MAT 540 : Problem Set 5

Due Thursday, October 17

1 Free presheaves

Let $\mathscr{C}$ be a category and R be a ring.

(a). (2 points) Show that the forgetful functor $\mathrm{PSh}(\mathscr{C}, R) \to \mathrm{PSh}(\mathscr{C})$ has a left adjoint $\mathscr{F} \longmapsto R^{(\mathscr{F})}$.

(b). (1 point) If X is an object of $\mathscr{C}$ and $h_X = \mathrm{Hom}_{\mathscr{C}}(\cdot, X)$ is the corresponding representabe presheaf, we write $R^{(X)}$ for $R^{(h_X)}$. Show that there is an isomorphism of additive functors from $\mathrm{PSh}(\mathscr{C}, R)$ to ${}_R\mathbf{Mod}$ (where $\mathscr{F}$ is the variable):

$$\mathrm{Hom}_{\mathrm{PSh}(\mathscr{C},R)}(R^{(X)}, \mathscr{F}) \simeq \mathscr{F}(X).$$

(c). (2 points) Suppose that $\mathscr{C}$ is equipped with a Grothendiech pretopology. If $\mathscr{F}$ is a sheaf for this pretopology, is $R^{(\mathscr{F})}$ always a sheaf ?

Solution.

(a). If $\mathscr{F}$ is a presheaf on $\mathscr{C}$, we define a presheaf $R^{(\mathscr{F})}$ by setting, for every $X \in \mathrm{Ob}(\mathscr{C})$, $R^{(\mathscr{F})}(X) = R^{(\mathscr{F}(X))}$; if $f : X \to Y$ is a morphism of $\mathscr{C}$, then we take for $R^{(\mathscr{F})}(f)$ the only R-linear extension of $\mathscr{F}(f)$. The presheaf $R^{(\mathscr{F})}$ is an object of $\mathrm{PSh}(\mathscr{C}, R)$, and its construction is clearly functorial in $\mathscr{F}$.

Now we show that the functor $\mathscr{F} \longmapsto R^{(\mathscr{F})}$ is left adjoint to the forgetful functor. Let $\mathscr{F}$ be a presheaf and $\mathscr{G}$ be a presheaf of R-modules. If $u : \mathscr{F} \to \mathscr{G}$ is a morphism of presheaves, then we define a morphism of presheaves $\alpha(u) : R^{(\mathscr{F})} \to \mathscr{G}$ by taking, for every $X \in \mathrm{Ob}(\mathscr{C})$, the morphism $\alpha(u)(X) : R^{(\mathscr{F}(X))} \to \mathscr{G}(X)$ to be the unique R-linear extension of $u(X) : \mathscr{F}(X) \to \mathscr{G}(X)$. By the universal property of the free R-module on a set, the map $\alpha : \mathrm{Hom}_{\mathrm{PSh}(\mathscr{C})}(\mathscr{F}, \mathscr{G}) \to \mathrm{Hom}_{\mathrm{PSh}(\mathscr{C},R)}(R^{(\mathscr{F})}, \mathscr{G})$ is bijective, and it is easy to check that it defines a morphism of functors on $\mathrm{PSh}(\mathscr{C})^{\mathrm{op}} \times \mathrm{PSh}(\mathscr{C}, R)$.

(b). We have an isomorphism of functors $\mathrm{Hom}_{\mathrm{PSh}(\mathscr{C},R)}(R^{(X)}, \mathscr{F}) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{PSh}(\mathscr{C})}(h_X, \mathscr{F})$ given by question (a), and an isomorphism of functors $\mathrm{Hom}_{\mathrm{PSh}(\mathscr{C})}(h_X, \mathscr{F}) \xrightarrow{\sim} \mathscr{F}(X)$ given by the Yoneda lemma.

(c). No. Let X be a topological space, let S be a singleton, and let $\mathscr{F}$ be the presheaf on X sending every open subset U of X to S. Then $R^{(\mathscr{F})}(U) = R$ for every open subset U of X, but a sheaf of R-modules on a topological space must take the value $\{0\}$ on $\varnothing$, so $R^{(\mathscr{F})}$ is not a sheaf (unless R is the zero ring).

$\square$

2 Constant presheaves and sheaves

Let $(\mathscr{C}, \mathscr{T})$ be a site. The *constant presheaf* on $\mathscr{C}$ with value S is the functor $\underline{S}_{psh} : \mathscr{C}^{\mathrm{op}} \to \mathbf{Set}$ sending any object to S and any morphism to id_S. The *constant sheaf* on $\mathscr{C}_{\mathscr{T}}$ with value S is the sheafification of the constant presheaf on $\mathscr{C}$ with value S; we will denote it by $\underline{S}$.

(a). (2 points) if $\mathscr{X} = (X_i \to X)_{i \in I}$ is a covering family, calculate $\check{\mathrm{H}}^0(\mathscr{X}, \underline{S}_{psh})$.

(b). (2 points) Suppose that $(\mathscr{C}, \mathscr{T})$ is the category of open subsets of the topological space $[0, 1]$, with the usual topology. Show that $(\underline{S}_{psh})^+$ is a sheaf if and only if $\mathrm{card}(S) \leq 1$.

(c). (2 points) Suppose that $(\mathscr{C}, \mathscr{T})$ is the category of open subsets of a locally connected topological space X. Show that, for every open subset U of X, we have $\underline{S}(U) = S^{\pi_0(U)}$.

Solution.

(a). Suppose that $I = \varnothing$. Then $\prod_{i \in I} \mathscr{F}(X_i)$ and $\prod_{i,j \in I} \mathscr{F}(X_i \times_X X_j)$ are both isomorphic to the terminal object of **Set**, i.e. to a singleton, so $\check{\mathrm{H}}^0(\mathscr{X}, \underline{S}_{psh})$ is a singleton.

Suppose that $I \neq \varnothing$. Then $\prod_{i \in I} \mathscr{F}(X_i) = S^I$. Also, for all $i, j \in I$, the maps $S = \mathscr{F}(X_i) \to \mathscr{F}(X_i \times_X X_j) = S$ and $S = \mathscr{F}(X_j) \to \mathscr{F}(X_i \times_X X_j) = S$ induced by the two projections are id_S. Let $s = (s_i)_{i \in I} \in S^I$. Then $s \in \check{\mathrm{H}}^0(\mathscr{X}, \mathscr{F})$ if and only if, for all $i, j \in I$, the images of s by the projections from S^I to its ith and jth factor are equal, that is, if and only if $s_i = s_j$ for all $i, j \in I$. So the diagonal embedding $S \subset S^I$ induces a bijection $S \xrightarrow{\sim} \check{\mathrm{H}}^0(\mathscr{X}, \mathscr{F})$.

(b). Let $\mathscr{F} = (\underline{S}_{psh})^+$, and let's pretend that we have not read the next question yet.

Suppose that $\mathrm{card}(S) \leq 1$. If S is a singleton, then, for every open cover $\mathscr{W} = (U_i)_{i \in I}$ of an open subset U of $[0, 1]$, the canonical map $\mathscr{F}(U) \to \prod_{i \in I} \mathscr{F}(U_i)$ and the two maps $\prod_{i \in I} \mathscr{F}(U_i) \to \prod_{i,j} \mathscr{F}(U_i \cap U_j)$ are isomorphisms, so $\mathscr{F}(U) \xrightarrow{\sim} \check{\mathrm{H}}^0(\mathscr{W}, \mathscr{F})$. If S is empty, this stays true as long as U and all the U_i are nonempty; as every open cover of a nonempty open set can be refined by an open cover that has only nonempty elements, we deduce again that $\mathscr{F}$ is a sheaf.

Suppose that $\mathscr{F}$ is a sheaf. Let $U_1 =]1/4, 1/2[$, $U_2 =]1/2, 3/4[$ and $U = U_1 \cup U_2$; we denote by $\mathscr{W}$ the open cover (U_1, U_2) of U. As $U_1 \cap U_2 \varnothing$, question (a) implies that $\check{\mathrm{H}}^0(\mathscr{W}, \mathscr{F}) = S \times S$, and that the canonical map $S = \mathscr{F}(U) \to \check{\mathrm{H}}^0(\mathscr{W}, \mathscr{F}) \to S \times S$ is the diagonal embedding. This is not bijection if $\mathrm{card}(S) \geq 2$, so we must have $\mathrm{card}(S) \leq 1$.

(c). Write $\mathscr{F} = (\underline{S}_{psh})^+$; by (a), the set $\mathscr{F}(\varnothing)$ is a singleton and we have $\mathscr{F}(V) = S$ for every nonempty open subset V of X.

Let$U$ be an open subset of X. If U is empty, we already know that $\underline{S}(U)$ is a singleton, hence isomorphic to $S^{\varnothing}$. Suppose that U is not empty. As U is locally connected, all its connected components are open (as well as closed), so we have $U = \coprod_{C \in \pi_0(U)} C$ as a topological space. Using the open cover $\{C \in \pi_0(U)\}$ of U, we see that the map $\underline{S}(U) \to \prod_{C \in \pi_0(U)} \underline{S}(C)$ must be bijective. So it suffices to show that, if U is connected and nonempty, then the canonical map $S = \mathscr{F}^+(U) \to \underline{S}(U)$ is bijective.

Let U be a nonempty connected subset of X, and let $\mathscr{W} = (U_i)_{i \in I}$ be an open cover of U. After replacing $\mathscr{W}$ by a refinement, we may assume that all the U_i are nonempty. For every $i \in I$, we denote by $I(i)$ the set of $j \in I$ such that there exists a sequence $i_0 = i, i_1, \ldots, i_n = j$ of elements of I such that $U_{i_{r-1}} \cap U_{i_r} \neq \varnothing$ for every $r \in \{1, \ldots, n\}$, and we set $V_i = \bigcup_{j \in I(i)} U_j$. Then the sets $I(i)$ form a partition of I. If we choose a subset K of I such that K intersects each $I(i)$ in a singleton, then $V_i \cap V_j = \varnothing$ if $i, j \in K$ and $i \neq j$, and $U = \bigcup_{i \in K} V_i$, so $U = \coprod_{i \in K} V_i$; but U is connected, so K has only on element. Now let $s = (s_i)_{i \in I} \in \prod_{i \in I} \mathscr{F}(U_i) = S^I$. If $i, j \in I$, the two images of s in $\mathscr{F}(U_i \cap U_j)$ are $s_{i|U_i \cap U_j}$ and $s_{j|U_i \cap U_j}$, so teh equality of these two images is an empty condition if $U_i \cap U_j = \varnothing$, and it equivalent to the condition that $s_i = s_j$ if $U_i \cap U_j \neq \varnothing$. But we have just shown that, for all $i, j \in I$, there exists a sequence $i_0 = i, i_1, \dots, i_n = j$ of elements of I such that $U_{i_{r-1}} \cap U_{i_r} \neq \varnothing$ for every $r \in \{1, \dots, n\}$, and we set $V_i = \bigcup_{j \in I(i)} U_j$. So $s \in \check{\mathrm{H}}^0(\mathscr{W}, \mathscr{F})$ if and only if $s_i = s_j$ for all $i, j \in I$; in other words, the map $S = \mathscr{F}(U) \to \check{\mathrm{H}}^0(\mathscr{W}, \mathscr{F})$ is bijective. So we conclude that $S = \mathscr{F}(U) \xrightarrow{\sim} \check{\mathrm{H}}^0(U, \mathscr{F}) = \underline{S}(U)$.

□

3 Points

Let $(\mathscr{C}, \mathscr{T})$ be a site. We are interested in the category $\mathrm{Points}(\mathscr{C}_{\mathscr{T}})$ whose objects are functors $\mathrm{Sh}(\mathscr{C}_{\mathscr{T}}) \to \mathbf{Set}$ that commutes with all small colimits and with finite limits, and whose morphisms are isomorphisms between such functors. [1]

A reference for many of the results of this problem is MacLane and Moerdijk, *Sheaves in geometry and logic* , especially Sections VII.5 and VII.6.

(a). Let $\mathscr{C}$ be an arbitrary category. Let $A : \mathscr{C} \to \mathbf{Set}$ be a functor. We denote by $\underline{\mathrm{Hom}}_{\mathscr{C}}(A, \cdot)$ the functor $\mathbf{Set} \to \mathrm{PSh}(\mathscr{C})$ sending a set S to the presheaf $X \longmapsto \mathrm{Hom}_{\mathbf{Set}}(A(X), S)$.

- (i) (1 point) Show that the functor $\underline{\mathrm{Hom}}_{\mathscr{C}}(A, \cdot)$ commutes with all limits.
- (ii) (3 points) Show that the functor $\underline{\mathrm{Hom}}_{\mathscr{C}}(A, \cdot)$ admits a left adjoint, which we will denote by $(\cdot) \otimes_{\mathscr{C}} A$, and that $((\cdot) \otimes_{\mathscr{C}} A) \circ h_{\mathscr{C}}$ is isomorphic to A. (Hint: First try to construct the adjoint on representable presheaves, and remember problem 2(a) of problem set 2.)

We say that the functor $A : \mathscr{C} \to \mathbf{Set}$ is *flat* if the functor $(\cdot) \otimes_{\mathscr{C}} A : \mathrm{PSh}(\mathscr{C}) \to \mathbf{Set}$ commutes with finite limits.

- (iii) (1 points) If A is flat, show that it commutes with all finite limits that exist in $\mathscr{C}$.
- (iv) (2 points) Suppose that $\mathscr{C}$ has all finite limits and that A commutes with finite limits. Let $\mathscr{F}$ be a presheaf on $\mathscr{C}$. If X, Y are objects of $\mathscr{C}/\mathscr{F}$, $x \in A(X)$ and $y \in A(Y)$, show that x and y represent the same element of $\mathscr{F} \otimes_{\mathscr{C}} A$ if and only if there exists an object Z of $\mathscr{C}$, morphisms $Z \to X$ and $Z \to Y$, and an element $z \in A(Z)$ whose images in $A(X)$ and $A(Y)$ are x and y respectively.
- (v) (3 points) If $\mathscr{C}$ has all finite limits, show that A is flat if and only if it commutes with finite limits. (Hint : To show that a functor commutes with finite limits, it suffices to show that it sends the final object to the final object and commutes with fibered products. You can admit this easy fact.)
- (vi) (2 points) Suppose that $\mathscr{C}$ has all finite limits. If $\mathscr{T}$ is the trivial pretopology on $\mathscr{C}$ (so that $\mathrm{Sh}(\mathscr{C}_{\mathscr{T}}) = \mathrm{PSh}(\mathscr{C})$), show that $\mathrm{Points}(\mathscr{C}_{\mathscr{T}})$ is equivalent to the category of flat functors $\mathscr{C} \to \mathbf{Set}$ (with morphisms being isomorphisms between these functors).

(b). Let $(\mathscr{C}, \mathscr{T})$ be a site. A flat functor $A : \mathscr{C} \to \mathbf{Set}$ is called *continuous* if, for every covering family $(X_i \to X)_{i \in I}$ in $\mathscr{C}$, the map $\coprod_{i \in I} A(X_i) \to A(X)$ is surjective.

[1] The idea of this definition is that we are abstracting the formal properties of stalk functors on the category of sheaves on a topological space.

For every $X \in \mathrm{Ob}(\mathscr{C})$, we denote by X^{sh} the sheafification of the representable presheaf $\mathrm{Hom}_{\mathscr{C}}(\cdot, X)$. This defines a functor $\mathscr{C} \to \mathrm{Sh}(\mathscr{C}_{\mathscr{T}})$, that commutes with finite limits.

(i) (2 points) Let $(f_i : X_i \to X)_{i \in I}$ be a covering family. We consider the morphisms

$$\coprod_{i,j \in I} (X_i \times_X X_j)^{\mathrm{sh}} \underset{g}{\overset{f}{\rightrightarrows}} \coprod_{i \in I} X_i^{\mathrm{sh}} \xrightarrow{h} X^{\mathrm{sh}}$$

where $h = \coprod_{i \in I} f_i^{\mathrm{sh}}$ and f (resp. g) is equal on $(X_i \times_X X_j)^{\mathrm{sh}}$ to the image by $(.)^{\mathrm{sh}} : \mathscr{C} \to \mathrm{Sh}(\mathscr{C}_{\mathscr{T}})$ of the first (resp. second) projection $X_i \times_X X_j \to X_i$ (resp. $X_i \times_X X_j \to X_j$).

Show that h is the cokernel of (f, g) in the category $\mathrm{Sh}(\mathscr{C}_{\mathscr{T}})$.

(ii) (1 point) Let $A : \mathscr{C} \to \mathbf{Set}$ be a flat functor, and suppose that $(\cdot) \otimes_{\mathscr{C}} A : \mathrm{PSh}(\mathscr{C}) \to \mathbf{Set}$ factors as $\mathrm{PSh}(\mathscr{C}) \overset{(\cdot)^{\mathrm{sh}}}{\to} \mathrm{Sh}(\mathscr{C}_{\mathscr{T}}) \overset{x_A}{\to} \mathbf{Set}$. Show that x_A is an object of $\mathrm{Points}(\mathscr{C}_{\mathscr{T}})$.

(iii) (2 points) If $A : \mathscr{C} \to \mathbf{Set}$ satisfies the hypothesis of the previous question, show that A is continuous. [2]

(c). Let $(C, \leq)$ be a preordered set. We see C as a category by taking $\mathrm{Hom}_C(a, b)$ to be a singleton if $a \leq b$, and empty otherwise.

(i) (1 point) Let $(a_i)_{i \in I}$ be a family of objects of C. Give a description of $\coprod_{i \in I} a_i$ and $\prod_{i \in I} a_i$ in (pre)ordered set terms.

(ii) (2 points) Give a similar translation of the property "C has all finite limits".

From now on, se suppose that C has all finite limits, and we fix a flat functor $A : C \to \mathbf{Set}$.

(iii) (2 points) Show that $\mathrm{card}(A(a)) \leq 1$ for every $a \in C$.

(iv) (2 points) Show that the set $I_A = \{a \in C \mid A(a) \neq \varnothing\}$ is a nonempty upper order ideal. (That is, if $a \in I_A$ and $a \leq b$, then $b \in I_C$.)

(v) (1 point) If $\mathscr{T}$ is any Grothendieck pretopology on C, show that the points of $C_{\mathscr{T}}$ don't have any nontrivial automorphisms.

(vi) (1 point) Suppose that any family $(a_i)_{i \in I}$ of elements of C has a least upper bound $\sup(a_i, i \in I)$. We say that a family of morphisms $(a_i \to a)_{i \in I}$ in $\mathscr{C}$ is covering if $a = \sup(a_i, i \in I)$. Suppose that this defines a pretopology on C. If A is continuous, show that I_A is a completely prime upper order ideal, that is, if $\sup(a_i, i \in I) \in I_A$, then at least one of the a_i is in I_A.

(d). Let X be a topological space, let $\mathscr{C} = \mathrm{Open}(X)$, and let $\mathscr{T}$ be the usual topology on X. Remember that a nonempty closed subset Z of X is called *irreducible* if, whenever $Z \subset Y_1 \cup Y_2$ with Y_1, Y_2 closed subsets of X, we have $Z \subset Y_1$ or $Z \subset Y_2$.

(i) (1 point) Show that a nonempty closed subset Z of X is irreducible if, for every open subset U of Z, the set $Z \cap U$ is either empty or dense in Z.

(ii) (1 point) Let Z be an irreducible closed subset of X, and let $\mathscr{U}_Z$ be the set of open subsets U of X such that $Z \cap U \neq \varnothing$. For every sheaf $\mathscr{F}$ on X, we set

$$\mathscr{F}_Z = \varinjlim_{U \in \mathrm{Ob}(\mathscr{U}_Z^{\mathrm{op}})} \mathscr{F}(U).$$

[2] In fact, the converse is true : points of $\mathscr{C}_{\mathscr{T}}$ correspond to flat continuous functors $\mathscr{C} \to \mathbf{Set}$.

Show that this defines a point of $\mathscr{C}_{\mathscr{T}}$.

(iii) (3 points) Let $x : \mathrm{Sh}(\mathscr{C}_{\mathscr{T}}) \to \mathbf{Set}$ be a point, and let $A : \mathscr{C} \to \mathbf{Set}$ be the corresponding flat continuous functor. Let

$$Z = X - \bigcup_{U \in \mathrm{Ob}(\mathscr{C}),\ A(U)=\varnothing} U.$$

Show that Z is an irreducible closed subset of X, and that x is isomorphic to the functor $\mathscr{F} \longmapsto \mathscr{F}_Z$. [3]

(iv) (1 point) If x_1 and x_2 are points of $\mathscr{C}_{\mathscr{T}}$ and Z_1 and Z_2 are the corresponding closed irreducible subsets of X, show that there exists a morphism from x_1 to x_2 if and only if $Z_1 \subset Z_2$.

(e). Let $X = [0,1]$ with the Lebesgue measure. We take $\mathscr{C}$ to be the category whose objects are Lebesgue-measurable subsets E of $[0,1]$, and such that $\mathrm{Hom}_{\mathscr{C}}(E, E')$ is a singleton if $E' - E$ has measure 0, and the empty set otherwise. We put the Grothendieck pretopology on $\mathscr{C}$ whose covering families are countable families $(E_n \to E)_{n\in\mathbb{N}}$ such that $E - \bigcup_{n\in\mathbb{N}} E_n$ has measure 0. (You can admit that this is a pretopology; it is not very hard.)

(i) (1 point) Show that the category $\mathrm{Sh}(\mathscr{C}_{\mathscr{T}})$ is not empty.

(ii) (2 points) Show that the category $\mathrm{Points}(\mathscr{C}_{\mathscr{T}})$ has no objects (that is, $\mathscr{C}_{\mathscr{T}}$ has no points).

Solution.

(a). (i) Let $\alpha : \mathscr{I} \to \mathbf{Set}$ be a functor, with $\mathscr{I}$ a small category. We want to show that the canonical morphism

$$\underline{\mathrm{Hom}}_{\mathscr{C}}(A, \cdot)(\varprojlim \alpha) \to \varprojlim(\underline{\mathrm{Hom}}_{\mathscr{C}}(A, \cdot) \circ \alpha)$$

is an isomorphism in $\mathrm{PSh}(\mathscr{C})$. For every $X \in \mathrm{Ob}(\mathscr{C})$, if we evaluate this morphism at X, we get the canonical morphism

$$\mathrm{Hom}_{\mathbf{Set}}(A(X), \varprojlim \alpha) \to \varprojlim_{i\in\mathrm{Ob}(\mathscr{I})} \mathrm{Hom}_{\mathbf{Set}}(A(X), \alpha(i))$$

(where we use Proposition I.5.3.1 of the notes to calculate the right-hand side), which is an isomorphism by definition of the limit.

(ii) By Proposition I.4.7 of the notes, it suffices to show that, for every presheaf $\mathscr{F}$ on $\mathscr{C}$, the functor $\mathbf{Set} \to \mathbf{Set}$, $S \longmapsto \mathrm{Hom}_{\mathrm{PSh}(\mathscr{C})}(\mathscr{F}, \underline{\mathrm{Hom}}_{\mathscr{C}}(A, S))$ is representable.

Suppose first that $\mathscr{F} = h_X = \mathrm{Hom}_{\mathscr{C}}(\cdot, X)$ is a representable presheaf. By the Yoneda lemma, for every set S, the map $\mathrm{Hom}_{\mathrm{PSh}(\mathscr{C})}(h_X, \mathrm{Hom}_{\mathscr{C}}(A, S)) \to \mathrm{Hom}_{\mathscr{C}}(A, S)(X) = \mathrm{Hom}_{\mathbf{Set}}(A(X), S)$ sending $u : h_X \to \mathrm{Hom}_{\mathscr{C}}(A, S)$ to $u(X)(\mathrm{id}_X)$ is bijective. An easy verification shows that this map defines an isomorphisms of functors. So the functor $S \longmapsto \mathrm{Hom}_{\mathrm{PSh}(\mathscr{C})}(h_X, \underline{\mathrm{Hom}}_{\mathscr{C}}(A, S))$ is represented by the set $A(X)$. Also, if $f : X \to Y$ is a morphism of $\mathscr{C}$ and $h_f : h_X \to h_Y$ is its image by the Yoneda

[3] So we have shown that points of $\mathscr{C}_{\mathscr{T}}$ correspond to closed irreducible subsets of X. If X is sober, that is, if every closed irreducible subset has a unique generic point, then points of $\mathscr{C}_{\mathscr{T}}$ correspond to points of X, but this is not true in general.

embedding, then we have a commutative diagram, for every set S:

$$\begin{tikzcd}
\mathrm{Hom}_{\mathrm{PSh}(\mathscr{C})}(h_Y, \mathrm{Hom}_{\mathscr{C}}(A,S)) \arrow[r, "\sim"] \arrow[d, "(\cdot)\circ h_f"'] & \mathrm{Hom}_{\mathbf{Set}}(A(Y),S) \arrow[d, "(\cdot)\circ A(f)"] \\
\mathrm{Hom}_{\mathrm{PSh}(\mathscr{C})}(h_X, \mathrm{Hom}_{\mathscr{C}}(A,S)) \arrow[r, "\sim"] & \mathrm{Hom}_{\mathbf{Set}}(A(X),S)
\end{tikzcd}$$

Indeed, let $u \in \mathrm{Hom}_{\mathrm{PSh}(\mathscr{C})}(h_Y, \mathrm{Hom}_{\mathscr{C}}(A,S))$. Then its image in $\mathrm{Hom}_{\mathbf{Set}}(A(X),S)$ by the upper right path of the diagram is $u(Y)(\mathrm{id}_Y)\circ A(f)$, and its image by the left bottom path of the diagram is $(u\circ h_f)(X)(\mathrm{id}_X) = u(X)(f) = u(X)(\mathrm{id}_Y \circ f)$. But these two are equal because, as u is a morphism of presheaves, we have a commutative diagram:

$$\begin{tikzcd}
h_Y(X) \arrow[r, equal] & \mathrm{Hom}_{\mathscr{C}}(Y,X) \arrow[r, "u(X)"] & \mathrm{Hom}_{\mathrm{S}}(A(X),S) \\
h_Y(Y) \arrow[r, equal] & \mathrm{Hom}_{\mathscr{C}}(X,X) \arrow[r, "u(Y)"'] \arrow[u, "(\cdot)\circ f"'] & \mathrm{Hom}_{\mathbf{Set}}(A(Y),S) \arrow[u, "(\cdot)\circ A(f)"']
\end{tikzcd}$$

It remains to show that the functor $S \longmapsto \mathrm{Hom}_{\mathrm{PSh}(\mathscr{C})}(\mathscr{F}, \underline{\mathrm{Hom}}_{\mathscr{C}}(A,S))$ is representable for an arbitrary presheaf $\mathscr{F}$ on $\mathscr{C}$. As in problem 2 of problem set 2, consider the category $\mathscr{C}/\mathscr{F}$ and the functor $G_{\mathscr{F}} : \mathscr{C}/\mathscr{F} \to \mathscr{C}$. We have shown in question (a) of that problem that there is a canonical isomorphism $\varinjlim(h_{\mathscr{C}}\circ G_{\mathscr{F}}) \overset{\sim}{\to} \mathscr{F}$. Let $\mathscr{F}\otimes_{\mathscr{C}} A = \varinjlim(A\circ G_{\mathscr{F}}) = \varinjlim_{X\in\mathrm{Ob}(\mathscr{C}/\mathscr{F})} A(X)$. Then we have, for every set S,

$$\begin{aligned}
\mathrm{Hom}_{\mathrm{PSh}(\mathscr{C})}(\mathscr{F}, \underline{\mathrm{Hom}}_{\mathscr{C}}(A,S)) &\simeq \mathrm{Hom}_{\mathrm{PSh}(\mathscr{C})}(\varinjlim(h_{\mathscr{C}}\circ G_{\mathscr{F}}), \mathrm{Hom}_{\mathscr{C}}(A,S)) \\
&\overset{\sim}{\to} \varprojlim_{X\in\mathrm{Ob}(\mathscr{C}/\mathscr{F})} \mathrm{Hom}_{\mathrm{PSh}(\mathscr{C})}(h_X, \mathrm{Hom}_{\mathscr{C}}(A,S)) \\
&\overset{\sim}{\to} \varprojlim_{X\in\mathrm{Ob}(\mathscr{C}/\mathscr{F})} \mathrm{Hom}_{\mathbf{Set}}(A(X),S)) \\
&\simeq \mathrm{Hom}_{\mathbf{Set}}(\mathscr{F}\otimes_{\mathscr{C}} A, S).
\end{aligned}$$

These isomorphisms are all easily seen to be functorial in S, so the set $\mathscr{F}\otimes_{\mathscr{C}} A$ represents the functor $S \longmapsto \mathrm{Hom}_{\mathrm{PSh}(\mathscr{C})}(\mathscr{F}, \underline{\mathrm{Hom}}_{\mathscr{C}}(A,S))$.

(iii) We know that $((\cdot)\otimes_{\mathscr{C}} A)\circ h_{\mathscr{C}} \simeq A$ by (ii), and that $h_{\mathscr{C}}$ commutes with all limits that exist in $\mathscr{C}$ by definition of limits, so, if $(\cdot)\otimes_{\mathscr{C}} A$ commutes with finite limits, so does A.

(iv) By Theorem I.5.2.1 of the notes, we have $\mathscr{F}\otimes_{\mathscr{C}} A = \coprod_{X\in\mathrm{Ob}(\mathscr{C}/\mathscr{F})} A(X)/\sim$, where $\sim$ is the equivalence relation generated by the relation R defined by: if $X, Y \in \mathrm{Ob}(\mathscr{C}/\mathscr{F})$ and $x \in A(X)$, $y \in A(Y)$, then xRy if there exists a morphism $f : X \to Y$ in $\mathscr{C}/\mathscr{F}$ such that $A(f)(x) = y$.

Let R' be the relation on $\coprod_{X\in\mathrm{Ob}(\mathscr{C}/\mathscr{F})} A(X)$ defined in the question. We clearly have $xRy \Rightarrow xR'y \Rightarrow x\sim y$ (with the same notation as in the previous paragraph), so it suffices to show that R' is an equivalence relation. It is clearly reflexive and symmetric. We show that it is transitive. Let X_1, X_2, X_3 be objects of $\mathscr{C}/\mathscr{F}$ and $x_1 \in A(X_1)$, $x_2 \in A(X_2)$, $x_3 \in A(X_3)$ such that $x_1R'x_2$ and $x_2R'x_3$. This means that we have $Y_1, Y_2 \in \mathrm{Ob}(\mathscr{C})$, morphisms $f_1 : Y_1 \to X_1$, $f_2 : Y_1 \to X_2$, $g_1 : Y_2 \to X_2$, $g_2 : Y_2 \to X_3$ in $\mathscr{C}/\mathscr{F}$ and elements $y_1 \in A(Y_1)$ and $y_2 \in A(Y_2)$ such that $A(f_1)(y_1) = x_1$, $A(f_2)(y_1) = x_2$, $A(g_1)(y_2) = x_2$ and $A(g_2)(y_2) = x_3$. Let $Z = Y_1 \times_{X_2} Y_2$, let

$p : Z \to Y_1$ and $q : Z \to Y_2$, and let $z = (y_1, y_2) \in A(Z) = A(Y_1) \times_{A(X_2)} A(Y_2)$.

$$\begin{tikzcd} & & Z \arrow[dl, "p"'] \arrow[dr, "q"] & & \\ & Y_1 \arrow[dl, "f_1"'] \arrow[dr, "f_2"] & & Y_2 \arrow[dl, "g_1"'] \arrow[dr, "g_2"] & \\ X_1 & & X_2 & & X_3 \end{tikzcd}$$

Then $A(f_1 \circ p)(z) = x_1$ and $A(g_2 \circ q) = x_3$, so $x_1 R' x_3$.

(v) Suppose that A commutes with finite limits. We want to show that it is flat.

Let $*$ be the final object of $\mathscr{C}$ (i.e. the limit of the unique functor $\varnothing \to \mathscr{C}$). Then the final object of $\mathrm{PSh}(\mathscr{C})$ is the presheaf h_*, which is also isomorphic to the constant presheaf with value a fixed singleton. The functor $G_{h_*} : \mathscr{C}/h_* \to \mathscr{C}$ is an isomorphism of categories, so, to show that $h_* \otimes_{\mathscr{C}} A$ is a final object of **Set**, we need to show that $S := \varinjlim_{X \in \mathrm{Ob}(\mathscr{C})} A(X)$ is a singleton. We have a morphism $A(*) \to S$ and $A(*)$ is a final object in **Set**, i.e. a singleton, so S is not empty. Let $X, Y \in \mathrm{Ob}(\mathscr{C})$, $x \in A(X)$ and $y \in A(Y)$. Then (x, y) is an element of $A(X \times Y) \simeq A(X) \times A(Y)$, and, if $p_1 : X \times Y \to X$ and $p_2 : X \times Y \to Y$ are the two projections, then $A(p_1)(x, y) = x$ and $A(p_2)(x, y) = y$. So $x \in A(X)$, $(x, y) \in A(X \times Y)$ and $y \in A(Y)$ define the same element of S. This shows that $\mathrm{card}(S) \leq 1$, hence that S is a singleton because S is not empty.

We now show that the functor $(\cdot) \otimes_{\mathscr{C}} A$ commutes with fiber products. Let $\mathscr{F} \to \mathscr{H}$ and $\mathscr{G} \to \mathscr{H}$ be morphisms in $\mathrm{PSh}(\mathscr{C})$, let $E = \mathscr{F} \otimes_{\mathscr{C}} A$, $E' = \mathscr{G} \otimes_{\mathscr{C}} A$, $E'' = \mathscr{H} \otimes_{\mathscr{C}} A$ and $F = (\mathscr{F} \times_{\mathscr{H}} \mathscr{G}) \otimes_{\mathscr{C}} A$. Applying the functor $(\cdot) \otimes_{\mathscr{C}} A$ to the commutative diagram

$$\begin{tikzcd} \mathscr{F} \times_{\mathscr{H}} \mathscr{G} \arrow[r] \arrow[d] & \mathscr{G} \arrow[d] \\ \mathscr{F} \arrow[r] & \mathscr{H} \end{tikzcd}$$

we get a commutative diagram

$$\begin{tikzcd} F \arrow[r] \arrow[d] & E' \arrow[d, "q"] \\ E \arrow[r, "p"'] & E'' \end{tikzcd}$$

and we want to show that this induces an isomorphism from F to the fiber product $E \times_{E''} E'$. So let S be another set, and let $u : S \to E$, $v : S \to E'$ be maps such that $p \circ u = q \circ v$. We want to show that these maps factor uniquely through a map $w : S \to F$. Let $s \in S$. To make the notation less cumbersome, we will use the Yoneda embedding to identify $\mathscr{C}$ to a full subcategory of $\mathrm{PSh}(\mathscr{C})$, so we write X instead of h_X if $X \in \mathrm{Ob}(\mathscr{C})$. Choose an object $X \to \mathscr{F}$ of $\mathscr{C}/\mathscr{F}$, an object $Y \to \mathscr{G}$ of $\mathscr{C}/\mathscr{G}$ and elements $x \in A(X)$ and $y \in A(Y)$ such that x represents $u(s)$ and y represents $v(s)$. The fact that $p(u(s)) = q(v(s)$ means that there exists an object Z of $\mathscr{C}$, a commutative diagram

$$\begin{tikzcd} Z \arrow[r] \arrow[dr] & Y \arrow[r] & \mathscr{G} \arrow[dr] & \\ & X \arrow[r] & \mathscr{F} \arrow[r] & \mathscr{H} \end{tikzcd}$$

in $\mathrm{PSh}(\mathscr{C})$ and $z \in A(Z)$ such that the images of z in $A(X)$ and $A(Y)$ are x and y. The diagram we just wrote gives a morphism $Z \to \mathscr{F} \times_{\mathscr{H}} \mathscr{G}$ in $\mathrm{PSh}(\mathscr{C})$, so we get an object of $\mathscr{C}/(\mathscr{F} \times_{\mathscr{H}} \mathscr{G})$, and, if $w : S \to F$ existed, we would necessarily have that $w(s)$ is the element of F represented by z. This proves the uniqueness of w. To prove its existence, we need to show that other choices of representatives of $u(s)$ and $v(s)$ would give the same element of F. So suppose that we have another commutative diagram

$$\begin{tikzcd}
Z' \arrow[r] \arrow[dr] & Y' \arrow[r] & \mathscr{G} \arrow[dr] & \\
& X' \arrow[r] & \mathscr{F} \arrow[r] & \mathscr{H}
\end{tikzcd}$$

and an element $z' \in A(Z')$ such that the image x' of z' in $A(X')$ is a representative of $u(s)$ and the image y' of z' in $A(Y')$ is a representative of $v(s)$. We must show that z and z' represent the same element of F. As x and x' represent the same element $u(s)$ of E, there exists an object X'' of $\mathscr{C}$, morphisms $X'' \to X$ and $X'' \to X'$ and an element $x'' \in A(X'')$ whose images in $A(X)$ and $A(X')$ are x and x' respectively. Similarly, we get $Y'' \to Y$, $Y'' \to Y'$ and $y'' \in A(Y'')$. Now replacing X by X'', Z by $X'' \times_X Z$, the morphism $Z \to X$ by the first projection $X'' \times_X Z \to X''$, the morphism $Z \to Y$ by the composition of the second projection $X'' \times_X Z \to Z$ and of $Z \to Y$, $x \in A(X)$ by $x'' \in A(X'')$ and $z \in A(Z)$ by $(x'', z) \in A(X'' \times_X Z) = A(X'') \times_{A(X)} A(Z)$, we may assume that there is a morphism $X \to X'$ such that the image of x in $A(X')$ is x'. Playing the same game with $Y'' \to Y$ (that is, replacing Z with $Y'' \times_Y Z$ etc), we may also assume that $Y'' = Y$ and $y'' = y$. We now have w commutative diagram

$$\begin{tikzcd}
Z' \arrow[rr, bend left] \arrow[drr] \arrow[ddrr] & Y \arrow[r] & Y' \arrow[r] & \mathscr{G} \arrow[dr] & \\
& & \mathscr{F} \times_{\mathscr{H}} \mathscr{G} \arrow[ur] \arrow[dr] & & \mathscr{H} \\
Z \arrow[uur] \arrow[urr] \arrow[r] & X \arrow[r] & X' \arrow[r] & \mathscr{F} \arrow[ur] &
\end{tikzcd}$$

and element $z \in A(Z)$, $z' \in A(z')$ such that the images of z in $A(X)$ and $A(Y)$ are x and y respectively, that the images of z' in $A(X')$ and $A(Y')$ are x' and y' respectively, the image of x in $A(X')$ is x' and the image of y in $A(Y')$ is y'.

$$\begin{tikzcd}
z' \arrow[rr, mapsto, bend left] \arrow[drr, mapsto] & y \arrow[r, mapsto] & y' \\
z \arrow[ur, mapsto] \arrow[r, mapsto] & x \arrow[r, mapsto] & x'
\end{tikzcd}$$

Let $Z'' = Z \times_{X' \times Y'} Z'$, and let $z'' = (z, z') \in A(Z \times_{X' \times Y'} Z') = A(Z) \times_{A(X') \times A(Y')} A(Z')$. To show that z, z' and z'' induce the same element of F (which will finish the proof), it suffices to show that the morphisms $Z'' \to Z \to \mathscr{F} \times_{\mathscr{H}} \mathscr{G}$ and $Z'' \to Z' \to \mathscr{F} \times_{\mathscr{H}} \mathscr{G}$ are equal. But these morphisms become equal after we compose them with the two projections from $\mathscr{F} \times_{\mathscr{H}} \mathscr{G}$ to $\mathscr{F}$ and $\mathscr{G}$, so they are equal by the universal property of the fiber product.

(vi) If $A : \mathscr{C} \to \mathbf{Set}$ is a flat functor, then the functor $x_A = (\cdot) \otimes_{\mathscr{C}} A : \mathrm{PSh}(\mathscr{C}) \to \mathbf{Set}$ commutes with all colimits (as a left adjoint) and with finite limits (by flatness of A), so it is an object of $\mathrm{Points}(\mathscr{C}_{\mathscr{T}})$. Also, the construction of $(\cdot) \otimes_{\mathscr{C}} A$ in the solution of (ii) is clearly functorial in A.

Conversely, let $x : \mathrm{PSh}(\mathscr{C}) \to \mathbf{Set}$ be an object of $\mathrm{Points}(\mathscr{C}_{\mathscr{T}})$, and let $A_x = x \circ h_{\mathscr{C}} : \mathscr{C} \to \mathbf{Set}$. Then A_x is a flat functor because both $h_{\mathscr{C}}$ and x commutes with finite limits, so we get a functor from $\mathrm{Points}(\mathscr{C}_{\mathscr{T}})$ to the category of flat functors $\mathscr{C} \to \mathbf{Set}$ (with isomorphisms of such functors as morphisms). Moreover, if x is a point, then it commutes with all colimits, so we have a canonical isomorphism for all $\mathscr{F}$:

$$x(\mathscr{F}) = x(\varinjlim_{X \in \mathrm{Ob}(\mathscr{C}/\mathscr{F})} h_X) \xrightarrow{\sim} \varinjlim_{X \in \mathrm{Ob}(\mathscr{C}/\mathscr{F})} x(h_X) = \varinjlim_{X \in \mathrm{Ob}(\mathscr{C}/\mathscr{F})} A(X) = \mathscr{F} \otimes_{\mathscr{C}} A = x_{A_x}(\mathscr{F}),$$

and this gives an isomorphism of functors $x \xrightarrow{\sim} x_{A_x}$.

Finally, if $A : \mathscr{C} \to \mathbf{Set}$ is a flat functor, we have already seen in (ii) that $x_A \circ h_{\mathscr{C}} \simeq A$; in other words, we have $A_{x_A} \simeq A$.

$\square$

4 G-sets

Let G be a finite group, let $\mathscr{C} = G - \mathbf{Set}$ be the category whose objects are sets with a left action of G and whose morphisms are G-equivariant maps. We consider the pretopology $\mathscr{T}$ on $\mathscr{C}$ for which a family $(f_i : X_i \to X)_{i \in I}$ is covering if and only if $X = \bigcup_{i \in I} f_i(X_i)$. [4]

Let A be G with its actionby left translations. More generally, for every subgroup H of G, we denote by A_H the set G/H with the action of G by left translations.

Useful fact: If $X \to Y$ is a surjective map in $\mathbf{Set}$ or $G - \mathbf{Set}$, then it is the cokernel of the two projections $X \times_Y X \to X$. (You still need to justify thisn if you want to use it.)

(a). (1 point) Show that every object of $G - \mathbf{Set}$ is a coproduct of objects isomorphic to some A_H.

(b). (1 point) Calculate $A \times_{A_H} A$ in the category $G - \mathbf{Set}$.

(c). (1 point) Show that every representable presheaf on $G - \mathbf{Set}$ is a sheaf.

(d). (1 point) Show the automorphisms of A in $G - \mathbf{Set}$ are exactly the maps $c_g : A \to A$, $a \longmapsto ag$, for $g \in G$.

(e). (1 point) If $\mathscr{F}$ is a presheaf on $G - \mathbf{Set}$, show the family $(\mathscr{F}(c_g))_{g \in G}$ defines a left action of G on $\mathscr{F}(A)$.

(f). (1 point) Consider the functor $\Phi : \mathrm{Sh}(\mathscr{C}_{\mathscr{T}}) \to G - \mathbf{Set}$ defined by $\Phi(\mathscr{F}) = \mathscr{F}(A)$ and the functor $\Psi : F - \mathbf{Set} \to \mathrm{Sh}(\mathscr{C}_{\mathscr{T}})$ fiven by $\Psi(X) = \mathrm{Hom}_{G-\mathbf{Set}}(\cdot, X)$. Show that $\Phi \circ \Psi \simeq \mathrm{id}_{G-\mathbf{Set}}$.

(g). (4 points) Show that $\Psi \circ \Phi \simeq \mathrm{id}_{\mathrm{Sh}(\mathscr{C}_{\mathscr{T}})}$. (Hint: For any G-set X, if $|X|$ is the set X with the trivial G-action, then we have a surjective G-equivariant map $A \times |X| \to X$, $(g, x) \longmapsto g \cdot x$, which induces an injection $\mathscr{F}(X) \to \prod_{x \in X} \mathscr{F}(A) = \mathrm{Hom}_{\mathbf{Set}}(X, \mathscr{F}(A))$.)

(h). (3 points) Let $x : \mathrm{Sh}(\mathscr{C}_{\mathscr{T}}) \to \mathbf{Set}$ be the functor $\mathscr{F} \longmapsto \mathscr{F}(A)$, where we forget the action of G on $\mathscr{F}(A)$ to see $\mathscr{F}(A)$ as a set. Show that every point of $\mathscr{C}_{\mathscr{T}}$ is isomorphic to x. (See the beginning of Problem 3 for the definition of points.) (Suggestion: if y is a point, calculate $y(\Psi(\{1\}))$, then $y(\Psi(A))$, then construct a morphism of functors $\mathrm{Hom}_{G-\mathbf{Set}}(A, \cdot) \to y \circ \Psi$, then show that it is an isomorphism.)

[4] It is very easy to check that this is a pretopology, you don't need to do it.

(i). (1 point) Show that the group of automorphisms of the point x is isomorphic to G.

Solution.

(a). Let X be a set with an action of G. Then X is the disjoint union of its G-orbits, and a G-orbit $G \cdot x$ is isomorphic to A_H, where H is the stabilizer of x.

(b). Let B be the set G with the trivial action of G. We have a G-equivariant bijection $A \times A \to A \times B$, $(x, y) \longmapsto (x, x^{-1}y)$. (Where $A \times A$ is the direct product in $G - \mathbf{Set}$, so G acts via $g \cdot (x, y) = (gx, gy)$.) If H is a subgroup of G, this bijection sends the G-subset $A \times_{A_H} A = \{(x, y) \in A \times A \mid x^{-1}y \in H\}$ to $A \times H$, where the factor H has the trivial action of G.

(c). This is exactly the content of the "useful fact" from the statement. Let's prove it. Let E be a G-set, and let $(f_i : X_i \to X)_{i \in I}$ be a covering family in $G - \mathbf{Set}$. Let $(u_i : X_i \to E)_{i \in I}$ be a family of G-equivariant maps such that, for all $i, j \in I$, the pullbacks of u_i and u_j to $X_i \times_X X_j$ (by the two projectins) agree. This means that, for every $x \in X$, if $x_i \in f_i^{-1}(x)$ and $x_j \in f_j^{-1}(x)$, then $u_i(x) = u_j(x)$. As $X = \bigcup_{i \in I} f_i(X_i)$, there exists a unique map $u : X \to E$ such that $u \circ f_i = u_i$ for every $i \in I$, and it suffices to check that u is G-equivariant. Let $x \in X$ and $g \in G$; choose $i \in I$ and $x_i \in X_i$ such that $x = f_i(x_i)$; then $g \cdot x = f_i(g \cdot x_i)$, so $u(g \cdot x) = u_i(g \cdot x_i) = g \cdot u(x_i) = g \cdot u(x)$.

(d). It is clear that the c_g are all automorphisms of A in $G - \mathbf{Set}$.

Conversely, let $\varphi : A \to A$ be an automorphism in $G - \mathbf{Set}$, and let $g = \varphi(1)$. Then, for everty $h \in A$, we have $\varphi(h) = \varphi(h \cdot 1) = h \cdot \varphi(1) = hg$. So $\varphi = c_g$.

(e). For every $g \in G$, the map $\mathscr{F}(c_g)$ is an automorphism of $\mathscr{F}(A)$ (in the category $\mathbf{Set}$), and we have $\mathscr{F}(c_1) = \mathrm{id}_{\mathscr{F}(A)}$ because $c_1 = \mathrm{id}_A$. If $g, h \in G$, we have $c_{gh} = c_h \circ c_g$, so $\mathscr{F}(c_{gh}) = \mathscr{F}(c_g) \circ \mathscr{F}(c_h)$. So we do get a left action of G on $\mathscr{F}(A)$.

(f). The functor Φ is well-defined, because, if $\alpha : \mathscr{F} \to \mathscr{G}$ is a morphism of sheaves and $g \in G$, then $\alpha(A) \circ \mathscr{F}(c_g) = \mathscr{G}(c_g) \circ \alpha(A)$, so $\alpha(A)$ is a G-equivariant map.

Let X be a G-set. Then we have a map $u(X) : \Phi(\Psi(X)) = \mathrm{Hom}_G(A, X) \to X$ sending $f : A \to X$ to $f(1)$, and this clearly defines a morphism of functors $u : \Phi \circ \Psi \to \mathrm{id}_{G-\mathbf{Set}}$. We show that it is an isomorphism. If $f, f' : A \to X$ are two G-equivariant maps such that $f(1) = f'(1)$, then, for every $g \in G$, we have $f(g) = f(g \cdot 1) = g \cdot f(1) = g \cdot f'(1) = f'(g)$. So $u(X)$ is injective. Let $x \in X$, and define a map $f : A \to X$ by $f(g) = g \cdot x$; then f is G-equivariant, and $u(X)(f) = x$; so $u(X)$ is surjective.

(g). If $\mathscr{F}$ is a sheaf, then $\Psi(\Phi(\mathscr{F})) = \mathrm{Hom}_{G-\mathbf{Set}}(\cdot, \mathscr{F}(A))$, so we must find an isomorphism of sheaves $\mathrm{Hom}_{G-\mathbf{Set}}(\cdot, \mathscr{F}(A)) \simeq \mathscr{F}$ that is functorial in $\mathscr{F}$.

Let $\mathscr{F}$ be a sheaf. For every G-set X, let $p_X : A \times |X| \to X$, $(g, x) \longmapsto g \cdot x$ be the G-equivariant surjection of the hint. It is a covering family in G-set, hence induces an injection $\iota(X, \mathscr{F}) : \mathscr{F}(X) \to \mathscr{F}(A \times |X|) = \mathscr{F}(\coprod_{x \in X} A) = \prod_{x \in X} \mathscr{F}(A) = \mathrm{Hom}_{\mathbf{Set}}(X, \mathscr{F}(A))$, that is a morphism of functors in X in $\mathscr{F}$. We first check that the image of $\iota(X, \mathscr{F})$ is contained in the set G-equivariant maps. Write $A \times |X| = \coprod_{x \in X} A_x$, with $A_x = A$ for every $x \in X$; we have $p_{X|A_x}(g) = g \cdot x$, for $g \in A$ and $x \in X$. Let $x \in X$ and $g \in G$; we set $y = g \cdot x$. Then we have a commutative diagram in $G - \mathbf{Set}$:

$$\begin{tikzcd} A \arrow[r, "c_g"] \arrow[d, "c_1"'] & A_x \arrow[d, "p_{X|A_x}"] \\ A_y \arrow[r, "p_{X|A_y}"'] & X \end{tikzcd}$$

So, if $e \in \mathscr{F}(X)$ and $u = \iota(X, \mathscr{F})(e) : X \to \mathscr{F}(A)$, then $\mathscr{F}(c_g)(u(x)) = \mathscr{F}(c_1)(u(g \cdot x))$. This shows that u is G-equivariant.

To finish the proof, we must show that $\iota(X, \mathscr{F})$ is surjective for every G-set X and every sheaf $\mathscr{F}$. Fix $\mathscr{F}$.

If $X = A$, then $p_A : A \times |A| \to A$ is the map $(g, h) \longmapsto gh$; if we write as before $A \times |A| = \coprod_{h \in G} A_h$ with $A_h = A$ for every h, then $p_{A|A_h} = c_h$ for every $h \in G$. So $\mathscr{F}(p_A) : \mathscr{F}(A) \to (\mathscr{F}(A))^A = \mathrm{Hom}_{\mathbf{Set}}(A, \mathscr{F}(A))$ is the map sending $e \in \mathscr{F}(A)$ to $A \to \mathscr{F}(A)$, $g \longmapsto \mathscr{F}(c_g)(e)$. It is easy to see that every G-equivariant map $u : A \to \mathscr{F}(A)$ is of this form (take $e = u(1)$). So $\iota(A, \mathscr{F})$ is surjective, hence bijective.

Note that the functors $\mathscr{F}$ and $\mathrm{Hom}_{G-\mathbf{Set}}(\cdot, \mathscr{F}(A))$ both send coproducts to products. For the second functor, this is by definition of a coproduct. For the first functor, suppose that $X = \coprod_{i \in I} X_i$. Then the then the family of injections $(X_i \to X)_{i \in I}$ is covering and $X_i \times_X X_j = \varnothing$ for $i \neq j$, so the morphism $\mathscr{F}(X) \to \prod_{i \in I} \mathscr{F}(X_i)$ is bijective. So if X is a disjoint union of copies of A, then $\iota(X, \mathscr{F})$ is a bijection.

Let X be an arbirtrary G-set. We have a surjective G-equivariant map $p_X : A \times |X| = \coprod_{x \in X} A_x \to X$, where $A_x = A$ for every $x \in X$ and $p_{A|A_x}$ sends $g \in A$ to $g \cdot x$. Let $x, y \in X$. Then we have a G-equivariant isomorphism $A_x \times_X A_y = \{(g, h) \in A \times A \mid g \cdot x = h \cdot y\} \xrightarrow{\sim} A \times G_{y,x}$, $(g, h) \mid (g, g^{-1}h)$, where $G_{x,y}$ is the set $\{g \in G \mid g \cdot y = x\}$ with the trivial action of G; in particular, $A_x \times_X A_y$ is a disjoint union of copies of A. So $P := (A \times |X|) \times_X (A \times |X|)$ also is a disjoint union of copies of A. Let $p_1, p_2 : P \to A \times |X|$ be the two projections. Then we have commutative diagrams

$$\begin{tikzcd}
\mathscr{F}(X) \arrow[r, "\mathscr{F}(p_X)"] \arrow[d, "\iota(X,\mathscr{F})"'] & \mathscr{F}(A \times |X|) \arrow[r, "\mathscr{F}(p_i)"] \arrow[d, "\iota(A \times |X|, \mathscr{F})"'] & \mathscr{F}(P) \arrow[d, "\iota(P,\mathscr{F})"'] \\
\mathrm{Hom}_{G-\mathbf{Set}}(X, \mathscr{F}(A)) \arrow[r, "p_X^*"'] & \mathrm{Hom}_{G-\mathbf{Set}}(A \times |X|, \mathscr{F}(A)) \arrow[r, "p_i^*"'] & \mathscr{F}(P)
\end{tikzcd}$$

for $i = 1, 2$, the maps $\mathscr{F}(p_X) : \mathscr{F}(X) \to \mathscr{F}(A \times |X|)$ and $p_X^* : \mathrm{Hom}_{G-\mathbf{Set}}(X, \mathscr{F}(A)) \to \mathrm{Hom}_{G-\mathbf{Set}}(A \times |X|, \mathscr{F}(A))$ are the kernels of $(\mathscr{F}(p_1), \mathscr{F}(p_2))$ and (p_1^*, p_2^*) respectively (because $\mathscr{F}$ and $\mathrm{Hom}_{G-\mathbf{Set}}(\cdot, \mathscr{F}(A))$ are sheaves), and the maps $\iota(A \times |X|, \mathscr{F})$ and $\iota(P, \mathscr{F})$ are bijective by the previous paragraph, so $\iota(X, \mathscr{F})$ is bijective.

(h). Note that $x \circ \Psi : G - \mathbf{Set} \to \mathbf{Set}$ is the functor $X \longmapsto \mathrm{Hom}_{G-\mathbf{Set}}(A, X)$. For every G-set X, we have a bijection $\mathrm{Hom}_{G-\mathbf{Set}}(A, X) \xrightarrow{\sim} X$ sending $u : A \to X$ to $u(1)$, and this gives an isomorphism from $x \circ \Psi$ to the forgetful functor $G - \mathbf{Set} \to \mathbf{Set}$. As Ψ is an equivalence of categories by (f) and (g), this shows that x commutes with all small limits and colimits, and in particular that it is a point.

Let $y : \mathrm{Sh}(\mathscr{C}_{\mathscr{T}}) \to \mathbf{Set}$ be a point, that is, a functor that commutes with all small colimits and all finite limits. The functor $F := y \circ \Psi : G - \mathbf{Set} \to \mathbf{Set}$ has the same property, so it sends the terminal object A_G of $G - \mathbf{Set}$ to a terminal object of $\mathbf{Set}$, i.e. a singleton. For every nonempty G-set X, the unique map $X \to A_G$ identifies A_G to the cokernel of the two projections $X \times X \to X$, so $F(A_G) \to F(X)$ is a kernel morphism, hence injective, and so $F(X)$ is not empty.

We calculate $F(A)$. We have an isomorphism of G-sets $A \times A \xrightarrow{\sim} \coprod_{x \in G} A_x$, where $A_x = A$ for every $x \in G$, sending $(g, h) \in A \times A$ to $g \in A_{g^{-1}h}$. Let $q_1, q_2 : \coprod_{x \in G} A_x \to A$ be the map corresponding to the two projections $p_1, p_2 : A \times A \to A$ by this isomorphism. Then, for every $x \in G$, we have $q_{1|A_x} = \mathrm{id}_A$ and $q_{2|A_x} = c_x$. Applying F and using the fact that F commutes with coproducts and finite products, we get two maps $F(q_1), F(q_2) : \coprod_{x\in G} F(A_x) \to F(A)$, such that $F(q_1)_{|F(A_x)} = \mathrm{id}_{F(A)}$ and $F(q_2)_{|F(A_x)} = F(c_x)$ for every $x \in G$, and such that the induced map $(F(q_1), F(q_2)) : \coprod_{x\in G} F(A_x) \to F(A) \times F(A)$ is bijective. Let $e \in F(A)$ (we know that $F(A) \neq \varnothing$ by the previous paragraph). Then (q_1, q_2) induces a bijection $\coprod_{x\in G}\{e\} \overset{\sim}{\to} \{e\} \times F(A)$, so we get a bijection $\iota : F(A) \overset{\sim}{\to} G = A$, and it is easy to see that $\iota \circ F(c_x) = c_x \circ \iota$ for every $x \in G$.

Now that we have an isomorphism $\iota : F(A) \overset{\sim}{\to} A$, we can construct a morphism of functors α from $x \circ \Psi = \mathrm{Hom}_{G-\mathbf{Set}}(A, \cdot)$ to F by sending $f : A \to X$ to $F(f)(\iota^{-1}(1)) \in F(X)$. We know that $\alpha(A)$ is bijective, so $\alpha(X)$ is bijective if the G-set X is a coproduct of copies of A, because both functors commute with coproducts. As every G-set is the cokernel of two G-equivariant maps between coproducts of copies of A (see the solution of (g)), and as both functors commute with cokernel, $\alpha(X)$ is an isomorphism for every X.

(i). As Ψ is an equivalence of categories, it suffices to calculate the group of automorphisms of $x \circ \Psi$. We can apply the other Yoneda lemma (see for example Corollary I.3.2.8): as $x \circ \Psi$ is a representable functor, every automorphism of this functor comes from an automorphism of the representing object, that is, of A. So, by question (d), every automorphism of $x \circ \Psi$ is of the form $\mathrm{Hom}_{G-\mathbf{Set}}(c_g, \cdot)$. If $g, h \in G$, then $c_{gh} = c_h \circ c_g$, so $\mathrm{Hom}_{G-\mathbf{Set}}(c_{gh}, \cdot) = \mathrm{Hom}_{G-\mathbf{Set}}(c_g, \cdot) \circ \mathrm{Hom}_{G-\mathbf{Set}}(c_h, \cdot)$. So we get an isomorphism $G \overset{\sim}{\to} \mathrm{Aut}(x \circ \Psi)$.

□