[d, "h"] & 0 \arrow[r] & 0 \\ N \arrow[r] & S[-1] \arrow[r] & C[-1] \arrow[r] & N[2] \end{tikzcd}$$

The hypothesis of (a) is satisfied, because the composition of h and of $S[-1] \to C[-1]$ is an element of $\mathrm{Hom}_{\mathscr{D}}(Z, C[-1]) = 0$.

(ii) The morphism f is admissible, because we take $S = Z$ in question (i), and then we have a distinguished triangle $0 \to S \to Z \to 0$. Similarly, the morphism g is admissible, because we can take $S = X[1]$ in (i), and then we have a distinguished triangle $X[1] \to S \to 0 \to X[2]$. Also, question (i) immediately implies that g is a cokernel of f and that f is a kernel of g.

(iii) Let $f : X \to Y$ be an admissible morphism in $\mathscr{C}$, and let $X \xrightarrow{f} Y \to S = Z \xrightarrow{+1}$ and $N[1] \to S \to C \xrightarrow{+1}$ be distinguished triangles as in question (i); by that question, we have $\mathrm{Ker}\, f = N$ and $\mathrm{Coker}\, f = C$. If f is a monomorphism, this implies that $N = 0$, so the morphism $S \to C$ is an isomorphism, which shows that S is isomorphic to the cokernel of f. If f is an epimorphism, then we have $C = 0$, so the morphism $N[1] \to S$ is an isomorphism, which shows that $S[-1]$ is isomorphic to the kernel of f.

(iv) By question (i), every morphism of $\mathscr{C}$ has a kernel and a cokernel. Let $f : X \to Y$ be a morphism of $\mathscr{C}$; we need to check that the canonical morphism $\mathrm{Coim}(f) \to \mathrm{Im}(f)$ is an isomorphism, or in other words that the canonical morphism $X \to \mathrm{Im}(f)$ is a cokernel of $\ker(f) \to X$. Let $X \xrightarrow{f} Y \to S \xrightarrow{+1}$ and $N[1] \to S \to C \xrightarrow{+1}$ be distinguished triangles as in question (i), and let $\alpha[-1] : N \to X$ and $\beta : Y \to C$ be the morphisms defined in that question. Applying the octahedral axiom to the morphisms $Y \to S$ and $S \to C$ and to their composition β, we get a commutative diagram where the rows and the third column are distinguished triangles:

$$\begin{tikzcd}
Y \arrow[r] \arrow[d, equal] & S \arrow[r] \arrow[d] & X[1] \arrow[r] \arrow[d] & Y[1] \arrow[d] \\
Y \arrow[r, "\beta"] \arrow[d] & C \arrow[r] \arrow[d, equal] & I[1] \arrow[r] \arrow[d] & Y[1] \arrow[d] \\
S \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & N[2] \arrow[r] \arrow[d] & S[1] \arrow[d] \\
Y[1] \arrow[r] & S[1] \arrow[r] & X[2] \arrow[r] & Y[2]
\end{tikzcd}$$

As β is the cokernel of f, it is an epimorphism, so, by question (iii), the morphism $I \to Y$ is isomorphic to $\mathrm{Ker}(\beta) \to Y$, that is, to $\mathrm{Im}(f) \to Y$. As we have a distinguished triangle $N \to X \to I \to N[1]$, question (iii) shows that $X \to I$ is isomorphic to $X \to \mathrm{Coker}(\alpha[-1])$, that is, to $X \to \mathrm{Coim}(f)$, so we are done.

Finally, we show that every short exact sequence of $\mathscr{C}$ is admissible. Let $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ be a short exact sequence in $\mathscr{C}$, and let $X \xrightarrow{f} Y \to S \xrightarrow{+1}$ be a distinguished triangle in $\mathscr{D}$. As f is an admissible monomorphism and $g : Y \to Z$ is a cokernel of f, question (iii) implies that there exists a commutative triangle

$$\begin{tikzcd}
Y \arrow[r, "g"] \arrow[dr] & Z \arrow[d, "\wr"] \\
& S
\end{tikzcd}$$

where $Z \to S$ is an isomorphism. This implies that $X \xrightarrow{f} Y \xrightarrow{g} Z$ extends to a distinguished triangle.

(v) Let $f : X \to Y$ be a morphism of $\mathscr{C}$. Let $N = \mathrm{Ker}(f)$, $C = \mathrm{Coker}(f)$ and $I = \mathrm{Im}(f)$. We have exact sequences $0 \to N \to X \to I \to 0$ and $0 \to I \to Y \to C \to 0$, that extend to distinguished triangles in $\mathscr{D}$ by the hypothesis. Applying the octohedral axiom to the morphism $X \to I$ and $I \to Y$ and to their composition $X \xrightarrow{f} Y$, we get a commutative diagram where the rows and the third column are distinguished triangles:

$$\begin{tikzcd}
X \arrow[r] \arrow[d, equal] & I \arrow[r] \arrow[d] & N[1] \arrow[r] \arrow[d] & X[1] \arrow[d, equal] \\
X \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d, equal] & S \arrow[r] \arrow[d] & X[1] \arrow[d] \\
I \arrow[r] & Y \arrow[r] & C \arrow[r] \arrow[d] & I[1] \\
& & N[2] &
\end{tikzcd}$$

This gives the two triangles of (i) and shows that f is admissible.

$\square$

2 t-structures

We use the convention of problem 1. A *t-structure* on $\mathscr{D}$ is the date of two full subcategories $\mathscr{D}^{\leq 0}$ and $\mathscr{D}^{\geq 0}$ such that (with the convention that $\mathscr{D}^{\leq n} = \mathscr{D}^{\leq 0}[-n]$ and $\mathscr{D}^{\geq n} = \mathscr{D}^{\geq 0}[-n]$);

(0) If $X \in \mathrm{Ob}(\mathscr{D})$ is isomorphic to an object of $\mathscr{D}^{\leq 0}$ (resp. $\mathscr{D}^{\geq 0}$), then X is in $\mathscr{D}^{\leq 0}$ (resp. $\mathscr{D}^{\geq 0}$).

(1) For every $X \in \mathrm{Ob}(\mathscr{D}^{\leq 0})$ and every $Y \in \mathrm{Ob}(\mathscr{D}^{\geq 1})$, we have $\mathrm{Hom}(X, Y) = 0$.

(2) We have $\mathscr{D}^{\leq 0} \subset \mathscr{D}^{\leq 1}$ and $\mathscr{D}^{\geq 0} \supset \mathscr{D}^{\geq 1}$.

(3) For every $X \in \mathrm{Ob}(\mathscr{D})$, there exists a distinguished triangle $A \to X \to B \overset{+1}{\to}$ with $A \in \mathrm{Ob}(\mathscr{D}^{\leq 0})$ and $B \in \mathrm{Ob}(\mathscr{D}^{\geq 1})$.

We fix a t-structure $(\mathscr{D}^{\leq 0}, \mathscr{D}^{\geq 0})$ on $\mathscr{D}$.

(a). (1 point) Show that the distinguished triangle of condition (3) is unique up to unique isomorphism.

(b). (3 points) For every $n \in \mathbb{Z}$, show that the inclusion functor $\mathscr{D}^{\leq n} \subset \mathscr{D}$ has a right adjoint $\tau^{\leq n}$ and the inclusion functor $\mathscr{D}^{\geq n} \subset \mathscr{D}$ has a left adjoint $\tau^{\geq n}$. (Hint: It suffice to treat the case $n = 0$.)

(c). (2 points) For every $n \in \mathbb{Z}$, show that there is a unique morphism $\delta : \tau^{\geq n+1}X \to (\tau^{\leq n}X)[1]$ such that the triangle $\tau^{\leq n}X \to X \to \tau^{\geq n+1}X \overset{\delta}{\to} (\tau^{\leq n}X)[1]$ is distinguished, where the other two morphisms are given by the counit and unit of the adjunctions of (b).

(d). (3 points) Let $a, b \in \mathbb{Z}$ such that $a \leq b$, and let $X \in \mathrm{Ob}(\mathscr{D})$. Show that there exists a unique morphism $\alpha : \tau^{\geq a}\tau^{\leq b}X \to \tau^{\leq b}\tau^{\geq a}X$ such that the following diagram commutes:

$$\begin{tikzcd} \tau^{\leq b}X \arrow[r] \arrow[d] & X \arrow[r] & \tau^{\geq a}X \\ \tau^{\geq a}\tau^{\leq b}X \arrow[rr, "\alpha"'] & & \tau^{\leq b}\tau^{\geq a}X \arrow[u] \end{tikzcd}$$

(where all the other morphisms are counit or unit morphisms of the adjunctions of (b)), and that α is an isomorphism. (Hint: Apply the octahedral axiom to $\tau^{\leq a-1}X \overset{f}{\to} \tau^{\leq b}X \overset{g}{\to} X$.)

(e). (1 points) If $a, b \in \mathbb{Z}$ are such that $a \leq b$, show that, for every $X \in \mathrm{Ob}(\mathscr{D})$, we have $\tau^{\geq a}\tau^{\leq b}X \in \mathrm{Ob}(\mathscr{D}^{\geq a}) \cap \mathrm{Ob}(\mathscr{D}^{\leq b})$.

Let $\mathscr{C} = \mathscr{D}^{\leq 0} \cap \mathscr{D}^{\geq 0}$; that is, $\mathscr{C}$ is the full subcategory of $\mathscr{D}$ such that $\mathrm{Ob}(\mathscr{C}) = \mathrm{Ob}(\mathscr{D}^{\leq 0}) \cap \mathrm{Ob}(\mathscr{D}^{\geq 0})$. We denote the functor $\tau^{\leq 0}\tau^{\geq 0} : \mathscr{D} \to \mathscr{C}$ by H^0. The category $\mathscr{C}$ is called the *heart* or *core* of the t-structure.

(f). (1 point) Show that $\mathscr{C}$ is an abelian category.

(g). (2 points) Show that, if $X \to Y \to Z \overset{+1}{\to}$ is a distinguished triangle in $\mathscr{D}$ such that $X, Z \in \mathrm{Ob}(\mathscr{C})$, then Y is also in $\mathscr{C}$.

(h). The goal of this question is to show that the functor $\mathrm{H}^0 : \mathscr{D} \to \mathscr{C}$ is a cohomological functor. Let $X \to Y \to Z \overset{+1}{\to}$ be a distinguished triangle in $\mathscr{D}$.

(i) (2 points) If $X, Y, Z \in \mathrm{Ob}(\mathscr{D}^{\leq 0})$, show that the sequence $\mathrm{H}^0(X) \to \mathrm{H}^0(Y) \to \mathrm{H}^0(Z) \to 0$ is exact in $\mathscr{C}$. (Hint: A sequence of morphisms $A \to B \to C \to 0$ in an abelian category $\mathscr{A}$ is exact if and only if, for every object D of $\mathscr{A}$, the sequence of abelian groups $0 \to \mathrm{Hom}_{\mathscr{A}}(C,D) \to \mathrm{Hom}_{\mathscr{A}}(B,D) \to \mathrm{Hom}_{\mathscr{A}}(A,D)$ is exact.)

(ii) (2 points) If $X \in \mathrm{Ob}(\mathscr{D}^{\leq 0})$, show that the sequence $\mathrm{H}^0(X) \to \mathrm{H}^0(Y) \to \mathrm{H}^0(Z) \to 0$ is exact in $\mathscr{C}$. (Hint: Construct a distinguished triangle $X \to \tau^{\leq 0}Y \to \tau^{\leq 0}Z \overset{+1}{\to}$.)

(iii) (1 point) If $Z \in \mathrm{Ob}(\mathscr{D}^{\geq 0})$, show that the sequence $0 \to \mathrm{H}^0(X) \to \mathrm{H}^0(Y) \to \mathrm{H}^0(Z)$ is exact in $\mathscr{C}$.

(iv) (2 points) In general, show that the sequence $\mathrm{H}^0(X) \to \mathrm{H}^0(Y) \to \mathrm{H}^0(Z)$ is exact in $\mathscr{C}$.

Solution.

(a). Suppose that we have two distinguished triangles $A \to X \to B \overset{+1}{\to}$ and $A' \to X \to B' \overset{+1}{\to}$ with $A, A' \in \mathrm{Ob}(\mathscr{D}^{\leq 0})$ and $B, B' \in \mathrm{Ob}(\mathscr{D}^{\geq 1})$. We have $B'[-1] \in \mathrm{Ob}(\mathscr{D}^{\geq 2}) \subset \mathrm{Ob}(\mathscr{D}^{\geq 1})$, so, by condition (1), $\mathrm{Hom}_{\mathscr{D}}(A, B') = 0$ and $\mathrm{Hom}^{-1}_{\mathscr{D}}(A, B') = \mathrm{Hom}_{\mathscr{D}}(A, B'[-1]) = 0$. So question (i) of problem 1 implies that id_X extends to a unique morphism of distinguished triangles

$$\begin{tikzcd} A \arrow[r] \arrow[d, "u"'] & X \arrow[r] \arrow[d, "\mathrm{id}_X"'] & B \arrow[r] \arrow[d, "v"'] & A[1] \arrow[d, "u[1]"] \\ A' \arrow[r] & X \arrow[r] & B' \arrow[r] & A'[1] \end{tikzcd}$$

Exchanging the roles of (A, B) and (A', B'), we get that id_X also extends to a unique morphism of distinguished triangles

$$\begin{tikzcd} A' \arrow[r] \arrow[d, "u'"'] & X \arrow[r] \arrow[d, "\mathrm{id}_X"'] & B' \arrow[r] \arrow[d, "v'"'] & A'[1] \arrow[d, "u'[1]"] \\ A \arrow[r] & X \arrow[r] & B \arrow[r] & A[1] \end{tikzcd}$$

So we have two endomorphisms of the distinguished triangle $A \to X \to B \overset{+1}{\to}$ extending id_X, the endomorphisms given by $(u' \circ u, \mathrm{id}_X, v' \circ v)$ and $(\mathrm{id}_A, \mathrm{id}_X, \mathrm{id}_B)$. For the same reason as before, these morphisms must be equal, so $u' \circ u = \mathrm{id}_A$ and $v' \circ v = \mathrm{id}_B$. We see similarly that $u \circ u' = \mathrm{id}_{A'}$ and $v \circ v' = \mathrm{id}_{B'}$.

(b). As the shift is an auto-equivalence of $\mathscr{D}$, we may assume that $n = 0$.

To show that the inclusion functor $\mathscr{D}^{\leq 0} \to \mathscr{D}$ has a right adjoint, it suffices by Proposition I.4.7 of the notes to show that the functor $F_Y : \mathrm{Hom}_{\mathscr{D}}(\cdot, Y) : (\mathscr{D}^{\leq 0})\mathrm{op} \to \mathbf{Set}$ is representable for every $Y \in \mathrm{Ob}(\mathscr{D})$. Let $Y \in \mathrm{Ob}(\mathscr{D})$, and let $A \to Y \to B \overset{+1}{\to}$ be a distinguished triangle with $A \in \mathrm{Ob}(\mathscr{D}^{\leq 0})$ and $B \in \mathrm{Ob}(\mathscr{D}^{\geq 1})$. Let $X \in \mathrm{Ob}(\mathscr{D})$. Then we have an exact sequence

$$\mathrm{Hom}_{\mathscr{D}}(X, B[-1]) \to \mathrm{Hom}_{\mathscr{D}}(X, A) \to \mathrm{Hom}_{\mathscr{D}}(X, Y) \to \mathrm{Hom}_{\mathscr{D}}(X, B).$$

If $X \in \mathrm{Ob}(\mathscr{D}^{\leq 0})$, then $\mathrm{Hom}_{\mathscr{D}}(X, B[-1]) = \mathrm{Hom}_{\mathscr{D}}(X, B) = 0$ by condition (1) (because $B[-1] \in \mathrm{Ob}(\mathscr{D}^{\geq 2}) \subset \mathrm{Ob}(\mathscr{D}^{\geq 1})$), so the morphism $\mathrm{Hom}_{\mathscr{D}}(X, A) \to \mathrm{Hom}_{\mathscr{D}}(X, Y)$ is an isomorphism. This shows that F_Y is representable by the couple $(A, A \to Y)$ (note that the morphism $A \to Y$ is an element of $F_Y(A)$).

Similary, To show that the inclusion functor $\mathscr{D}^{\geq 0} \to \mathscr{D}$ has a left adjoint, it suffices by Proposition I.4.7 of the notes to show that the functor $G_X : \mathrm{Hom}_{\mathscr{D}}(X, \cdot) : \mathscr{D}^{\geq 0} \to \mathbf{Set}$

is representable for every $X \in \mathrm{Ob}(\mathscr{D})$. As in the previous paragraph, we see that, if $A \to X \to B \overset{+1}{\to}$ is a distinguished triangle with $A \in \mathrm{Ob}(\mathscr{D}^{\leq -1})$ and $B \in \mathrm{Ob}(\mathscr{D}^{\geq 0})$ (to get such a triangle, use condition (3) for $X[-1]$ and then apply the functor [1]), then G_X is representable by the pair $(B, X \to B)$.

(c). As in question (b), it suffices to treat the case $n = 0$. Let $X \in \mathrm{Ob}(\mathscr{D})$, and let $A \to X \to B \to A[1]$ be a distinguished triangle such that $A \in \mathrm{Ob}(\mathscr{D}^{\geq 0})$ and $B \in \mathrm{Ob}(\mathscr{D}^{\geq 1})$. We have seen in the solution of question (b) that the morphism $\tau^{\leq 0}X \to X$ is isomorphic to $A \to X$, and the morphism $X \to \tau^{\geq 1}X$ is isomorphic to $X \to B$, so the morphism $B \to A[1]$ induces a morphism $\delta : \tau^{\geq 1}X \to (\tau^{\leq 0}X)[1]$ that makes the triangle $\tau^{\leq 0}X \to X \to \tau^{\geq 1}X \overset{\delta}{\to} (\tau^{\leq 0}X)[1]$ distinguished.

(d). Let $X \in \mathrm{Ob}(\mathscr{D})$. As $\mathscr{D}^{\leq a} \subset \mathscr{D}^{\leq b}$, the canonical morphism $\tau^{\leq b}X \to X$ induces an isomorphism $\mathrm{Hom}_{\mathscr{D}}(\tau^{\leq a}X, \tau^{\leq b}X) \overset{\sim}{\to} \mathrm{Hom}_{\mathscr{D}}(\tau^{\leq a}X, \tau^{\leq b}X)$, so the canonical morphism $\tau^{\leq a}X \to X$ factors through a morphism $\tau^{\leq a}X \to \tau^{\leq b}X$; applying the functor $\tau^{\leq a}$, we get a sequence of morphisms

$$\tau^{\leq a}X \to \tau^{\leq a}\tau^{\leq b}X \to \tau^{\leq b}X \to X.$$

Hence, if Y is an object of $\mathscr{D}^{\leq a}$, then the bijection $\mathrm{Hom}_{\mathscr{D}}(Y, \tau^{\leq a}X) \overset{\sim}{\to} \mathrm{Hom}_{\mathscr{D}}(Y, X)$ is equal to the composition

$$\mathrm{Hom}_{\mathscr{D}}(Y, \tau^{\leq a}X) \to \mathrm{Hom}_{\mathscr{D}}(Y, \tau^{\leq a}\tau^{\leq b}X) \to \mathrm{Hom}_{\mathscr{D}}(Y, \tau^{\leq b}X) \to \mathrm{Hom}_{\mathscr{D}}(Y, X),$$

where the second and third maps are bijection. This shows that $\mathrm{Hom}_{\mathscr{D}}(Y, \tau^{\leq a}X) \to \mathrm{Hom}_{\mathscr{D}}(Y, \tau^{\leq a}\tau^{\leq b}X)$ is bijective for every $Y \in \mathrm{Ob}(\mathscr{D}^{\leq a})$, i.e. that the morphism $\tau^{\leq a}X \to \tau^{\leq a}\tau^{\leq b}X$ is an isomorphism. Similarly, we have a canonical isomorphism $\tau^{\geq b}\tau^{\geq a}X \overset{\sim}{\to} \tau^{\geq b}X$ for every $X \in \mathrm{Ob}(\mathscr{D})$.

Note also that, by question (c), if $c \in \mathbb{Z}$, then an object X of $\mathscr{D}$ is in $\mathscr{D}^{\leq c}$ (resp. $\mathscr{D}^{\geq c}$) if and only if $\tau^{\geq c+1}X = 0$ (resp. $\tau^{\leq c-1}X = 0$). In particular, if $X \in \mathrm{Ob}(\mathscr{D})$, then we have $\tau^{\geq b+1}\tau^{\geq a}\tau^{\leq b}X = \tau^{\geq b+1}\tau^{\leq b}X = 0$ and $\tau^{\leq a-1}\tau^{\leq b}\tau^{\geq a}X = \tau^{\leq a-1}\tau^{\geq a}X = 0$ (where the first isomorphisms are proved in the previous paragraph), so $\tau^{\geq a}\tau^{\leq b}X \in \mathscr{D}^{\leq b}$ and $\tau^{\leq b}\tau^{\geq a}X \in \mathrm{Ob}(\mathscr{D}^{\geq a})$.

Now let $X \in \mathrm{Ob}(\mathscr{D})$. By definition of $\tau^{\geq a}$, the morphism $\tau^{\leq b}X \to X \to \tau^{\geq a}X$ factors uniquely as

$$\tau^{\leq b}X \to \tau^{\geq a}\tau^{\leq b}X \overset{(1)}{\to} \tau^{\geq a}X.$$

As $\tau^{\geq a}\tau^{\leq b}X \in \mathrm{Ob}(\mathscr{D}^{\leq b})$, the morphism (1) factors uniquely as

$$\tau^{\geq a}\tau^{\leq b}X \overset{(2)}{\to} \tau^{\leq b}\tau^{\geq a}X \to \tau^{\geq a}X.$$

It remains to show that (2) is an isomorphism. Applying the octahedral axiom to the canonical morphism $\tau^{\leq a-1}X \to \tau^{\leq b}X \to X$ (and their composition), we get a commutative diagram whose rows and third column are distinguished triangles:

$$\begin{tikzcd}
\tau^{\leq a-1}X \arrow[r] \arrow[d, equal] & \tau^{\leq b}X \arrow[r] \arrow[d] & \tau^{\geq a}\tau^{\leq b}X \arrow[r, "+1"] \arrow[d] & {} \\
\tau^{\leq a-1}X \arrow[r] \arrow[d] & X \arrow[r] \arrow[d, equal] & \tau^{\geq a}X \arrow[r, "+1"] \arrow[d] & {} \\
\tau^{\leq b}X \arrow[r] & X \arrow[r] & \tau^{\geq b+1}X \arrow[r, "+1"] \arrow[d, "+1"] & {} \\
 & & {} &
\end{tikzcd}$$

So, by question (c), we have a morphism of distinguished triangles

$$\begin{tikzcd}
\tau^{\geq a}\tau^{\leq b}X \arrow[r] \arrow[d, "(2)"'] & \tau^{\geq a}X \arrow[r] \arrow[d, equal] & \tau^{\geq b+1}X \arrow[r, "+1"] \arrow[d, equal] & {} \\
\tau^{\leq b}\tau^{\geq a}X \arrow[r] & \tau^{\geq a}X \arrow[r] & \tau^{\geq b+1}X \arrow[r, "+1"] & {}
\end{tikzcd}$$

This shows that (2) is an isomorphism.

(e). We already showed this in the solution of question (d).

(f). We already know that $\mathscr{C}$ is a full addditive subcategory of $\mathscr{D}$, because it is the intersection of two full additive subcategories. We also have $\mathrm{Hom}^n(X,Y) = 0$ if $X, Y \in \mathrm{Ob}(\mathscr{C})$ and $n < 0$ by properties (1) and (2) of a t-structure. So, by question (b)(iv) of problem 1, it suffices to show that every morphism of $\mathscr{C}$ is admissible. Let $f : X \to Y$ be a morphism of $\mathscr{C}$, and complete it to a distinguished triangle $X \to Y \to S \overset{+1}{\to}$. Let $N = \tau^{\leq -1}S[-1]$ and $C = \tau^{\geq 0}S$. By question (c), we have a distinguished triangle $N[1] \to S \to C \overset{+1}{\to}$, so it suffices to show that N and C are in $\mathscr{C}$. By question (e), it suffices to show that $S \in \mathrm{Ob}(\mathscr{D}^{\leq 0} \cap \mathscr{D}^{\geq -1})$.

Note that we have a distinguished triangle $Y \to S \to X[1] \overset{+1}{\to}$. Let $S' = \tau^{\geq 1}S$. As $Y \in \mathrm{Ob}(\mathscr{D}^{\leq 0})$ and $X[1] \in \mathrm{Ob}(\mathscr{D}^{\leq -1}) \subset \mathrm{Ob}(\mathscr{D}^{\leq 0})$, condition (1) in the definition of a t-structure implies that $\mathrm{Hom}_{\mathscr{D}}(Y, S') = \mathrm{Hom}_{\mathscr{D}}(X[1], S') = 0$, and, applying the cohomological functor $\mathrm{Hom}_{\mathscr{D}}(\cdot, S')$ to the distinguished triangle $Y \to S \to X[1] \overset{+1}{\to}$, we deduce that $\mathrm{Hom}_{\mathscr{D}}(S, S') = 0$. As $\mathrm{Hom}_{\mathscr{D}}(S, S') = \mathrm{Hom}_{\mathscr{D}^{\geq 1}}(S', S')$, this implies that $S' = 0$, hence that $S \in \mathrm{Ob}(\mathscr{D}^{\leq 0})$. The proof that $S \in \mathrm{Ob}(\mathscr{D}^{\geq -1})$ is similar.

(g). We showed in the solution of (f) that, if X and Z are in $\mathscr{D}^{\leq 0}$ (resp. in $\mathscr{D}^{\geq 0}$), then so is Y. This immediately implies the result.

(h). (i) We first prove the hint. If $A \to B \to C \to 0$ is exact, the exactness of $0 \to \mathrm{Hom}_{\mathscr{A}}(C, D) \to \mathrm{Hom}_{\mathscr{A}}(B, D) \to \mathrm{Hom}_{\mathscr{A}}(A, D)$ for every D simply follows from the left exactness of the functor $\mathrm{Hom}_{\mathscr{A}}(\cdot, D)$. Suppose that we have morphisms $A \overset{f}{\to} B \overset{g}{\to} C$ such that $0 \to \mathrm{Hom}_{\mathscr{A}}(C, D) \to \mathrm{Hom}_{\mathscr{A}}(B, D) \to \mathrm{Hom}_{\mathscr{A}}(A, D)$ is exact for every D. Taking $D = \mathrm{Coker}\, g$, we see that the canonical morphism $u : C \to \mathrm{Coker}\, g$ is sent to $0 = u \circ g \in \mathrm{Hom}_{\mathscr{A}}(B, \mathrm{Coker}\, g)$, so $u = 0$, so $\mathrm{Coker}\, g = 0$ and g is surjective. Also, taking $D = C$, we see that id_C goes to $g \in \mathrm{Hom}_{\mathscr{D}}(B, C)$, then to $g \circ f \in \mathrm{Hom}_{\mathscr{A}}(A, C)$, so we have $g \circ f = 0$. It remains to show that the inclusion $\mathrm{Im}\, f \subset \mathrm{Ker}\, g$ is an isomorphism. Take $D = B/\,\mathrm{Im}\, f$ and let $v : B \to D$ be the canonical projection. Then $v \circ f = 0$, so, by hypothesis, there exists a morphism $w : C \to D$ such that $v = w \circ g$. In particular, we have $\mathrm{Ker}\, g \subset \mathrm{Ker}\, v = \mathrm{Im}\, f$.

Now we prove the statement of (i). As $X \in \mathscr{D}^{\leq 0}$, then we have $\mathrm{H}^0(X) = \tau^{\geq 0}X$, hence $\mathrm{Hom}_{\mathscr{C}}(\mathrm{H}^0(X), D) \simeq \mathrm{Hom}_{\mathscr{D}}(X, D)$ for every $D \in \mathrm{Ob}(\mathscr{C})$, and similarly for Y and Z. Also, if $D \in \mathrm{Ob}(\mathscr{C})$, then axiom (1) of t-structures implies that $\mathrm{Hom}_{\mathscr{D}}(X[1], D) = 0$. So, if $D \in \mathrm{Ob}(\mathscr{C})$, applying the cohomological functor $\mathrm{Hom}_{\mathscr{D}}(\cdot, D)$ to the distinguished triangle $X \to Y \to Z \overset{+1}{\to}$ gives an exact sequence

$$\mathrm{Hom}_{\mathscr{D}}(X[1], D) = 0 \to \mathrm{Hom}_{\mathscr{C}}(\mathrm{H}^0(Z), D) \to \mathrm{Hom}_{\mathscr{C}}(\mathrm{H}^0(Y), D) \to \mathrm{Hom}_{\mathscr{D}}(\mathrm{H}^0(X), D).$$

This shows that the sequence $\mathrm{H}^0(X) \to \mathrm{H}^0(Y) \to \mathrm{H}^0(Z) \to 0$ is exact in $\mathscr{C}$.

(ii) For every $T \in \mathrm{Ob}(\mathscr{D}^{\geq 1})$, applying the cohomological functor $\mathrm{Hom}_{\mathscr{D}}(\cdot, T)$ to $X \to Y \to Z \overset{+1}{\to}$ and using the fact that $\mathrm{Hom}_{\mathscr{D}}(X, T) = \mathrm{Hom}_{\mathscr{D}}(X[1], T) = 0$ (because $X, X[1] \in \mathrm{Ob}(\mathscr{D}^{\leq 0})$) gives an isomorphism $\mathrm{Hom}_{\mathscr{D}}(Z, T) \overset{\sim}{\to} \mathrm{Hom}_{\mathscr{D}}(Y, T)$, hence an isomorphism $\mathrm{Hom}_{\mathscr{D}}(\tau^{\geq 1}Z, T) \xrightarrow{\sim} \mathrm{Hom}_{\mathscr{D}}(\tau^{\geq 1}Y, T)$. This implies that the functor $\tau^{\geq 1}$ sends the morphism $Y \to Z$ to an isomorphism. Applying the octahedral axiom to the morphisms $Y \to Z \to \tau^{\geq 1}Z$, we get a commutative diagram whose rows and third column are distinguished triangles:

$$\begin{tikzcd}
Y \arrow[r] \arrow[d, equal] & Z \arrow[r] \arrow[d] & X[1] \arrow[r, "+1"] \arrow[d] & {} \\
Y \arrow[r] \arrow[d] & \tau^{\geq 1}Z \arrow[r] \arrow[d, equal] & \tau^{\geq 0}Y[1] \arrow[r, "+1"] \arrow[d] & {} \\
Z \arrow[r] & \tau^{\geq 1}Z \arrow[r] & \tau^{\leq 0}Z \arrow[r, "+1"] \arrow[d, "+1"] & {} \\
 & & {} &
\end{tikzcd}$$

So we have a distinguished triangle $X \to \tau^{\leq 0}Y \to \tau^{\leq 0}Z \xrightarrow{+1}$. Applying question (i) gives an exact sequence

$$\mathrm{H}^0(X) \to \mathrm{H}^0(\tau^{\leq 0}Y) \to \mathrm{H}^0(\tau^{\leq 0}Z) \to 0$$

in $\mathscr{C}$. As the morphism $\mathrm{H}^0(\tau^{\leq 0}Y) \to \mathrm{H}^0(Y)$ induced by $\tau^{\leq 0}Y \to Y$ is an isomorphism (by definition of H^0) and similarly for Z, we are done.

(iii) This is just the result of question (ii) in the opposite category. (Note that $(\mathscr{D}^{\geq 0}, \mathscr{D}^{\leq 0})$ is a t-structure on $\mathscr{D}^{\mathrm{op}}$.)

(iv) Applying the octahedral axiom to the morphisms $\tau^{\leq 0}X \to X \to Y$, we get a commutative diagram whose rows and third column are distinguished triangles:

$$\begin{tikzcd}
\tau^{\leq 0}X \arrow[r] \arrow[d, equal] & X \arrow[r] \arrow[d] & \tau^{\geq 1}X \arrow[r, "+1"] \arrow[d] & {} \\
\tau^{\leq 0}X \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d, equal] & T \arrow[r, "+1"] \arrow[d] & {} \\
X \arrow[r] & Y \arrow[r] & Z \arrow[r, "+1"] \arrow[d, "+1"] & {} \\
 & & {} &
\end{tikzcd}$$

Question (ii) for the second row gives an exact sequence

$$\mathrm{H}^0(\tau^{\leq 0}X) = \mathrm{H}^0(X) \to \mathrm{H}^0(Y) \to \mathrm{H}^0(T) \to 0,$$

and question (iii) for the distinguished triangle $T \to Z \to \tau^{\geq 1}X[1] \xrightarrow{+1}$ gives an exact sequence

$$0 \to \mathrm{H}^0(T) \to \mathrm{H}^0(Z).$$

Putting these two sequences together, we see that the sequence

$$\mathrm{H}^0(X) \to \mathrm{H}^0(Y) \to \mathrm{H}^0(Z)$$

is exact.

$\square$

3 The canonical t-structure

Let $\mathscr{A}$ be an abelian category.

(a). (2 points) Let $n \in \mathbb{Z}$. If $X \in \mathrm{Ob}(\mathrm{D}^{\leq n}(\mathscr{A}))$ and $Y \in \mathrm{Ob}(\mathrm{D}^{\geq n+1}(\mathscr{A}))$, show that $\mathrm{Hom}_{\mathrm{D}(\mathscr{A})}(X, Y) = 0$.

(b). (3 points) Show that $(\mathrm{D}^{\leq 0}(\mathscr{A}), \mathrm{D}^{\geq 0}(\mathscr{A}))$ is a t-structure on $\mathrm{D}(\mathscr{A})$, that its heart is equivalent to $\mathscr{A}$, and that the associated functor $\mathrm{H}^0 : \mathrm{D}(\mathscr{A}) \to \mathscr{A}$ is the 0th cohomology functor.

Solution.

(a). After replacing X and Y by isomorphic objects in $\mathrm{D}(\mathscr{A})$, we may assume that $X^m = 0$ for $m > n$ and $X^m = 0$ for $m \leq n$. Let $u : X \to Y$ be a morphism in $\mathrm{D}(\mathscr{A})$. Then we have morphisms $f : X \to Z$ and $s : Y \to Z$ in $K(\mathscr{A})$ such that s is a quasi-isomorphism and $u = s^{-1} \circ f$ in $\mathrm{D}(\mathscr{A})$. As $Y^m = 0$ for $m \leq n$, the morphism $s' = \tau^{\geq n+1} s : Y = \tau^{\geq n+1} Y \to Z' = \tau^{\geq n+1} Z$ is also a quasi-isomorphism, and we have a commutative diagram:

$$\begin{tikzcd}
& Z & \\
X \arrow[ur, "f"] \arrow[dr, "\tau^{\geq n+1} f"'] & & Y \arrow[ul, "s"'] \arrow[dl, "s'"] \\
& Z' &
\end{tikzcd}$$

By Theorem V.2.2.4 of the notes, this implies that $s^{-1} \circ f = s'^{-1} \circ \tau^{\geq n+1} f$ as morphisms in $\mathrm{D}(\mathscr{A})$. But $X^m = 0$ for $m \geq n+1$, so $\tau^{\geq n+1} f = 0$, and finally $u = 0$.

(b). Let $\mathscr{D}^{\leq 0} = \mathrm{D}^{\leq 0}(\mathscr{A})$ and $\mathscr{D}^{\geq 0} = \mathrm{D}^{\geq 0}(\mathscr{A})$. Note that, for every $n \in \mathbb{Z}$, we have $\mathscr{D}^{\leq n} = \mathrm{D}^{\leq n}(\mathscr{A})$ and $\mathscr{D}^{\geq n} = \mathrm{D}^{\geq n}(\mathscr{A})$. We check conditions (0)-(3) in the definition of a t-structure. Condition (0) is clear, condition (1) follows from question (a), condition (2) follows from the description of $\mathscr{D}^{\leq 1}$ and $\mathscr{D}^{\geq 1}$ that we just gave, and condition (3) follows from Proposition V.4.2.7(i) of the notes.

The fact that the heart of the t-structure $(\mathscr{D}^{\leq 0}, \mathscr{D}^{\geq 0})$ is canonically equivalent to $\mathscr{A}$ is proved in Remark V.4.2.5 of the notes. Finally, the isomorphisms of functors $\mathrm{H}^0 \simeq \tau^{\leq 0} \tau^{\geq 0}$ is Proposition V.4.2.7(ii) of the notes.

□

4 Torsion

Let $\mathscr{D} = \mathscr{D}(\mathbf{Ab})$, and let

$$^*\mathrm{D}^{\leq 0} = \{X \in \mathscr{D} \mid \mathrm{H}^i(X) = 0 \text{ for } i > 1, \text{ and } \mathrm{H}^1(X) \text{ is torsion}\}$$

and

$$^*\mathrm{D}^{\geq 0} = \{X \in \mathscr{D} \mid \mathrm{H}^i(X) = 0 \text{ for } i < 0, \text{ and } \mathrm{H}^0(X) \text{ is torsionfree}\}.$$

Let $\mathscr{C} = {}^*\mathrm{D}^{\leq 0} \cap {}^*\mathrm{D}^{\geq 0}$.

(a). Show that $({}^*\mathrm{D}^{\leq 0}, {}^*\mathrm{D}^{\geq 0})$ is a t-structure on $\mathscr{D}$.(2 points for condition (1), 1 for condition (2) and 2 for condition (3)) (b). Let $f : A \to B$ be a morphism of torsionfree abelian groups. We can see A and B as objects of $\mathscr{C}$ (concentrated in degree 0), and then f is also a morphism of $\mathscr{C}$.

(i) (2 points) Show that f is a monomorphism in $\mathscr{C}$ if and only if f is injective (and **Ab**) and $B/f(A)$ is torsionfree.

(ii) (1 point) Show that f is an epimorphism in $\mathscr{C}$ if and only if $f \otimes_{\mathbb{Z}} \mathbb{Q}$ is surjective.

(iii) (3 points) Calculate the kernel, the cokernel and the image of f in $\mathscr{C}$.

(c). (1 points) For every $n \geq 1$, show that $\mathrm{Ext}^1_{\mathbf{Ab}}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}) \simeq \mathbb{Z}/n\mathbb{Z}$.

(d). (1 point) If A and B are finitely generated abelian groups, show that $\mathrm{Ext}^n_{\mathbf{Ab}}(A, B) = 0$ for every $n \geq 2$. [1]

(e). (2 points) Let $X \in \mathrm{Ob}(\mathscr{C})$. Suppose that $\mathrm{H}^i(X)$ is a finitely generated abelian group for every $i \in \mathbb{Z}$. If $\mathrm{Hom}_{\mathscr{D}}(X, \mathbb{Z}) = 0$, show that $X = 0$.

(f). (1 point) Give an example of a nonzero $X \in \mathrm{Ob}(\mathscr{C})$ such that $\mathrm{Hom}_{\mathscr{D}}(X, \mathbb{Z}) = 0$.

(g). (2 points) Let $X \in \mathrm{Ob}(\mathscr{D})$. If $X \in \mathrm{Ob}({}^*\mathrm{D}^{\leq 0})$ (resp $X \in \mathrm{Ob}({}^*\mathrm{D}^{\geq 0} \cap \mathrm{D}^b(\mathbf{Ab}))$ and $\mathrm{H}^i(X)$ is finitely generated for every $i \in \mathbb{Z}$), show that $R\,\mathrm{Hom}_{\mathbf{Ab}}(X, \mathbb{Z})$ is in $\mathrm{D}^{\geq 0}(\mathbf{Ab})$ (resp. $\mathrm{D}^{\leq 0}(\mathbf{Ab})$).

(h). (3 points) Let $X \in \mathrm{Ob}(\mathscr{D})$ be a bounded complex of finitely generated abelian groups. If $R\,\mathrm{Hom}_{\mathbf{Ab}}(X, \mathbb{Z})$ is in $\mathrm{D}^{\geq 0}(\mathbf{Ab})$ (resp. $\mathrm{D}^{\leq 0}(\mathbf{Ab})$), show that $X \in \mathrm{Ob}({}^*\mathrm{D}^{\leq 0})$ (resp $X \in \mathrm{Ob}({}^*\mathrm{D}^{\geq 0})$).

Solution.

(a). Note that ${}^*\mathrm{D}^{\leq 0} \subset \mathrm{D}^{\leq 1}(\mathscr{A})$ and ${}^*\mathrm{D}^{\geq 0} \subset \mathrm{D}^{\geq 0}(\mathscr{A})$.

Condition (0) is obvious. Let $X \in \mathrm{Ob}({}^*\mathrm{D}^{\leq 0})$ and $Y \in \mathrm{Ob}({}^*\mathrm{D}^{\geq 1})$. Then $X \in \mathrm{D}^{\leq 1}(\mathscr{A})$ and $Y \in \mathrm{D}^{\geq 1}(\mathscr{A})$, so we have isomorphisms

$$\mathrm{Hom}_{\mathscr{D}}(X, Y) \xrightarrow{\sim} \mathrm{Hom}_{\mathscr{D}}(\tau^{\geq 1}X, Y) \xrightarrow{\sim} \mathrm{Hom}_{\mathscr{D}}(\tau^{\geq 1}X, \tau^{\leq 1}Y) = \mathrm{Hom}_{\mathbf{Ab}}(\mathrm{H}^1(X), \mathrm{H}^1(Y)).$$

As $\mathrm{H}^1(X)$ is torsion and $\mathrm{H}^1(Y)$ is torsionfree, this last group is equal to 0. This proves condition (1). Condition (2) is clear.

If $X \in \mathrm{Ob}(\mathscr{C}(\mathbf{Ab}))$, we set

$$B^1(X)' = \{z \in Z^1(X) \mid \exists n \in \mathbb{Z} - \{0\},\ nz \in B^1(X)\},$$

and we define ${}^*\tau^{\leq 0}X$ and ${}^*\tau^{\geq 1}X$ by

$${}^*\tau^{\leq 0}(X) = (\ldots \to X^{-2} \to X^{-1} \to X^0 \to B^1(X)' \to 0 \to \ldots)$$

and

$${}^*\tau^{\geq 1}(X) = (\ldots \to 0 \to 0 \to X^1/B^1(X)' \to X^2 \to X^3 \to \ldots).$$

These constructions are clearly functorial in X, and we have obvious morphisms ${}^*\tau^{\leq 0}X \to X$ and $X \to {}^*\tau^{\geq 1}X$. If we apply the functor H^n to the first morphism, then we get the identity of $\mathrm{H}^n(X)$ if $n \leq 0$, the inclusion $0 \to \mathrm{H}^n(X)$ if $n \geq 2$, and the inclusion $\mathrm{H}^1(X)_{\mathrm{tors}} \to \mathrm{H}^1(X)$ if $n = 1$. If we apply the functor H^n to the second morphism, then we get the identity of $\mathrm{H}^n(X)$ if $n \geq 2$, the unique map $\mathrm{H}^n(X) \to 0$ if $n \leq 0$, and the projection $\mathrm{H}^1(X) \to \mathrm{H}^1(X)/\mathrm{H}^1(X)_{\mathrm{tors}}$ if $n = 1$. In particular, if $X \to Y$ is a quasi-isomorphism, then so are the morphisms ${}^*\tau^{\leq 0}X \to {}^*\tau^{\leq 0}Y$ and ${}^*\tau^{\geq 1}X \to {}^*\tau^{\geq 1}Y$, so the functors ${}^*\tau^{\leq 0}$

[1] This actually holds for any abelian groups.

and ${}^*\tau^{\geq 1}$ induce endofunctors of $\mathrm{D}(|Ab)$, that we will still write ${}^*\tau^{\leq 0}$ and ${}^*\tau^{\geq 1}$. Finally, for every $X \in \mathrm{Ob}(\mathscr{C}(\mathbf{Ab}))$, the sequence $0 \to {}^*\tau^{\leq 0}X \to X \to {}^*\tau^{\geq 1}X \to 0$ is exact in $\mathscr{C}(\mathbf{Ab})$, so it induces a distinguished triangle ${}^*\tau^{\leq 0}X \to X \to {}^*\tau^{\geq 1}X \xrightarrow{+1}$ in $\mathrm{D}(\mathbf{Ab})$. As ${}^*\tau^{\leq 0}X \in \mathrm{Ob}({}^*\mathrm{D}^{\leq 0})$ and ${}^*\tau^{\geq 1}X \in \mathrm{Ob}({}^*\mathrm{D}^{\geq 1})$ by construction, this proves condition (3).

(b). Let $f : A \to B$ be a morphism of torsionfree abelian groups. We denote by $\mathrm{Ker}_{\mathscr{C}} f$, $\mathrm{Coker}_{\mathscr{C}} f$ etc the kernel, cokernel etc of f in the category $\mathscr{C}$, and by $\mathrm{Ker}\, f$, $\mathrm{Coker}\, f$ etc the kernel, cokernel etc of f in the category $\mathbf{Ab}$.

We solve question (iii) first, by using the formulas for $\mathrm{Ker}_{\mathscr{C}} f$ and $\mathrm{Coker}_{\mathscr{C}} f$ from question (b)(i) of problem 1. First we complete $f : A \to B$ to the distinguished triangle $A \xrightarrow{f} B \to \mathrm{Mc}(f) \xrightarrow{+1}$. By definition of the mapping cone, the complex $\mathrm{Mc}(f)$ is equal to

$$\ldots \to 0 \to A \xrightarrow{f} B \to 0 \to \ldots,$$

with B in degree 0. Then we have $\mathrm{Ker}_{\mathscr{C}} f[1] = {}^*\tau^{\leq -1}\,\mathrm{Mc}(f)$ and $\mathrm{Coker}_{\mathscr{C}} f = {}^*\tau^{\geq 0}\,\mathrm{Mc}(f)$. By the formulas that we proved in the solution of question (a), this shows that $\mathrm{Ker}_{\mathscr{C}} f$ is the complex

$$\ldots \to 0 \to A \to I \to 0 \to \ldots$$

with A in degree 0, and $\mathrm{Coker}_{\mathscr{C}} f$ is the complex

$$\ldots \to 0 \to B/I \to 0 \to \ldots$$

with B/I in degree 0, where

$$I = \{x \in B \mid \exists n \in \mathbb{Z} - \{0\},\ nx \in \mathrm{Im}\, f\}.$$

Note that the abelian group B/I is torsionfree, so we can apply what we just did to calculate the kernel (in $\mathscr{C}$) of the canonical projection $B \to B/I$, which is $\mathrm{Im}_{\mathscr{C}} f$. We get that

$$\mathrm{Im}_{\mathscr{C}} f = (\ldots \to 0 \to B \to B/I \to 0 \to \ldots),$$

where B is in degree 0; this is quasi-isomorphic to the object I of $\mathbf{Ab}$, seen as complex concentrated in degree 0 (the quasi-isomorphism is given by the inclusion $I \subset B$); note that this is an object of $\mathscr{C}$ because I is torsionfree.

Now we can solve (i) and (ii) easily. For example, the morphism f is an epismorphism in $\mathscr{C}$ if and only if $I = \mathrm{Im}\, f = B$, that is, if and only if $B/\mathrm{Im}\, f$ is torsion, which is equivalent to the fact that $f \otimes_{\mathbb{Z}} \mathbb{Z}$ is surjective (in the category of $\mathbb{Q}$-vector spaces). On the other hand, the morphism f is a monomorphism in $\mathscr{C}$ if and only if $\mathrm{Ker}_{\mathscr{C}} f = 0$, which means that the complex $\ldots \to 0 \to A \to I \to 0 \to \ldots$ is quasi-isomorphic to 0, i.e. has zero cohomology. This happens if and only if the morphism $A \to I$ is injective (i.e., as $I \subset B$, the morphism f itself is injective in $\mathbf{Ab}$) and $I = \mathrm{Im}\, f$ (i.e. $B/\mathrm{Im}\, f$ is torsionfree).

(c). The exact sequence

$$0 \to \mathbb{Z} \xrightarrow{\cdot n} \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \to 0$$

is a projective resolution of $\mathbb{Z}/n\mathbb{Z}$ in $\mathbf{Ab}$, so we can calculate the $\mathrm{Ext}^i_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z})$ by applying the functor $\mathrm{Hom}_{\mathbb{Z}}(\cdot, \mathbb{Z})$ to the complex $\ldots \to 0 \to \mathbb{Z} \xrightarrow{\cdot n} \mathbb{Z} \to 0 \to \ldots$ (with the second $\mathbb{Z}$ in degree 0). We get $\mathrm{Ext}^0_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}) = \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}) = \mathrm{Ker}(\mathbb{Z} \xrightarrow{\cdot n} \mathbb{Z}) = 0$, $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}) = \mathrm{Coker}(\mathbb{Z} \xrightarrow{\cdot n} \mathbb{Z}) = \mathbb{Z}/n\mathbb{Z}$, and $\mathrm{Ext}^i_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}) = 0$ if $i \not\in \{0, 1\}$.

(d). Using the resolution of $\mathbb{Z}/n\mathbb{Z}$ from the solution of question (c), we get that, if B is any abelian group, then $\mathrm{Ext}^0_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, B) = \{x \in B \mid nx = 0\}$, $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, B) = B/nB$, and $\operatorname{Ext}^i_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, B) = 0$ if $i \notin \{0,1\}$. Also, as $\mathbb{Z}$ is a projective $\mathbb{Z}$-module, we have $\operatorname{Ext}^0_{\mathbb{Z}}(\mathbb{Z}, B) = \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}, B) = B$ and $\operatorname{Ext}^i_{\mathbb{Z}}(\mathbb{Z}, B) = 0$ for $i \neq 0$.

Let A be a finitely generated abelian group and B be an abelian group. Then $A = A_0 \oplus A_1$ where A_0 is finitely generated free abelian group and A_1 is a finitely generated torsion abelian group, i.e. a direct sum of groups $\mathbb{Z}/n\mathbb{Z}$ for $n \geq 1$. So, if $i \geq 1$, we have $\operatorname{Ext}^i_{\mathbb{Z}}(A, B) = \operatorname{Ext}^i_{\mathbb{Z}}(A_1, B)$, and $\operatorname{Ext}^i_{\mathbb{Z}}(A_1, B) = 0$ if $i \geq 2$.

(e). As $\mathrm{H}^i(X) = 0$ for $i \notin \{0,1\}$, we have $\tau^{\leq -1}X \simeq 0$ and $\tau^{\geq 2}X \simeq 0$, so, by Proposition V.4.2.7 of the notes, the canonical morphisms $\tau^{\leq 0}X \to \mathrm{H}^0(X)$ and $\mathrm{H}^1(X)[-1] \to \tau^{\geq 1}X$ are isomorphisms. In particular, using the remark after Definition V.4.5.1 of the notes, we get, for every $i \in \mathbb{Z}$,

$$\operatorname{Hom}_{\mathscr{D}}(\tau^{\geq 1}X[-i], \mathbb{Z}) = \operatorname{Hom}_{\mathscr{D}}(\mathrm{H}^1(X), \mathbb{Z}[1+i]) = \operatorname{Ext}^{1+i}_{\mathbb{Z}}(\mathrm{H}^1(X), \mathbb{Z}) = 0,$$

which is 0 by question (d) if $i \geq 1$. On the other hand, we have

$$\operatorname{Hom}_{\mathscr{D}}(\tau^{\leq 0}X[i], \mathbb{Z}) = \operatorname{Ext}^{-i}_{\mathbb{Z}}(\mathrm{H}^0(X), \mathbb{Z}),$$

which is equal to 0 if $i \geq 1$. So applying $\operatorname{Hom}_{\mathscr{D}}(\cdot, \mathbb{Z})$ to the distinguished triangle $\tau^{\leq 0}X \to X \to \tau^{\geq 1}X \xrightarrow{+1}$ gives an exact sequence

$$0 \to \operatorname{Ext}^1_{\mathbb{Z}}(\mathrm{H}^1(X), \mathbb{Z}) \to \operatorname{Hom}_{\mathscr{D}}(X, \mathbb{Z}) \to \operatorname{Hom}_{\mathbb{Z}}(\mathrm{H}^0(X), \mathbb{Z}) \to 0.$$

Hence, of $\operatorname{Hom}_{\mathscr{D}}(X, \mathbb{Z}) = 0$, then $\operatorname{Hom}_{\mathbb{Z}}(H^0(X), \mathbb{Z}) = \operatorname{Ext}^1_{\mathbb{Z}}(\mathrm{H}^1(X), \mathbb{Z}) = 0$.

As X is an object of $\mathscr{C}$, we know that $\mathrm{H}^0(X)$ is torsionfree and $\mathrm{H}^1(X)$ is torsion. Moreover, by assumption, both $\mathrm{H}^0(X)$ and $\mathrm{H}^1(X)$ are finitely generated. So we have $\mathrm{H}^0(X) \simeq \mathbb{Z}^n$ for some $n \in \mathbb{N}$, and $\operatorname{Hom}_{\mathbb{Z}}(\mathrm{H}^0(X), \mathbb{Z}) \simeq \mathbb{Z}^n \simeq \mathrm{H}^0(X)$ (non canonically). On the other hand, we have $\mathrm{H}^1(X) \simeq \bigoplus_{s=1}^r \mathbb{Z}/n_s\mathbb{Z}$ for some integers $n_1, \ldots, n_r \geq 2$. By question (c), we get that $\operatorname{Ext}^1_{\mathbb{Z}}(\mathrm{H}^1(X), \mathbb{Z}) \simeq \mathrm{H}^1(X)$ (also non canonically). So, if $\operatorname{Hom}_{\mathscr{D}}(X, \mathbb{Z}) = 0$, then $\mathrm{H}^0(X) = 0$ and $\mathrm{H}^1(X) = 0$, which shows that $X \simeq 0$ in $\mathscr{D}$, hence in $\mathscr{C}$.

(f). Let $X = \mathbb{Q}$ (concentrated in degree 0). Then $X \not\simeq 0$, but $\operatorname{Hom}_{\mathscr{D}}(X, \mathbb{Z}) = \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Z}) = 0$.

(g). Suppose that $X \in \operatorname{Ob}({}^*\mathrm{D}^{\leq 0})$. Then $X \in \operatorname{Ob}(\mathrm{D}^{\leq 1}(\mathbf{Ab}))$, so we have an exact triangle

$$\tau^{\leq 0}X \to X \to \tau^{\geq 1}X \simeq \mathrm{H}^1(X)[-1] \xrightarrow{+1}.$$

Applying the triangulated functor $R\operatorname{Hom}_{\mathbf{Ab}}(\cdot, \mathbb{Z})$, we get an exact triangle in $\mathrm{D}(\mathbf{Ab})$:

$$R\operatorname{Hom}_{\mathscr{D}}(\mathrm{H}^1(X)[-1], \mathbb{Z}) = R\operatorname{Hom}_{\mathscr{D}}(\mathrm{H}^1(X), \mathbb{Z})[1] \to R\operatorname{Hom}_{\mathscr{D}}(X, \mathbb{Z}) \to R\operatorname{Hom}_{\mathscr{D}}(\tau^{\leq 0}X, \mathbb{Z}) \xrightarrow{+1}.$$

If $i \leq -1$, then $\mathbb{Z}[-i] \in \mathrm{D}^{\geq 1}(\mathbf{Ab})$, so $\mathrm{H}^i(R\operatorname{Hom}_{\mathscr{D}}(\tau^{\leq 0}X, \mathbb{Z})) = \operatorname{Hom}_{\mathscr{D}}(\tau^{\leq 0}X, \mathbb{Z}[-i]) = 0$. This shows that $R\operatorname{Hom}_{\mathscr{D}}(\tau^{\leq 0}X, \mathbb{Z}) \in \operatorname{Ob}(\mathrm{D}^{\geq 0}(\mathbf{Ab}))$. For every $i \in \mathbb{Z}$, we have

$$\mathrm{H}^i(R\operatorname{Hom}_{\mathscr{D}}(\tau^{\geq 1}X, \mathbb{Z})) = \operatorname{Ext}^{i+1}_{\mathbb{Z}}(\mathrm{H}^1(X), \mathbb{Z}).$$

This is equal to 0 if $i \leq -2$; if $i = -1$, then $\operatorname{Ext}^{i+1}_{\mathbb{Z}}(\mathrm{H}^1(X), \mathbb{Z}) = \operatorname{Hom}_{\mathbb{Z}}(\mathrm{H}^1(X), \mathbb{Z})$ is also equal to 0, since $\mathrm{H}^1(X)$ is torsion. So $R\operatorname{Hom}_{\mathscr{D}}(\tau^{\geq 1}X, \mathbb{Z})$ is also in $\operatorname{Ob}(\mathrm{D}^{\geq 0}(\mathbf{Ab}))$. As we have an exact sequence

$$\mathrm{H}^i(R\operatorname{Hom}_{\mathscr{D}}(\tau^{\geq 1}X, \mathbb{Z})) \to \mathrm{H}^i(R\operatorname{Hom}_{\mathscr{D}}(X, \mathbb{Z})) \to \mathrm{H}^i(R\operatorname{Hom}_{\mathscr{D}}(\tau^{\leq 0}X, \mathbb{Z}))$$

for every $i \in \mathbb{Z}$, we conclude that $\mathrm{H}^i(R\operatorname{Hom}_{\mathscr{D}}(X, \mathbb{Z})) = 0$ for $i \leq -1$, i.e. that $R\operatorname{Hom}_{\mathscr{D}}(X, \mathbb{Z})$ is in $\mathrm{D}^{\geq 0}(\mathbf{Ab})$.

Suppose that $X \in \mathrm{Ob}({}^*\mathrm{D}^{\geq 0} \cap \mathrm{D}^b(\mathbf{Ab}))$ and that the $\mathrm{H}^i(X)$ are finitely generated. We have $\mathrm{H}^i(X) = 0$ if $i \leq -1$ or if i is big enough, and $\mathrm{H}^0(X)$ is torsionfree. In particular, the canonical morphism $X \to \tau^{\geq 0}X$ is an isomorphism and $\tau^{\geq i}X \simeq 0$ for i big enough. So it suffices to prove that, if $i \geq 0$ is an integer such that $R\operatorname{Hom}_{\mathscr{D}}(\tau^{\geq i+1}X, \mathbb{Z})$ is in $\mathrm{D}^{\leq 0}(\mathbf{Ab})$, then $R\operatorname{Hom}_{\mathscr{D}}(\tau^{\geq i}X, \mathbb{Z})$ is also in $\mathrm{D}^{\leq 0}(\mathbf{Ab})$. We have an exact triangle

$$\mathrm{H}^i(X)[-i] \to \tau^{\geq i}X \to \tau^{\geq i+1}X \overset{+1}{\to},$$

so we get an exact triangle

$$R\operatorname{Hom}_{\mathscr{D}}(\tau^{\geq i+1}X, \mathbb{Z}) \to R\operatorname{Hom}_{\mathscr{D}}(\tau^{\geq i}X, \mathbb{Z}) \to R\operatorname{Hom}_{\mathscr{D}}(\mathrm{H}^i(X)[-i], \mathbb{Z}) \overset{+1}{\to},$$

and it suffices to prove that $R\operatorname{Hom}_{\mathscr{D}}(\mathrm{H}^i(X)[-i], \mathbb{Z})$ is in $\mathrm{D}^{\leq 0}(\mathbf{Ab})$. Let $j \geq 1$. Then

$$\mathrm{H}^j(R\operatorname{Hom}_{\mathscr{D}}(\mathrm{H}^i(X)[-i], \mathbb{Z})) = \mathrm{H}^j(R\operatorname{Hom}_{\mathscr{D}}(\mathrm{H}^i(X), \mathbb{Z}[i])) = \operatorname{Ext}^{i+j}_{\mathbb{Z}}(\mathrm{H}^i(X), \mathbb{Z}).$$

If $i \geq 1$, then $i + j \geq 2$, so this group is zero by question (d). If $i = 0$, then $\mathrm{H}^i(X)$ is a free $\mathbb{Z}$-module, so $\operatorname{Ext}^j_{\mathbb{Z}}(\mathrm{H}^i(X), \mathbb{Z}) = 0$ for every $j \geq 1$. In both cases, we get that $\mathrm{H}^j(R\operatorname{Hom}_{\mathscr{D}}(\mathrm{H}^i(X)[-i], \mathbb{Z})) = 0$.

(h). If $n \in \mathbb{Z}$, let ${}^*\tau^{\leq n}$ and ${}^*\tau^{\geq n}$ be the truncation functors for the t-structure $({}^*\mathrm{D}^{\leq n}, {}^*\mathrm{D}^{\geq n})$, and define ${}^*\mathrm{H}^n : \mathscr{D} \to \mathscr{C}$ by ${}^*\mathrm{H}^n(X) = ({}^*\tau^{\leq n}{}^*\tau^{\geq n}X)[n] = {}^*\mathrm{H}^0(X[n])$.

Let $X \in \mathrm{Ob}(\mathscr{D})$ satisfying the conditions of the question. Then $\mathrm{H}^n(X) = 0$ for all but finitely many $n \in \mathbb{Z}$, so there exists $N \in \mathbb{N}$ such that $\mathrm{H}^n(X) = 0$ for $|n| \geq N$. Then $X \in \mathrm{Ob}({}^*\mathrm{D}^{\leq N})$ and $X \in \mathrm{Ob}({}^*\mathrm{D}^{\geq -N})$, so ${}^*\tau^{\leq n}X \overset{\sim}{\to} X$ for $n \geq N+1$ and $X \overset{\sim}{\to} {}^*\tau^{\geq n}X$ for $n \geq -N-1$.

First we show the following claim: If $R\operatorname{Hom}_{\mathscr{D}}(X, \mathbb{Z}) = 0$, then $X = 0$. Indeed, suppose that $X \neq 0$, and let n be the biggest integer such that ${}^*\tau^{\leq n}X \to X$ is not an isomorphism (such a n exists because ${}^*\tau^{\leq n}X = 0$ for n small enough). We have an exact triangle

$${}^*\tau^{\leq n}X \to X \to {}^*\mathrm{H}^{n+1}(X)[-n-1] \overset{+1}{\to}$$

with ${}^*\mathrm{H}^{n+1}(X) \neq 0$, hence an exact triangle

$$R\operatorname{Hom}_{\mathscr{D}}({}^*\mathrm{H}^{n+1}(X)[-n-1], \mathbb{Z}) \to R\operatorname{Hom}_{\mathscr{D}}(X, \mathbb{Z}) \to R\operatorname{Hom}_{\mathscr{D}}({}^*\tau^{\leq n}X, \mathbb{Z}) \overset{+1}{\to}.$$

By question (g), we have $R\operatorname{Hom}_{\mathscr{D}}({}^*\tau^{\leq n}X, \mathbb{Z}) \in \mathrm{Ob}(\mathrm{D}^{\geq -n}(\mathbf{Ab}))$, so the morphism

$$\mathrm{H}^{-n-1}(R\operatorname{Hom}_{\mathscr{D}}({}^*\mathrm{H}^{n+1}(X)[-n-1], \mathbb{Z})) \to \mathrm{H}^{-n-1}(R\operatorname{Hom}_{\mathscr{D}}(X, \mathbb{Z}))$$

is an isomorphism. As $\mathrm{H}^{-n-1}(R\operatorname{Hom}_{\mathscr{D}}({}^*\mathrm{H}^{n+1}(X)[-n-1], \mathbb{Z})) = \operatorname{Hom}_{\mathscr{D}}({}^*\mathrm{H}^{n+1}(X), \mathbb{Z}) \neq 0$ by question (e), we conclude that $\mathrm{H}^{-n-1}(R\operatorname{Hom}_{\mathscr{D}}(X, \mathbb{Z})) \neq 0$, hence $R\operatorname{Hom}_{\mathscr{D}}(X, \mathbb{Z}) \neq 0$.

Suppose that $R\operatorname{Hom}_{\mathscr{D}}(X, \mathbb{Z})$ is in $\mathrm{D}^{\geq 0}(\mathbf{Ab})$. We want to show that $X \in \mathrm{Ob}({}^*\mathrm{D}^{\leq 0})$. We have a distinguished triangle

$${}^*\tau^{\leq 0}X \to X \to {}^*\tau^{\geq 1}X \overset{+1}{\to},$$

hence a distinguished triangle

$$R\operatorname{Hom}_{\mathscr{D}}({}^*\tau^{\geq 1}X, \mathbb{Z}) \to R\operatorname{Hom}_{\mathscr{D}}(X, \mathbb{Z}) \to R\operatorname{Hom}_{\mathscr{D}}({}^*\tau^{\leq 0}X, \mathbb{Z}) \overset{+1}{\to}.$$

Also, by question (g), we have $R\operatorname{Hom}_{\mathscr{D}}({}^*\tau^{\geq 1}X, \mathbb{Z}) \in \mathrm{D}^{\leq -1}(\mathbf{Ab})$ and $R\operatorname{Hom}_{\mathscr{D}}({}^*\tau^{\leq 0}X, \mathbb{Z}) \in \mathrm{D}^{\geq 0}(\mathbf{Ab})$. In particular, if $i \leq -1$, then

$\mathrm{H}^i(R\,\mathrm{Hom}_{\mathscr{D}}({}^*\tau^{\geq 1}X,\mathbb{Z})) \overset{\sim}{\to} \mathrm{H}^i(R\,\mathrm{Hom}_{\mathscr{D}}(X,\mathbb{Z})) = 0$. This implies that $R\,\mathrm{Hom}_{\mathscr{D}}({}^*\tau^{\geq 1}X,\mathbb{Z}) = 0$, hence that ${}^*\tau^{\geq 1}X = 0$ by the claim we proved in the previous paragraph. So ${}^*\tau^{\leq 0}X \to X$ is an isomorphism.

Now suppose that $R\,\mathrm{Hom}_{\mathscr{D}}(X,\mathbb{Z})$ is in $\mathrm{D}^{\leq 0}(\mathbf{Ab})$. We want to show that $X \in \mathrm{Ob}({}^*\mathrm{D}^{\geq 0})$. We have a distinguished triangle

$$
{}^*\tau^{\leq -1}X \to X \to {}^*\tau^{\geq 0}X \xrightarrow{+1},
$$

hence a distinguished triangle

$$
R\,\mathrm{Hom}_{\mathscr{D}}({}^*\tau^{\geq 0}X,\mathbb{Z}) \to R\,\mathrm{Hom}_{\mathscr{D}}(X,\mathbb{Z}) \to R\,\mathrm{Hom}_{\mathscr{D}}({}^*\tau^{\leq -1}X,\mathbb{Z}) \xrightarrow{+1}.
$$

Also, by question (g), we have $R\,\mathrm{Hom}_{\mathscr{D}}({}^*\tau^{\leq -1}X,\mathbb{Z}) \in \mathrm{D}^{\geq 1}(\mathbf{Ab})$ and $R\,\mathrm{Hom}_{\mathscr{D}}({}^*\tau^{\geq 0}X,\mathbb{Z}) \in \mathrm{D}^{\leq 0}(\mathbf{Ab})$. In particular, if $i \geq 1$, then $0 = \mathrm{H}^i(R\,\mathrm{Hom}_{\mathscr{D}}(X,\mathbb{Z})) \overset{\sim}{\to} \mathrm{H}^i(R\,\mathrm{Hom}_{\mathscr{D}}({}^*\tau^{\leq -1}X,\mathbb{Z}))$. This implies that $R\,\mathrm{Hom}_{\mathscr{D}}({}^*\tau^{\leq -1}X,\mathbb{Z}) = 0$, hence that ${}^*\tau^{\geq 1}X = 0$. So ${}^*\tau^{\leq 0}X \to X$ is an isomorphism.

□

5 Weights

Let $\mathscr{A}$ be an abelian category. Suppose that we have a family $(\mathscr{A}_n)_{n\in\mathbb{Z}}$ of full abelian subcategories of $\mathscr{A}$ such that:

(1) If $n \neq m$, then $\mathrm{Hom}_{\mathscr{A}}(A,B) = 0$ for any $A \in \mathrm{Ob}(\mathscr{A}_n)$ and $B \in \mathrm{Ob}(\mathscr{A}_m)$.

(2) Any object A of $\mathscr{A}$ has a *weight filtration*, that is, an increasing filtration $\mathrm{Fil}_\bullet A$ such that $\mathrm{Fil}_n A = 0$ for $n << 0$, $\mathrm{Fil}_n A = A$ for $n >> 0$ and $\mathrm{Fil}_n A/\mathrm{Fil}_{n+1} A \in \mathrm{Ob}(\mathscr{A}_n)$ for every $n \in \mathbb{Z}$.

For every $n \in \mathbb{Z}$, we denote by $\mathscr{A}_{\leq n}$ (resp. $\mathscr{A}_{\geq n+1}$) the full subcategory of $\mathscr{A}$ whose objects are the $A \in \mathrm{Ob}(\mathscr{A})$ having a weight filtration $\mathrm{Fil}_\bullet A$ such that $\mathrm{Fil}_n A = A$ (resp. $\mathrm{Fil}_n A = 0$).

(a). (1 point) If $A \in \mathrm{Ob}(\mathscr{A}_{\leq n})$ and $B \in \mathrm{Ob}(\mathscr{A}_{\geq n+1})$, show that $\mathrm{Hom}_{\mathscr{A}}(A,B) = 0$.

(b). (2 points) Show that the inclusion functor $\mathscr{A}_{\leq n} \subset \mathscr{A}$ has a right adjoint ${}^w\tau^{\leq n}$, and that the inclusion functor $\mathscr{A}_{\geq n} \subset \mathscr{A}$ has a left adjoint ${}^w\tau^{\geq n}$.

(c). (2 points) If $A \in \mathrm{Ob}(\mathscr{A}_{\leq n})$ and $B \in \mathrm{Ob}(\mathscr{A}_{\geq n+1})$, show that $\mathrm{Ext}^i_{\mathscr{A}}(A,B) = 0$ for every $i \in \mathbb{Z}$.

(d). (4 points) Define two full subcategories ${}^w\mathrm{D}^{\leq n}$ and ${}^w\mathrm{D}^{\geq n}$ of $\mathrm{D}^b(\mathscr{A})$ by:

$$
\mathrm{Ob}({}^w\mathrm{D}^{\leq n}) = \{X \in \mathrm{Ob}(\mathrm{D}^b(\mathscr{A})) \mid \forall i \in \mathbb{Z},\ \mathrm{H}^i(X) \in \mathscr{A}_{\leq n}\}
$$

and

$$
\mathrm{Ob}({}^w\mathrm{D}^{\geq n+1}) = \{X \in \mathrm{Ob}(\mathrm{D}^b(\mathscr{A})) \mid \forall i \in \mathbb{Z},\ \mathrm{H}^i(X) \in \mathscr{A}_{\geq n+1}\}.
$$

Show that $({}^w\mathrm{D}^{\leq n}, {}^w\mathrm{D}^{\geq n+1})$ is a t-structure on $\mathrm{D}^b(\mathscr{A})$, and that the heart of this t-structure is $\{0\}$.

Solution.

(a). If $\mathrm{Fil}_\bullet A$ is a filtration on an object A of $\mathscr{A}$ such that $\mathrm{Fil}_n A = A$ for $n >> 0$ and $\mathrm{Fil}_n A = 0$ for $n << 0$, the length of $\mathrm{Fil}_\bullet$ is by definition the integer $n_1 - n_2$, where n_1 is the smallest integer such that $\mathrm{Fil}_{n_1} A = A$ and n_2 is the biggest integer such that $\mathrm{Fil}_{n_2} A = 0$. For example, if the length of $\mathrm{Fil}_\bullet A$ is 0, then there exists $n \in \mathbb{Z}$ such that $\mathrm{Fil}_n A = A$ and $\mathrm{Fil}_n A = 0$, so $A = 0$.

If A has a weight filtration $\mathrm{Fil}_\bullet A$ of length 1, then there exists $n \in \mathbb{Z}$ such that $\mathrm{Fil}_n A = A$ and $\mathrm{Fil}_{n-1} A = 0$, so $A = \mathrm{Fil}_n A / \mathrm{Fil}_{n-1} A \in \mathrm{Ob}(\mathscr{A}_n)$. Conversely, if $A \in \mathrm{Ob}(\mathscr{A}_n)$ for some $n \in \mathbb{Z}$, then it has a weight filtration $\mathrm{Fil}_\bullet A$ of length 1, given by $\mathrm{Fil}_k A = A$ for $k \geq n$ and $\mathrm{Fil}_k A = 0$ for $k \leq n-1$.

For every subset I of $\mathbb{N}$, we denote by $\mathscr{A}_I$ the full subcategory of $\mathscr{A}$ whose objects are the $A \in \mathrm{Ob}(\mathscr{A})$ having a weight filtration $\mathrm{Fil}_\bullet A$ such that $\mathrm{Fil}_n A / \mathrm{Fil}_{n-1} A = 0$ if $n \notin I$.

We prove a more general statement than that of the question: if I and J are disjoint subsets of $\mathbb{N}$, if $A \in \mathrm{Ob}(\mathscr{A}_I)$ and $B \in \mathrm{Ob}(\mathscr{A}_J)$, then $\mathrm{Hom}_{\mathscr{A}}(A, B) = 0$. Choose weight filtrations $\mathrm{Fil}_\bullet A$ and $\mathrm{Fil}_\bullet B$ on A and B such that $\mathrm{Fil}_n A = \mathrm{Fil}_{n-1} A$ if $n \notin I$ and $\mathrm{Fil}_n B = \mathrm{Fil}_{n-1} B$ if $n \notin J$. We prove that $\mathrm{Hom}_{\mathscr{A}}(A, b) = 0$ by induction on the sum of the lengths ℓ_A and ℓ_B of $\mathrm{Fil}_\bullet A$ and $\mathrm{Fil}_\bullet B$. If $\ell_A + \ell_B \leq 1$, then one of the filtrations has length 0, so one of A of B is 0, so the result if clear. If $\ell_A + \ell_B \geq 3$, then one of the filtrations has length ≥ 2. If for example $\ell_A \geq 2$, then $\mathrm{Fil}_\bullet A$ induces a weight filtration of length $\ell_A - 1$ on $A' = \mathrm{Fil}_{n-1} A$, and $A'' = \mathrm{Fil}_n A / \mathrm{Fil}_{n-1} A \in \mathrm{Ob}(\mathscr{A}_n)$ has a weight filtration of length 1. As A' and A'' are both in $\mathscr{A}_I$, the induction hypothesis implies that $\mathrm{Hom}_{\mathscr{A}}(A', B) = \mathrm{Hom}_{\mathscr{A}}(A'', B) = 0$. Moreover, the exact sequence

$$0 \to A' \to A \to A'' \to 0$$

induces an exact sequence

$$\mathrm{Hom}_{\mathscr{A}}(A'', B) \to \mathrm{Hom}_{\mathscr{A}}(A, B) \to \mathrm{Hom}_{\mathscr{A}}(A', B),$$

so $\mathrm{Hom}_{\mathscr{A}}(A, B) = 0$. The case where $\ell_B \geq 2$ is similar. It remains to treat the case where $\ell_A + \ell_B = 2$. If $\ell_A = 0$ (resp. $\ell_B = 0$), then $A = 0$ (resp. $B = 0$), so the result is obvious. Finally, suppose that $\ell_A = 1$ and $\ell_B = 1$. Then there exist $n_A \in I$ and $n_B \in J$ such that $A \in \mathrm{Ob}(\mathscr{A}_{n_A})$ and $B \in \mathrm{Ob}(\mathscr{A}_{n_B})$; as $I \cap J = \varnothing$, we have $n_A \neq n_B$, so $\mathrm{Hom}_{\mathscr{A}}(A, B) = 0$ by assumption (1).

(b). We show the existence of ${}^w\tau^{\leq n}$. It suffices to show that, for every $B \in \mathrm{Ob}(\mathscr{A})$, the functor $\mathscr{A}_{\leq n} \to \mathbf{Set}$, $A \mapsto \mathrm{Hom}_{\mathscr{A}}(A, B)$ is representable. Fix $B \in \mathrm{Ob}(\mathscr{A})$, and let $\mathrm{Fil}_\bullet B$ be a weight filtration on B. Then $\mathrm{Fil}_\bullet B$ induces a weight filtration on $B/\mathrm{Fil}_n B$, which shows that $B/\mathrm{Fil}_n B \in \mathrm{Ob}(\mathscr{A}_{\geq n+1})$. Let $A \in \mathrm{Ob}(\mathscr{A}_{\leq n})$. Applying $\mathrm{Hom}_{\mathscr{A}}(A, \cdot)$ to the exact sequence

$$0 \to \mathrm{Fil}_n B \to B \to B/\mathrm{Fil}_n B \to 0$$

and using question (a), we see that the canonical morphism $\mathrm{Hom}_{\mathscr{A}}(A, \mathrm{Fil}_n B) \to \mathrm{Hom}_{\mathscr{A}}(A, B)$ is an isomorphism. This shows that the couple $(\mathrm{Fil}_n B, \mathrm{Fil}_n B \subset B)$ represents the functor $\mathscr{A}_{\leq n} \to \mathbf{Set}$, $A \mapsto \mathrm{Hom}_{\mathscr{A}}(A, B)$. In particular, we get ${}^w\tau^{\leq n} B = \mathrm{Fil}_n B$. By uniqueness of the right adjoint, this implies that the weight filtration on B is unique.

If $A \in \mathrm{Ob}(\mathscr{A})$ and $\mathrm{Fil}_\bullet A$ is its weight filtration, a similar proof shows that the pair $(A/\mathrm{Fil}_{n-1} A, A \to A/\mathrm{Fil}_{n-1} A)$ represents the functor $\mathscr{A}_{\geq n} \to \mathbf{Set}$, $B \mapsto \mathrm{Hom}_{\mathscr{A}}(A, B)$. This shows the existence of ${}^w\tau^{\geq n}$ and the fact that ${}^w\tau^{\geq n} A = A/\mathrm{Fil}_{n-1} A$.

Note also that the formulas for ${}^w\tau^{\leq n}$ and ${}^w\tau^{\geq n+1}$ imply that, for every $A \in \mathrm{Ob}(\mathscr{A})$, the following sequence is exact:

$$0 \to {}^w\tau^{\leq n} A \to A \to {}^w\tau^{\geq n+1} A \to 0.$$

(c). For later use, we prove that the functors ${}^w\tau^{\leq n}$ are exact. Let $f : A \to B$ be a morphism of $\mathscr{A}$; we want to prove that, for every $n \in \mathbb{Z}$, the canonical morphisms ${}^w\tau^{\leq n}(\operatorname{Ker} f) \to \operatorname{Ker}({}^w\tau^{\leq n} f)$ and $\operatorname{Coker}({}^w\tau^{\leq n} f) \to {}^w\tau^{\leq n}(\operatorname{Coker} f)$ are isomorphisms; this implies in particular that $\operatorname{Ker}({}^w\tau^{\leq n}), \operatorname{Coker}({}^w\tau^{\leq n}) \in \operatorname{Ob}(\mathscr{A}_{\leq n})$ and that $\operatorname{Ker}({}^w\tau^{\geq n+1}), \operatorname{Coker}({}^w\tau^{\geq n+1}) \in \operatorname{Ob}(\mathscr{A}_{\geq n+1})$, so that $\mathscr{A}_{\leq n}$ and $\mathscr{A}_{\geq n+1}$ are abelian subcategories of $\mathscr{A}$. We prove the claim by induction on $\ell_A + \ell_B$, where ℓ_A (resp. ℓ_B) is the length of the weight filtration of A (resp. B). If $n \in \mathbb{Z}$, applying the snake lemma to the commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & {}^w\tau^{\leq n}A \arrow[r] \arrow[d] & A \arrow[r] \arrow[d, "f"] & {}^w\tau^{\geq n+1}A \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & {}^w\tau^{\leq n}B \arrow[r] & B \arrow[r] & {}^w\tau^{\geq n+1}B \arrow[r] & 0
\end{tikzcd}$$

we get an exact sequence

$$0 \to \operatorname{Ker}({}^w\tau^{\leq n} f) \to \operatorname{Ker}(f) \to \operatorname{Ker}({}^w\tau^{\geq n+1} f) \xrightarrow{\delta} \operatorname{Coker}({}^w\tau^{\leq n}) \to \operatorname{Coker}(f) \to \operatorname{Coker}({}^w\tau^{\geq n+1}) \to 0.$$

The claim that we want to prove is equivalent to the fact that $\delta = 0$. Indeed, if $\delta = 0$ then we immediately get the result, and if $\operatorname{Coker}({}^w\tau^{\leq n} f) \in \operatorname{Ob}(\mathscr{A}_{\leq n})$ and $\operatorname{Ker}({}^w\tau^{\geq n+1} f) \in \operatorname{Ob}(\mathscr{A}_{\geq n+1})$ then the solution of (a) implies that $\delta = 0$. We first show that the result holds if at least two of ${}^w\tau^{\leq n}A$, ${}^w\tau^{\geq n+1}A$, ${}^w\tau^{\leq n}B$ or ${}^w\tau^{\geq n+1}B$ are 0. If ${}^w\tau^{\geq n+1}A$ or ${}^w\tau^{\leq n}B$ is 0, then $\delta = 0$. Suppose that ${}^w\tau^{\leq n}A = 0$ and ${}^w\tau^{\geq n+1}B = 0$; then $A \in \operatorname{Ob}(\mathscr{A}_{\geq n+1})$ and $B \in \operatorname{Ob}(\mathscr{A}_{\leq n})$, so $f = 0$ by the solution of (a), and the result is clear. If $\ell_A, \ell_B \leq 1$, then there exist $n_A, n_B \in \mathbb{Z}$ such that $A \in \operatorname{Ob}(\mathscr{A}_{n_A})$ and $B \in \operatorname{Ob}(\mathscr{A}_{n_B})$, and then, for every $n \in \mathbb{Z}$, at least two of ${}^w\tau^{\leq n}A$, ${}^w\tau^{\geq n+1}A$, ${}^w\tau^{\leq n}B$ or ${}^w\tau^{\geq n+1}B$ are 0, so we are done. Suppose that $\ell_A \geq 2$, and let $n \in \mathbb{Z}$. If both ${}^w\tau^{\leq n}A$ and ${}^w\tau^{\geq n+1}A$ are nonzero, then they both have weight filtrations of lengths $< \ell_A$; by the induction hypothesis, applied to ${}^w\tau^{\leq n} f$ and ${}^w\tau^{\geq n+1} f$, we have $\operatorname{Ker}({}^w\tau^{\geq n+1} f) \in \operatorname{Ob}(\mathscr{A}_{\geq n+1})$ and $\operatorname{Coker}({}^w\tau^{\leq n} f) \in \operatorname{Ob}(\mathscr{A}_{\leq n})$, so $\delta = 0$ and we are done. If ${}^w\tau^{\geq n+1}A = 0$, then $\delta = 0$, and again we are done. Suppose that ${}^w\tau^{\leq n}A = 0$. If ${}^w\tau^{\leq n}B$ and ${}^w\tau^{\geq n+1}B$ are both nonzero, then again we can use the induction hypothesis to finish the proof; if at least one of them is), then at least two of ${}^w\tau^{\leq n}A$, ${}^w\tau^{\geq n+1}A$, ${}^w\tau^{\leq n}B$ or ${}^w\tau^{\geq n+1}B$ are 0, so we are done. The case where $\ell_B \geq 2$ is similar.

Now fix $n \in \mathbb{Z}$ and let $A \in \operatorname{Ob}(\mathscr{A}_{\leq n})$ and $B \in \operatorname{Ob}(\mathscr{A}_{\geq n+1})$. If $i \leq -1$, then $\operatorname{Ext}^i_{\mathscr{A}}(A, B) = \operatorname{Hom}_{\mathrm{D}(\mathscr{A})}(A, B[i]) = 0$ by Corollary V.4.2.8 of the notes. If $i = 0$, then $\operatorname{Ext}^i_{\mathscr{A}}(A, B) = \operatorname{Hom}_{\mathscr{A}}(A, B) = 0$ by question (a). Suppose that $i \geq 1$. We use the description of $\operatorname{Ext}^i_{\mathscr{A}}(A, B)$ given by Proposition V.4.5.3 of the notes. So let $x \in \operatorname{Ext}^i_{\mathscr{A}}(A, B)$, and let $c = (0 \to B \xrightarrow{f} E_{i-1} \xrightarrow{f_{i-1}} \dots \xrightarrow{f_1} E_0 \xrightarrow{f_0} A \to 0)$ be a Yoneda extension of A by B representing x. Applying ${}^w\tau^{\leq n}$ to this exact sequence, we get an exact sequence $0 \to 0 \to F_{i-1} \to \dots \to F_0 \to A \to 0$, where $F_j = {}^w\tau^{\leq n}E_j$ for every $j \in \{0, \dots, i-1\}$. We denote the obvious inclusion $F_j \to E_j$ by u_j. So we have a commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & B \arrow[r, "f"] & E_{i-1} \arrow[r] & E_{i-2} \arrow[r] & \dots \arrow[r] & E_0 \arrow[r] & A \arrow[r] & 0 \\
0 \arrow[r] & B \arrow[r, "\mathrm{id}_B + 0"'] \arrow[u, equal] & B \oplus F_{i-1} \arrow[r] \arrow[u, "f+u_{i-1}"] & F_{i-2} \arrow[r] \arrow[u, "u_{i-2}"] & \dots \arrow[r] & F_0 \arrow[r] \arrow[u, "u_0"] & A \arrow[r] \arrow[u, equal] & 0
\end{tikzcd}$$

where the morphism $B \oplus F_{i-1} \to F_{i-2}$ is equal to 0 on B and to ${}^w\tau^{\leq n} f_{i-1}$ on F_{i-1}. So the second row also represents $x \in \operatorname{Ext}^i_{\mathscr{A}}(A, B)$. To show that $x = 0$, it suffices to show that the morphism

$$g : (\ldots \to 0 \to B \to B \oplus F_{i-1} \to F_{i-2} \ldots F_0 \to 0 \to \ldots) \to B[i]$$

(with F_0 in degree 0 on the left hand side) is equal to 0. But the complex $(\ldots \to 0 \to B \to B \oplus F_{i-1} \to F_{i-2} \ldots F_0 \to 0 \to \ldots)$ is the direct sum of $(\ldots \to 0 \to B \to B \to 0 \ldots 0 \to 0 \to \ldots)$ (with the first B in degree $-i$) and of $(\ldots \to 0 \to 0 \to F_{i-1} \to F_{i-2} \ldots F_0 \to 0 \to \ldots)$ (with F_0 in degree 0), the morphism g is 0 on the second of these summands, and the first of these summands is quasi-isomorphic to 0, so $g = 0$ in $\mathrm{D}(\mathscr{A})$.

(d). Fix $n \in \mathbb{Z}$. Note that we have proved in the solution of (c) that $\mathscr{A}_{\leq n}$ and $\mathscr{A}_{\geq n+1}$ are abelian subcategories of $\mathscr{A}$.

By condition (1), if $m \in \mathbb{Z}$ and $A \in \mathrm{Ob}(\mathscr{A})$ is isomorphic to an object of $\mathscr{A}_m$, then $A \in \mathrm{Ob}(\mathscr{A}_m)$. By the existence of weight filtrations (condition (2)), if $A \in \mathrm{Ob}(\mathscr{A})$ is isomorphic to an object of $\mathscr{A}_{\leq n}$ (resp. $\mathscr{A}_{\geq n+1}$), then A is in $\mathscr{A}_{\leq n}$ (resp. $\mathscr{A}_{\geq n+1}$). This implies that $({}^w\mathrm{D}^{\leq n}, {}^w\mathrm{D}^{\geq n+1})$ satisfies condition (0) of the definition of a t-structure.

We clearly have ${}^w\mathrm{D}^{\leq n}[k] = {}^w\mathrm{D}^{\leq n}$ and ${}^w\mathrm{D}^{\geq n+1}[k] = {}^w\mathrm{D}^{\geq n+1}[k]$ for every $k \in \mathbb{Z}$, so condition (2) of the definition of a t-structure is clear.

For every $A \in \mathrm{Ob}(\mathrm{D}^b(\mathscr{A}))$, we define the cohomological amplitude of A to be $n_1 - n_2$, where n_1 (resp. n_2) is the biggest (resp. smallest) integer $n \in \mathbb{Z}$ such that $\mathrm{H}^n(A) \neq 0$. If the cohomological amplitude of A is 0 then $A = 0$, and if it is 1, then there exists $n \in \mathbb{Z}$ such that $\mathrm{H}^i(A) = 0$ for $i \neq 0$, so that $A \simeq \mathrm{H}^n(A)[-n]$.

Let $A \in \mathrm{Ob}({}^w\mathrm{D}^{\leq n})$ and $B \in \mathrm{Ob}({}^w\mathrm{D}^{\geq n+1})$. We claim that $\mathrm{Ext}^i_{\mathscr{A}}(A, B) = 0$ for every $i \in \mathbb{Z}$. (In particular, we get condition (1) of the definition of a t-structure.) We prove this by induction on $c_A + c_B$, where c_A (resp. c_B) is the cohomological amplitude of A (resp. B). If $c_A, c_B \leq 1$, then the claim follows from question (c). Suppose that $c_A \geq 2$. Then there exists $n \in \mathbb{Z}$ such that $\tau^{\leq n}A \to A$ and $A \to \tau^{\geq n+1}A$ are not isomorphisms, hence $\tau^{\leq n}A$, $\tau^{\geq n+1}A$ have cohomological amplitude $< c_A$. Let $i \in \mathbb{Z}$. Applying the cohomological functor $\mathrm{Ext}^i_{\mathscr{A}}(\cdot, B) = \mathrm{Hom}_{\mathrm{D}(\mathscr{A})}(\cdot, B[i])$ to the exact triangle

$$\tau^{\leq A} \to A \to \tau^{\geq n+1}A \xrightarrow{+1},$$

we get an exact sequence

$$\mathrm{Ext}^i_{\mathscr{A}}(\tau^{\geq n+1}A, B) \to \mathrm{Ext}^i_{\mathscr{A}}(A, B) \to \mathrm{Ext}^i_{\mathscr{A}}(\tau^{\leq n}A, B).$$

As $\mathrm{Ext}^i_{\mathscr{A}}(\tau^{\leq 1}A, B) = \mathrm{Ext}^i_{\mathscr{A}}(\tau^{\geq n+1}A, B) = 0$ by the induction hypothesis, this implies that $\mathrm{Ext}^i_{\mathscr{A}}(A, B) = 0$. The case wgere $c_B \geq 2$ is similar.

We check condition (3) of the definition of a t-structure. Let $X \in \mathrm{Ob}(\mathrm{D}^b(\mathscr{A}))$. We start with a remark: Suppose that there exists an exact triangle $(*)\quad A \to X \to B \xrightarrow{+1}$ with $A \in \mathrm{Ob}({}^w\mathrm{D}^{\leq n})$ and $B \in \mathrm{Ob}({}^w\mathrm{D}^{\geq n+1})$. Let $i \in \mathbb{Z}$. Then we have an exact sequence

$$\mathrm{H}^{i-1}(B) \to \mathrm{H}^i(A) \to \mathrm{H}^i(X) \to \mathrm{H}^i(B) \to \mathrm{H}^{i+1}(A),$$

in which $\mathrm{H}^i(A)$, $\mathrm{H}^{i+1}(A)$ are in $\mathscr{A}_{\leq n}$ and $\mathrm{H}^{i-1}(B)$, $\mathrm{H}^i(B)$ are in $\mathscr{A}_{\geq n+1}$. By the solution of question (a), the morphisms $\mathrm{H}^{i-1}(B) \to \mathrm{H}^i(A)$ and $\mathrm{H}^i(B) \to \mathrm{H}^{i+1}(A)$ are zero, so we get an exact sequence

$$0 \to \mathrm{H}^i(A) \to \mathrm{H}^i(X) \to \mathrm{H}^i(b) \to 0,$$

which proves that $\mathrm{H}^i(A) = {}^w\tau^{\leq n}(\mathrm{H}^i(X))$ and $\mathrm{H}^i(B) = {}^w\tau^{\geq n+1}(\mathrm{H}^i(X))$.

Now we prove by induction on the cohomological amplitude c of X that there exists an exact triangle $(*)$. If $c = 0$, then $X = 0$ and we can take $A = B = 0$. Suppose that $c = 1$. Then there exists $m \in \mathbb{Z}$ such that $X \simeq \mathrm{H}^m(X)[-m]$. As ${}^w\mathrm{D}^{\leq n}$ and ${}^w\mathrm{D}^{\geq n+1}$ are stable by all functor $[k]$, it suffices to prove the existence of the exact triangle $(*)$ for $\mathrm{H}^m(X)$, so we may assume that $X \in \mathrm{Ob}(\mathscr{A})$. Then we can take for $(*)$ the exact triangle associated to the exact sequence $0 \to {}^w\tau^{\leq n}X \to X \to {}^w\tau^{\geq n+1}X \to 0$. Suppose that $c \geq 2$. Then there exists $m \in \mathbb{Z}$ such that $\tau^{\leq m}X \to X$ and $X \to \tau^{\geq m+1}X$ are not isomorphisms, hence $X' = \tau^{\leq m}X$, $X'' = \tau^{\geq m+1}X$ have cohomological amplitude $< c$. By the induction hypothesis, we have exact triangles $A' \to X' \to B' \overset{+1}{\to}$ and $A'' \to X'' \to B'' \overset{+1}{\to}$, with $A', A'' \in \mathrm{Ob}({}^w\mathrm{D}^{\leq n})$ and $B', B'' \in \mathrm{Ob}({}^w\mathrm{D}^{\geq n+1})$. By question (a) of problem 1, there exists a unique morphism of exact triangles

$$\begin{tikzcd}
A'' \arrow[r] \arrow[d] & X'' \arrow[r] \arrow[d] & B'' \arrow[r, "+1"] \arrow[d] & {} \\
A'[1] \arrow[r] & X'[1] \arrow[r] & B'[1] \arrow[r, "+1"] & {}
\end{tikzcd}$$

extending the morphism $X'' \to X'[1]$. We complete the morphism $A'' \to A'[1]$ to an exact triangle $A' \to A \to A'' \to A'[1]$. By axiom (TR4) of triangulated categories, we can find a morphism $A \to X$ such that the diagram

$$\begin{tikzcd}
A' \arrow[r] \arrow[d] & A \arrow[r] \arrow[d] & A'' \arrow[r, "+1"] \arrow[d] & {} \\
X' \arrow[r] & X \arrow[r] & X'' \arrow[r, "+1"] & {}
\end{tikzcd}$$

is a morphism of exact triangles. Finally, we complete the morphism $A \to X$ to an exact triangle $A \to X \to B \overset{+1}{\to}$. We claim that this is the desired exact triangle $(*)$. To prove this claim, it suffices to show that $\mathrm{H}^i(A) = {}^w\tau^{\leq n}(\mathrm{H}^i(X))$ for every $i \in \mathbb{Z}$; indeed, by the long exact sequence of cohomology, this implies that, for every $i \in \mathbb{Z}$, the morphism $\mathrm{H}^i(X) \to \mathrm{H}^i(B)$ is surjective and identifies $\mathrm{H}^i(B)$ to ${}^w\tau^{\geq n+1}(\mathrm{H}^i(X))$, and so we will have $A \in \mathrm{Ob}({}^w\mathrm{D}^{\leq n})$ and $B \in \mathrm{Ob}({}^w\mathrm{D}^{\geq n+1})$. To prove the claim, let $i \in \mathbb{Z}$. We have a commutative diagram with exact rows

$$\begin{tikzcd}
\mathrm{H}^{i-1}(A'') \arrow[r] \arrow[d] & \mathrm{H}^i(A') \arrow[r] \arrow[d] & \mathrm{H}^i(A) \arrow[r] \arrow[d] & \mathrm{H}^i(A'') \arrow[r] \arrow[d] & \mathrm{H}^{i+1}(A') \arrow[d] \\
\mathrm{H}^{i-1}(X'') \arrow[r] & \mathrm{H}^i(X') \arrow[r] & \mathrm{H}^i(X) \arrow[r] & \mathrm{H}^i(X'') \arrow[r] & \mathrm{H}^{i+1}(X')
\end{tikzcd}$$

If $i \leq m$, then $\mathrm{H}^i(X'') = \mathrm{H}^{i-1}(X'') = 0$, so $\mathrm{H}^j(A'') = {}^w\tau^{\leq n}\mathrm{H}^j(X'') = 0$ for $j \in \{i, i-1\}$, so the diagram becomes a commutative square whose horizontal arrows are isomorphisms:

$$\begin{tikzcd}
{}^w\tau^{\leq n}\mathrm{H}^i(X') \arrow[r, "\sim"] \arrow[d] & \mathrm{H}^i(A) \arrow[d] \\
\mathrm{H}^i(X') \arrow[r, "\sim"] & \mathrm{H}^i(X)
\end{tikzcd}$$

which shows that $\mathrm{H}^i(A) = {}^w\tau^{\leq n}\mathrm{H}^i(X)$. If $i \geq m+1$, then $\mathrm{H}^i(X') = \mathrm{H}^{i+1}(X') = 0$, so $\mathrm{H}^j(A') = {}^w\tau^{\leq n}\mathrm{H}^j(X') = 0$ for $j \in \{i, i+1\}$, so the diagram becomes a commutative square whose horizontal arrows are isomorphisms:

$$\begin{tikzcd}
\mathrm{H}^i(A) \arrow[r, "\sim"] \arrow[d] & {}^w\tau^{\leq n}\mathrm{H}^i(X'') \arrow[d] \\
\mathrm{H}^i(X) \arrow[r, "\sim"] & \mathrm{H}^i(X'')
\end{tikzcd}$$

which shows again that $\mathrm{H}^i(A) = {}^w\tau^{\leq n}\mathrm{H}^i(X)$.

Finally, we calculate the heart of the t-structure $({}^w\mathrm{D}^{\leq n}, {}^w\mathrm{D}^{\geq n+1})$. Let $X \in \mathrm{Ob}({}^w\mathrm{D}^{\leq n}) \cap \mathrm{Ob}({}^w\mathrm{D}^{\geq n+1})$. For every $i \in \mathbb{Z}$, the object $\mathrm{H}^i(X)$ of $\mathscr{A}$ is in $\mathrm{Ob}(\mathscr{A}_{\leq n}) \cap \mathrm{Ob}(\mathscr{A}_{\geq n+1})$, so $\mathrm{id}_{\mathrm{H}^i(X)} = 0$ by question (a), so $\mathrm{H}^i(X) = 0$. This shows that $X = 0$.

$\square$