MAT 540 : Problem Set 6

Due Sunday, November 3

We make the following useful convention: if $(x_0, \dots, x_n)$ is some list and if $i \in \{0, \dots, n\}$, then $(x_0, \dots, \hat{x}_i, \dots, x_n)$ means $(x_0, \dots, x_{i-1}, x_{i+1}, \dots, x_n)$.

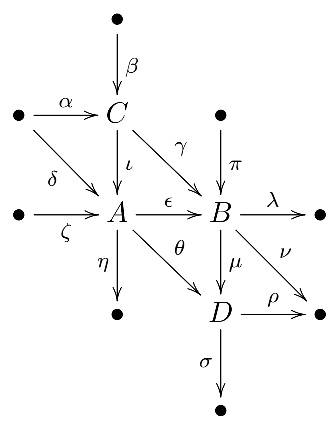
Also, if S is a set and $\mathbb{Z}^{(S)}$ is the free $\mathbb{Z}$-module on S, we denote the canonical basis of this free module by $(e_s)_{s \in S}$.

1 Salamander lemma (5 points)

Prove the salamander lemma (Theorem IV.2.1.3 of the notes).

Solution. If we turn the complex of (ii) 90 degree to the left and see it as a complex in the opposite category of $\mathscr{A}$, then we are exactly in the situation of (i). So it suffices to prove (i).

We give names to some morphisms of the complex



We check the exactness of the sequence at each object. By the Freyd-Mitchell embedding theorem (Theorem III.3.1 of the notes), we may assume that $\mathscr{A}$ is a category of left R-modules. (Hence take elements in the objects of $\mathscr{A}$.)

In ${}_{=}A = \operatorname{Ker}\epsilon / \operatorname{Im}\gamma$, the subobject $\operatorname{Im}(1)$ is the image of $\iota(\operatorname{Ker}\gamma) \subset A$, and $\operatorname{Ker}(2) = (\operatorname{Ker}\epsilon \cap (\operatorname{Im}\iota + \operatorname{Im}\zeta)) / \operatorname{Im}\zeta$. So $\operatorname{Im}(1) \subset \operatorname{Ker}(2)$. Conversely, take an element of $\operatorname{Ker}(2)$, lift it to $x \in \operatorname{Ker}\epsilon$, and choose $y \in C$ such that $x \in \iota(y) + \operatorname{Im}\zeta$. Then $\gamma(y) \in \epsilon(x) + \epsilon(\operatorname{Im}\zeta) = 0$, so y defines an element of $C_\square = \operatorname{Ker}\gamma / (\operatorname{Im}\alpha + \operatorname{Im}\beta)$, so $y \in \operatorname{Im}(1)$.

In $A_\square = \operatorname{Ker}\theta / (\operatorname{Im}\iota + \operatorname{Im}\zeta)$, the subobject $\operatorname{Im}(2)$ is the image of $\operatorname{Ker}\epsilon \subset A$, and $\operatorname{Ker}(3)$ is the set of elements that have a lift $x \in \operatorname{Ker}\theta$ such that $\epsilon(x) \in \operatorname{Im}(\gamma)$. So we clearly have $\operatorname{Im}(2) \subset \operatorname{Ker}(3)$. Consider an element of $\operatorname{Ker}(3)$, choose a lift $x \in \operatorname{Ker}\theta$ of that element such that $\epsilon(x) = \gamma(y)$, for some $y \in C$. Then $x - \iota(y)$ and x have the same image in $A_\square$, and $\epsilon(x - \iota(y)) = 0$, so the image of x in $A_\square$ is in $\mathrm{Im}(2)$.

In ${}^\square B = (\mathrm{Ker}\,\lambda \cap \mathrm{Ker}\,\mu)/\mathrm{Im}\,\gamma$, the subobject $\mathrm{Im}(3)$ is the image of $\epsilon(\mathrm{Ker}\,\theta) \subset B$, and $\mathrm{Ker}(4)$ is the set of elements of ${}^\square B$ that have a lift $x \in (\mathrm{Ker}\,\lambda \cap \mathrm{Ker}\,\mu) \cap \mathrm{Im}\,\epsilon$. So we clearly have $\mathrm{Im}(3) \subset \mathrm{Ker}(4)$. Conversely, consider an element of $\mathrm{Ker}(4)$, and choose a lift $x \in \mathrm{Ker}\,\lambda \cap \mathrm{Ker}\,\mu$ of this element such that we can write $x = \epsilon(y)$, with $y \in A$. Then $\theta(y) = \mu(x) = 0$, so $x \in \epsilon(\theta(y))$, and its image in ${}^\square B$ is in $\mathrm{Im}(3)$.

In ${}_=B = \mathrm{Ker}\,\lambda/\mathrm{Im}\,\epsilon$, the subobject $\mathrm{Im}(4)$ is the image of $\mathrm{Ker}\,\lambda \cap \mathrm{Ker}\,\mu \subset B$, and $\mathrm{Ker}(5)$ is the set of elements of ${}_=B$ that have a lift $x \in \mathrm{Ker}\,\lambda$ such that $\mu(x) \in \mathrm{Im}(\theta)$. So we clearly have $\mathrm{Im}(4) \subset \mathrm{Ker}(5)$. Conversely, consider an element of $\mathrm{Ker}(5)$, and choose a lift $x \in \mathrm{Ker}\,\lambda$ of this element such that we can write $\mu(x) = \theta(y)$, with $y \in A$. Then $\lambda(x - \epsilon(y)) = 0$, the elements x and $x - \epsilon(y)$ of $\mathrm{Ker}\,\lambda$ have the same image in ${}_=B$, and $\mu(x - \epsilon(y)) = 0$, so $x - \epsilon(y) \in \mathrm{Ker}\,\lambda \cap \mathrm{Ker}\,\mu$, and its image in ${}_=B$ is in $\mathrm{Im}(4)$.

□

2 Some bar resolutions

(a). (3 points) Let S be a nonempty set. We define a complex of $\mathbb{Z}$-modules $X^\bullet$ by:

- $X^n = 0$ and $d_X^n = 0$ if $n \geq 2$;
- $X^1 = \mathbb{Z}$ and $d_X^1 = 0$;
- $X^0 = \mathbb{Z}^{(S)}$ and $d_X^0 : X^0 \to X^1 = \mathbb{Z}$ sends every e_s to 1;
- if $n \geq 1$, then $X^{-n} = \mathbb{Z}^{(S^{n+1})}$ and $d^{-n} : \mathbb{Z}^{(S^{n+1})} \to \mathbb{Z}^{(S^n)}$ sends $e_{(s_0,\ldots,s_n)}$ to $\sum_{i=0}^n (-1)^i e_{(s_0,\ldots,\hat{s}_i,\ldots,s_n)}$, for all $s_0, \ldots, s_n \in S$.

Show that $X^\bullet$ is indeed a complex (i.e. that $d_X^{n+1} \circ d_X^n = 0$ for every $n \in \mathbb{Z}$), and that it is acyclic. (Hint: Fix $s \in S$. If $n \geq -1$, consider the morphism $t^{-n} : X^{-n} \to X^{-n-1}$ sending $e_{(s_0,\ldots,s_n)}$ to $e_{(s,s_0,\ldots,s_n)}$.)

(b). Let G be a group. For every $n \geq 0$, let $X_n(G) = \mathbb{Z}^{(G^{n+1})}$. By (a), we have an acyclic complex of $\mathbb{Z}$-modules $X^\bullet$, where $X^1 = \mathbb{Z}$, $X^{-n} = X_n(G)$ if $n \geq 0$, $X^n = 0$ if $n \geq 2$, and the differentials are as in (a).

  (i) (2 points) We make G act as $X_n(G)$ by $g \cdot e_{(g_0,\ldots,g_n)} = e_{(gg_0,\ldots,gg_n)}$, and we make G act trivially on $\mathbb{Z}$. Show that $X^\bullet$ is an acyclic complex of $\mathbb{Z}[G]$-modules.

  (ii) (2 points) Show that $X_n(G)$ is a free $\mathbb{Z}[G]$-module for every $n \geq 0$. [1]

Let I_n be the $\mathbb{Z}$-submodule of $X_n(G)$ generated by the $e_{(g_0,\ldots,g_n)}$ such that $g_i = g_{i+1}$ for some $i \in \{0, \ldots, n-1\}$.

  (iii) (2 points) Show that I_n is a free $\mathbb{Z}[G]$-submodule of $X_n(G)$ and that $d^{-n}(I_n) \subset I_{n-1}$ for $n \geq 0$, with $I_{-1} = \{0\}$.

  (iv) (2 points) By the previous question, we get a complex of $\mathbb{Z}[G]$-modules $Y^\bullet$ such that $Y^n = 0$ for ≥ 2, $Y^1 = \mathbb{Z}$, $Y^{-n} = X_n(G)/I_n$ if $n \geq 0$ and d_Y^n is the morphism induced by d_X^n for every $n \in \mathbb{Z}$. Show that $Y^\bullet$ is acyclic. (Hint: Try to imitate the method of (a).)

[1]The complex $X^\bullet$ is called the *unnormalized bar resolution* of $\mathbb{Z}$ as a $\mathbb{Z}[G]$-module.

[2]

Solution.

(a). If we set $S^0 = \{()\}$ (the set whose only element is the empty sequence of elements of S), then we can see X^1 as the free $\mathbb{Z}$-module on S^0, with basis element $1 = e_{()}$. In this way, the formula for d^{-n} also works if $n = 0$.

We prove that $X^\bullet$ is a complex. If $n \geq 0$, then $d^{n+1} \circ d^n = 0$ because $d^{n+1} = 0$. We assume that $n \geq 1$ and we calculate $d^{-n+1} \circ \circ d^{-n} : \mathbb{Z}^{(S^{n+1})} \to \mathbb{Z}^{(S^{n-1})}$. Let $s_0, \dots, s_n \in S$. Then

$$\begin{aligned} d^{-n+1} \circ d^{-n}(e_{(s_0,\dots,s_n)}) &= \sum_{i=0}^{n} (-1)^i d^{-n+1}(e_{s_0,\dots,\hat{s}_i,\dots,s_n}) \\ &= \sum_{i=0}^{n} \sum_{j=0}^{i-1} (-1)^{i+j} e_{(s_0,\dots,\hat{s}_j,\dots,\hat{s}_i,\dots,s_n)} + \sum_{i=0}^{n} \sum_{j=i+1}^{n} (-1)^{i+j-1} e_{(s_0,\dots,\hat{s}_i,\dots,\hat{s}_j,\dots,s_n)} \\ &= \sum_{j=0}^{n} \sum_{i=j+1}^{n} (-1)^{i+j} e_{(s_0,\dots,\hat{s}_j,\dots,\hat{s}_i,\dots,s_n)} + \sum_{i=0}^{n} \sum_{j=i+1}^{n} (-1)^{i+j-1} e_{(s_0,\dots,\hat{s}_i,\dots,\hat{s}_j,\dots,s_n)} \\ &= 0. \end{aligned}$$

We fix $s \in S$. We define $t^m : X^m \to X^{m-1}$ by $t^m = 0$ for $m \geq 2$, and $t^{-n} : \mathbb{Z}^{(S^{n+1})} \to \mathbb{Z}^{(S^{n+2})}$, $e_{(s_0,\dots,s_n)} \longmapsto e_{(s,s_0,\dots,s_n)}$ if $n \geq -1$. We want to prove that $(t^m)_{m \in \mathbb{Z}}$ is a homotopy between $\mathrm{id}_{X^\bullet}$ and 0. We have to check that $\mathrm{id}_{X^m} = t^{m+1} \circ d^m + d^{m-1} \circ t^m$ for every $m \in \mathbb{Z}$. If $m \geq 2$, then both sides are equal to 0. If $m = 1$, then we want to check that $\mathrm{id}_{\mathbb{Z}} = d^0 \circ t^1$; the right hand side sends $e_{()}$ to $d^0(e_s) = e_{()}$, so we get the desired identity. Suppose that $m \geq 0$, and write $n = -m$. Let $(s_0, \dots, s_n) \in S^{n+1}$. Then $(t^{-n+1} \circ d^{-n} + d^{-n-1} \circ t^{-n})(e_{(s_0,\dots,s_n)})$ is equal to

$$\begin{aligned} &(t^{-n+1} \circ d^{-n} + d^{-n-1} \circ t^{-n})(e_{(s_0,\dots,s_n)}) \\ &= \sum_{i=0}^{n} (-1)^i e_{(s,s_0,\dots,\hat{s}_i,\dots,s_n)} + d^{-n}(e_{(s,s_0,\dots,s_n)}) \\ &= \sum_{i=0}^{n} (-1)^i e_{(s,s_0,\dots,\hat{s}_i,\dots,s_n)} + e_{(s_0,\dots,s_n)} + \sum_{i=0}^{n} (-1)^{i+1} e_{(s,s_0,\dots,\hat{s}_i,\dots,s_n)} \\ &= e_{(s_0,\dots,s_n)}. \end{aligned}$$

So $t^{-n+1} \circ d^{-n} + d^{-n-1} \circ t^{-n} = \mathrm{id}_{X^{-n}}$.

(b). (i) As the formation of kernels and cokernsl commutes with the forgetful functor from ${}_{\mathbb{Z}[G]}\mathbf{Mod}$ to $\mathbf{Ab}$, and as we know that $X^\bullet$ is an acyclic complex of $\mathbb{Z}$-modules by (a), it suffices to show that $X^\bullet$ is a complex of $\mathbb{Z}[G]$-modules, i.e. that its differentials are $\mathbb{Z}[G]$-linear. But this is clear from the definitions of the differentials and of the action of $\mathbb{Z}[G]$.

(ii) It suffices to find a $\mathbb{Z}[G]$-basis of $X_n(G)$. If $(g_1, \dots, g_n) \in G^n$, then the morphism $\mathbb{Z}[G] \to X_n(G)$, $a \longmapsto a \cdot e_{(1,g_1,g_1g_2\dots,g_1g_2\dots g_n)}$ is injective with image $V_{g_1,\dots,g_n} := \mathrm{Span}(\{e_{(h_0,h_1,\dots,h_n)},\ h_{i-1}^{-1}h_i = g_i \text{ for } 1 \leq i \leq n\}$. As $X_n(G) = \bigoplus_{(g_1,\dots,g_n) \in G^n} V_{(g_1,\dots,g_n)}$ (because these subspaces are generated by mutually disjoint subsets of the canonical basis of $X_n(G)$), we deduce that the family $(e_{(1,g_1,g_1g_2,\dots,g_1g_2\dots g_n)})_{(g_1,\dots,g_n) \in G^n}$ is a $\mathbb{Z}[G]$-basis of $X_n(G)$.

[2]The complex $Y^\bullet$ is called the *normalized bar resolution* of $\mathbb{Z}$ as a $\mathbb{Z}[G]$-module.

(iii) We have found a $\mathbb{Z}[G]$-basis $(e_{(1,g_1,g_1g_2,\dots,g_1g_2\dots g_n)})_{(g_1,\dots,g_n)\in G^n}$ of $X_n(G)$ in (ii), and the calculation of $V_{g_1,\dots,g_n} = \mathbb{Z}[G]\cdot e_{(1,g_1,g_1g_2,\dots,g_1g_2\dots g_n)}$ in the proof of that question show that $V_{g_1,\dots,g_n}$ is included in I_n if one of the g_i is equal to 1, and that $V_{g_1,\dots,g_n}\cap I_n = \{0\}$ otherwise. So I_n is the $\mathbb{Z}[G]$-submodule of $X_n(G)$ generated by the $e_{(1,g_1,g_1g_2,\dots,g_1g_2\dots g_n)}$ such that at least one of the g_i is equal to 1, and in particular it is a free $\mathbb{Z}[G]$-submodule of $X_n(G)$.

We check that $d^{-n}(I_n)\subset I_{n-1}$. Let $(g_0,\dots,g_n)\in G^{n+1}$, and suppose that $d_i = d_{i+1}$ for some $i\in\{0,\dots,n-1\}$. Then

$$\begin{aligned} d^{-n}(e_{(g_0,\dots,g_n)}) &= \sum_{j\in\{0,\dots,n\}-\{i,i+1\}} (-1)^j e_{(g_0,\dots,\hat{g}_j,\dots,g_n)} + (-1)^i e_{(g_0,\dots,g_{i-1},g_i,g_{i+2},\dots,g_n)} \\ &\quad + (-1)^{i+1} e_{(g_0,\dots,g_{i-1},g_{i+1},g_{i+2},\dots,g_n)} \\ &= \sum_{j\in\{0,\dots,n-\{i,i+1\}} (-1)^j e_{(g_0,\dots,\hat{g}_j,\dots,g_n)}. \end{aligned}$$

The last sum is clearly in I_{n-1}.

(iv) It suffices to show that $Y^\bullet$ is acyclic as a complex of $\mathbb{Z}$-modules. Let $t^m : X^m \to X^{m-1}$ be the morphisms of (a), for example for $s=1$ (the unit element of G). Then, if $n\geq 0$, $t^{-n} : X_n(G)\to X_{n+1}(G)$ sends I_n to I_{n+1}, so it induces a morphism $\bar{t}_n : Y^{-n}\to Y^{-n+1}$. We also denote by $\bar{t}^1$ the morphism $t^1 : Y^1=\mathbb{Z}\to Y^0 = X_0(G)$ (note that $I_0=\{0\}$) and set $\bar{t}^m = 0$ for $m\geq 2$. Then, by (a), the family $(t^m)_{m\in\mathbb{Z}}$ defines a homotopy between $\mathrm{id}_{Y^\bullet}$ and 0.

□

3 Čech cohomology, part 1

This problem uses problem 1 of problem set 5.

Let $\mathscr{C}$ be a category that admits fiber products, and let $\mathscr{X} = (f : X_i\to X)_{i\in I}$ be a family of morphisms of $\mathscr{C}$. If $i_0,\dots,i_p\in I$, we write $X_{i_0,\dots,i_p} = X_{i_0}\times_X X_{i_1}\times_X\dots\times_X X_{i_p}$. For every $p\in\mathbb{Z}$, we define an abelian presheaf $\mathscr{C}_p(\mathscr{X})\in\mathrm{Ob}(\mathrm{PSh}(\mathscr{C},\mathbb{Z}))$ in the following way:

- if $p<0$, then $\mathscr{C}_p = 0$;
- if $p\geq 0$, then
$$\mathscr{C}_p(\mathscr{X}) = \bigoplus_{i_0,\dots,i_p\in I}\mathbb{Z}^{(X_{i_0,\dots,i_p})}.$$

We also define a morphism of presheaves $d_p : \mathscr{C}_p(\mathscr{X})\to\mathscr{C}_{p-1}(\mathscr{X})$ in the following way:

- if $p\leq 0$, then $d_p = 0$;
- if $p\geq 1$, then d_p is given on the component $\mathbb{Z}^{(X_{i_0,\dots,i_p})}$ by the morphism $\mathbb{Z}^{(X_{i_0,\dots,i_p})}\to\bigoplus_{q=0}^p\mathbb{Z}^{(X_{i_0,\dots,i_{q-1},i_{q+1},\dots,i_p})}\subset\mathscr{C}_{p-1}(\mathscr{X})$ equal to $\sum_{q=0}^p(-1)^q\delta^q_{i_0,\dots,i_p}$, where $\delta^q_{i_0,\dots,i_p} : \mathbb{Z}^{(X_{i_0,\dots,i_p})}\to\mathbb{Z}^{(X_{i_0,\dots,i_{q-1},i_{q+1},\dots,i_p})}$ is the image of the canonical projection $X_{i_0,\dots,i_p}\to X_{i_0,\dots,i_{q-1},i_{q+1},\dots,i_p}$ by the functor $\mathscr{C}\overset{h_{\mathscr{C}}}{\to}\mathrm{PSh}(\mathscr{C})\overset{\mathbb{Z}^{(\cdot)}}{\to}\mathrm{PSh}(\mathscr{C},\mathbb{Z})$.

(a). (4 points) Show that $\mathrm{Ker}(d_p)\supset\mathrm{Im}(d_{p+1})$ for every $p\in\mathbb{Z}$ and that this is an equality for $p\neq 0$.

Hint: For every object Y of $\mathscr{C}$, we have

$$\mathrm{Hom}_{\mathscr{C}}(Y, X_{i_0,\ldots,i_p}) = \coprod_{h\in\mathrm{Hom}_{\mathscr{C}}(Y,X)} \mathrm{Hom}_{\mathscr{C}}(Y, X_{i_0})_h \times \ldots \times \mathrm{Hom}_{\mathscr{C}}(Y, X_{i_p})_h,$$

where, for every $i \in I$, $\mathrm{Hom}_{\mathscr{C}}(Y, X_i)_h = \{g \in \mathrm{Hom}_{\mathscr{C}}(Y, X_i) \mid f_i \circ g = h\}$. Set $S_h = \coprod_{i\in I} \mathrm{Hom}_{\mathscr{C}}(Y, X_i)_h$ and think of question 2(a).

(b). (1 point) Let $\varepsilon : \mathscr{C}_0(\mathscr{X}) \to \mathbb{Z}^{(X)}$ be the morphism that is equal on the component $\mathbb{Z}^{(X_i)}$ to the image of $f_i : X_i \to X$ by the functor $\mathscr{C} \overset{h_{\mathscr{C}}}{\to} \mathrm{PSh}(\mathscr{C}) \overset{\mathbb{Z}^{(\cdot)}}{\to} \mathrm{PSh}(\mathscr{C}, \mathbb{Z})$. Show that $\mathrm{Ker}(\varepsilon) = \mathrm{Im}(d_1)$.

For every $p \in \mathbb{Z}$, we define a functor $\check{\mathscr{C}}^p(\mathscr{X}, \cdot) : \mathrm{PSh}(\mathscr{C}, \mathbb{Z}) \to \mathbf{Ab}$ by $\check{\mathscr{C}}^p(\mathscr{X}, \mathscr{F}) = \mathrm{Hom}_{\mathrm{PSh}(\mathscr{C},\mathbb{Z})}(\mathscr{C}^p(\mathscr{X}), \mathscr{F})$, and a morphism of functors $d^p : \check{\mathscr{C}}^p(\mathscr{X}, \cdot) \to \check{\mathscr{C}}^{p+1}(\mathscr{X}, \cdot)$ by $d^p = \mathrm{Hom}_{\mathrm{PSh}(\mathscr{C},\mathbb{Z})}(d_{p+1}, \cdot)$. The family $(\check{\mathscr{C}}^p(\mathscr{X}, \mathscr{F}), d^p)_{p\in\mathbb{Z}}$ is called the *Čech complex of $\mathscr{F}$ (relative to the family $\mathscr{X}$)*. For every $p \geq 0$, we set $\check{\mathrm{H}}^p(\mathscr{X}, \mathscr{F}) = \mathrm{Ker}(d^p(\mathscr{F}))/\mathrm{Im}(d^{p-1}(\mathscr{F}))$. This is called the *pth Čech cohomology group of $\mathscr{F}$ (relative to the family $\mathscr{X}$)*. Note that the definition of $\check{\mathrm{H}}^p(\mathscr{X}, \mathscr{F})$ is functorial in $\mathscr{F}$, so $\check{\mathrm{H}}^p(\mathscr{X}, \cdot)$ is a functor from $\mathrm{PSh}(\mathscr{C}, \mathbb{Z})$ to $\mathbf{Ab}$.

(c). (2 points) Show that, for every abelian presheaf $\mathscr{F}$ and every $p \geq 0$, we have

$$\check{\mathscr{C}}^p(\mathscr{X}, \mathscr{F}) = \prod_{i_0,\ldots,i_p} \mathscr{F}(X_{i_0,\ldots,i_p}),$$

and that the definition of $\check{\mathrm{H}}^0(\mathscr{X}, \mathscr{F})$ given here generalizes that of Definition III.2.2.4 of the notes.

(d). (1 point) If $\mathscr{F}$ is an injective object of $\mathrm{PSh}(\mathscr{C}, \mathbb{Z})$, show that $\check{\mathrm{H}}^p(\mathscr{X}, \mathscr{F}) = 0$ for every $p \geq 1$.

(e). (2 points) Suppose that we have a Grothendieck topology $\mathscr{T}$ on $\mathscr{C}$, that $\mathscr{X}$ is a covering family, and that $\mathscr{F}$ is an injective object of $\mathrm{Sh}(\mathscr{C}_{\mathscr{T}}, \mathbb{Z})$. Show that $\check{\mathrm{H}}^p(\mathscr{X}, \mathscr{F}) = 0$ for $p \geq 1$ and that $\check{\mathrm{H}}^0(\mathscr{X}, \mathscr{F}) = \mathscr{F}(X)$.

(f). (2 points) Let $\mathscr{F} \in \mathrm{Ob}(\mathrm{PSh}(\mathscr{C}, \mathbb{Z}))$, let $\mathscr{F} \to \mathscr{I}^\bullet$ be an injective resolution of $\mathscr{F}$ in $\mathrm{PSh}(\mathscr{C}, \mathbb{Z})$. Show that we have canonical isomorphisms

$$\mathrm{H}^n(\check{\mathrm{H}}^0(\mathscr{X}, \mathscr{I}^\bullet)) \simeq \check{\mathrm{H}}^n(\mathscr{X}, \mathscr{F}).$$

[3]

Solution.

(a). If $p = 0$, then $d_p = 0$, so $\mathrm{Ker}(d_p) \supset \mathrm{Im}(d_{p+1})$. If $p \leq -1$, then $\mathscr{C}_p(\mathscr{X})(Y) = 0$, so $\mathrm{Ker}(d_p(Y)) = \mathrm{Im}(d_{p+1}(Y)) = 0$. To treat the other cases, it suffices to prove that, for every $Y \in \mathrm{Ob}(\mathscr{C})$, we have $\mathrm{Ker}(d_p(Y)) = \mathrm{Im}(d_{p+1}(Y))$ for $p \geq 1$.

We fix $Y \in \mathrm{Ob}(\mathscr{C})$, and we use the notation of the hint. For every $h \in \mathrm{Hom}_{\mathscr{C}}(Y, X)$, let $S_h = \coprod_{i\in I} \mathrm{Hom}_{\mathscr{C}}(Y, X_i)_h$. Fix $p \geq 0$. The fact that

$$\mathrm{Hom}_{\mathscr{C}}(Y, X_{i_0,\ldots,i_p}) = \coprod_{h\in\mathrm{Hom}_{\mathscr{C}}(Y,X)} \mathrm{Hom}_{\mathscr{C}}(Y, X_{i_0})_h \times \ldots \times \mathrm{Hom}_{\mathscr{C}}(Y, X_{i_p})_h$$

[3] In other words, $\check{\mathrm{H}}^n(\mathscr{X}, \cdot)$ is the nth right derived functor of $\check{\mathrm{H}}^0(\mathscr{X}, \cdot)$.

for all $i_0, \dots, i_p$ is obvious, so we get

$$\coprod_{(i_0,\dots,i_p)\in I^{p+1}} \mathrm{Hom}_{\mathscr{C}}(Y, X_{i_0,\dots,i_p}) = \coprod_{h\in \mathrm{Hom}_{\mathscr{C}}(Y,X)} S_h^{p+1},$$

and

$$\begin{aligned}\mathscr{C}_p(\mathscr{X})(Y) &= \bigoplus_{(i_0,\dots,i_p)\in I^{p+1}} \mathbb{Z}^{(\mathrm{Hom}_{\mathscr{C}}(Y,X_{i_0,\dots,i_p}))} \\ &= \bigoplus_{h\in\mathrm{Hom}_{\mathscr{C}}(Y,X)} \mathbb{Z}^{(S_h^{p+1})}.\end{aligned}$$

So $\mathscr{C}_p(\mathscr{X})(Y)$ is the direct sum indexed by $h \in \mathrm{Hom}_{\mathscr{C}}(Y, X)$ of the terms of degree $-p$ of the complex of Problem 2(a) for $S = S_h$, and $d_p : \mathscr{C}_p(\mathscr{X})(Y) \to \mathscr{C}_{p-1}(\mathscr{X})(Y)$ is the direct sum of the differentials of this complex if $p \geq 1$ (this follows immediately from the definition of d_p). As the complex of 2(a) is acyclic, this implies that $\mathrm{Ker}(d_p(Y)) = \mathrm{Im}(d_{p+1}(Y))$ if $p \geq 1$.

(b). Let $Y \in \mathrm{Ob}(\mathscr{C})$. We use the same notation as in the solution of (a). Then

$$\mathscr{C}_0(\mathscr{X})(Y) = \bigoplus_{h\in\mathrm{Hom}_{\mathscr{C}}(Y,X)} \mathbb{Z}^{(S_h)},$$

$$\mathbb{Z}^{(X)}(Y) = \mathbb{Z}^{(\mathrm{Hom}_{\mathscr{C}}(Y,X))} = \bigoplus_{h\in\mathrm{Hom}_{\mathscr{C}}(X,Y)} \mathbb{Z},$$

and $\varepsilon(Y)$ is the sum of the morphisms $d_0 : \mathbb{Z}^{(S_h)} \to \mathbb{Z}$ from Problem 2(a). So the result follows again from Problem 2(a).

(c). The $\check{\mathscr{C}}^p(\mathscr{X}, \mathscr{F}) = \prod_{i_0,\dots,i_p} \mathscr{F}(X_{i_0,\dots,i_p})$ follows immediately from the definition of $\mathscr{C}_p(\mathscr{X})$, the universal property of the direct sum and question (b) of Problem 1 of problem set 5.

In particular, we have $\check{\mathscr{C}}^0(\mathscr{X}, \mathscr{F}) = \prod_{i\in I} \mathscr{F}(X_i)$ and $\check{\mathscr{C}}^1(\mathscr{X}, \mathscr{F}) = \prod_{i,j\in I} \mathscr{F}(X_i \times_X X_j)$, and (by definition of $d_1 : \mathscr{C}_1(\mathscr{X}) \to \mathscr{C}_0(\mathscr{X})$) $d^0 : \check{\mathscr{C}}^0(\mathscr{X}, \mathscr{F}) \to \check{\mathscr{C}}^1(\mathscr{X}, \mathscr{F})$ sends a family $(s_i)_{i\in I}$ to $(p_{i,ij}^* s_i - p_{j,ij}^* s_j)_{i,j\in I}$, where $p_{i,ij} : X_i \times_X X_j \to X_i$ and $p_{j,ij} : X_i \times_X X_j \to X_j$ are the two projections. So $\check{\mathrm{H}}^0(\mathscr{X}, \mathscr{F}) = \mathrm{Ker}(d^0)$ is equal to the set $\check{\mathrm{H}}^0(\mathscr{X}, \mathscr{F})$ of Definition III.2.2.4 of the notes.

(d). If $\mathscr{F}$ is an injective object of $\mathrm{PSh}(\mathscr{C}, \mathbb{Z})$, then the functor $\mathrm{Hom}_{\mathrm{PSh}(\mathscr{C},\mathbb{Z})}(\cdot, \mathscr{F})$ is exact, so the statement follows from (a).

(e). The fact that $\check{\mathrm{H}}^0(\mathscr{X}, \mathscr{F}) = \mathscr{F}(X)$ follows from the end of (c) and from the definition of a sheaf (see Remark III.2.2.5 of the notes).

The inclusion functor $\Phi : \mathrm{Sh}(\mathscr{C}_{\mathscr{T}}, \mathbb{Z}) \subset \mathrm{PSh}(\mathscr{C}, \mathbb{Z})$ is right adjoint to the sheafification functor and the sheafification functor is exact, so Φ sends injective objects of $\mathrm{Sh}(\mathscr{C}_{\mathscr{T}}, \mathbb{Z})$ to injective objects of $\mathrm{PSh}(\mathscr{C}, \mathbb{Z})$ by Lemma II.2.4.4 of the notes. So the fact that $\check{\mathrm{H}}^p(\mathscr{X}, \mathscr{F}) = 0$ for $p \geq 1$ follows from (e).

(f). Applying the functors $\check{\mathscr{C}}^p(\mathscr{X}, \cdot)$ to the complex $\mathscr{F} \to \mathscr{I}^\bullet$, we get a double complex in $\mathrm{PSh}(\mathscr{C}, \mathbb{Z})$, whose pth row is $\check{\mathscr{C}}^p(\mathscr{X}, \mathscr{F}) \to \check{\mathscr{C}}^p(\mathscr{X}, \mathscr{I}^\bullet)$, whose (-1)th column is the complex $\check{\mathscr{C}}^\bullet(\mathscr{X}, \mathscr{F})$ and whose nth column is the complex $\check{\mathscr{C}}^\bullet(\mathscr{X}, \mathscr{I}^n)$ for $n \geq 0$ (the other columns are 0).

We consider the double complex, where we write $\check{\mathscr{C}}^p(\cdot)$ and $\check{\mathrm{H}}^0(\cdot)$ for $\check{\mathscr{C}}^p(\mathscr{X}, \cdot)$ and

$\check{\mathrm{H}}^0(\mathscr{X},\cdot)$:

$$\begin{array}{ccccccccccc}
 & & 0 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & 0 & \longrightarrow & \check{\mathrm{H}}^0(\mathscr{I}^0) & \longrightarrow & \check{\mathrm{H}}^0(\mathscr{I}^1) & \longrightarrow & \check{\mathrm{H}}^0(\mathscr{I}^2) & \longrightarrow & \dots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \check{\mathscr{C}}^0(\mathscr{F}) & \longrightarrow & \check{\mathscr{C}}^0(\mathscr{I}^0) & \longrightarrow & \check{\mathscr{C}}^0(\mathscr{I}^1) & \longrightarrow & \check{\mathscr{C}}^0(\mathscr{I}^2) & \longrightarrow & \dots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \check{\mathscr{C}}^1(\mathscr{F}) & \longrightarrow & \check{\mathscr{C}}^1(\mathscr{I}^0) & \longrightarrow & \check{\mathscr{C}}^1(\mathscr{I}^1) & \longrightarrow & \check{\mathscr{C}}^1(\mathscr{I}^2) & \longrightarrow & \dots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \check{\mathscr{C}}^2(\mathscr{F}) & \longrightarrow & \check{\mathscr{C}}^2(\mathscr{I}^0) & \longrightarrow & \check{\mathscr{C}}^2(\mathscr{I}^1) & \longrightarrow & \check{\mathscr{C}}^2(\mathscr{I}^2) & \longrightarrow & \dots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & & \vdots & &
\end{array}$$

Every column of this double complex except for the first one is exact by (d). Also, every row except for the first one is exact, because the functor $\mathrm{PSh}(\mathscr{C},\mathbb{Z}) \to \mathbf{Ab}$, $\mathscr{G} \to \mathscr{G}(Y)$ is exact for every object Y of $\mathscr{C}$, and direct products of exact sequences in $\mathbf{Ab}$ are exact. So the $\infty \times \infty$ lemma (Corollary IV.2.2.4 of the notes) gives a canonical isomorphism between the cohomology of the first row and the cohomology of the first column, which is exactly what the question is asking for.

□

4 The fpqc topology is subcanonical

Let A be a commutative ring and B be a commutative A-algebra. For every $n \geq 1$, we write $B^{\otimes n}$ for the n-fold tensor product $B \otimes_A B \otimes_A \ldots \otimes_A B$. We consider the following sequence $\mathscr{A}_{B/A}$ of morphisms of A-modules:

$$0 \to A \overset{d^0}{\to} B \overset{d^1}{\to} B^{\otimes 2} \overset{d^2}{\to} B^{\otimes 3} \to \ldots$$

where the morphism $\mathscr{A}^0_{B/A} = A \to \mathscr{A}^1_{B/A} = B$ is the structural morphism and $d^n : \mathscr{A}^n_{B/A} = B^{\otimes n} \to \mathscr{A}^{n+1}_{B/A} = B^{\otimes(n+1)}$ is defined by

$$d^n(b_1 \otimes \ldots \otimes b_n) = \sum_{i=1}^{n+1} (-1)^{i+1} b_1 \otimes \ldots \otimes b_{i-1} \otimes 1 \otimes b_i \otimes \ldots \otimes b_n.$$

For example, $d^1(b) = 1 \otimes b - b \otimes 1$ and $d^2(b_1 \otimes b_2) = 1 \otimes b_1 \otimes b_2 - b_1 \otimes 1 \otimes b_2 + b_1 \otimes b_2 \otimes 1$.

(a). (1 point) Show that $\mathscr{A}_{B/A}$ is a complex. [4]

(b). (2 points) Suppose that the morphism of A-algebras $A \to B$ has a section, that is, that there exists a morphism of A-algebras $s : B \to A$ such that $s \circ d^0 = \mathrm{id}_A$. Show that $\mathscr{A}_{B/A}$ is homotopic to 0 as a complex of A-modules.

[4] It is called the *Amitsur complex*, hence the notation.

(c). (1 point) Under the hypothesis of (b), show that $\mathscr{A}_{B/A} \otimes_A M$ is acyclic for every A-module M.

(d). (1 point) We don't assume that $A \to B$ has a section anymore. Let M be a A-module. Show that we have a canonical isomorphism

$$B \otimes_A (\mathscr{A}_{B/A} \otimes_A M) \xrightarrow{\sim} \mathscr{A}_{B \otimes_A B/B} \otimes_B (M \otimes_A B),$$

where we see $B \otimes_A B$ as a B-algebra via the morphism $b \longmapsto b \otimes 1$.

(e). (2 points) Suppose that the morphism $A \to B$ is faithfully flat. Show that the complex $\mathscr{A}_{B/A} \otimes_A M$ is acyclic fo every A-module M.

Remark If $(f_i)_{i \in I}$ is a family of elements generating the unit ideal of A, then $B := \prod_{i \in I} A_{f_i}$ is a faithfully flat A-algebra, and, for any A-module M, the complex $\mathscr{A}_{B/A} \otimes_A M$ is the Čech complex of the quasi-coherent sheaf on $\operatorname{Spec} A$ corresponding to M for the open cover $(D_{f_i})_{i \in I}$. Applying the result of (e), we see that the Čech cohomology of any quasi-coherent sheaf on $\operatorname{Spec} A$ for the open cover $(D_{f_i})_{i \in I}$ is zero in degree ≥ 1.

Let $A - \mathbf{CAlg}$ be the the category of commutative A-algebras, and $\mathscr{C} = (A - \mathbf{CAlg})^{\mathrm{op}}$; to distinguish between objects of $A - \mathbf{CAlg}$ and $\mathscr{C}$, we write $\operatorname{Spec} B$ for the object of $\mathscr{C}$ corresponding to a commutative A-algebra B. We consider the fpqc topology on $\mathscr{C}$; this means that covering families in $\mathscr{C}$ are morphisms $\operatorname{Spec} C \to \operatorname{Spec} B$ such that $B \to C$ is a faithfully flat A-algebra morphism; also, if $B = 0$, then the empty family covers $\operatorname{Spec} B$.

(f). (1 point) Show that this is a Grothendieck pretopology on $\mathscr{C}$.

(g). (1 point) Let M be a A-module. We define a presheaf $\mathscr{F}_M$ on $\mathscr{C}$ by $\mathscr{F}_M(\operatorname{Spec} B) = B \otimes_A M$; if $\operatorname{Spec} C \to \operatorname{Spec} B$ is a morphism of $\mathscr{C}$, corresponding to a morphism of A-algebras $u : B \to C$, then $\mathscr{F}_M(\operatorname{Spec} B) = B \otimes_A M \to \mathscr{F}_M(\operatorname{Spec} C) = C \otimes_A M$ sends $b \otimes m$ to $u(b) \otimes m$. Show that $\mathscr{F}_M$ is a sheaf.

(h). (2 points) Show that every representable presheaf on $\mathscr{C}$ is a sheaf.

Solution.

(a). This is very similar to the beginning of 2(a). If $a \in A$, then $1 \otimes a = a \otimes 1$ in $B \otimes_A B$, so $d^1 \circ d^0(a) = 0$. Suppose that $n \geq 1$, and let $b_1, \ldots, b_n \in B$. Then

$$\begin{aligned}
d^{n+1} \circ d^n(b_1 \otimes b_n) &= d^{n+1}(\sum_{i=1}^{n+1} (-1)^{i+1} b_1 \otimes \ldots \otimes b_{i-1} \otimes 1 \otimes b_i \otimes \ldots b_n) \\
&= \sum_{i=1}^{n+1} \sum_{j=1}^{i} (-1)^{i+j} b_1 \otimes \ldots \otimes b_{j-1} \otimes 1 \otimes b_j \otimes \ldots b_{i-1} \otimes 1 \otimes b_i \otimes \ldots b_n \\
&\quad + \sum_{i=1}^{n+1} \sum_{j=i}^{n+1} (-1)^{i+j+1} b_1 \otimes \ldots \otimes b_{i-1} \otimes 1 \otimes b_i \otimes \ldots b_{j-1} \otimes 1 \otimes b_j \otimes \ldots b_n \\
&= \sum_{j=1}^{n+1} \sum_{i=j}^{n+1} (-1)^{i+j} b_1 \otimes \ldots \otimes b_{j-1} \otimes 1 \otimes b_j \otimes \ldots b_{i-1} \otimes 1 \otimes b_i \otimes \ldots b_n \\
&\quad + \sum_{i=1}^{n+1} \sum_{j=i}^{n+1} (-1)^{i+j+1} b_1 \otimes \ldots \otimes b_{i-1} \otimes 1 \otimes b_i \otimes \ldots b_{j-1} \otimes 1 \otimes b_j \otimes \ldots b_n \\
&= 0.
\end{aligned}$$

(b). We write $C^n = B^{\otimes n}$ for $n \geq 1$, $C^0 = A$, $C^n = 0$ for $n \leq -1$, and we denote $d^n : C^n \to C^{n+1}$ the morphism defined in the beginning. We define $s^n : C^n \to C^{n-1}$ in the following way:

- if $n \leq 0$, then $s^n = 0$;
- $s^1 = s : B \to A$;
- if $n \geq 2$, then $s^n : B^{\otimes n} \to B^{\otimes(n-1)}$ sends $b_1 \otimes \ldots \otimes b_n$ to $(-1)^{n-1}s(b_n)(b_1 \otimes \ldots \otimes s_{n-1})$ (this is A-linear in each b_i, hence does define a morphism on the tensor product).

We claim that $(s^n)_{n\in\mathbb{Z}}$ is a homotopy between $\mathrm{id}_{C^\bullet}$ and 0. To prove this claim, we have to calculate the morphism $g^n := d^{n-1} \circ s^n + s^{n+1} \circ d^n$ for every $n \in \mathbb{Z}$. If $n \leq -1$, then $g^n = 0 = \mathrm{id}_{C^n}$. If $n = 0$, then $g^n = s \circ d^0 = \mathrm{id}_A$. Suppose that $n \geq 1$. Then, for all $b_1, \ldots, b_n \in B$, we have that $g^n(b_1 \otimes \ldots \otimes b_n)$ is equal to

$$(-1)^{n-1}s(b_n)d^{n-1}(b_1 \otimes \ldots \otimes b_{n-1}) + s^{n+1}(\sum_{i=1}^{n+1}(-1)^{i+1}b_1 \otimes \ldots \otimes b_{i-1} \otimes 1 \otimes b_i \otimes \ldots \otimes b_n)$$

$$= s(b_n)\sum_{i=1}^{n}(-1)^{i+n}b_1 \otimes \ldots \otimes b_{i-1} \otimes 1 \otimes b_i \otimes \ldots \otimes b_{n-1}$$

$$+ \sum_{i=1}^{n}(-1)^{n+i+1}s(b_n)(b_1 \otimes \ldots \otimes b_{i-1} \otimes 1 \otimes b_i \otimes \ldots \otimes b_{n-1}) + s(1)(b_1 \otimes \ldots \otimes b_n)$$

$$= b_1 \otimes \ldots \otimes b_n.$$

So $g^n = \mathrm{id}_{C^n}$.

Note that the homotopy that we just constructed is A-linear, so $\mathscr{A}_{B/A}$ is homotopic to 0 as a complex of A-modules.

(c). As the functor $(\cdot) \otimes_A M : {}_A\mathbf{Mod} \to {}_A\mathbf{Mod}$ is additive and the complex of A-modules $\mathscr{A}_{B/A}$ is homotopic to 0, the complex $\mathscr{A}_{B/A} \otimes_A M$ is also homotopic to 0, and in particular acyclic.

(d). In degree 0, this isomorphism is the isomorphism $B \otimes_A (A \otimes_A M) \simeq B \otimes_B (M \otimes_A B)$ sending $B \otimes (1 \otimes m)$ to $b \otimes (m \otimes 1) = 1 \otimes (m \otimes b)$. If $n \geq 1$, we have morphism $u : B \otimes_A (\mathscr{A}^n_{B/A} \otimes_A M) \to \mathscr{A}^n_{B\otimes_A B/B} \otimes_B M$ and $v : \mathscr{A}^n_{B\otimes_A B/B} \otimes_B M \to B \otimes_A (\mathscr{A}^n_{B/A} \otimes_A M)$ defined by

$$u(b_0 \otimes (b_1 \otimes \ldots \otimes b_n \otimes m)) = ((1 \otimes b_1) \otimes \ldots \otimes (1 \otimes b_n)) \otimes (m \otimes b_0)$$

and

$$v((b'_1 \otimes b_1) \otimes \ldots \otimes (b'_n \otimes b_n) \otimes (m \otimes b_0)) = (b_0 b'_1 \ldots b'_n) \otimes (b_1 \otimes \ldots \otimes b_n \otimes m)$$

if $b_0, b_1, b'_1, \ldots, b_n, b'_n \in B$ and M. It is easy to check that these morphisms are well-defined and inverses of each other.

(e). Note that the structural morphism $B \to B \otimes_A B$, $b \longmapsto b \otimes 1$ has a section $B \otimes_A B \to B$, $b_1 \otimes b_2 \longmapsto b_1 b_2$ which is a morphism of B-algebras. So, by (c) and (d), the complex of B-modules $B \otimes_A (\mathscr{A}_{B/A} \otimes_A M)$ is acyclic. As B is a faithfully flat A-algebra, this implies that the complex of A-modules $\mathscr{A}_{B/A} \otimes_A M$ is acyclic.

(f). We check the axioms of Definition III.2.1.1 of the notes. Axiom (CF3) is clear, because an isomorphism of rings is faithfully flat. Axiom (CF2) says that the composition os two faithfully flat morphisms of A-algebras is also faithfully flat, which is also true. Axiom (CF1) says that, if $B \to C$ and $B \to D$ are faithfully flat morphisms of A-algebras, then $B \to C \otimes_B D$ is also faithfully flat, which is also true.

(g). The sheaf condition says that:

(1) If $B = 0$, then the sequence $0 \to B \otimes_A M \to 0$ is exact, which is certainly true.

(2) For every faithfully flat A-algebra morphism $B \to C$, the sequence

$$0 \to M' \xrightarrow{f} C \otimes_B M' \xrightarrow{g} (C \otimes_B C) \otimes_B M'$$

is exact, where $M' = B \otimes_A M$, f sends $m \in M'$ to $1 \otimes m \in C \otimes_B M$, and g sends $c \otimes m \in C \otimes_B M$ to $(1 \otimes c) \otimes m - (c \otimes 1) \otimes m$. This exactness follows from question (e).

(h). Let D be a commutative A-algebra. We want to show that the presheaf $\mathrm{Hom}_{\mathscr{C}}(\cdot, \operatorname{Spec} D)$ is a sheaf. If we consider the empty cover of $\operatorname{Spec}(0)$, the sheaf condition says that $\mathrm{Hom}_{\mathscr{C}}(\operatorname{Spec}(0), \operatorname{Spec}(D)) = \mathrm{Hom}_{A-\mathbf{CAlg}}(D, 0)$ should be a singleton, which is true. Let $u : B \to C$ be a faithfully flat morphism of commutative A-algebras. The sheaf condition for the covering family $\operatorname{Spec} C \to \operatorname{Spec} B$ says that:

(1) The map $\mathrm{Hom}_{A-\mathbf{CAlg}}(D, B) \to \mathrm{Hom}_{A-\mathbf{CAlg}}(D, C)$, $v \longmapsto u \circ v$ is injective; this is true because u, being faithfully flat, is injective.

(2) If $f : D \to C$ is a morphism of A-algebras such that $f(c) \otimes 1 = 1 \otimes f(c)$ in $C \otimes_B C$ for every $c \in C$, then there exists a morphism of A-algebras $v : D \to B$ such that $f = u \circ v$.

We prove (2). By (e), the kernel of the morphism $g : C \to C \otimes_B C$, $c \longmapsto 1 \otimes c - c \otimes 1$ is $u(B)$. The condition on f says that $g \circ f = 0$; as u is injective, it implies that we can write $f = u \circ v$, for a uniquely determined A-linear morphism $v : D \to B$. As u is an injective morphism of A-algebras and f is a morphism of A-algebras, the map v is also a morphism of A-algebras.

□

5 Čech cohomology, part 2

Let X be a topological space.

(a). (2 points) Let $0 \to \mathscr{F} \to \mathscr{G} \to \mathscr{H} \to 0$ be a short exact sequence of abelian sheaves on X, and let U be an open subset of X. Suppose that every open cover of U has a refinement $\mathscr{U}$ such that $\check{\mathrm{H}}^1(\mathscr{U}, \mathscr{F}) = 0$. Show that the sequence

$$0 \to \mathscr{F}(U) \to \mathscr{G}(U) \to \mathscr{H}(U) \to 0$$

is exact.

(b). Let $\mathscr{B}$ be a basis of the topology of X, and **Cov** be a set of open covers of open subsets of X, such that:

(1) If $(U_i)_{i \in I}$ is in **Cov**, then $\bigcup_{i \in I} U_i$ and all the $U_{i_0} \cap \ldots \cap U_{i_p}$ are in $\mathscr{B}$, for $p \in \mathbb{N}$ and $i_0, \ldots, i_p \in I$.

(2) If $U \in \mathscr{B}$, then any open cover of U has a refinement in **Cov**.

Let $\mathscr{I}$ be the full category of injective objects in $\mathrm{Sh}(X, \mathbb{Z})$, and $\mathscr{C}$ be the full category whose objects are abelian sheaves $\mathscr{F}$ such that $\check{\mathrm{H}}^n(\mathscr{U}, \mathscr{F}) = 0$ for every $\mathscr{U} \in$ **Cov** and every $n \geq 1$. [5]

[5]For example, if X is a scheme, we could take $\mathscr{B}$ to be the set of open affine subschemes of X and $\mathscr{U}$ to be the

(i) (2 points) Show that $\mathscr{C}$ contains $\mathscr{I}$ and is stable by taking cokernels of injective morphisms.

(ii) (2 points) If $\mathscr{F}$ is an object of $\mathscr{C}$, show that, for every $U \in \mathscr{B}$, we have $\mathrm{H}^1(U, \mathscr{F}) = 0$.

(iii) (2 points) Show by induction on n that, for every $n \geq 1$, every $U \in \mathscr{B}$ and every object $\mathscr{F}$ of $\mathscr{C}$, we have $\mathrm{H}^n(U, \mathscr{F}) = 0$.

(iv) (2 points) Let $\mathscr{F}$ be an object of $\mathscr{C}$ and $\mathscr{X} = (U_i)_{i\in I}$ be an open cover of X such that, for every $p \in \mathbb{N}$ and all $i_0, \dots, i_p$, we have $U_{i_0} \cap \dots \cap U_{i_p} \in \mathscr{B}$. Show that the canonical morphism $\check{\mathrm{H}}^n(\mathscr{X}, \mathscr{F}) \to \mathrm{H}^n(X, \mathscr{F})$ of Example IV.4.1.12(2) of the notes is an isomorphism for every $n \geq 0$. [6]

Solution.

(a). We give names to the morphisms of the exact sequence: $0 \to \mathscr{F} \xrightarrow{f} \mathscr{G} \xrightarrow{g} \mathscr{H} \to 0$. Let U be an open subset of X. We know that the sequence $0 \to F(U) \to \mathscr{G}(U) \to \mathrm{H}(U)$ is exact, so it suffices to show that $\mathscr{G}(U) \to \mathrm{H}(U)$ is surjective.

Let $s \in \mathscr{F}(U)$. Choose an open cover $\mathscr{U} = (U_i)_{i\in I}$ such that, for every $i \in I$, there exists $s_i \in \mathscr{G}(U_i)$ such that $g(s_i) = s_{|U_i}$. By the hypothesis, after replacing $\mathscr{U}$ by a refinement, we may assume that $\check{\mathrm{H}}^1(\mathscr{U}, \mathscr{F}) = 0$. For $i, j \in I$, let $s_{ij} = s_{i|U_i\cap U_j} - s_{j|U_i\cap U_j}$. As $g(s_i)$ and $g(s_j)$ are equal on $U_i \cap U_i$, there exists $t_{ij} \in \mathscr{F}(U_i \cap U_j)$ such that $f(t_{ij}) = s_{ij}$. If $i, j, k \in I$, then we have

$$\begin{aligned} s_{ij|U_i\cap U_j\cap U_k} - s_{ik|U_i\cap U_j\cap U_k} + s_{jk|U_i\cap U_j\cap U_k} &= s_{i|U_i\cap U_j\cap U_k} - s_{j|U_i\cap U_j\cap U_k} \\ &\quad - (s_{i|U_i\cap U_j\cap U_k} - s_{k|U_i\cap U_j\cap U_k}) + s_{j|U_i\cap U_j\cap U_k} - s_{k|U_i\cap U_j\cap U_k} \\ &= 0, \end{aligned}$$

so the family $(t_{ij})_{(i,j)\in I^2} \in \check{\mathscr{C}}^1(\mathscr{U}, \mathscr{F})$ is in the kernel of d^1. As $\check{\mathrm{H}}^1(\mathscr{U}, \mathscr{F}) = 0$, there exists $(t_i)_{i\in I} \in \check{\mathscr{C}}^0(\mathscr{U}, \mathscr{F}) = \prod_{i\in I} \mathscr{F}(U_i)$ such that $d^0((t_i)_{I\in I}) = (t_{ij}))$, that is, $t_{ij} = t_{i|U_i\cap U_j} - t_{j|U_i\cap U_j}$. For every $i \in I$, let $s'_i = s_i - f(t_i)$. Then, for $i, j \in I$, we have

$$s'_{i|U_i\cap U_j} - s'_{j|U_i\cap U_j} = s_{ij} - f(t_{ij}) = 0.$$

So there exists $s' \in \mathscr{G}(U)$ such that $s'_{|U_i} = s'_i$ for every $i \in I$. Moreover, we have $g(s')_{|U_i} = g(s'_i) = g(s_i) = s_{|U_i}$ for every $i \in I$, so $g(s') = s$.

(b). (i) Let $\mathscr{F}$ be an object of $\mathscr{I}$. We know that $\check{\mathrm{H}}^p(\mathscr{U}, \mathscr{F}) = 0$ for every covering family family $\mathscr{U}$ of an open subset of X and for every $p \geq 1$ by question (e) of problem 3, so $\mathscr{F}$ is in $\mathscr{C}$.

Now let $0 \to \mathscr{F} \to \mathscr{G} \to \mathscr{H} \to 0$ be an exact sequence of abelian sheaves on X, and suppose that $\mathscr{F}$ and $\mathscr{G}$ are in $\mathscr{C}$. By question (a), the sequence $0 \to \mathscr{F} \to \mathscr{G} \to \mathscr{H} \to 0$ is also exact as a sequence of abelian presheaves. Let $\mathscr{U} \in$ **Cov** By problem 2, the functors $\check{\mathrm{H}}^n(\mathscr{U}, \cdot)$ are the right derived functors of $\check{\mathrm{H}}^0(\mathscr{U}, \cdot)$ on the category $\mathrm{PSh}(X, \mathbb{Z})$, so we have a long exact sequence

$$\dots \to \check{\mathrm{H}}^n(\mathscr{U}, \mathscr{G}) \to \check{\mathrm{H}}^n(\mathscr{U}, \mathscr{H}) \to \check{\mathrm{H}}^{n+1}(\mathscr{U}, \mathscr{F}) \to \check{\mathrm{H}}^{n+1}(\mathscr{U}, \mathscr{G}) \to \dots$$

If $n \geq 1$, then $\check{\mathrm{H}}^n(\mathscr{U}, \mathscr{G}) = 0$ and $\check{\mathrm{H}}^{n+1}(\mathscr{U}, \mathscr{F}) = 0$ by the hypothesis on $\mathscr{F}$ and $\mathscr{G}$, so $\check{\mathrm{H}}^n(\mathscr{U}, \mathscr{H}) = 0$. This shows that $\mathscr{H}$ is an object of $\mathscr{C}$.

set of open covers of open affine subschemes of X by principal open affines, and then $\mathscr{C}$ would contain all the quasi-coherent sheaves on X.

[6] If X is a scheme, this shows that, for every quasi-coherent sheaf $\mathscr{F}$ on X, the cohomology of $\mathscr{F}$ is isomorphism to its Čech cohmology relative to any open affine cover of X.

(ii) Let $\mathscr{F}$ be an object of $\mathscr{C}$, let $f : \mathscr{F} \to \mathscr{G}$ be an injective morphism of abelian sheaves with $\mathscr{G}$ an injective object of $\mathrm{Sh}(X, \mathbb{Z})$, and $\mathscr{H} = \mathrm{Coker}(f)$. Let $U \in \mathscr{B}$. Then we have an exact sequence

$$0 \to \mathscr{F}(U) \to \mathscr{G}(U) \to \mathscr{H}(U) \to \mathrm{H}^1(U, \mathscr{F}) \to \mathrm{H}^1(U, \mathscr{G}) \to \dots$$

But $\mathrm{H}^1(U, \mathscr{G}) = 0$ because $\mathscr{G}$ is injective, and the morphism $\mathscr{G}(U) \to \mathscr{H}(U)$ is surjective by (i), so $\mathrm{H}^1(U, \mathscr{F}) = 0$.

(iii) We already know that the result holds for $n = 1$ by question (iii). Suppose that it holds for some $n \geq 1$. Let let $\mathscr{F}$ be an object of $\mathscr{C}$. Choose an injective morphism $f : \mathscr{F} \to \mathscr{G}$ with $\mathscr{G}$ an object of $\mathscr{I}$, and let $\mathscr{H} = \mathrm{Coker}\, f$. Let $U \in \mathscr{B}$. We have a long exact sequence of cohomology

$$\dots \mathrm{H}^n(U, \mathscr{H}) \to \mathrm{H}^{n+1}(U, \mathscr{F}) \to \mathrm{H}^{n+1}(U, \mathscr{G}) \to \dots$$

By question (ii), the sheaf $\mathscr{H}$ is an object of $\mathscr{C}$, so $\mathrm{H}^n(U, \mathscr{H}) = 0$ by the induction hypothesis. Moreover, as $\mathscr{G}$ is an injective object of $\mathrm{Sh}(X, \mathbb{Z})$, we have $\mathrm{H}^{n+1}(U, \mathscr{G}) = 0$. So $\mathrm{H}^{n+1}(U, \mathscr{F}) = 0$.

(iv) We use the notation of Example IV.4.1.12(2) of the notes. By question (iii), for every $p \in \mathbb{N}$, all $i_0, \dots, i_p \in I$, and every $q \geq 1$, we have

$$R^q\Phi(\mathscr{F})(U_{i_0} \cap \dots \cap U_{i_p}) = \mathrm{H}^q(U_{i_0} \cap \dots \cap U_{i_p}, \mathscr{F}) = 0.$$

By definition of Čech cohomology, this implies that, for every $p \in \mathbb{N}$ and every $q \geq 1$, we have $\check{\mathrm{H}}^p(\mathscr{X}, R^q\Phi(\mathscr{F})) = 0$. Let

$$E_2^{pq} = \check{\mathrm{H}}^p(\mathscr{X}, R^q\Phi(\mathscr{F})) \Rightarrow \mathrm{H}^{p+q}(X, \mathscr{F})$$

be the Čech cohomology to cohomology spectral sequence for the open cover $\mathscr{X}$. By the calculation we just did, we have $E_2^{pq} = 0$ if $q \geq 1$, so the spectral sequence degenerates at E_2 and $E_\infty^{pq} = E_2^{pq}$ is zero unless $q = 0$. So for every $p \in \mathbb{N}$, the subobject $E_\infty^{p,0} = E_2^{p,0} = \check{\mathrm{H}}^p(\mathscr{X}, \mathscr{F})$ of $\mathrm{H}^p(X, \mathscr{F})$ is actually equal to $\mathrm{H}^p(X, \mathscr{F})$, which shows that the morphism $\check{\mathrm{H}}^p(\mathscr{X}, \mathscr{F}) \to \mathrm{H}^p(\mathscr{X}, \mathscr{F})$ is an isomorphism.

$\square$