MAT 540 : Problem Set 8

Due Thursday, November 14

1 Right multiplicative systems

Let $\mathscr{C}$ be a category and W be a set of morphisms of $\mathscr{C}$. Let $\mathscr{I}$ be a full subcategory of $\mathscr{C}$ and $W_{\mathscr{I}}$ be the set of morphisms of $\mathscr{I}$ that are in W. Suppose that W is a right multiplicative system and that, for every $s : X \to Y$ in W such that $X \in \mathrm{Ob}(\mathscr{I})$, there exists a morphism $f : Y \to Z$ with $Z \in \mathrm{Ob}(\mathscr{I})$ and $f \circ s \in W$.

Show that $W_{\mathscr{I}}$ is a right multiplicative system. (1 point for (S1)+(S2), 1 point each for (S3) and (S4))

2 Isomorphisms in triangulated categories

(4 points)

Let $(\mathscr{D}, T)$ be a triangulated category, and let $f : X \to Y$ be a morphism of $\mathscr{D}$. Show that f is an isomorphism if and only if there exists a distinguished triangle $X \xrightarrow{f} Y \to Z \to T(X)$ with $Z = 0$.

3 Null systems

Let $(\mathscr{D}, T)$ be a triangulated. Remember that a null system in $\mathscr{D}$ is a set $\mathscr{N}$ of objects of $\mathscr{D}$ such that:

(N1) $0 \in \mathscr{N}$;

(N2) for every $X \in \mathrm{Ob}(\mathscr{C})$, we have $X \in \mathscr{N}$ if and only if $T(X) \in \mathscr{N}$;

(N3) if $X \to Y \to Z \to T(X)$ is a distinguished triangle and if $X, Y \in \mathscr{N}$, then $Z \in \mathscr{N}$.

We fix a null system $\mathscr{N}$, and we denote by $W_{\mathscr{N}}$ the set of morphisms $f : X \to Y$ in $\mathscr{D}$ such that there exists a distinguished triangle $X \xrightarrow{f} Y \to Z \to T(X)$ with $Z \in \mathscr{N}$.

(a). (1 point) If $X \in \mathscr{N}$ and Y is isomorphic to X, show that $Y \in \mathscr{N}$.

(b). (1 point) Show that $W_{\mathscr{N}}$ contains all the isomorphisms of $\mathscr{D}$.

(c). (2 points) Show that $W_{\mathscr{N}}$ is stable by composition.

(d). (4 points) Show that $W_{\mathscr{N}}$ satisfies conditions (S3) and (S4) of Definition V.2.2.1 of the notes.

(e). (2 points) Show that $W_{\mathscr{N}}$ is also a left multiplicative system.

4 Localization of functors

Let $\mathscr{C}$ be a category, let W be a set of morphisms of $\mathscr{C}$, and let $\mathscr{I}$ be a full subcategory of $\mathscr{C}$; denote by $W_{\mathscr{I}}$ the set of morphisms of $\mathscr{I}$ that are in W. We fix a localization $Q : \mathscr{C} \to \mathscr{C}[W^{-1}]$ of $\mathscr{C}$ by W, and we denote by $\iota : \mathscr{I} \to \mathscr{C}$ the inclusion functor. Let $F : \mathscr{C} \to \mathscr{D}$ be a functor. Suppose that:

(a) W is a right multiplicative system;

(b) for every $X \in \mathrm{Ob}(\mathscr{C})$, there exists a morphism $s : X \to Y$ in W such that $Y \in \mathrm{Ob}(\mathscr{I})$;

(c) for every $s \in W_{\mathscr{I}}$, the morphism $F(s)$ is an isomorphism.

Show that, for every functor $G : \mathscr{C}[W^{-1}] \to \mathscr{D}$, the map

$$\alpha : \mathrm{Hom}_{\mathrm{Func}(\mathscr{C},\mathscr{D})}(F, G \circ Q) \to \mathrm{Hom}_{\mathrm{Func}(\mathscr{I},\mathscr{D})}(F \circ \iota, G \circ Q \circ \iota)$$

induced by composition on the right by ι is bijective. (2 points for injectivity, 3 points for surjectivity)

5 Localization of a triangulated category

Let $(\mathscr{D}, T)$ be a triangulated category, let $\mathscr{N}$ be a null system in $\mathscr{D}$, and let $W = W_{\mathscr{N}}$ be the corresponding multiplicative system. (See problem 3.) We write $Q : \mathscr{D} \to \mathscr{D}/\mathscr{N}$ for $Q : \mathscr{D} \to \mathscr{D}[W^{-1}]$.

(a). (1 point) Show that there exists an auto-equivalence $T : \mathscr{D}/\mathscr{N} \to \mathscr{D}/\mathscr{N}$ such that $T \circ Q \simeq Q \circ T$.

We say that a triangle in $\mathscr{D}/\mathscr{N}$ is distinguished if it is isomorphic to the image by Q of a distinguished triangle of $\mathscr{D}$. Axiom (TR0) of Definition V.1.1.4 of the notes is obvious.

(b). (5 points: 1 point per axiom) Show that axioms (TR1)-(TR5) also hold.

6 More group cohomology

The description of group cohomology in Subsection IV.3.5 of the notes can be useful in this problem.

We define elements u, v, r and s of the symmetric group $\mathfrak{S}_4$ by $u = (12)(34)$, $v = (14)(23)$, $r = (123)$ and $s = (13)$. The Klein four group is the normal subgroup K of $\mathfrak{S}_4$ generated by u and v.

Let k be a field of characteristic 2.

(a). (2 points) Show that $\mathfrak{S}_4 / K \simeq \mathfrak{S}_3$.

(b). (2 points) Show that there is a unique representation $\tau : \mathfrak{S}_4 \to \mathrm{GL}_2(k)$ such that $\tau(u) = \tau(v) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\tau(r) = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$ and $\tau(s) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

Let $M = M_2(k)$, with the action of $\mathfrak{S}_4$ given by $g \cdot A = \tau(g) A \tau(g)^{-1}$, for $g \in \mathfrak{S}_4$ and $A \in M_2(k)$. We identify $\mathfrak{S}_3$ with the subgroup of $\mathfrak{S}_4$ generated by r and s. We have a short exact sequence of groups

$$1 \to \mathbb{Z}/3\mathbb{Z} \to \mathfrak{S}_3 \to \mathbb{Z}/2\mathbb{Z} \to 1,$$

where the generator $1 \in \mathbb{Z}/3\mathbb{Z}$ is sent to $r \in \mathfrak{S}_3$.

(c). (2 points) If N is any representation of $\mathbb{Z}/3\mathbb{Z}$ on a k-vector space, show that $\mathrm{H}^p(\mathbb{Z}/3\mathbb{Z}, N) = 0$ for every $p \geq 1$. (You might find Remark IV.3.5.1 of the notes useful.)

(d). (1 point) If N is any representation of $\mathfrak{S}_3$ on a k-vector space, show that we have canonical isomorphisms $\mathrm{H}^p(\mathbb{Z}/2\mathbb{Z}, N^{\mathbb{Z}/3\mathbb{Z}}) \xrightarrow{\sim} \mathrm{H}^p(\mathfrak{S}_3, N)$ for every $p \geq 0$.

(e). (2 points) Show that $\mathrm{H}^p(\mathfrak{S}_3, M) = 0$ for every $p \geq 1$.

(f). (1 point) Show that we have canonical isomorphisms

$$\mathrm{H}^p(\mathbb{Z}/2\mathbb{Z}, \mathrm{H}^1(K, M)^{\mathbb{Z}/3\mathbb{Z}}) \xrightarrow{\sim} \mathrm{H}^p(\mathfrak{S}_3, \mathrm{H}^1(K, M)),$$

for every $p \geq 0$.

(g). (1 points) Show that $\mathrm{H}^1(K, M) = \mathrm{Hom}_{\mathbf{Grp}}(K, M)$, and that the action of $\mathfrak{S}_3$ on $\mathrm{H}^1(K, M)$ is given by $(g \cdot \varphi)(x) = g \cdot \varphi(g^{-1}xg)$, if $g \in \mathfrak{S}_3$, $x \in K$ and $\varphi \in \mathrm{H}^1(K, M)$.

(h). (3 points) Show that $\mathrm{H}^0(\mathfrak{S}_3, \mathrm{H}^1(K, M))$ is a 1-dimensional k-vector space, and that $\mathrm{H}^p(\mathfrak{S}_3, \mathrm{H}^1(K, M)) = 0$ if $p \geq 1$.

(i). (2 points) Show that we have canonical isomorphisms $\mathrm{H}^1(\mathfrak{S}_4, M) \xrightarrow{\sim} \mathrm{H}^1(K, M)^{\mathfrak{S}_3}$ and $\mathrm{H}^2(\mathfrak{S}_4, M) \xrightarrow{\sim} \mathrm{H}^2(K, M)^{\mathfrak{S}_3}$.

(j). (3 points) Let N be a k-vector space with trivial action of K. Show that the map $Z^2(K, N) \to N^3$ sending a 2-cocycle $\eta : K^2 \to N$ to $(\eta(u, u) - \eta(1, 1), \eta(v, v) - \eta(1, 1), \eta(uv, uv) - \eta(1, 1))$ induces an isomorphism $\mathrm{H}^2(K, N) \xrightarrow{\sim} N^3$.

(k). (2 points) Show that $\mathrm{H}^2(\mathfrak{S}_4, M)$ is a 2-dimensional k-vector space.