MAT 540 : Problem Set 6

Due Sunday, November 3

We make the following useful convention: if $(x_0, \ldots, x_n)$ is some list and if $i \in \{0, \ldots, n\}$, then $(x_0, \ldots, \hat{x}_i, \ldots, x_n)$ means $(x_0, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$.

Also, if S is a set and $\mathbb{Z}^{(S)}$ is the free $\mathbb{Z}$-module on S, we denote the canonical basis of this free module by $(e_s)_{s \in S}$.

1 Salamander lemma (5 points)

Prove the salamander lemma (Theorem IV.2.1.3 of the notes).

2 Some bar resolutions

(a). (3 points) Let S be a nonempty set. We define a complex of $\mathbb{Z}$-modules $X^\bullet$ by:

- $X^n = 0$ and $d_X^n = 0$ if $n \geq 2$;
- $X^1 = \mathbb{Z}$ and $d_X^1 = 0$;
- $X^0 = \mathbb{Z}^{(S)}$ and $d_X^0 : X^0 \to X^1 = \mathbb{Z}$ sends every e_s to 1;
- if $n \geq 1$, then $X^{-n} = \mathbb{Z}^{(S^{n+1})}$ and $d^{-n} : \mathbb{Z}^{(S^{n+1})} \to \mathbb{Z}^{(S^n)}$ sends $e_{(s_0,\ldots,s_n)}$ to $\sum_{i=0}^n (-1)^i e_{(s_0,\ldots,\hat{s}_i,\ldots,s_n)}$, for all $s_0, \ldots, s_n \in S$.

Show that $X^\bullet$ is indeed a complex (i.e. that $d_X^{n+1} \circ d_X^n = 0$ for every $n \in \mathbb{Z}$), and that it is acyclic. (Hint: Fix $s \in S$. If $n \geq -1$, consider the morphism $t^{-n} : X^{-n} \to X^{-n-1}$ sending $e_{(s_0,\ldots,s_n)}$ to $e_{(s,s_0,\ldots,s_n)}$.)

(b). Let G be a group. For every $n \geq 0$, let $X_n(G) = \mathbb{Z}^{(G^{n+1})}$. By (a), we have an acyclic complex of $\mathbb{Z}$-modules $X^\bullet$, where $X^1 = \mathbb{Z}$, $X^{-n} = X_n(G)$ if $n \geq 0$, $X^n = 0$ if $n \geq 2$, and the differentials are as in (a).

(i) (2 points) We make G act as $X_n(G)$ by $g \cdot e_{(g_0,\ldots,g_n)} = e_{(gg_0,\ldots,gg_n)}$, and we make G act trivially on $\mathbb{Z}$. Show that $X^\bullet$ is an acyclic complex of $\mathbb{Z}[G]$-modules.

(ii) (2 points) Show that $X_n(G)$ is a free $\mathbb{Z}[G]$-module for every $n \geq 0$.

Let I_n be the $\mathbb{Z}$-submodule of $X_n(G)$ generated by the $e_{(g_0,\ldots,g_n)}$ such that $g_i = g_{i+1}$ for some $i \in \{0, \ldots, n-1\}$.

(iii) (2 points) Show that I_n is a free $\mathbb{Z}[G]$-submodule of $X_n(G)$ and that $d^{-n}(I_n) \subset I_{n-1}$ for $n \geq 0$, with $I_{-1} = \{0\}$.

(iv) (2 points) By the previous question, we get a complex of $\mathbb{Z}[G]$-modules $Y^\bullet$ such that $Y^n = 0$ for ≥ 2, $Y^1 = \mathbb{Z}$, $Y^{-n} = X_n(G)/I_n$ if $n \geq 0$ and d_Y^n is the morphism induced by d_X^n for every $n \in \mathbb{Z}$. Show that $Y^\bullet$ is acyclic. (Hint: Try to imitate the method of (a).)

3 Čech cohomology, part 1

This problem uses problem 1 of problem set 5.

Let $\mathscr{C}$ be a category that admits fiber products, and let $\mathscr{X} = (f : X_i \to X)_{i \in I}$ be a family of morphisms of $\mathscr{C}$. If $i_0, \ldots, i_p \in I$, we write $X_{i_0,\ldots,i_p} = X_{i_0} \times_X X_{i_1} \times_X \ldots \times_X X_{i_p}$. For every $p \in \mathbb{Z}$, we define an abelian presheaf $\mathscr{C}_p(\mathscr{X}) \in \mathrm{Ob}(\mathrm{PSh}(\mathscr{C}, \mathbb{Z}))$ in the following way:

- if $p < 0$, then $\mathscr{C}_p = 0$;
- if $p \geq 0$, then
$$\mathscr{C}_p(\mathscr{X}) = \bigoplus_{i_0,\ldots,i_p \in I} \mathbb{Z}^{(X_{i_0,\ldots,i_p})}.$$

We also define a morphism of presheaves $d_p : \mathscr{C}_p(\mathscr{X}) \to \mathscr{C}_{p-1}(\mathscr{X})$ in the following way:

- if $p \leq 0$, then $d_p = 0$;
- if $p \geq 1$, then d_p is given on the component $\mathbb{Z}^{(X_{i_0,\ldots,i_p})}$ by the morphism $\mathbb{Z}^{(X_{i_0,\ldots,i_p})} \to \bigoplus_{q=0}^p \mathbb{Z}^{(X_{i_0,\ldots,i_{q-1},i_{q+1},\ldots,i_p})} \subset \mathscr{C}_{p-1}(\mathscr{X})$ equal to $\sum_{q=0}^p (-1)^q \delta^q_{i_0,\ldots,i_p}$, where $\delta^q_{i_0,\ldots,i_p} : \mathbb{Z}^{(X_{i_0,\ldots,i_p})} \to \mathbb{Z}^{(X_{i_0,\ldots,i_{q-1},i_{q+1},\ldots,i_p})}$ is the image of the canonical projection $X_{i_0,\ldots,i_p} \to X_{i_0,\ldots,i_{q-1},i_{q+1},\ldots,i_p}$ by the functor $\mathscr{C} \xrightarrow{h_{\mathscr{C}}} \mathrm{PSh}(\mathscr{C}) \xrightarrow{\mathbb{Z}^{(\cdot)}} \mathrm{PSh}(\mathscr{C}, \mathbb{Z})$.

(a). (4 points) Show that $\mathrm{Ker}(d_p) \supset \mathrm{Im}(d_{p+1})$ for every $p \in \mathbb{Z}$ and that this is an equality for $p \neq 0$.

<u>Hint</u>: For every object Y of $\mathscr{C}$, we have
$$\mathrm{Hom}_{\mathscr{C}}(Y, X_{i_0,\ldots,i_p}) = \coprod_{h \in \mathrm{Hom}_{\mathscr{C}}(Y,X)} \mathrm{Hom}_{\mathscr{C}}(Y, X_{i_0})_h \times \ldots \times \mathrm{Hom}_{\mathscr{C}}(Y, X_{i_p})_h,$$
where, for every $i \in I$, $\mathrm{Hom}_{\mathscr{C}}(Y, X_i)_h = \{g \in \mathrm{Hom}_{\mathscr{C}}(Y, X_i) \mid f_i \circ g = h\}$. Set $S_h = \coprod_{i \in I} \mathrm{Hom}_{\mathscr{C}}(Y, X_i)_h$ and think of question 2(a).

(b). (1 point) Let $\varepsilon : \mathscr{C}_0(\mathscr{X}) \to \mathbb{Z}^{(X)}$ be the morphism that is equal on the component $\mathbb{Z}^{(X_i)}$ to the image of $f_i : X_i \to X$ by the functor $\mathscr{C} \xrightarrow{h_{\mathscr{C}}} \mathrm{PSh}(\mathscr{C}) \xrightarrow{\mathbb{Z}^{(\cdot)}} \mathrm{PSh}(\mathscr{C}, \mathbb{Z})$. Show that $\mathrm{Ker}(\varepsilon) = \mathrm{Im}(d_1)$.

For every $p \in \mathbb{Z}$, we define a functor $\check{\mathscr{C}}^p(\mathscr{X}, \cdot) : \mathrm{PSh}(\mathscr{C}, \mathbb{Z}) \to \mathbf{Ab}$ by $\check{\mathscr{C}}^p(\mathscr{X}, \mathscr{F}) = \mathrm{Hom}_{\mathrm{PSh}(\mathscr{C},\mathbb{Z})}(\mathscr{C}^p(\mathscr{X}), \mathscr{F})$, and a morphism of functors $d^p : \check{\mathscr{C}}^p(\mathscr{X}, \cdot) \to \check{\mathscr{C}}^{p+1}(\mathscr{X}, \cdot)$ by $d^p = \mathrm{Hom}_{\mathrm{PSh}(\mathscr{C},\mathbb{Z})}(d_{p+1}, \cdot)$. The family $(\check{\mathscr{C}}^p(\mathscr{X}, \mathscr{F}), d^p)_{p \in \mathbb{Z}}$ is called the *Čech complex of $\mathscr{F}$ (relative to the family $\mathscr{X}$)*. For every $p \geq 0$, we set $\check{\mathrm{H}}^p(\mathscr{X}, \mathscr{F}) = \mathrm{Ker}(d^p(\mathscr{F}))/\mathrm{Im}(d^{p-1}(\mathscr{F}))$. This is called the *pth Čech cohomology group of $\mathscr{F}$ (relative to the family $\mathscr{X}$)*. Note that the definition of $\check{\mathrm{H}}^p(\mathscr{X}, \mathscr{F})$ is functorial in $\mathscr{F}$, so $\check{\mathrm{H}}^p(\mathscr{X}, \cdot)$ is a functor from $\mathrm{PSh}(\mathscr{C}, \mathbb{Z})$ to $\mathbf{Ab}$.

(c). (2 points) Show that, for every abelian presheaf $\mathscr{F}$ and every $p \geq 0$, we have
$$\check{\mathscr{C}}^p(\mathscr{X}, \mathscr{F}) = \prod_{i_0,\ldots,i_p} \mathscr{F}(X_{i_0,\ldots,i_p}),$$
and that the definition of $\check{\mathrm{H}}^0(\mathscr{X}, \mathscr{F})$ given here generalizes that of Definition III.2.2.4 of the notes.

(d). (1 point) If $\mathscr{F}$ is an injective object of $\mathrm{PSh}(\mathscr{C},\mathbb{Z})$, show that $\check{\mathrm{H}}^p(\mathscr{X},\mathscr{F}) = 0$ for every $p \geq 1$.

(e). (2 points) Suppose that we have a Grothendieck topology $\mathscr{T}$ on $\mathscr{C}$, that $\mathscr{X}$ is a covering family, and that $\mathscr{F}$ is an injective object of $\mathrm{Sh}(\mathscr{C}_{\mathscr{T}},\mathbb{Z})$. Show that $\check{\mathrm{H}}^p(\mathscr{X},\mathscr{F}) = 0$ for $p \geq 1$ and that $\check{\mathrm{H}}^0(\mathscr{X},\mathscr{F}) = \mathscr{F}(X)$.

(f). (2 points) Let $\mathscr{F} \in \mathrm{Ob}(\mathrm{PSh}(\mathscr{C},\mathbb{Z}))$, let $\mathscr{F} \to \mathscr{I}^\bullet$ be an injective resolution of $\mathscr{F}$ in $\mathrm{PSh}(\mathscr{C},\mathbb{Z})$. Show that we have canonical isomorphisms

$$\mathrm{H}^n(\check{\mathrm{H}}^0(\mathscr{X},\mathscr{I}^\bullet)) \simeq \check{\mathrm{H}}^n(\mathscr{X},\mathscr{F}).$$

4 The fpqc topology is subcanonical

Let A be a commutative ring and B be a commutative A-algebra. For every $n \geq 1$, we write $B^{\otimes n}$ for the n-fold tensor product $B \otimes_A B \otimes_A \ldots \otimes_A B$. We consider the following sequence $\mathscr{A}_{B/A}$ of morphisms of A-modules:

$$0 \to A \overset{d^0}{\to} B \overset{d^1}{\to} B^{\otimes 2} \overset{d^2}{\to} B^{\otimes 3} \to \ldots$$

where the morphism $\mathscr{A}^0_{B/A} = A \to \mathscr{A}^1_{B/A} = B$ is the structural morphism and $d^n : \mathscr{A}^n_{B/A} = B^{\otimes n} \to \mathscr{A}^{n+1}_{B/A} = B^{\otimes(n+1)}$ is defined by

$$d^n(b_1 \otimes \ldots \otimes b_n) = \sum_{i=1}^{n+1} (-1)^{i+1} b_1 \otimes \ldots \otimes b_{i-1} \otimes 1 \otimes b_i \otimes \ldots \otimes b_n.$$

For example, $d^1(b) = 1 \otimes b - b \otimes 1$ and $d^2(b_1 \otimes b_2) = 1 \otimes b_1 \otimes b_2 - b_1 \otimes 1 \otimes b_2 + b_1 \otimes b_2 \otimes 1$.

(a). (1 point) Show that $\mathscr{A}_{B/A}$ is a complex.

(b). (2 points) Suppose that the morphism of A-algebras $A \to B$ has a section, that is, that there exists a morphism of A-algebras $s : B \to A$ such that $s \circ d^0 = \mathrm{id}_A$. Show that $\mathscr{A}_{B/A}$ is homotopic to 0 as a complex of A-modules.

(c). (1 point) Under the hypothesis of (b), show that $\mathscr{A}_{B/A} \otimes_A M$ is acyclic for every A-module M.

(d). (1 point) We don't assume that $A \to B$ has a section anymore. Let M be a A-module. Show that we have a canonical isomorphism

$$B \otimes_A (\mathscr{A}_{B/A} \otimes_A M) \overset{\sim}{\to} \mathscr{A}_{B \otimes_A B/B} \otimes_B (M \otimes_A B),$$

where we see $B \otimes_A B$ as a B-algebra via the morphism $b \longmapsto b \otimes 1$.

(e). (2 points) Suppose that the morphism $A \to B$ is faithfully flat. Show that the complex $\mathscr{A}_{B/A} \otimes_A M$ is acyclic fo every A-module M.

Let $A - \mathbf{CAlg}$ be the the category of commutative A-algebras, and $\mathscr{C} = (A - \mathbf{CAlg})^{\mathrm{op}}$; to distinguish between objects of $A - \mathbf{CAlg}$ and $\mathscr{C}$, we write $\operatorname{Spec} B$ for the object of $\mathscr{C}$ corresponding to a commutative A-algebra B. We consider the fpqc topology on $\mathscr{C}$; this means that covering families in $\mathscr{C}$ are morphisms $\operatorname{Spec} C \to \operatorname{Spec} B$ such that $B \to C$ is a faithfully flat A-algebra morphism; also, if $B = 0$, then the empty family covers $\operatorname{Spec} B$.

(f). (1 point) Show that this is a Grothendieck pretopology on $\mathscr{C}$.

(g). (1 point) Let M be a A-module. We define a presheaf $\mathscr{F}_M$ on $\mathscr{C}$ by $\mathscr{F}_M(\operatorname{Spec} B) = B \otimes_A M$; if $\operatorname{Spec} C \to \operatorname{Spec} B$ is a morphism of $\mathscr{C}$, corresponding to a morphism of A-algebras $u : B \to C$, then $\mathscr{F}_M(\operatorname{Spec} B) = B \otimes_A M \to \mathscr{F}_M(\operatorname{Spec} C) = C \otimes_A M$ sends $b \otimes m$ to $u(b) \otimes m$. Show that $\mathscr{F}_M$ is a sheaf.

(h). (2 points) Show that every representable presheaf on $\mathscr{C}$ is a sheaf.

5 Čech cohomology, part 2

Let X be a topological space.

(a). (2 points) Let $0 \to \mathscr{F} \to \mathscr{G} \to \mathscr{H} \to 0$ be a short exact sequence of abelian sheaves on X, and let U be an open subset of X. Suppose that every open cover of U has a refinement $\mathscr{U}$ such that $\check{\mathrm{H}}^1(\mathscr{U}, \mathscr{F}) = 0$. Show that the sequence

$$0 \to \mathscr{F}(U) \to \mathscr{G}(U) \to \mathscr{H}(U) \to 0$$

is exact.

(b). Let $\mathscr{B}$ be a basis of the topology of X, and **Cov** be a set of open covers of open subsets of X, such that:

(1) If $(U_i)_{i \in I}$ is in **Cov**, then $\bigcup_{i \in I} U_i$ and all the $U_{i_0} \cap \ldots \cap U_{i_p}$ are in $\mathscr{B}$, for $p \in \mathbb{N}$ and $i_0, \ldots, i_p \in I$.

(2) If $U \in \mathscr{B}$, then any open cover of U has a refinement in **Cov**.

Let $\mathscr{I}$ be the full category of injective objects in $\mathrm{Sh}(X, \mathbb{Z})$, and $\mathscr{C}$ be the full category whose objects are abelian sheaves $\mathscr{F}$ such that $\check{\mathrm{H}}^n(\mathscr{U}, \mathscr{F}) = 0$ for every $\mathscr{U} \in$ **Cov** and every $n \geq 1$.

(i) (2 points) Show that $\mathscr{C}$ contains $\mathscr{I}$ and is stable by taking cokernels.

(ii) (2 points) If $\mathscr{F}$ is an object of $\mathscr{C}$, show that, for every $U \in \mathscr{B}$, we have $\mathrm{H}^1(U, \mathscr{F}) = 0$.

(iii) (2 points) Show by induction on n that, for every $n \geq 1$, every $U \in \mathscr{B}$ and every object $\mathscr{F}$ of $\mathscr{C}$, we have $\mathrm{H}^n(U, \mathscr{F}) = 0$.

(iv) (2 points) Let $\mathscr{F}$ be an object of $\mathscr{C}$ and $\mathscr{X} = (U_i)_{i \in I}$ be an open cover of X such that, for every $p \in \mathbb{N}$ and all $i_0, \ldots, i_p$, we have $U_{i_0} \cap \ldots \cap U_{i_p} \in \mathscr{B}$. Show that the canonical morphism $\check{\mathrm{H}}^n(\mathscr{X}, \mathscr{F}) \to \mathrm{H}^n(X, \mathscr{F})$ of Example IV.4.1.12(2) of the notes is an isomorphism for every $n \geq 0$.