MAT 540 : Problem Set 3

Due Friday, October 4

1. Free preadditive and additive categories. (extra credit)

Remember that **Cat** is the category of category (the objects of **Cat** are categories, and the morphisms of **Cat** are functors). Let **PreAdd** be the category whose objects are preadditive categories and whose morphisms are additive functors; let **Add** be the full subcategory of **PreAdd** whose objects are additive categories. We have a (faithful) forgetful functor For : **PreAdd** $\to$ **Cat**; we also denote the inclusion functor from **Add** to **PreAdd** by F.

(a). (2 points) Show that For has a left adjoint, that we will denote by $\mathscr{C} \longmapsto \mathbb{Z}[\mathscr{C}]$.

(b). (4 points) Show that F has a left adjoint, that we will denote by $\mathscr{C} \longmapsto \mathscr{C}^{\oplus}$. (Hint : If $\mathscr{C}$ is preadditive, consider the category $\mathscr{C}^{\oplus}$ whose objects are 0 and finite sequences $(X_1, \dots, X_n)$ of objects of $\mathscr{C}$, where a morphism from $(X_1, \dots, X_n)$ to $(Y_1, \dots, Y_m)$ is a $m \times n$ matrix of morphisms $X_i \to Y_j$, and where the only from 0 to any object and from any object to 0 is 0.)

2. Pseudo-abelian completion. Let $\mathscr{C}$ be an additive category. If X is an object of $\mathscr{C}$, an endomorphism $p \in \mathrm{End}_{\mathscr{C}}(X)$ is called a *projector* or *idempotent* if $p \circ p = p$. A *pseudo-abelian* (or *Karoubian*) category is a preadditive category in which every projector has a kernel.

(a). (3 points) Let $\mathscr{C}$ be a category and $p \in \mathrm{End}_{\mathscr{C}}(X)$ be a projector. Show that :

- $\mathrm{Ker}(p, \mathrm{id}_X)$ exists if and only if $\mathrm{Coker}(p, \mathrm{id}_X)$ exists;
- if $u : Y \to X$ is a kernel of (p, id_X) and $v : X \to Z$ is a cokernel of (p, id_X), then there exists a unique morphism $f : Z \to Y$ such that $u \circ f \circ v = p$, and this morphism f is an isomorphism.

(b). (3 points) If $\mathscr{C}$ is a pseudo-abelian category, show that every projector has a kernel, a cokernel, a coimage and an image and that, if $p \in \mathrm{End}_{\mathscr{C}}(X)$ is a projector, then the canonical morphisms $\mathrm{Ker}(p) \to X$ and $\mathrm{Im}(p) \to X$ make X into a coproduct of $(\mathrm{Ker}(p), \mathrm{Im}(p))$. (In other words, the coproduct $\mathrm{Ker}(p) \oplus \mathrm{Im}(p)$ exists, and it is canonically isomorphic to X.)

(c). (3 points) Let $\mathscr{C}$ be a category. Its *pseudo-abelian completion* (or *Karoubi envelope*) is the category $\mathrm{kar}(\mathscr{C})$ defined by :

- $\mathrm{Ob}(\mathrm{kar}(\mathscr{C})) = \{(X, p) \mid X \in \mathrm{Ob}(\mathscr{C}),\ p \in \mathrm{End}_{\mathscr{C}}(X) \text{ is a projector}\}$;
- $\mathrm{Hom}_{\mathrm{kar}(\mathscr{C})}((X, p), (Y, q)) = \{f \in \mathrm{Hom}_{\mathscr{C}}(X, Y) \mid q \circ f = f \circ p = f\}$;
- the composition is given by that of $\mathscr{C}$, and the identity morphism of (X, p) is p.

Show that $\mathrm{kar}(\mathscr{C})$ is a pseudo-abelian category, and that the functor $\mathscr{C} \to \mathrm{kar}(\mathscr{C})$ sending X to (X, id_X) is additive and fully faithful.

(d). (2 points) If $\mathscr{C}$ is an additive category, show that $\mathrm{kar}(\mathscr{C})$ is also additive.

(e). (3 points) Let **PseuAb** be the full subcategory of **PreAdd** (see problem 1) whose objects are pseudo-abelian categories. Show that the inclusion functor $\mathbf{PseuAb} \to \mathbf{PreAdd}$ has a left adjoint.

3. Torsionfree abelian groups (extra credit)

Let $\mathbf{Ab}_{\mathrm{tf}}$ be the full subcategory of $\mathbf{Ab}$ whose objects are torsionfree abelian groups.

(a). (2 points) Give formulas for kernels, cokernels, images and coimages in $\mathbf{Ab}_{\mathrm{tf}}$.

(b). (2 points) Show that the inclusion functor $\iota : \mathbf{Ab}_{\mathrm{tf}} \to \mathbf{Ab}$ admits a left adjoint κ, and give this left adjoint.

4. Filtered R-modules Let R be a ring, and let $\mathrm{Fil}({}_R\mathbf{Mod})$ be the category of filtered R-modules $(M, \mathrm{Fil}_* M)$ (see Example II.1.4.3 of the notes) such that $M = \bigcup_{n\in\mathbb{Z}} \mathrm{Fil}_n M$. [1]

(a). (2 points) Give formulas for kernels, cokernels, images and coimages in $\mathrm{Fil}({}_R\mathbf{Mod})$.

(b). (2 points) Let $\iota : \mathrm{Fil}({}_R\mathbf{Mod}) \to \mathrm{Func}(\mathbb{Z}, {}_R\mathbf{Mod})$ be the functor sending a filtered R-module $(M, \mathrm{Fil}_* M)$ to the functor $\mathbb{Z} \to {}_R\mathbf{Mod}$, $n \longmapsto \mathrm{Fil}_n M$. Show that ι is fully faithful.

(c). (3 points) Show that ι has a left adjoint κ, and give a formula for κ.

(d). (2 points) Show that every object of the abelian category $\mathrm{Func}(\mathbb{Z}, {}_R\mathbf{Mod})$ is isomorphic to the cokernel of a morphism between objects in the essential image of ι.

5. Admissible "topology" on $\mathbb{Q}$ If you have not seen sheaves (on a topological space) in a while, you might want to go read about them a bit, otherwise (b) will be very hard, and (f) won't be as shocking as it should be. Also, if the construction of sheafification that you learned used stalks, you should go and read a construction that uses open covers instead; see for example Section III.1 of the notes.

If $a, b \in \mathbb{R}$, we write

$$[a, b] = \{x \in \mathbb{R} \mid a \leq x \leq b\}$$

and

$$]a, b[= \{x \in \mathbb{R} \mid a < x < b\}.$$

Consider the space $\mathbb{Q}$ with its usual topology. An *open rational interval* is an open subset of $\mathbb{Q}$ of the form $\mathbb{Q} \cap]a, b[$ with $a, b \in \mathbb{Q}$. An *closed rational interval* is a closed subset of $\mathbb{Q}$ of the form $\mathbb{Q} \cap [a, b]$, with $a, b \in \mathbb{Q}$.

We say that an open subset U of $\mathbb{Q}$ is *admissible* if we can write U as a union $\bigcup_{i\in I} A_i$ of open rational intervals such that, for every closed rational interval $B = \mathbb{Q} \cap [a, b] \subset U$, there exists a finite subset J of I and closed rational intervals $B_j \subset A_j$, for $j \in J$, such that $B \subset \bigcup_{j\in J} B_j$.

If U is an admissible open subset of $\mathbb{Q}$ and $U = \bigcup_{i\in I} U_i$ is an open cover of U, we say that this cover is *admissible* if, for every closed rational interval $B = \mathbb{Q} \cap [a, b] \subset U$, there exist a finite subset J of I and closed rational intervals $B_j \subset U_j$, for $j \in J$, such that $B \subset \bigcup_{j\in J} B_j$.

Let Open_a be the poset of admissible open subsets of $\mathbb{Q}$ (ordered by inclusion), and let $\mathrm{PSh}_a = \mathrm{Func}(\mathrm{Open}_a, \mathbf{Ab})$. This is called the category of presheaves of abelian groups on the

[1] We say that the filtration is *exhaustive*.

admissible topology of $\mathbb{Q}$. If $F : \mathrm{Open}_a^{\mathrm{op}} \to \mathbf{Set}$ is a presheaf and $U \subset V$ are admissible open subsets of $\mathbb{Q}$, we denote the map $F(V) \to F(U)$ by $s \longmapsto s_{|U}$.

We say that a presheaf $F : \mathrm{Open}_a^{\mathrm{op}} \to \mathbf{Ab}$ is a *sheaf* if, for every admissible open subset U of $\mathbb{Q}$ and for every admissible cover $(U_i)_{i\in I}$ of U, the following two conditions hold :

(1) the map $F(U) \to \prod_{i\in I} F(U_i)$, $s \longmapsto (s_{|U_i})$ is injective;

(2) if $(s_i) \in \prod_{i\in I} F(U_i)$ is such that $s_{i|U_i\cap U_j} = s_{j|U_i\cap U_j}$ for all $i, j \in I$, then there exists $s \in F(U)$ such that $s_i = s_{|U_i}$ for every $i \in I$.

The full subcategory Sh_a of PSh_a whose objects are sheaves is called the category of sheaves of abelian groups on the admissible topology of $\mathbb{Q}$. [2]

(a). Let U be an open subset of $\mathbb{Q}$, and let $V(U)$ be the union of all the open subsets V of $\mathbb{R}$ such that $V \cap \mathbb{Q} = U$. Show that $V(U)$ is the union of all the intervals $[a, b]$, for $a, b \in \mathbb{Q}$ such that $\mathbb{Q} \cap [a, b] \subset U$.

(b). (2 points) Show that every open set in $\mathbb{Q}$ is admissible. [3]

(c). (1 point) Give an open cover of an open subset of $\mathbb{Q}$ that is not an admissible open cover.

(d). (3 points) Show that the inclusion functor $\mathrm{Sh}_a \to \mathrm{PSh}_a$ has a left adjoint $F \longmapsto F^{sh}$. (The sheafification functor.)

(e). (4 points) Show that Sh_a is an abelian category.

(f). (3 points) Show that the inclusion $\mathrm{Sh}_a \to \mathrm{PSh}_a$ is left exact but not exact, and that the sheafification functor $\mathrm{PSh}_a \to \mathrm{Sh}_a$ is exact.

For every $x \in \mathbb{Q}$ and every presheaf $F \in \mathrm{Ob}(\mathrm{PSh}_a)$, we define the *stalk* of F at x to be $F_x = \varinjlim_{U \ni x} F(U)$, that is, the colimit of the functor $\phi : \mathrm{Open}_a(\mathbb{Q}, x)^{\mathrm{op}} \to \mathbf{Ab}$, where $\mathrm{Open}_a(\mathbb{Q}, x)$ is the full subcategory of Open_a of admissible open subsets containing x and ϕ is the restriction of F.

(g). (2 points) For every $x \in \mathbb{Q}$, show that the functor $\mathrm{Sh}_a \to \mathbf{Ab}$, $F \longmapsto F_x$ is exact.

(h). (4 points) Let PSh (resp. Sh) be the usual category of presheaves (resp. sheaves) of abelian groups on $\mathbb{R}$. Show that the functor $\mathrm{Sh} \to \mathrm{PSh}_a$ sending a sheaf F on $\mathbb{R}$ to the presheaf $U \longmapsto F(V(U))$ on $\mathbb{Q}$ is fully faithful, that its essential image is Sh_a, and that it is exact as a functor from Sh_a to Sh.

(i). (2 points) Find a nonzero object F of Sh_a such that $F_x = 0$ for every $x \in \mathbb{Q}$.

6. Canonical topology on an abelian category Let $\mathscr{A}$ be an abelian category. Let $\mathrm{PSh} = \mathrm{Func}(\mathscr{A}^{\mathrm{op}}, \mathbf{Ab})$ be the category of presheaves of abelian groups on $\mathscr{A}$. We say that a presheaf $F : \mathscr{A}^{\mathrm{op}} \to \mathbf{Ab}$ is a *sheaf* (in the canonical topology) if, for every epimorphism $f : X \to Y$ in $\mathscr{A}$, the following sequence of abelian groups is exact :

$$0 \longrightarrow F(Y) \xrightarrow{F(g)} F(X) \xrightarrow{F(p_1)-F(p_2)} F(X \times_Y X)\ ,$$

where $p_1, p_2 : X \times_Y X \to X$ are the two projections. We denote by Sh the full subcategory of PSh whose objects are the sheaves.[4]

[2] Of course, we could also define presheaves and sheaves with values in **Set**.

[3] There is a similar notion of admissible open subset in $\mathbb{Q}^n$, where intervals are replaced by products of intervals, and this result does not hold for $n \geq 2$.

[4] Again, we could define sheaves of sets.

(a). (2 points) If $f : X \to Y$ is an epimorphism in $\mathscr{A}$, show that it is the cokernel of the morphism $p_1 - p_2 : X \times_Y X \to X$, where $p_1, p_2 : X \times_Y X \to X$ are the two projections as before.

(b). (2 points) Let $f : X \to Y$ be an epimorphism in $\mathscr{A}$ and $g : Z \to Y$ be a morphism. Consider the second projection $p_Z : X \times_Y Z \to Z$. Show that p_Z is an epimorphism.

(c). (1 point) Show that every representable presheaf on $\mathscr{A}$ is a sheaf.

(d). (3 points) Show that the inclusion functor $\mathrm{Sh} \to \mathrm{PSh}$ has a left adjoint $F \longmapsto F^{sh}$. (The sheafification functor.)

(e). (4 points) Show that Sh is an abelian category.

(f). (3 points) Show that the inclusion $\mathrm{Sh} \to \mathrm{PSh}$ is left exact but not exact, and that the sheafification functor $\mathrm{PSh} \to \mathrm{Sh}$ is exact.