MAT 540 : Problem Set 7

Due Thursday, November 7

1 Diagram chasing lemmas via spectral sequences

This problem will ask to reprove some of the diagram chasing lemmas using the two spectral sequences of a double complex. This is circular, because of course the diagram chasing lemmas are used to establish the existence of the spectral sequences. The goal is just to get you used to manipulating spectral sequences on simple examples.

(a). **The $\infty \times \infty$ lemma**: (2 points) Suppose that we have a double complex $X = (X^{n,m}, d_1^{n,m}, d_2^{n,m})$ such that $X^{n,m} = 0$ if $n < 0$ or $m < 0$. Suppose also that the complexes $(X^{\bullet,n}, d_{1,X}^{\bullet,n})$ and $(X^{n,\bullet}, d_{2,X}^{n,\bullet})$ are exact if $n \neq 0$. Using the two spectral sequences of the double complex, prove that we have canonical isomorphisms

$$\mathrm{H}^n(X^{\bullet,0}, d_{1,X}^{\bullet,0}) \simeq \mathrm{H}^n(X^{0,\bullet}, d_{2,X}^{0,\bullet}).$$

(Hint: Both spectral sequences degenerate at the first page.)

(b). **The four lemma**: Consider a commutative diagram with exact rows in $\mathscr{A}$:

(*)

$$\begin{array}{ccccccc} A' & \longrightarrow & B' & \xrightarrow{g} & C' & \longrightarrow & D' \\ \uparrow{\scriptstyle u} & & \uparrow{\scriptstyle v} & & \uparrow{\scriptstyle w} & & \uparrow{\scriptstyle t} \\ A & \longrightarrow & B & \xrightarrow[f]{} & C & \longrightarrow & D \end{array}$$

Suppose that u is surjective and t is injective. We want to show that $f(\mathrm{Ker}\, v) = \mathrm{Ker}\, w$ and that $\mathrm{Im}\, v = g^{-1}(\mathrm{Im}\, w)$.

(i) (1 point) Show that $\mathrm{Im}\, v = g^{-1}(\mathrm{Im}\, w)$ if and only if the morphism $\mathrm{Coker}\, v \to \mathrm{Coker}\, w$ induced by g is injective.

We consider the double complex X represented on diagram (*), with the convention that all the objects that don't appear are 0, the object A is in bidegree $(0,0)$, the differential $d_{1,X}$ is horizontal and the differential $d_{2,X}$ is vertical. (So, for example, $X^{3,0} = C$, $X^{1,0} = A'$ and $X^{2,2} = 0$.) Let ${}^{I}E$ and ${}^{II}E$ the two spectral sequences of this double complex.

(ii) (2 points) Show that ${}^{II}E$ degenerates at the second page.

(iii) (1 point) Show that $\mathrm{H}^2(\mathrm{Tot}(X)) = 0$.

(iv) (1 point) Write the first page of ${}^{I}E$.

(v) (1 point) Show that ${}^{I}E$ degenerates at the second page.

(vi) (2 points) Show that $f(\mathrm{Ker}\, v) = \mathrm{Ker}\, w$ and that $\mathrm{Im}\, v = g^{-1}(\mathrm{Im}\, w)$.

(c). **The long exact sequence of cohomology**: We consider a short exact sequence of complexes $0 \to A^\bullet \to B^\bullet \to C^\bullet \to 0$; to simplify the notation, we will assume that $A^n = B^n = C^n = 0$ for $n < 0$. Consider the following double complex X:

$$\begin{array}{ccccccc}
0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \dots \\
\uparrow & & \uparrow & & \uparrow & & \\
C^0 & \longrightarrow & C^1 & \longrightarrow & C^2 & \longrightarrow & \dots \\
\uparrow & & \uparrow & & \uparrow & & \\
B^0 & \longrightarrow & B^1 & \longrightarrow & B^2 & \longrightarrow & \dots \\
\uparrow & & \uparrow & & \uparrow & & \\
A^0 & \longrightarrow & A^1 & \longrightarrow & A^2 & \longrightarrow & \dots
\end{array}$$

where $X^{0,0} = A$, the differential $d_{1,X}$ (resp. $d_{2,X}$) is represented horizontally (resp. vertically), and $X^{n,m} = 0$ if $n < 0$, $m < 0$ or $m \geq 3$. Let ${}^{I}E$ and ${}^{II}E$ be the two spectral sequences of X.

(i) (1 point) Show that ${}^{I}E$ degenerates at the first page and that $\mathrm{H}^n(\mathrm{Tot}(X)) = 0$ for every $n \in \mathbb{Z}$.

(ii) (1 point) Calculate ${}^{II}E_1$.

(iii) (1 point) Show that ${}^{II}E$ degenerates at the third page.

(iv) (2 points) Show that ${}^{II}E_2^{00} = {}^{II}E_\infty^{00}$ and that ${}^{II}E_2^{1q} = {}^{II}E_\infty^{1q}$ for every $q \geq 0$.

(v) (1 point) Show that $d_2^{0q} : {}^{II}E_2^{0q} \to {}^{II}E_2^{2,q-1}$ is an isomorphism for every $q \geq 1$.

(vi) (1 point) Show that we have a long exact sequence

$$\dots \to \mathrm{H}^n(A^\bullet) \to \mathrm{H}^n(B^\bullet) \to \mathrm{H}^n(C^\bullet) \xrightarrow{\delta^n} \mathrm{H}^{n+1}(A^\bullet) \to \mathrm{H}^{n+1}(B^\bullet) \to \dots$$

where δ^n comes from a differential of the spectral sequence ${}^{II}E$.

2 Group cohomology

(a). **Cohomology of cyclic groups**: If G is a group, $a \in \mathbb{Z}[G]$ and M is a left $\mathbb{Z}[G]$-module, we denote by $a : M \to M$ the $\mathbb{Z}[C_n]$-linear map $x \longmapsto a \cdot x$. For every $n \geq 1$, we denote by C_n the cyclic group of order n and by σ a generator of C_n, and we write $N = 1+\sigma+\sigma^2+\dots+\sigma^{n-1}$. We also write $C_\infty = \mathbb{Z}$ and $\sigma = 1 \in C_\infty$. If $n \in \{1, 2, \dots\} \cup \{\infty\}$, we have a $\mathbb{Z}[C_n]$-linear map $\epsilon : \mathbb{Z}[C_\infty] \to \mathbb{Z}$ sending each element of C_n to $1 \in \mathbb{Z}$.

(i) (2 points) If $n \geq 1$, show that:

$$\dots \to \mathbb{Z}[C_n] \xrightarrow{N} \mathbb{Z}[C_n] \xrightarrow{\sigma-1} \mathbb{Z}[C_n] \xrightarrow{N} \mathbb{Z}[C_n] \xrightarrow{\sigma-1} \mathbb{Z}[C_n] \xrightarrow{\epsilon} \mathbb{Z} \to 0$$

is an exact sequence.

(ii) (2 points) If M is a $\mathbb{Z}[C_n]$-module, show that:

$$\mathrm{H}^q(C_n, M) = \begin{cases} M^{C_n} & \text{if } q = 0 \\ M^{C_n}/N \cdot M & \text{if } q \geq 2 \text{ is even} \\ \{x \in M \mid N \cdot x = 0\}/(\sigma - 1) \cdot M & \text{if } q \text{ is odd.} \end{cases}$$

(iii) (1 point) Show that

$$0 \to \mathbb{Z}[C_\infty] \stackrel{\sigma-1}{\to} \mathbb{Z}[C_\infty] \stackrel{\epsilon}{\to} \mathbb{Z} \to 0$$

is an exact sequence.

(iv) (2 points) If M is a $\mathbb{Z}[C_\infty]$-module, show that:

$$\mathrm{H}^q(C_\infty, M) = \begin{cases} \{x \in M \mid \sigma \cdot x = x\} & \text{if } q = 0 \\ M/(\sigma - 1) \cdot M & \text{if } q = 1 \\ 0 & \text{if } q \geq 2. \end{cases}$$

(b). Let n be a integer, and let $G = C_n \rtimes C_2$ be the dihedral group of order $2n$, where the nontrivial element of C_2 acts on C_n by multiplication by -1. Then $K = C_n$ is a normal subgroup of G, and $G/K \simeq C_2$.

(i) (3 points) Show that

$$\mathrm{H}^q(C_n, \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } q = 0 \\ \mathbb{Z}/n\mathbb{Z} & \text{if } q \geq 2 \text{ is even} \\ 0 & \text{if } q \text{ is odd,} \end{cases}$$

and show that the nontrivial element of C_2 acts by $(-1)^{q/2}$ on $\mathrm{H}^q(C_n, \mathbb{Z})$ if q is even.

(ii) (2 points) Calculate $\mathrm{H}^p(C_2, \mathrm{H}^q(C_n, \mathbb{Z}))$ for all $p, q \geq 0$.

(iii) (2 points) If n is odd, show that

$$\mathrm{H}^m(G, \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } m = 0 \\ \mathbb{Z}/2\mathbb{Z} & \text{if } m = 2 \mod 4 \\ \mathbb{Z}/2n\mathbb{Z} & \text{if } m > 0 \text{ and } m = 0 \mod 4 \\ 0 & \text{if } m \text{ is odd.} \end{cases}$$

(c). Let G be a group, and suppose that G has a normal subgroup K such that $G/K \simeq \mathbb{Z}$. Let M be a $\mathbb{Z}[G]$-module.

(i) (1 point) Show that the Hochschild-Serre spectral sequence degenerates at E_2.

(ii) (2 points) We fix a generator σ of G/K and, for every $q \in \mathbb{N}$, we write $\mathrm{H}^q(K, M)^\sigma = \{x \in \mathrm{H}^q(K, M) \mid \sigma(x) = x\}$ and $\mathrm{H}^q(K, M)_\sigma = \mathrm{H}^q(K, M)/(\sigma - 1) \cdot H^q(K, M)$.

Show that $\mathrm{H}^0(G, M) = \mathrm{H}^0(K, M)^\sigma$, and the we have short exact sequences

$$0 \to \mathrm{H}^{m-1}(K, M)_\sigma \to \mathrm{H}^m(G, M) \to \mathrm{H}^m(K, M)^\sigma \to 0$$

for every $m \geq 1$.

(d). Let G be a group.

(i) (2 points) If K is a central subgroup of G, show that G/K acts trivially on $\mathrm{H}_*(K, \mathbb{Z})$ and on $\mathrm{H}^*(K, \mathbb{Z})$.

Let σ be an element of infinite order in the center of G, and $K = \langle \sigma \rangle$. Let M be a $\mathbb{Z}[G]$-module. We write $M^\sigma = \{x \in M \mid \sigma \cdot x = x\}$ and $M_\sigma = M/(\sigma - 1) \cdot M$.

(ii) (1 point) Show that the Hochschild-Serre spectral sequence calculating $\mathrm{H}^*(G, M)$ degenerates at E_3.

(iii) (2 points) Show that $\mathrm{H}^0(G,M) = \mathrm{H}^0(G/K, M^\sigma)$, and that we have a long exact sequence:

$$0 \to \mathrm{H}^1(G/K, M^\sigma) \to \mathrm{H}^1(G,M) \to \mathrm{H}^0(G/K, M_\sigma) \to \mathrm{H}^2(G/K, M^\sigma) \\ \to \mathrm{H}^2(G,M) \to \mathrm{H}^1(G/K, M_\sigma) \to \mathrm{H}^3(G/K, M^\sigma) \to \dots$$

3 Flabby and soft sheaves

Let X be a topological space. If $\mathscr{F}$ is a sheaf on X and Y is a subset of X, we set

$$\mathscr{F}(Y) = \varinjlim_{Y \subset U \in \mathrm{Open}(X)^{\mathrm{op}}} \mathscr{F}(U).$$

If $Y \subset Y'$, we have a map $\mathscr{F}(Y') \to \mathscr{F}(Y)$ induced by the restriction maps of $\mathscr{F}$.

We say that $\mathscr{F}$ is *flabby* (or *flasque*) if, for every open subset U of X, the restriction map $\mathscr{F}(X) \to \mathscr{F}(U)$ is surjective. We say that $\mathscr{F}$ is *soft* if, for every *closed* subset F, the map $\mathscr{F}(X) \to \mathscr{F}(F)$ is surjective.

Let R be a ring. If M is a left R-module and $x \in X$, we write $S_{x,M}$ for the presheaf on X given by $S_{x,M}(U) = M$ if $x \in U$ and $S_{x,M}(U) = 0$ if $x \notin U$, with the obvious restriction maps (equal to 0 or id_M). It is easy to see that this is a sheaf, and we call it the *skryscraper sheaf* at x with value M.

(a). (1 point) Show that any flabby sheaf is soft.

(b). (2 points) Let $d \geq 1$, and let $\mathscr{F}$ be the sheaf $U \longmapsto C^\infty(U, \mathbb{C})$ on $\mathbb{R}^d$. Show that the sheaf $\mathscr{F}$ is soft.

(c). (1 point) For every $x \in X$, show that the functor ${}_R\mathbf{Mod} \to \mathrm{Sh}(\mathscr{F}, R)$, $M \longmapsto S_{x,M}$ is right adjoint to the functor $\mathscr{F} \longmapsto \mathscr{F}_x$.

(d). (2 points) If $(M_x)_{x \in X}$ is a family of R-modules, show that $\prod_{x \in X} S_{x,M_x}$ is a flabby sheaf, and that it is an injective sheaf if every M_x is an injective R-module.

(e). (1 point) For every sheaf of R-modules $\mathscr{F}$ on X, we set $G(\mathscr{F}) = \prod_{x \in X} S_{x,\mathscr{F}_x}$. Show that the canonical morphism $\mathscr{F} \to G(\mathscr{F})$ (sending any $s \in \mathscr{F}(U)$ to the family $(s_x)_{x \in U}$) is injective.

(f). (2 points) Show that sheaves on R-modules on X have a functorial resolution by flabby injective sheaves.

(g). (2 points) Let $0 \to \mathscr{F} \to \mathscr{G} \to \mathscr{H} \to 0$ be an exact sequence in $\mathrm{Sh}(X, R)$, with $\mathscr{F}$ flabby. Show that the sequence $0 \to \mathscr{F}(X) \to \mathscr{G}(X) \to \mathscr{H}(X) \to 0$ is exact.

An open cover $(U_i)_{i \in I}$ of X is called *locally finite* if every point of X has a neighborhood that meets only finitely many of the U_i. We say that X is *paracompact* if every open cover of X has a locally finite refinement. We admit the following facts:

(1) A metric space is paracompact.

(2) If X is paracompact and $(U_i)_{i \in I}$ is an open cover of X, then there exists an open cover $(V_i)_{\in I}$ of X such that $\overline{V_i} \subset U_i$ for every $i \in I$.

(h). Suppose that X is a separable metric space. [1] Let $0 \to \mathscr{F} \xrightarrow{f} \mathscr{G} \xrightarrow{g} \mathscr{H} \to 0$ be a short sequence of sheaves of R-modules on X, with $\mathscr{F}$ soft. The goal of this question is to prove that the sequence $0 \to \mathscr{F}(X) \to \mathscr{G}(X) \to \mathscr{H}(X) \to 0$ is exact.

[1] It would be enough to assume that X is paracompact.

(i) (1 point) Let $s \in \mathscr{H}(X)$. Show that there exists a locally finite open cover $(U_n)_{n\in\mathbb{N}}$ and sections $t_n \in \mathscr{G}(U_n)$ such that $g(t_n) = s_{|U_n}$ for every $n \in \mathbb{N}$.

(ii) (2 points) Take an open cover $(V_n)_{n\in\mathbb{N}}$ of X such that $F_n := \overline{V_n} \subset U_n$ for every $n \in \mathbb{N}$. Prove by induction on n that, for every $n \geq 0$, there exists a section $t_n \in \mathscr{F}(F_0 \cup \ldots \cup F_n)$ such that $g(t_n) = s_{|F_0\cup\ldots\cup F_n}$.

(iii) (1 point) Show that s has a preimage in $\mathscr{G}(X)$.

(i). (3 points) If $\mathscr{F}$ is a flabby sheaf of R-modules on a topological space X, or a soft sheaf of R-modules on a separable metric space X, show that $\mathrm{H}^n(X, \mathscr{F}) = 0$ for every $n \geq 1$. (Hint: Try to adapt the strategy of Problem 5(b) of problem set 6.)