Let $G$ be a topological group and $(\pi, V)$ be a unitary representation of $G$. A matrix coefficient of $\pi$ is a function $G \to \mathbb{C}$ of the form $x \mapsto \langle \pi(x)(v), w \rangle$, with $v, w \in V$. Note that these functions are all continuous. We say that the matrix coefficient is diagonal if $v = w$; a diagonal matrix coefficient is a function of positive type by proposition III.2.4, and we call it a function of positive type associated to $\pi$. We say that a function of positive type is normalized if it is of the form $x \mapsto \langle \pi(x)(v), v \rangle$ with $\|v\| = 1$. We denote by $P(\pi)$ the set of functions of positive type associated to $\pi$.

Remember also that, if $G$ is locally compact (and $\mu$ is a left Haar measure on $G$), then the left regular representation $\pi_L$ is the representation of $G$ on $L^2(G) := L^2(G, \mu)$ given by $\pi_L(x)(f) = L_x f$, for $x \in G$ and $f \in L^2(G)$. In this problem, we’ll just call $\pi_L$ the regular representation of $G$.

1. Let $(\pi_1, V_1), (\pi_2, V_2)$ be unitary representations of $G$.
   a) (3) Show that the algebraic tensor product $V_1 \otimes_{\mathbb{C}} V_2$ has a Hermitian inner product, uniquely determined by $\langle v_1 \otimes v_2, w_1 \otimes w_2 \rangle = \langle v_1, w_1 \rangle \langle v_2, w_2 \rangle$.
   b) (2) We denote the completion of $V_1 \otimes_{\mathbb{C}} V_2$ for this inner form by $V_1^\hat{\otimes} C V_2$. Show that the formula $(x, v_1 \otimes v_2) \mapsto \pi_1(x)(v_1) \otimes \pi_2(x)(v_2)$ defines a unitary representation of $G$ on $V_1^\hat{\otimes} C V_2$. (This is called the tensor product representation and usually denoted by $\pi_1 \otimes \pi_2$.)
   c) (2) If $V_1$ and $V_2$ are finite-dimensional, show that, for every $x \in G$, we have
      $$\text{Tr}(\pi_1 \otimes \pi_2(x)) = \text{Tr}(\pi_1(x)) \text{Tr}(\pi_2(x)).$$

2. Let $G$ be a discrete group, and let $\varphi = \mathbf{1}_{\{1\}}$.
   a) (1) Show that $\varphi$ is a function of positive type on $G$.
   b) (2) Show that $V_\varphi$ is equivalent to the regular representation of $G$.
   c) (2, extra credit) If $G \neq \{1\}$, show that the representations $V_\varphi$ and $V_\varphi \otimes_{\mathbb{C}} V_\varphi$ are not equivalent. (You can use problem 12.)

3. Let $G$ be a locally compact group.
   a) (1) If $f, g \in C_c(G)$, show that $f * g \in C_c(G)$.
   b) (3) Show that every matrix coefficient of the regular representation of $G$ vanishes at $\infty$.
   c) (2) Suppose that $G$ is not compact. If $(\pi, V)$ is a finite-dimensional unitary representation of $G$, show that it has a matrix coefficient that does not vanish at $\infty$.
   d) (1) If $G$ is not compact, show that its regular representation has no finite-dimensional subrepresentation.
4. (extra credit, 3) Let $V$ be a locally convex topological $\mathbb{C}$-vector space, $K$ be a compact convex subset of $V$, and $F \subset K$ be such that $K$ is the closure of the convex hull of $F$. Show that every extremal point of $K$ is in the closure of $F$. (This is known as Milman’s theorem.)

We will see soon that, if $G$ is compact, then the regular representation of $G$ contains all the irreducible representations of $G$ (which are all finite-dimensional); in fact, it is the closure of the direct sum of all its irreducible subrepresentations. On the other hand, if $G$ is abelian, then its regular representation is the direct integral of all the irreducible representations of $G$. Let $(\pi, V)$ be unitary representations of $G$. We say that $\pi$ is weakly contained in $\pi'$, and write $\pi \prec \pi'$, if $\mathcal{P}(\pi)$ is contained in the closure of the set of finite sums of elements of $\mathcal{P}(\pi')$ for the topology of convergence on compact subsets of $G$. In other words, $\pi \prec \pi'$ if, for every $v \in V$, for every $K \subset G$ compact and every $c > 0$, there exist $v'_1, \ldots, v'_n \in V'$ such that

$$
\sup_{x \in K} |\langle \pi(x)(v), v \rangle - \sum_{i=1}^{n} \langle \pi'(x)(v'_i), v'_i \rangle | < c.
$$

5. Let $(\pi, V)$ and $(\pi', V')$ be unitary representations of $G$. Let $C \subset V$ such that $\text{Span}(\pi(x)(v), x \in G, v \in C)$ is dense in $V$. Suppose that every function $x \mapsto \langle \pi(x)(v), v \rangle$, for $v \in C$, is in the closure of the set of finite sums of elements of $\mathcal{P}(\pi')$ (still for the topology of convergence on compact subsets of $G$). The goal of this problem is to show that this implies $\pi \prec \pi'$.

Let $X$ be the set of $v \in V$ such that $x \mapsto \langle \pi(x)(v), v \rangle$ is in the closure of the set of finite sums of elements of $\mathcal{P}(\pi')$ (for the same topology as above).

a) (1) Show that $X$ is stable by all the $\pi(x), x \in G$, and under scalar multiplication.
b) (1) If $v \in X$ and $x_1, x_2 \in G$, show that $\pi(x_1)(v) + \pi(x_2)(v) \in X$.
c) (1) Show that $X$ is closed in $V$.
d) (1) If $v \in X$, show that the smallest closed $G$-invariant subspace of $V$ containing $v$ is contained in $X$.
e) Let $v_1, v_2 \in X$, and let $W_1$ (resp. $W_2$) be the smallest closed $G$-invariant subspace of $V$ containing $v_1$ (resp. $v_2$). Let $W = \overline{W_1 + W_2}$, and denote by $T : W \to W_1$ the orthogonal projection, where we take the orthogonal complement of $W_1$ in $W$.

i. (1) Show that $T$ is $G$-equivariant and that $T(W_2)$ is dense in $W_1^\perp$.

ii. (2) Show that $W_1^\perp \subset X$.

iii. (1) Show that $v_1 + v_2 \in X$. (Hint : Use $T(v_1 + v_2)$ and $(v_1 + v_2) - T(v_1 + v_2)$.)
f) (1) Show that $\pi \prec \pi'$.

6. Let $(\pi, V)$ and $(\pi', V')$ be two unitary representations of $G$ such that $\pi \prec \pi'$. Let $C$ be the closure in the weak* topology on $L^\infty(G)$ of the convex hull of the set of normalized functions of positive type associated to $\pi'$.

a) (1) Show that every normalized function of positive type associated to $\pi$ is in $C$.

---

1. For example, if $V$ is cyclic, $C$ could just contain a cyclic vector for $V$. 
b) (3) If π is irreducible, show that every normalized function of positive type associated to π is a limit in the topology of convergence on compact subsets of G of normalized functions of positive type associated to π′. (Hint: problem 4.)

c) (2) If π is the trivial representation of G, show that, for every compact subset K of G and every c > 0, there exists v′ ∈ V′ such that ∥v′∥ = 1 and that

\[ \sup_{x \in K} \| \pi'(x)(v') - v' \| < c. \]

d) (1) Conversely, suppose that, for every compact subset K of G and every c > 0, there exists v′ ∈ V′ such that ∥v′∥ = 1 and that

\[ \sup_{x \in K} \| \pi'(x)(v') - v' \| < c. \]

Show that the trivial representation is weakly contained in π′.

7. (3) Let G be a finitely generated discrete group, and let S be a finite set of generators for G. Show that the trivial representation of G is weakly contained in the regular representation of G if and only, for every ε > 0, there exists f ∈ L²(G) such that

\[ \sup_{x \in S} \| L_x f - f \|_2 < \varepsilon \| f \|_2. \]

8. (2) Let G = Z. Show that the trivial representation of G is weakly contained in the regular representation of G.

9. Let G = R.
   a) (2) Show that the trivial representation of G is weakly contained in the regular representation of G.
   b) (1) Show that every irreducible unitary representation of G is weakly contained in the regular representation of G. ²

10. (extra credit, 4) Let G be the free (nonabelian) group on two generators, with the discrete topology. Show that the trivial representation of G is not weakly contained in the regular representation of G.

11. (2) If π₁, π₂, π₁', π₂' are unitary representations of G such that π₁ ≪ π₁' and π₂ ≪ π₂, show that π₁ ⊗ π₂ ≪ π₁' ⊗ π₂'.

12. Suppose that G is discrete. For every x ∈ G, we denote by δₓ ∈ L²(G) the characteristic function of \{x\}.

Let (π, V) be a unitary representation of G, and let (π₀, V) be the trivial representation of G on V (i.e. π₀(x) = id_V for every x ∈ G).
   a) (3) Show that the formula v ⊗ f → \( \sum_{x \in G} f(x)(π(x)^{-1}(v)) \otimes δ_x \) gives a well-defined and continuous \( C \)-linear transformation from \( V \otimes_C L²(G) \) to itself.
   b) (2) Show that the representations π ⊗ π_L and π₀ ⊗ π_L are equivalent (remember that π_L is the left regular representation of G).

13. (extra credit, 5) Generalize the result of 12(b) to non-discrete locally compact groups.

14. (2) Show that the following are equivalent:
   (i) The trivial representation of G is weakly contained in π_L.
   (ii) Every unitary representation of G is weakly contained in π_L.

²We will see later that this is true for every abelian locally compact group.