MAT 540 : Problem Set 5

Due Thursday, October 17

1 Free presheaves

Let $\mathscr{C}$ be a category and R be a ring.

(a). (2 points) Show that the forgetful functor $\mathrm{PSh}(\mathscr{C}, R) \to \mathrm{PSh}(\mathscr{C})$ has a left adjoint $\mathscr{F} \longmapsto R^{(\mathscr{F})}$.

(b). (1 point) If X is an object of $\mathscr{C}$ and $h_X = \mathrm{Hom}_{\mathscr{C}}(\cdot, X)$ is the corresponding representabe presheaf, we write $R^{(X)}$ for $R^{(h_X)}$. Show that there is an isomorphism of additive functors from $\mathrm{PSh}(\mathscr{C}, R)$ to ${}_R\mathbf{Mod}$ (where $\mathscr{F}$ is the variable):

$$\mathrm{Hom}_{\mathrm{PSh}(\mathscr{C},R)}(R^{(X)}, \mathscr{F}) \simeq \mathscr{F}(X).$$

(c). (2 points) Suppose that $\mathscr{C}$ is equipped with a Grothendiech pretopology. If $\mathscr{F}$ is a sheaf for this pretopology, is $R^{(\mathscr{F})}$ always a sheaf ?

2 Constant presheaves and sheaves

Let $(\mathscr{C}, \mathscr{T})$ be a site. The *constant presheaf* on $\mathscr{C}$ with value S is the functor $\underline{S}_{psh} : \mathscr{C}^{\mathrm{op}} \to \mathbf{Set}$ sending any object to S and any morphism to id_S. The *constant sheaf* on $\mathscr{C}_{\mathscr{T}}$ with value S is the sheafification of the constant presheaf on $\mathscr{C}$ with value S; we will denote it by $\underline{S}$.

(a). (2 points) if $\mathscr{X} = (X_i \to X)_{i \in I}$ is a covering family, calculate $\check{\mathrm{H}}^0(\mathscr{X}, \underline{S}_{psh})$.

(b). (2 points) Suppose that $(\mathscr{C}, \mathscr{T})$ is the category of open subsets of the topological space $[0, 1]$, with the usual topology. Show that $(\underline{S}_{psh})^+$ is a sheaf if and only if $\mathrm{card}(S) \leq 1$.

(c). (2 points) Suppose that $(\mathscr{C}, \mathscr{T})$ is the category of open subsets of a locally connected topological space X. Show that, for every open subset U of X, we have $\underline{S}(U) = S^{\pi_0(U)}$.

3 Points

Let $(\mathscr{C}, \mathscr{T})$ be a site. We are interested in the category $\mathrm{Points}(\mathscr{C}_{\mathscr{T}})$ whose objects are functors $\mathrm{Sh}(\mathscr{C}_{\mathscr{T}}) \to \mathbf{Set}$ that commutes with all small colimits and with finite limits, and whose morphisms are isomorphisms between such functors. [1]

(a). Let $\mathscr{C}$ be an arbitrary category. Let $A : \mathscr{C} \to \mathbf{Set}$ be a functor. We denote by $\underline{\mathrm{Hom}}_{\mathscr{C}}(A, \cdot)$ the functor $\mathbf{Set} \to \mathrm{PSh}(\mathscr{C})$ sending a set S to the presheaf $X \longmapsto \mathrm{Hom}_{\mathbf{Set}}(A(X), S)$.

[1] The idea of this definition is that we are abstracting the formal properties of stalk functors on the category of sheaves on a topological space.

(i) (1 point) Show that the functor $\underline{\mathrm{Hom}}_{\mathscr{C}}(A, \cdot)$ commutes with all limits.

(ii) (3 points) Show that the functor $\underline{\mathrm{Hom}}_{\mathscr{C}}(A, \cdot)$ admits a left adjoint, which we will denote by $(\cdot) \otimes_{\mathscr{C}} A$, and that $((\cdot) \otimes_{\mathscr{C}} A) \circ h_{\mathscr{C}}$ is isomorphic to A. (Hint: First try to construct the adjoint on representable presheaves, and remember problem 2(a) of problem set 2.)

We say that the functor $A : \mathscr{C} \to \mathbf{Set}$ is *flat* if the functor $(\cdot) \otimes_{\mathscr{C}} A : \mathrm{PSh}(\mathscr{C}) \to \mathbf{Set}$ commutes with finite limits.

(iii) (1 points) If A is flat, show that it commutes with all finite limits that exist in $\mathscr{C}$.

(iv) (2 points) Suppose that $\mathscr{C}$ has all finite limits and that A commutes with finite limit. Let $\mathscr{F}$ be a presheaf on $\mathscr{C}$. If X, Y are objects of $\mathscr{C}/\mathscr{F}$, $x \in A(X)$ and $y \in A(Y)$, show that x and y represent the same element of $\mathscr{F} \otimes_{\mathscr{C}} A$ if and only if there exists an object Z of $\mathscr{C}$, morphisms $Z \to X$ and $Z \to Y$, and an element $z \in A(Z)$ whose images in $A(X)$ and $A(Y)$ are x and y respectively.

(v) (3 points) If $\mathscr{C}$ has all finite limits, show that A is flat if and only if it commutes with finite limits. (Hint : To show that a functor commutes with finite limits, it suffices to show that it sends the final object to the final object and commutes with fibered products. You can admit this easy fact.)

(vi) (2 points) Suppose that $\mathscr{C}$ has all finite limits. If $\mathscr{T}$ is the trivial pretopology on $\mathscr{C}$ (so that $\mathrm{Sh}(\mathscr{C}_{\mathscr{T}}) = \mathrm{PSh}(\mathscr{C})$), show that $\mathrm{Points}(\mathscr{C}_{\mathscr{T}})$ is equivalent to the category of flat functors $\mathscr{C} \to \mathbf{Set}$ (with morphisms being isomorphisms between these functors).

(b). Let $(\mathscr{C}, \mathscr{T})$ be a site. A flat functor $A : \mathscr{C} \to \mathbf{Set}$ is called *continuous* if, for every covering family $(X_i \to X)_{i \in I}$ in $\mathscr{C}$, the map $\coprod_{i \in I} A(X_i) \to A(X)$ is surjective.

For every $X \in \mathrm{Ob}(\mathscr{C})$, we denote by X^{sh} the sheafification of the representable presheaf $\mathrm{Hom}_{\mathscr{C}}(\cdot, X)$. This defines a functor $\mathscr{C} \to \mathrm{Sh}(\mathscr{C}_{\mathscr{T}})$, that commutes with finite limits.

(i) (2 points) Let $(f_i : X_i \to X)_{i \in I}$ be a covering family. We consider the morphisms

$$\coprod_{i,j \in I} (X_i \times_X X_j)^{\mathrm{sh}} \overset{f}{\underset{g}{\rightrightarrows}} \coprod_{i \in I} X_i^{\mathrm{sh}} \xrightarrow{h} X^{\mathrm{sh}}$$

where $h = \coprod_{i \in I} f_i^{\mathrm{sh}}$ and f (resp. g) is equal on $(X_i \times_X X_j)^{\mathrm{sh}}$ to the image by $(.)^{\mathrm{sh}} : \mathscr{C} \to \mathrm{Sh}(\mathscr{C}_{\mathscr{T}})$ of the first (resp. second) projection $X_i \times_X X_j \to X_i$ (resp. $X_i \times_X X_j \to X_j$).

Show that h is the cokernel of (f, g) in the category $\mathrm{Sh}(\mathscr{C}_{\mathscr{T}})$.

(ii) (1 point) Let $A : \mathscr{C} \to \mathbf{Set}$ be a flat functor, and suppose that $(\cdot) \otimes_{\mathscr{C}} A : \mathrm{PSh}(\mathscr{C}) \to \mathbf{Set}$ factors as $\mathrm{PSh}(\mathscr{C}) \xrightarrow{(\cdot)^{\mathrm{sh}}} \mathrm{Sh}(\mathscr{C}_{\mathscr{T}}) \xrightarrow{x_A} \mathbf{Set}$. Show that x_A is an object of $\mathrm{Points}(\mathscr{C}_{\mathscr{T}})$.

(iii) (2 points) If $A : \mathscr{C} \to \mathbf{Set}$ satisfies the hypothesis of the previous question, show that A is continuous. [2]

(c). Let $(C, \leq)$ be a preordered set. We see C as a category by taking $\mathrm{Hom}_C(a, b)$ to be a singleton if $a \leq b$, and empty otherwise.

(i) (1 point) Let $(a_i)_{i \in I}$ be a family of objects of C. Give a description of $\coprod_{i \in I} a_i$ and $\prod_{i \in I} a_i$ in (pre)ordered set terms.

[2] In fact, the converse is true : points of $\mathscr{C}_{\mathscr{T}}$ correspond to flat continuous functors $\mathscr{C} \to \mathbf{Set}$.

(ii) (2 points) Give a similar translation of the property "C has all finite limits".

From now on, se suppose that C has all finite limits, and we fix a flat functor $A : C \to \mathbf{Set}$.

(iii) (2 points) Show that $\mathrm{card}(A(a)) \leq 1$ for every $a \in C$.

(iv) (2 points) Show that the set $I_A = \{a \in C \mid A(a) \neq \varnothing\}$ is a nonempty upper order ideal. (That is, if $a \in I_A$ and $a \leq b$, then $b \in I_C$.)

(v) (1 point) If $\mathscr{T}$ is any Grothendieck pretopology on C, show that the points of $C_{\mathscr{T}}$ don't have any nontrivial automorphisms.

(vi) (1 point) Suppose that any family $(a_i)_{i \in I}$ of elements of C has a least upper bound $\sup(a_i, i \in I)$. We say that a family of morphisms $(a_i \to a)_{i \in I}$ in $\mathscr{C}$ is covering if $a = \sup(a_i, i \in I)$. Suppose that this defines a pretopology on C. If A is continuous, show that I_A is a completely prime upper order ideal, that is, if $\sup(a_i, i \in I) \in I_A$, then at least one of the a_i is in I_A.

(d). Let X be a topological space, let $\mathscr{C} = \mathrm{Open}(X)$, and let $\mathscr{T}$ be the usual topology on X. Remember that a nonempty closed subset Z of X is called *irreducible* if, whenever $Z \subset Y_1 \cup Y_2$ with Y_1, Y_2 closed subsets of X, we have $Z \subset Y_1$ or $Z \subset Y_2$.

(i) (1 point) Show that a nonempty closed subset Z of X is irreducible if, for every open subset U of Z, the set $Z \cap U$ is either empty or dense in Z.

(ii) (1 point) Let Z be an irreducible closed subset of X, and let $\mathscr{U}_Z$ be the set of open subsets U of X such that $Z \cap U \neq \varnothing$. For every sheaf $\mathscr{F}$ on X, we set

$$\mathscr{F}_Z = \varinjlim_{U \in \mathrm{Ob}(\mathscr{U}_Z^{\mathrm{op}})} \mathscr{F}(U).$$

Show that this defines a point of $\mathscr{C}_{\mathscr{T}}$.

(iii) (3 points) Let $x : \mathrm{Sh}(\mathscr{C}_{\mathscr{T}}) \to \mathbf{Set}$ be a point, and let $A : \mathscr{C} \to \mathbf{Set}$ be the corresponding flat continuous functor. Let

$$Z = X - \bigcup_{U \in \mathrm{Ob}(\mathscr{C}),\ A(U) = \varnothing} U.$$

Show that Z is an irreducible closed subset of X, and that x is isomorphic to the functor $\mathscr{F} \longmapsto \mathscr{F}_Z$.

(iv) (1 point) If x_1 and x_2 are points of $\mathscr{C}_{\mathscr{T}}$ and Z_1 and Z_2 are the corresponding closed irreducible subsets of X, show that there exists a morphism from x_1 to x_2 if and only if $Z_1 \subset Z_2$.

(e). Let $X = [0, 1]$ with the Lebesgue measure. We take $\mathscr{C}$ to be the category whose objects are Lebesgue-measurable subsets E of $[0, 1]$, and such that $\mathrm{Hom}_{\mathscr{C}}(E, E')$ is a singleton if $E' - E$ has measure 0, and the empty set otherwise. We put the Grothendieck pretopology on $\mathscr{C}$ whose covering families are countable families $(E_n \to E)_{n \in \mathbb{N}}$ such that $E - \bigcup_{n \in \mathbb{N}} E_n$ has measure 0. (You can admit that this is a pretopology; it is not very hard.)

(i) (1 point) Show that the category $\mathrm{Sh}(\mathscr{C}_{\mathscr{T}})$ is not empty.

(ii) (2 points) Show that the category $\mathrm{Points}(\mathscr{C}_{\mathscr{T}})$ has no objects (that is, $\mathscr{C}_{\mathscr{T}}$ has no points).

4 G-sets

Let G be a finite group, let $\mathscr{C} = G - \mathbf{Set}$ be the category whose objects are sets with a left action of G and whose morphisms are G-equivariant maps. We consider the pretopology $\mathscr{T}$ on $\mathscr{C}$ for which a family $(f_i : X_i \to X)_{i \in I}$ is covering if and only if $X = \bigcup_{i \in I} f_i(X_i)$. [3]

Let A be G with its actionby left translations. More generally, for every subgroup H of G, we denote by A_H the set G/H with the action of G by left translations.

Useful fact: If $X \to Y$ is a surjective map in **Set** or $G - \mathbf{Set}$, then it is the cokernel of the two projections $X \times_Y X \to X$. (You still need to justify thisn if you want to use it.)

(a). (1 point) Show that every object of $G - \mathbf{Set}$ is a coproduct of objects isomorphic to some A_H.

(b). (1 point) Calculate $A \times_{A_H} A$ in the category $G - \mathbf{Set}$.

(c). (1 point) Show that every representable presheaf on $G - \mathbf{Set}$ is a sheaf.

(d). (1 point) Show the automorphisms of A in $G - \mathbf{Set}$ are exactly the maps $c_g : A \to A$, $a \longmapsto ag$, for $g \in G$.

(e). (1 point) If $\mathscr{F}$ is a presheaf on $G - \mathbf{Set}$, show the family $(\mathscr{F}(c_g))_{g \in G}$ defines a left action of G on $\mathscr{F}(A)$.

(f). (1 point) Consider the functor $\Phi : \mathrm{Sh}(\mathscr{C}_{\mathscr{T}}) \to G - \mathbf{Set}$ defined by $\Phi(\mathscr{F}) = \mathscr{F}(A)$ and the functor $\Psi : F - \mathbf{Set} \to \mathrm{Sh}(\mathscr{C}_{\mathscr{T}})$ fiven by $\Psi(X) = \mathrm{Hom}_{G-\mathbf{Set}}(\cdot, X)$. Show that $\Phi \circ \Psi \simeq \mathrm{id}_{G-\mathbf{Set}}$.

(g). (4 points) Show that $\Psi \circ \Phi \simeq \mathrm{id}_{\mathrm{Sh}(\mathscr{C}_{\mathscr{T}})}$. (Hint: For any G-set X, if $|X|$ is the set X with the trivial G-action, then we have a surjective G-equivariant map $A \times |X| \to X$, $(g, x) \longmapsto g \cdot x$, which induces an injection $\mathscr{F}(X) \to \prod_{x \in X} \mathscr{F}(A) = \mathrm{Hom}_{\mathbf{Set}}(X, \mathscr{F}(A))$.)

(h). (3 points) Let $x : \mathrm{Sh}(\mathscr{C}_{\mathscr{T}}) \to \mathbf{Set}$ be the functor $\mathscr{F} \longmapsto \mathscr{F}(A)$, where we forget the action of G on $\mathscr{F}(A)$ to see $\mathscr{F}(A)$ as a set. Show that every point of $\mathscr{C}_{\mathscr{T}}$ is isomorphic to x. (See the beginning of Problem 3 for the definition of points.) (Suggestion: if y is a point, calculate $y(\Psi(\{1\}))$, then $y(\Psi(A))$, then construct a morphism of functors $\mathrm{Hom}_{G-\mathbf{Set}}(A, \cdot) \to y \circ \Psi$, then show that it is an isomorphism.)

(i). (1 point) Show that the group of automorphisms of the point x is isomorphic to G.

[3] It is very easy to check that this is a pretopology, you don't need to do it.