1 Abelian subcategories of triangulated categories

Let $D$ be a triangulated category. We denote the shift functors by $X \mapsto X[1]$, and we write triangles as $X \to Y \to Z \to X[1]$ or $X \to Y \to Z \to X[1]$. For every $X,Y \in \text{Ob}(D)$ and every $n \in \mathbb{Z}$, we write $\text{Hom}^n_D(X,Y) = \text{Hom}(X,Y[n])$.

(a). Let $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$ and $X' \xrightarrow{u'} Y' \xrightarrow{v'} Z' \xrightarrow{w'} X'[1]$ be two distinguished triangles of $D$, and let $g : Y \to Y'$ be a morphism.

(i) (2 points) Show that the following conditions are equivalent:

1. $v' \circ g \circ u = 0$;
2. there exists $f : X \to X'$ such that $u' \circ f = g \circ u$;
3. there exists $h : Z \to Z'$ such that $h \circ v = v' \circ g$;
4. there exist $f : X \to X'$ and $h : Z \to Z'$ such that $(f,g,h)$ is a morphism of triangles.

(ii) (1 point) Suppose that the conditions (i) hold and that $\text{Hom}^{-1}_D(X,Z') = 0$. Show that the morphisms $f$ and $h$ of (i)(2) and (i)(3) are unique.

(b). Let $C$ be a full subcategory of $D$, and suppose that $\text{Hom}^n(X,Y) = 0$ if $X,Y \in \text{Ob}(C)$ and $n < 0$.

(i) (2 points) Let $f : X \to Y$ be a morphism of $C$. Take a distinguished triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$ in $D$, and suppose that we have a distinguished triangle $N[1] \xrightarrow{\alpha} S \xrightarrow{\beta} C \xrightarrow{\gamma} X'[1]$ with $N,C \in \text{Ob}(\mathcal{C})$. In particular, we get morphisms $\alpha : N[1] \to S \to X[1]$ and $\beta : Y \to S \to X$.

Show that $\alpha[-1] : N \to X$ is a kernel of $f$ and that $\beta : Y \to C$ is a cokernel of $f$.

We say that a morphism $f$ of $\mathcal{C}$ is admissible if there exist distinguished triangles satisfying the conditions of (i). We say that a sequence of morphisms of $\mathcal{C}$ $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ is an admissible short exact sequence if there exists a distinguished triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$ in $D$.

(ii) (2 points) Suppose that $\mathcal{C}$ as a zero object. If $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$ is a distinguished triangle in $D$ with $X,Y,Z \in \text{Ob}(\mathcal{C})$, show that $f$ and $g$ are admissible, that $f$ is a kernel of $g$ and that $g$ is a cokernel of $f$.

(iii) (2 points) If $f : X \to Y$ is an admissible monomorphism (resp. epimorphism) in $\mathcal{C}$.
and $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{+1}$ is a distinguished triangle in $\mathcal{D}$, show that $Z$ (resp. $Z[-1]$) is isomorphic to an object of $\mathcal{C}$ and $Z = \text{Coker}(f)$ (resp. $Z = \text{Ker}(f)$).

(iv) (4 points) Suppose that every morphism of $\mathcal{C}$ is admissible and $\mathcal{C}$ is an additive subcategory of $\mathcal{D}$. Show that $\mathcal{C}$ is an abelian category and that every short exact sequence in $\mathcal{C}$ is admissible.

(v) (3 points) Suppose that $\mathcal{C}$ is an abelian category and that every short exact sequence in $\mathcal{C}$ is admissible. Show that every morphism of $\mathcal{C}$ is admissible.

2 t-structures

We use the convention of problem 1. A $t$-structure on $\mathcal{D}$ is the data of two full subcategories $\mathcal{D}^{\leq 0}$ and $\mathcal{D}^{\geq 0}$ such that (with the convention that $\mathcal{D}^{\leq n} = \mathcal{D}^{\leq 0}[-n]$ and $\mathcal{D}^{\geq n} = \mathcal{D}^{\geq 0}[-n]$):

(0) If $X \in \text{Ob}(\mathcal{D})$ is isomorphic to an object of $\mathcal{D}^{\leq 0}$ (resp. $\mathcal{D}^{\geq 0}$), then $X$ is in $\mathcal{D}^{\leq 0}$ (resp. $\mathcal{D}^{\geq 0}$).

(1) For every $X \in \text{Ob}(\mathcal{D}^{\leq 0})$ and every $Y \in \text{Ob}(\mathcal{D}^{\geq 1})$, we have $\text{Hom}(X, Y) = 0$.

(2) We have $\mathcal{D}^{\leq 0} \subset \mathcal{D}^{\leq 1}$ and $\mathcal{D}^{\geq 0} \supset \mathcal{D}^{\geq 1}$.

(3) For every $X \in \text{Ob}(\mathcal{D})$, there exists a distinguished triangle $A \to X \to B \xrightarrow{+1}$ with $A \in \text{Ob}(\mathcal{D}^{\leq 0})$ and $B \in \text{Ob}(\mathcal{D}^{\geq 1})$.

We fix a $t$-structure $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ on $\mathcal{D}$.

(a). (1 point) Show that the distinguished triangle of condition (3) is unique up to unique isomorphism.

(b). (3 points) For every $n \in \mathbb{Z}$, show that the inclusion functor $\mathcal{D}^{\leq n} \subset \mathcal{D}$ has a right adjoint $\tau^{\leq n}$ and the inclusion functor $\mathcal{D}^{\geq n} \subset \mathcal{D}$ has a left adjoint $\tau^{\geq n}$. (Hint: It suffice to treat the case $n = 0$.)

(c). (2 points) For every $n \in \mathbb{Z}$, show that there is a unique morphism $\delta : \tau^{\geq n+1}X \to (\tau^{\leq n}X)[1]$ such that the triangle $\tau^{\leq n}X \to X \to \tau^{\geq n+1}X \xrightarrow{\delta} (\tau^{\leq n}X)[1]$, where the other two morphisms are given by the counit and unit of the adjunctions of (b).

(d). (3 points) Let $a, b \in \mathbb{Z}$ such that $a \leq b$, and let $X \in \text{Ob}(\mathcal{D})$. Show that there exists a unique morphism $\alpha : \tau^{\geq a} \tau^{\leq b}X \to \tau^{\leq b} \tau^{\geq a}X$ such that the following diagram commutes:

$$
\begin{array}{ccc}
\tau^{\leq b}X & \xrightarrow{\delta} & \tau^{\geq a}X \\
\tau^{\geq a} \tau^{\leq b}X & \xrightarrow{\alpha} & \tau^{\leq b} \tau^{\geq a}X \\
\end{array}
$$

(where all the other morphisms are counit or unit morphisms of the adjunctions of (b)), and that $\alpha$ is an isomorphism. (Hint: Apply the octahedral axiom to $\tau^{\leq a}X \xrightarrow{f} \tau^{\leq b}X \xrightarrow{g} X$.)

(e). (1 points) If $a, b \in \mathbb{Z}$ are such that $a \leq b$, show that, for every $X \in \text{Ob}(\mathcal{D})$, we have $\tau^{\geq a} \tau^{\leq b}X \in \text{Ob}(\mathcal{D}^{\leq a}) \cap \text{Ob}(\mathcal{D}^{\geq b})$.

Let $\mathcal{C} = \mathcal{D}^{\leq 0} \cap \mathcal{D}^{\geq 0}$; that is, $\mathcal{C}$ is the full subcategory of $\mathcal{D}$ such that $\text{Ob}(\mathcal{C}) = \text{Ob}(\mathcal{D}^{\leq 0}) \cap \text{Ob}(\mathcal{D}^{\geq 0})$. We denote the functor $\tau^{\leq 0} \tau^{\geq 0} : \mathcal{D} \to \mathcal{C}$ by $H^0$. The category $\mathcal{C}$ is called the heart or core of the $t$-structure.

(f). (1 point) Show that $\mathcal{C}$ is an abelian category.
(g). (2 points) Show that, if \( X \rightarrow Y \rightarrow Z \xrightarrow{+1} \) is a distinguished triangle in \( \mathcal{D} \) such that \( X, Z \in \text{Ob}(\mathcal{C}) \), then \( Y \) is also in \( \mathcal{C} \).

(h). The goal of this question is to show that the functor \( H^0 : \mathcal{D} \rightarrow \mathcal{C} \) is a cohomological functor. Let \( X \rightarrow Y \rightarrow Z \xrightarrow{+1} \) be a distinguished triangle in \( \mathcal{D} \).

(i) (2 points) If \( X, Y, Z \in \text{Ob}(\mathcal{D}_{\leq 0}) \), show that the sequence \( H^0(X) \rightarrow H^0(Y) \rightarrow H^0(Z) \rightarrow 0 \) is exact in \( \mathcal{C} \). (Hint: A sequence of morphisms \( A \rightarrow B \rightarrow C \rightarrow 0 \) in an abelian category \( \mathcal{A} \) is exact if and only if, for every object \( D \) of \( \mathcal{A} \), the sequence of abelian groups \( \text{Hom}(D, A) \rightarrow \text{Hom}(D, B) \rightarrow \text{Hom}(D, C) \rightarrow 0 \) is exact.)

(ii) (2 points) If \( X \in \text{Ob}(\mathcal{D}_{\leq 0}) \), show that the sequence \( H^0(X) \rightarrow H^0(Y) \rightarrow H^0(Z) \rightarrow 0 \) is exact in \( \mathcal{C} \). (Hint: Construct a distinguished triangle \( X \rightarrow \tau_{\leq 0} Y \rightarrow \tau_{\leq 0} Z \xrightarrow{+1} \).)

(iii) (1 point) If \( Z \in \text{Ob}(\mathcal{D}_{\geq 0}) \), show that the sequence \( 0 \rightarrow H^0(X) \rightarrow H^0(Y) \rightarrow H^0(Z) \rightarrow 0 \) is exact in \( \mathcal{C} \).

(iv) (2 points) In general, show that the sequence \( H^0(X) \rightarrow H^0(Y) \rightarrow H^0(Z) \rightarrow 0 \) is exact in \( \mathcal{C} \).

3 The canonical t-structure

Let \( \mathcal{A} \) be an abelian category.

(a). (2 points) Let \( n \in \mathbb{Z} \). If \( X \in \text{Ob}(\mathcal{D}_{\leq n}(\mathcal{A})) \) and \( Y \in \text{Ob}(\mathcal{D}_{\geq n+1}(\mathcal{A})) \), show that \( \text{Hom}_{\mathcal{D}(\mathcal{A})}(X, Y) = 0 \).

(b). (3 points) Show that \( (\mathcal{D}_{\leq 0}(\mathcal{A}), \mathcal{D}_{\geq 0}(\mathcal{A})) \) is a t-structure on \( \mathcal{D}(\mathcal{A}) \), that its heart is equivalent to \( \mathcal{A} \), and that the associated functor \( H^0 : \mathcal{D}(\mathcal{A}) \rightarrow \mathcal{A} \) is the 0th cohomology functor.

4 Torsion

Let \( \mathcal{D} = \mathcal{D}(\mathbb{A}) \), and let

\[ *\mathcal{D}_{\leq 0} = \{ X \in \mathcal{D} \mid H^i(X) = 0 \text{ for } i > 1, \text{ and } H^1(X) \text{ is torsion} \} \]

and

\[ *\mathcal{D}_{\geq 0} = \{ X \in \mathcal{D} \mid H^i(X) = 0 \text{ for } i < 0, \text{ and } H^0(X) \text{ is torsionfree} \}. \]

Let \( \mathcal{C} = *\mathcal{D}_{\leq 0} \cap *\mathcal{D}_{\geq 0} \).

(a). Show that \((*\mathcal{D}_{\leq 0}, *\mathcal{D}_{\geq 0})\) is a t-structure on \( \mathcal{D} \). (2 points for condition (1), 1 for condition (2) and 2 for condition (3))

(b). Let \( f : A \rightarrow B \) be a morphism of torsionfree abelian groups. We can see \( A \) and \( B \) as objects of \( \mathcal{C} \) (concentrated in degree 0), and then \( f \) is also a morphism of \( \mathcal{C} \).

(i) (2 points) Show that \( f \) is a monomorphism in \( \mathcal{C} \) if and only if \( f \) is injective (and \( \mathbb{A} \)) and \( B/f(A) \) is torsionfree.

(ii) (1 point) Show that \( f \) is an epimorphism in \( \mathcal{C} \) if and only if \( f \otimes_{\mathbb{Z}} \mathbb{Q} \) is surjective.

(iii) (3 points) Calculate the kernel, the cokernel and the image of \( f \) in \( \mathcal{C} \).

(c). (1 points) For every \( n \geq 1 \), show that \( \text{Ext}^1_{\mathbb{A}}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}) \simeq \mathbb{Z}/n\mathbb{Z} \).
(d) (1 point) If $A$ and $B$ are finitely generated abelian groups, show that $\text{Ext}^n_{\text{Ab}}(A, B) = 0$ for every $n \geq 2$.  

(e) (2 points) Let $X \in \text{Ob}({\mathcal C})$. Suppose that $H^i(X)$ is a finitely generated abelian groups for every $i \in \mathbb{Z}$. If $\text{Hom}_A(X, Z) = 0$, show that $X = 0$.

(f) (1 point) Give an example of a $X \in \text{Ob}({\mathcal C})$ nonzero such that $\text{Hom}_A(X, Z) = 0$.

(g) (2 points) Let $X \in \text{Ob}(\mathcal{D})$. If $X \in \text{Ob}(\mathcal{D}_{\leq 0})$ (resp $X \in \text{Ob}(\mathcal{D}_{\geq 0})$), show that $R\text{Hom}_{\text{Ab}}(X, Z)$ is in $\mathcal{D}_{\geq 0}(\text{Ab})$ (resp. $\mathcal{D}_{\leq 0}(\text{Ab})$).

(h) (3 points) Let $X$ be a complex of finitely generated abelian groups. If $R\text{Hom}_{\text{Ab}}(X, Z)$ is in $\mathcal{D}_{\geq 0}(\text{Ab})$ (resp. $\mathcal{D}_{\leq 0}(\text{Ab})$), show that $X \in \text{Ob}(\mathcal{D}_{\leq 0})$ (resp. $X \in \text{Ob}(\mathcal{D}_{\geq 0})$).

5 Weights

Let $\mathcal{A}$ be an abelian category. Suppose that we have a family $(\mathcal{A}_n)_{n \in \mathbb{Z}}$ of full abelian subcategories of $\mathcal{A}$ such that:

1. If $n \neq m$, then $\text{Hom}_\mathcal{A}(A, B) = 0$ for any $A \in \text{Ob}(\mathcal{A}_n)$ and $B \in \text{Ob}(\mathcal{A}_m)$.

2. Any object $A$ of $\mathcal{A}$ has a weight filtration, that is, an increasing filtration $\text{Fil}_nA$ such that $\text{Fil}_nA = 0$ for $n << 0$, $\text{Fil}_nA = A$ for $n >> 0$ and $\text{Fil}_nA/\text{Fil}_{n+1}A \in \text{Ob}(\mathcal{A}_n)$ for every $n \in \mathbb{Z}$.

For every $n \in \mathbb{Z}$, we denote by $\mathcal{A}_{\leq n}$ (resp. $\mathcal{A}_{\geq n}$) the full subcategory of $\mathcal{A}$ whose objects are the $A \in \text{Ob}(\mathcal{A})$ having a weight filtration $\text{Fil}_nA$ such that $\text{Fil}_nA = A$ (resp. $\text{Fil}_nA = 0$).

(a) (1 point) If $A \in \text{Ob}(\mathcal{A}_{\leq n})$ and $B \in \text{Ob}(\mathcal{A}_{\geq n+1})$, show that $\text{Hom}_\mathcal{A}(A, B) = 0$.

(b) (2 points) Show that the inclusion functor $\mathcal{A}_{\leq n} \subset \mathcal{A}$ has a right adjoint $\mathcal{W}_{\leq n}$, and that the inclusion functor $\mathcal{A}_{\geq n} \subset \mathcal{A}$ has a left adjoint $\mathcal{W}_{\geq n}$.

(c) (2 points) If $A \in \text{Ob}(\mathcal{A}_{\leq n})$ and $B \in \text{Ob}(\mathcal{A}_{\geq n+1})$, show that $\text{Ext}^i_\mathcal{A}(B, A) = 0$ for every $i \in \mathbb{Z}$.

(d) (4 points) Define two full subcategories $\mathcal{W}D_{\leq n}$ and $\mathcal{W}D_{\geq n}$ of $\text{D}^b(\mathcal{A})$ by:

$$\text{Ob}(\mathcal{W}D_{\leq n}) = \{ X \in \text{Ob}(\text{D}^b(\mathcal{A})) \mid \forall i \in \mathbb{Z}, \ H^i(X) \in \mathcal{A}_{\leq n} \}$$

and

$$\text{Ob}(\mathcal{W}D_{\geq n+1}) = \{ X \in \text{Ob}(\text{D}^b(\mathcal{A})) \mid \forall i \in \mathbb{Z}, \ H^i(X) \in \mathcal{A}_{\geq n+1} \}.$$ 

Show that $(\mathcal{W}D_{\leq n}, \mathcal{W}D_{\geq n+1})$ is a t-structure on $\text{D}^b(\mathcal{A})$, and that the heart of this t-structure is $\{0\}$.  

\footnote{This actually holds for any abelian groups.}