More examples of Haar measures

1. Let $G$ be a locally compact group, and let $H$ be a closed subgroup of $G$. We write $\pi$ for the quotient map from $G$ to $G/H$. We denote by $\Delta_G$ (resp. $\Delta_H$) the modular function of $G$ (resp. $H$), and we assume that $\Delta_{G|H} = \Delta_H$. We fix left Haar measures $\mu_G$ and $\mu_H$ on $G$ and $H$.

   a) (1) Show that, for every compact subset $K'$ of $G/H$, there exists a compact subset $K$ of $G$ such $\pi(K) = K'$.

   b) (1) Let $f \in L^1(G)$. Show that the function $G \to \mathbb{C}$, $x \mapsto \int_H f(xh) d\mu_H(h)$ is invariant by right translations by elements of $H$. Hence it defines a function $G/H \to \mathbb{C}$, that we will denote by $f^H$.

   c) (2) If $f \in C_c(G)$, show that $f^H \in C_c(G/H)$.

   d) (2) Show that the map $C_c(G) \to C_c(G/H)$, $f \mapsto f^H$ is surjective. (Hint: You may use the fact that, for every compact subset $K$ of $G$, there exists a function $\varphi \in C^+_c(G)$ such that $\varphi(x) > 0$ for every $x \in K$.)

   e) (2) If $f \in C_c(G)$ is such that $f^H = 0$, show that $\int_G f(x) d\mu_G(x) = 0$. (Hint: use a function in $C_c(G/H)$ that is equal to 1 on $\pi(\text{supp}(f))$, and proposition 2.12 of the notes. (Sorry.))

   f) (2) Show that there exists a unique regular Borel measure $\mu_{G/H}$ on $G/H$ that is invariant by left translations by elements of $G$ and such that, for every $f \in C_c(G)$, we have $\int_G f(x) d\mu_G(x) = \int_{G/H} f^H(y) d\mu_{G/H}(y)$.

   g) (1) If $P$ is a closed subgroup of $G$ such that $\pi$ induces a homeomorphism $P \sim G/H$, show that the inverse image of $\mu_{G/H}$ by this homeomorphism is a left Haar measure on $P$.

   h) (2) If $P$ is a closed subgroup of $G$ such that the map $P \times H \to G$, $(p, h) \mapsto ph$ is a homeomorphism, and if $d\mu_P$ is a left Haar measure on $P$, show that the linear functional $C_c(G) \to \mathbb{C}$, $f \mapsto \int_P \int_H f(ph) d\mu_P(p) d\mu_H(h)$ defines a left Haar measure on $G$.

Solution.

a) Let $V$ be a compact neighborhood of 1 in $G/H$. Then $\pi(V)$ is a compact neighborhood of $\pi(1)$ in $G/H$. We have $K' \subset \bigcup_{x \in \pi^{-1}(K')} \pi(xV)$. As $K'$ is compact, we can find $x_1, \ldots, x_n$ such that $K' \subset \bigcup_{i=1}^n \pi(x_iV)$. Let $K = \pi^{-1}(K') \cap (\bigcup_{i=1}^n x_iV)$. Then $K$ is a closed subset of the compact set $\bigcup_{i=1}^n x_iV$, hence it is compact, and we have $\pi(K) = K'$. 
b) Let $x \in H$. Then, for every $g \in G$, we have
\[
\int_H f(gxh) d\mu_H(h) = \int_H f(gh) d\mu_H(h)
\]
by the left invariance of $\mu_H$.

c) We need to show that $f^H$ is continuous and that it has compact support.

Fix a symmetric compact neighborhood $V_0$ of 1, and note that $A := \text{supp } f \cup V_0(\text{supp } f)$ is compact. Let $\varepsilon > 0$. As $f$ is left uniformly continuous, there exists a neighborhood $V \subset V_0$ of 1 such that, for every $x \in G$ and every $y \in V$, we have $|f(yx) - f(x)| \leq \varepsilon$. Then, for every $x \in G$ and every $y \in V$, we have
\[
|f^H(\pi(yx)) - f^H(x)| = \left| \int_H (f(yxh) - f(gh)) d\mu_H(h) \right| \leq \varepsilon \mu_H(x^{-1}A \cap H),
\]
because $f(yxh) = f(xh) = 0$ unless $y \in (x^{-1} \text{supp } f) \cup (x^{-1}y^{-1} \text{supp } f) \subset x^{-1}A$. As $x^{-1}A \cap H$ is compact, it has finite measure, and the calculation above implies that $f^H$ is continuous at the point $\pi(x)$.

Now we show that $f^H$ has compact support. By definition of $f^H$, we have $f^H(\pi(x)) = 0$ if $x \notin KH$. So the support of $f^H$ is contained in $\pi(KH) = \pi(K)$, hence it is compact.

d) Let $g \in C_c(G/H)$, and let $K'$ be its support. By question (a), there exists a compact subset $K$ of $G$ such that $\pi(K) = K'$. Let $\phi \in C^+_c(G)$ be such that $\phi(x) > 0$ for every $x \in K$. We show that $\phi^H(y) > 0$ for every $y \in K'$. Let $y \in K'$, write $y = \pi(x)$ with $x \in K$. As $\phi(x) > 0$ and $\phi$ is continuous, we can find an open neighborhood $V$ of 1 in $G$ and a $c \in \mathbb{R}_{>0}$ such that $\phi(x') \geq c$ for every $x' \in xV$. In particular,
\[
\phi^H(y) = \int_H \phi(xh) d\mu_H(h) \geq \int_{H \cap V} \phi(xh) d\mu_H(h) \geq c \cdot \mu_H(U \cap H) > 0
\]
(as $U \cap H$ is a nonempty open subset of $H$, we have $\mu_H(U \cap H) > 0$).

We define a function $F : G \to \mathbb{C}$ in the following way:
\[
F(x) = \begin{cases} 
\frac{g(\pi(x))}{\phi^H(\pi(x))} & \text{if } \phi^H(\pi(x)) > 0 \\
0 & \text{otherwise.}
\end{cases}
\]
Note that $F$ is continuous on the open subsets $U_1 = \{ x \in G | \phi^H(\pi(x)) > 0 \}$ and $U_2 = G - \text{supp}(g \circ \pi)$ (on the second subset, it is identically zero). As $U_1 \supset \pi^{-1}(K')$ and $\pi^{-1}(K') = \text{supp}(g \circ \pi)$, we have $U_1 \cup U_2 = G$, the function $F$ is continuous on $G$.

Finally, we take $f = F \phi$. Then $f \in C_c(G)$, and we just need to show that $f^H = g$.

Let $x \in G$. If $\phi^H(\pi(x)) = 0$, then $f(xh) = 0$ for every $h \in H$, so $f^H(\pi(x)) = 0$. We have seen that $\phi^H$ takes positive values on $K' = \text{supp}(g)$, so we also have $x \notin \text{supp}(g)$, i.e., $g(x) = 0 = f^H(x)$. Now assume that $\phi^H(\pi(x)) > 0$. Note that the function $H \to \mathbb{C}$, $h \mapsto F(xh)$ is constant. So
\[
f^H(\pi(x)) = F(x) \int_H \phi(xh) d\mu_H(h) = \frac{g(\pi(x))}{\phi^H(\pi(x))} \phi^H(\pi(x)) = g(\pi(x)).
\]
Finally, note that $f \in C^+_c(G)$ if $g \in C^+_c(G/H)$, and that we also proved along the way that $f^H \in C^+_c(G/H)$ if $f \in C^+_c(G)$ (we proved this for $\phi$).
e) Let \( \psi \in \mathcal{C}_c(G/H) \) be such that \( \psi(y) = 1 \) for every \( y \in \pi(\text{supp } f) \). By question (d), there exists \( \varphi \in \mathcal{C}_c(G) \) such that \( \varphi^H = \psi \). We have

\[
\int_G f(x)d\mu_G(x) = \int_G f(x)\varphi^H(\pi(x))d\mu_G(x)
\]

\[
= \int_{G \times H} f(x)\varphi(xh)d\mu_G(x)d\mu_H(h)
\]

\[
= \int_H (\int_G f(x)\varphi(xh)d\mu_G(x))d\mu_H(h)
\]

\[
= \int_H (\Delta_G(h)^{-1}\int_G f(xh^{-1})\varphi(x)d\mu_G(x))d\mu_H(h)
\]

\[
= \int_G \varphi(x)(\int_H \Delta_H(h)^{-1}f(xh^{-1})d\mu_H(h))d\mu_G(x)
\]

\[
= \int_G \varphi(x)(\int_H f(xh)d\mu_H(h))d\mu_G(x) \quad \text{(by proposition 2.12 of the notes)}
\]

\[
= 0 \quad \text{(because } f^H = 0) .
\]

f) By question (e), the positive linear function \( \mathcal{C}_c(G) \to \mathbb{C}, f \mapsto \int_G f d\mu_G \) factors through the linear map \( \mathcal{C}_c(G) \to \mathcal{C}_c(G/H), f \mapsto f^H \). By question (d) (and the remark at the end of its solution), it defines a positive linear functional \( \mathcal{C}_c(G/H) \to \mathbb{C} \). By the Riesz representation theorem, this comes from a regular Borel measure \( \mu_{G/H} \) on \( G/H \). Unravelling the definition, we get, for every \( f \in \mathcal{C}_c(G) \),

\[
\int_G f d\mu_G = \int_{G/H} f^H d\mu_{G/H}.
\]

By the left invariance of \( \mu_G \) and question (d), we have, if \( f \in \mathcal{C}_c(G/H) \) and \( x \in G \),

\[
\int_{G/H} f(xy)d\mu_{G/H}(y) = \int_{G/H} f(y)d\mu_{G/H}.
\]

Using the uniqueness part of the Riesz representation theorem (as we did in class), we see that \( \mu_{G/H}(xE) = \mu_{G/H}(E) \) for every Borel subset \( E \) of \( G/H \).

g) Let \( \nu \) be the inverse image of \( \mu_{G/H} \) by the homeomorphism \( \alpha : P \to G/H \). It is a regular Borel measure because \( \alpha \) is a homeomorphism. Also, note that \( \alpha(xy) = x\alpha(y) \) for every \( x \in P \) (this is obvious on the definition of \( \alpha \)). As \( \mu_{G/H} \) is invariant by left translations by elements of \( P \), so is \( \nu \).

h) The hypothesis implies that \( \pi \) induces a homeomorphism \( P \to G/H \), hence we get a left Haar measure \( \nu \) on \( P \) as in question (g). By the uniqueness of left Haar measures, we have \( \mu_P = c\nu \) for some \( c \in \mathbb{R}_{>0} \). Hence, for every \( f \in \mathcal{C}_c(G) \),

\[
\int_H \int_P f(ph)d\mu_P(p)d\mu_H(h) = c \int_P (\int_H f(ph)d\mu_H(h))d\nu(p) =
\]

\[
c \int_{G/H} f^H(y)d\mu_{G/H}(y) = c \int_G f(x)d\mu_G(x).
\]

So the functional \( f \mapsto \int_H \int_P f(ph)d\mu_P(p)d\mu_H(h) \) is positive and corresponds to the left Haar measure \( c\mu_G \) on \( G \).
2. Let $G$ be a locally compact group. Let $A$ and $N$ be two closed subgroups of $G$ such that $A \times N \to G$, $(a,n) \mapsto an$ is a homeomorphism and that $A$ normalizes $N$ (i.e. for every $a \in A$ and $n \in N$, we have $ana^{-1} \in N$).

a) (2) If $\mu_A$ and $\mu_N$ are left Haar measures on $A$ and $N$, show that the linear functional $C_c(G) \to \mathbb{C}$, $f \mapsto \int_A \int_N f(an)d\mu_A(a)d\mu_N(n)$ defines a left Haar measure on $G$.

b) (1) Let $a \in A$. Show that there exists $\alpha(a) \in \mathbb{R}_{>0}$ such that, for every $f \in C_c(N)$, we have

$$\int_N f(ana^{-1})d\mu_N(n) = \alpha(a) \int_N f(n)d\mu_N(n).$$

c) (1) If $\Delta_G$, $\Delta_A$ and $\Delta_N$ are the modular functions of $G$, $A$ and $N$ respectively, show that $\Delta_G(an) = \alpha(a)\Delta_A(a)\Delta_N(n)$ if $a \in A$ and $n \in N$.

**Solution.**

a) The setup is very similar to that of problem 1 (with for example $N$ playing the role of $H$), with the difference that we don’t make any assumption on the modular functions. Still, the results questions (a)-(d) of problem 1 still stay true, since their proof doesn’t use the assumption on the modular functions. In particular, we get a surjective linear transformation $f \mapsto f^N$ from $C_c(G)$ to $C_c(G/N) \cong C_c(A)$, and it sends $C^*_c(G)$ onto $C^*_c(A)$. The linear functions of the statement sends $f \in C_c(G)$ to $\int_A f^N(a)d\mu_A(a)$, so it is positive, and the Riesz representation theorem says that there is a unique regular Borel measure $\mu_G$ on $G$ such that, for every $f \in C_c(G)$, we have

$$\int_G f d\mu_G = \int_A \int_N f(an)d\mu_A(a)d\mu_N(n).$$

As $\mu_A$ is a left Haar measure on $A$, the formula above implies that $\int_G L_a f d\mu_G = \int_G f d\mu_G$ for every $f \in C_c(G)$ and every $a \in A$. We show that $\mu_G$ is left invariant by $N$. Let $x \in N$ and $f \in C_c(G)$. Then we have

$$\int_G L_x f d\mu_G = \int_A \int_N f(xan)d\mu_A(a)d\mu_N(n) = \int_A \left( \int_N f(a(a^{-1}xa)n)d\mu_N(n) \right)d\mu_A(a)$$

$$= \int_A \int_N f(an)d\mu_N(n))d\mu_A(a) \quad \text{because } a^{-1}xa \in N \text{ and } \mu_N \text{ is left invariant}$$

$$= \int_G f d\mu_G.$$

As $G = AN$, this implies that $\int_G L_x g d\mu_G = \int_G f d\mu_G$ for every $x \in G$ and every $f \in C_c(G)$. By proposition 2.6 of the notes, $\mu_G$ is a left Haar measure on $G$.

b) Note that the map $N \to N$, $n \mapsto a^{-1}na$ is a homeomorphism. Hence the formula $E \mapsto \mu_N(a^{-1}Ea)$ defines a regular Borel measure on $N$, which we denote by $\nu$. If $E$ is a Borel subset and $n \in N$, then

$$\nu(nE) = \mu(a^{-1}nEa) = \mu((a^{-1}na)a^{-1}Ea) = \mu(a^{-1}Ea) = \nu(E).$$

Hence $\nu$ is a left Haar measure on $N$, and so there exists $\alpha(a) \in \mathbb{R}_{>0}$ such that $\nu = \alpha(a)\mu_N$. Now, if $E$ is Borel subset of $N$ and $f = \mathbf{1}_E$, the function $n \mapsto f(ana^{-1})$ is the characteristic function of $a^{-1}Ea$, so

$$\int_N f(ana^{-1})d\mu_N(n) = \mu(a^{-1}Ea) = \alpha(a)\mu(E) = \alpha(a) \int_N f d\mu_N.$$

This extends in the usual way to all the functions $f \in L^1(N)$, and in particular to $f \in C_c(N)$. 

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c) Let \( a \in A \) and \( n \in N \), and fix \( f \in C^+_c(G) \). Then we have
\[
\Delta_G(an)^{-1} \int_G f \, d\mu_G = \int_G R_{an}(f) \, d\mu_G = \int_A \int_N f(bman) \, d\mu_A(b) \, d\mu_N(m) \\
= \int_A (\int_N f(b(a^{-1}ma)n) \, d\mu_N(m)) \, d\mu_A(b) \\
= \alpha(a)^{-1} \int_A (\int_N f(bamn) \, d\mu_A(b)) \, d\mu_N(m) \\
= \alpha(a)^{-1} \Delta_N(n)^{-1} \int_N (\int_A f(bamN(b)) \, d\mu_A(b)) \, d\mu_N(m) \\
= \alpha(a)^{-1} \Delta_N(n)^{-1} \Delta_A(a)^{-1} \int_N (\int_A f(bm) \, d\mu_A(b)) \, d\mu_N(m) \\
= \alpha(a)^{-1} \Delta_N(n)^{-1} \Delta_A(a)^{-1} \int_G f \, d\mu_G.
\]
As \( \int_G f \, d\mu_G > 0 \), this implies that \( \Delta_G(an) = \alpha(a) \Delta_A(a) \Delta_N(n) \) .

\[\square\]

3. Let \( G = \text{SL}_n(\mathbb{R}) \), \( H = \text{SO}(n) \), and let \( P \subset G \) be the subgroup of upper triangular matrices with positive entries on the diagonal (and determinant 1).

a) (4) Show that the map \( P \times H \to G \), \( (p, h) \mapsto ph \) is a homeomorphism. (Hint: Gram-Schmidt.)

b) (3) Give a formula for a left Haar measure on \( P \) similar to the formula in problem 6(d) of problem set 1.

c) (4) Calculate the modular function of \( P \).

d) (2) Show that \( G \) is unimodular. (There are several ways to do this.)

e) (2) If \( n = 2 \), show that \( \text{SO}(n) \simeq S^1 \) (the circle group), and give a left Haar measure on \( G \).

Solution.

a) In this problem, we denote the usual Euclidian inner product on \( \mathbb{R}^n \) by \( \langle ., . \rangle \), and the associated norm by \( ||.|| \).

We denote the map \( P \times H \to G \) of the statement by \( \alpha \). This map is continuous because \( \text{SL}_n(\mathbb{R}) \) is a topological group. We first show that it is injective. Suppose that we have \( p, p' \in P \) and \( h, h' \in H \) such that \( ph = p'h' \). Then \( p^{-1}p' = h(h')^{-1} \in P \cap H \) is a special orthogonal matrix that is upper triangular with positive entries on the diagonal. Such a matrix has to be the identity. Indeed, let \( (v_1, \ldots, v_n) \) be its columns, and let \( (e_1, \ldots, e_n) \) be the canonical basis of \( \mathbb{R}^n \). We want to show that \( (v_1, \ldots, v_n) = (e_1, \ldots, e_n) \). As \( v_1 \) is a norm 1 vector and a positive multiple of \( e_1 \), we must have \( v_1 = e_1 \). As the vectors \( v_2, \ldots, v_n \) are orthogonal to \( v_1 \), their first entries are all 0. So \( v_2 \) is a positive multiple of \( e_2 \); as \( v_2 \) is norm 1, we must have \( v_2 = e_2 \). Now the vectors \( v_3, \ldots, v_n \) are orthogonal to \( v_2 \), so their second entries are zero, so \( v_3 \) is a positive multiple of \( e_3 \) etc.

Now remember the Gram-Schmidt orthonormalization process. If \( (v_1, \ldots, v_n) \) is a basis of \( \mathbb{R}^n \), it produces an orthogonal basis \( (w_1, \ldots, w_n) \) and an orthonormal basis \( (u_1, \ldots, u_n) \) in the following way :
b) Note that $P$ is an open subset of the $\mathbb{R}$-vector space $V$ of upper triangular matrices in $M_n(\mathbb{R})$. Moreover, for every $p \in P$, left translation by $p$ on $P$ is the restriction of the linear endomorphism $T_p : V \to V$, $x \mapsto px$. So we can apply problem 5 of problem set 1 to define a Haar measure on $P$ as $|\det(T_p)|^{-1}dV(p)$, where $dV$ is Lebesgue measure on $V$.

We still need to calculate $\det(T_p)$ for $p \in P$. Let $p \in P$, and let $a_1, \ldots, a_n$ be its diagonal entries. Let $(e_1, \ldots, e_n)$ be the canonical basis of $\mathbb{R}^n$ as before, and let $V_i = \text{Span}(e_1, \ldots, e_i) \subset \mathbb{R}^n$ for $1 \leq i \leq n$. Note that the action of $p \in GL_n(\mathbb{R})$ preserves the subspace $V_1, \ldots, V_n$, and that their determinant of the endomorphism of $V_i$ induced by $p$ is $a_1 \ldots a_i$. By decomposing $V$ using the columns of the matrices (as in the solution of problem 6(c) of problem set 1), we get an isomorphism $V \simeq V_1 \oplus V_2 \oplus \ldots \oplus V_n$ such that the endomorphism $T_p$ corresponds to the action of $p$ on each $V_i$. So we get

$$\det(T_p) = \prod_{i=1}^{n} \prod_{r=1}^{i} a_r = a_1^n a_2^{n-2} \cdots a_{n-1}^2 a_n = \prod_{i=1}^{n} a_i^{n+1-i}.$$  

c) Daniel: I'm not even sure of my own signs here, so don't take points off for a sign mistakes.

We will use problem 2, with $G = P$, $N$ the group of unipotent upper triangular matrices (i.e. of upper triangular matrices with ones on the diagonal) and $A$ the group of diagonal matrices with positive diagonal entries. Let $\alpha : A \times N \to P$ be the map defined by $\alpha(a, n) = an$. Let’s show that $\alpha$ is a homeomorphism. The map $\alpha$ is obviously continuous, and it is injective because $N \cap A = \{1\}$. Let $x \in P$, and let $a \in A$ be the matrix with the same diagonal entries as $x$. Then $n := a^{-1}x$ is in $N$, and $\alpha(a, n) = x$. Hence $\alpha$ is bijective. Moreover, the matrix $a$ depends continuously on $x$, hence so does $n$, so the inverse of $\alpha$ is continuous, and finally $\alpha$ is a homeomorphism.

We want to apply question 2(c). For this, we need to calculate the modular functions of $A$ and $N$ and the function $\alpha : A \to \mathbb{R}_{>0}$.

First, as $A$ is commutative, we have $\Delta_A = 1$.

For $N$, there are several ways to proceed. For example, you may notice that $N$ is obviously homeomorphic (as a topological space only) to the $\mathbb{R}$-vector space $W$ of upper triangular matrices in $M_n(\mathbb{R})$ with zeroes on the diagonal. (Just forget the
diagonal terms of the matrices.) Moreover, for \( n \in \mathbb{N} \), left translation by \( n \) on \( N \) corresponds to the linear endomorphism \( U_n \) of \( W \) given by \( U_n(X) = nX \), for \( X \in W \). Note that \( W \) is a subspace of the space \( V \) of the previous question, and that \( U_n \) is the restriction of \( T_n \). So we can use the same method as in the previous question to calculate \( \det(U_n) \), and we get \( \det(U_n) = 1 \). Hence Lebesgue measure on \( W \) is a left Haar measure on \( N \). We can redo everything using right translations instead of left translations, and we get that Lebesgue measure on \( W \) is also a right Haar measure on \( N \). This means that \( N \) is unimodular, so \( \Delta_N = 1 \).

Finally, we need to calculate the function \( \alpha \). Remember that it is defined by

\[
\int_N f(ana^{-1})dn = \alpha(a) \int_N f(n)dn
\]

for every \( f \in \mathcal{C}_c(N) \), where \( dn \) is Lebesgue measure on \( W \) (which we have just seen is a Haar measure on \( N \)). Note that \( c_a : X \mapsto aXa^{-1} \) is a linear endomorphism of \( W \), so we can calculate \( \int_N f(ana^{-1})dn \) using the change of variables formula once we know \( \det(c_a) \). We get \( \det(c_a) \int_N f(ana^{-1})dn = \int_N f(n)dn \), hence \( \alpha(a) = \det(c_a)^{-1} \).

But is easy to see that, if the diagonal entries of \( a \) are \((a_1, \ldots, a_n)\), then

\[
\det(c_a) = a_1^{n-1}a_2^{n-3} \ldots a_n^{1-n} = \prod_{i=1}^{n} a_i^{n-2i+1}.
\]

Hence finally, for \( p \in P \),

\[
\Delta_p(p) = a_1^{1-n}a_2^{3-n} \ldots a_n^{n-1} = \prod_{i=1}^{n} a_i^{2i-n-1},
\]

where \( a_1, \ldots, a_n \) are the diagonal entries of \( p \).

d) If you know (or know how to prove) that \( \text{SL}_n(\mathbb{R}) \) is equal to its commutator subgroup, then this isn easy. Here is another way: Let \( \text{GL}_n(\mathbb{R})^+ \) be the group of \( n \times n \) matrices with positive determinant. This is an open subgroup of \( \text{GL}_n(\mathbb{R}) \) (it’s the inverse image of \( \mathbb{R}^+ \) by the continuous group morphism \( \det : \text{GL}_n(\mathbb{R}) \rightarrow \mathbb{R}^\times \), so, if \( \mu \) is a Haar measure on \( \text{GL}_n(\mathbb{R}) \) (remember that \( \text{GL}_n(\mathbb{R}) \) is unimodular by problem 6(c) of problem set 1), its restriction to \( \text{GL}_n(\mathbb{R})^+ \) is a nonzero regular Borel measure, and it is obviously a left and right Haar measure on \( \text{GL}_n(\mathbb{R})^+ \). Now note that we have an isomorphism of topological groups \( \mathbb{R}^0 \times \text{SL}_n(\mathbb{R}) \rightarrow \text{GL}_n(\mathbb{R})^+ \), \((\lambda, x) \mapsto \lambda x \) (whose inverse is given by \( x \mapsto (\det(x)^{1/n}, \det(x)^{-1/n}) \)), so we can apply problem 2 with \( G = \text{GL}_n(\mathbb{R})^+, A = \mathbb{R}^0 I_n \) and \( N = \text{SL}_n(\mathbb{R}) \). As \( A \) and \( N \) commute, we have \( \alpha = 1 \). We know that \( A \) is unimodular because it is commutative, and we have just seen that \( \text{GL}_n(\mathbb{R})^+ \) is unimodular, hence 2(c) implies that \( \text{SL}_n(\mathbb{R}) \) is also unimodular.

e) It is well-known that the group of rotations in \( \mathbb{R}^2 \) (i.e. \( \text{SO}(2) \)) is isomorphic to the circle group \( S^1 \). The isomorphism sends \( e^{i2\pi \theta} \in S^1 \) to the matrix \( \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \).

Also, we have seen in class that we can define a Haar measure on \( S^1 \) by the linear functional sending \( f \in \mathcal{C}_c(S^1) \) to \( \int_0^1 f(e^{i2\pi \theta})d\theta \), where \( d\theta \) is Lebesgue measure on \( \mathbb{R} \).

The point of this, of course, is that problem 1 now allows you to define a Haar measure on \( \text{SL}_2(\mathbb{R}) \). To treat the case of \( \text{SL}_n(\mathbb{R}) \), we need a Haar measure on \( \text{SO}(n) \). An example of such a measure is given in problem 6.

\[\square\]
4. (Remember problems 4, 5, 6, 8 of problem set 1.) We denote by $dx$ a Haar measure on the additive group $\mathbb{Q}_p$. We also denote by $dx$ (resp. $dA$) the product measure on $\mathbb{Q}_p^n$ (resp. $M_n(\mathbb{Q}_p) \simeq \mathbb{Q}_p^{n^2}$); note that it is a Haar measure for the corresponding additive group.

a) (2) Show that, for every $f \in L^1(\mathbb{Q}_p)$ and every $a \in \mathbb{Q}_p^\times$, $b \in \mathbb{Q}_p$, we have

$$\int_{\mathbb{Q}_p} f(x)dx = |a|_p \int_{\mathbb{Q}_p} f(ax + b)dx.$$

b) (3) Let $n \geq 1$. Show that, if $f \in L^1(\mathbb{Q}_p^n)$, $A \in \text{GL}_n(\mathbb{Q}_p)$ and $b \in \mathbb{Q}_p^n$, we have

$$\int_{\mathbb{Q}_p^n} f(x)dx = |\det(A)|_p \int_{\mathbb{Q}_p^n} f(Ax + b)dx.$$

c) (2) Show that $|\det(A)|_p^{-n}dA$ is a left and right Haar measure on $\text{GL}_n(\mathbb{Q}_p)$.

d) (3) Let $B$ be the group of upper triangular matrices in $\text{GL}_n(\mathbb{Q}_p)$. Find a left Haar measure on $B$ and calculate the modular function of $B$.

Solution.

a) First, using the invariance by translation of $dx$, we see that

$$\int_{\mathbb{Q}_p} f(ax + b)dx = \int_{\mathbb{Q}_p} f(ax)dx$$

for every $f \in L^1(\mathbb{Q}_p)$ and $a, b \in \mathbb{Q}_p$.

Let $a \in \mathbb{Q}_p^\times$. We use the notation of problem 8 of problem set 1. If $x \in \mathbb{Q}_p$ and $m \in \mathbb{Z}$, then

$$aB(x, p^m) = \{ay \mid |x - y|_p \leq p^m\} = \{y \in \mathbb{Q}_p \mid |ax - y|_p \leq |a|_p p^m\} = B(ax, |a|_p p^m),$$

and so, by 8(a) of problem set 1, $\text{vol}(aB(x, p^m)) = |a|_p \text{vol}(B(x, p^m))$. Using question (b) of the same problem, we get $\text{vol}(aE) = |a|_p \text{vol}(E)$ for every Borel subset $E$ of $\mathbb{Q}_p$. Suppose that $f = 1_E$, with $E$ a Borel subset of $\mathbb{Q}_p$. Then

$$\int_{\mathbb{Q}_p} f(ax)dx = \text{vol}(a^{-1} E) = |a|_p^{-1} \int_{\mathbb{Q}_p} f(x)dx,$$

so we get the desired result for this function $f$. The result now follows for every $f \in L^1(\mathbb{Q}_p)$ by linearity and continuity of the integral.

b) Using the translation invariance of $dx$ as in question (a), we see that it suffices to prove the result in the case $b = 0$. Let $A \in \text{GL}_n(\mathbb{Q}_p)$. First note that $A = A_1 A_2$ and if we know the result for $A_1$ and $A_2$, then we know it for $A$; indeed, for every $f \in L^1(\mathbb{Q}_p^n)$, we’ll have

$$\int_{\mathbb{Q}_p^n} f(x)dx = |\det(A)|_p \int_{\mathbb{Q}_p^n} f(A_1 x)dx =$$

$$|\det(A_1)|_p |\det(A_2)|_p \int_{\mathbb{Q}_p^n} f(A_1(A_2 x))dx = |\det(A)|_p \int_{\mathbb{Q}_p^n} f(Ax)dx.$$
to multiplying on the left by a permutation matrix). So, by the observation above, it suffices to prove the result for upper and lower triangular matrices and for permutation matrices.

Suppose first that $A$ is a permutation matrix. So there exists a permutation $\sigma \in S_n$ such that, for every $x = (x_1, \ldots, x_n) \in \mathbb{Q}_p^n$, $Ax = (x_{\sigma(1)}, \ldots, x_{\sigma(n)})$. As $dx$ is the product of identical measures on the $n$ factors $\mathbb{Q}_p$ of $\mathbb{Q}_p^n$, we have, for every $f \in L^1(\mathbb{Q}_p)$, $\int_{\mathbb{Q}_p^n} f(Ax)dx = \int_{\mathbb{Q}_p^n} f(x)dx$. The result now follows from the fact that $\det(A) = \pm 1$.

Suppose that $A$ is upper triangular, and write $A = (a_{ij})_{1 \leq i,j \leq n}$. Let $f \in L^1(\mathbb{Q}_p)$. Then

$$\int_{\mathbb{Q}_p^n} f(A(x_1, \ldots, x_n)) = \int_{\mathbb{Q}_p} \cdots \int_{\mathbb{Q}_p} f(a_{11}x_1 + \cdots + a_{1n}x_n, \ldots, a_{n-1,n-1}x_{n-1} + a_{n-1,n}x_n, a_{nn}x_n)dx_n \cdots dx_1.$$  

Using question (a), we see that this last integral is equal to

$$|a_{11}|_p^{-1} \cdots |a_{n-1,n-1}|_p^{-1} |a_{nn}|_p^{-1} \int_{\mathbb{Q}_p^n} f(x)dx = |\det(A)|_p^{-1} \int_{\mathbb{Q}_p^n} f(x)dx.$$

The case of lower triangular matrices is similar (just put the $dx_i$ reverse order).

c) Once we have the change of variables formula of question (b), we can replace $\mathbb{R}$ by $\mathbb{Q}_p$ in problems 5 and 6 of problem set 1 and all the results will stay true, with exactly the same proofs. (Except 6(b), which doesn’t make sense for $\mathbb{Q}_p$. In particular, we get that $|\det(A)|_p^{-n}dA$ is a left and right Haar measure on $\text{GL}_n(\mathbb{Q}_p)$.

d) Again, we can just apply the proofs of questions (b) and (c) of problem 3 (and the analogue for $\mathbb{Q}_p$, problem 5 of problem set 1) to get the result. Assuming that there is no sign mistake in problem 3, a left Haar measure on $B$ is $\prod_{i=1}^n |a_{ii}|_p^{-n-1}dA$, where $dA$ is the product measure on the $\mathbb{Q}_p$-vector space $V \simeq \mathbb{Q}_p^{n(n+1)/2}$ of upper triangular matrices and the $a_{ij}$ are the entries of the matrix. And the modular function of $B$ is given by

$$\Delta(A) = \prod_{i=1}^n |a_{ii}|_p^{2i-n-1}.$$  

$\square$

5. (extra credit) The goal of this problem is to give a formula for a Haar measure on $\text{SO}(n)$. (We could do something similar for the unitary group $U(n)$.)

a) (1) For $X \in M_n(\mathbb{R})$, we set $\Phi(X) = (I_n - X)(I_n + X)^{-1}$. Show that this is well-defined if $-1$ is not an eigenvalue of $X$, and that we have $\Phi(\Phi(X)) = X$ whenever this makes sense.

b) (2) We denote by $A_n$ the $\mathbb{R}$-vector space of $n \times n$ antisymmetric matrices (i.e. of $X \in M_n(\mathbb{R})$ such that $X^T = -X$) and by $U$ the set of elements of $\text{SO}(n)$ that don’t have $-1$ as an eigenvalue. Show that $U$ is an open dense subset of $\text{SO}(n)$, and that $\Phi$ induces a homeomorphism $A_n \to U$.

c) (2) Let $X \in A_n$. Show that there exist open dense subsets $V$ and $W$ of $A_n$ such that the formula $\Phi(L_XY) = \Phi(X)\Phi(Y)$ defines a diffeomorphism $L_X : V \to W$, and that $0 \in V$. 
d) Let \( dX \) be Lebesgue measure on \( A_n \). For every \( X \in A_n \) and every \( Y \in A_n \) on which \( L_X \) is defined, we denote by \( L_X(Y) \) the differential at \( Y \) of \( L_X \). It is a linear transformation from \( A_n \) to \( A_n \) such that, for every \( H \in A_n \),
\[
L_X(Y + tH) = L_X(Y) + tL_X(Y)(H) + o(t).
\]

Fix \( X \in A_n \). We want to compute \( \det(L_X'(0)) \). Remember that \( L_X'(0) \) is a linear endomorphism of \( A_n \), and note that \( A_n \otimes_R \mathbb{C} \) is the space of antisymmetric matrices in \( M_n(\mathbb{C}) \).

i. (1) Show that \( \det(L_X'(0)) \) is well-defined and nonzero.
ii. (1) Show that we have
\[
L_X'(0)(H) = (I_n - X)H(I_n + X),
\]
for every \( H \in A_n \).
iii. (1) Show that \( X \) has a basis of (complex) eigenvectors \( (v_1, \ldots, v_n) \) such that the corresponding eigenvalues are of the form \( i\lambda_1, \ldots, i\lambda_n \), with \( \lambda_1, \ldots, \lambda_n \in \mathbb{R} \).
iv. (1) For \( j, k \in \{1, \ldots, n\} \), we set \( Y_{jk} = v_j v_k^T - v_k v_j^T \). Show that \( Y_{jk} \in A_n \otimes_R \mathbb{C} \), and that it is an eigenvector for \( L_X'(0) \), with corresponding eigenvalue \((1 - i\lambda_j)(1 - i\lambda_k)\).
v. (1) Show that \( (Y_{jk})_{1 \leq j < k \leq n} \) is a basis of \( A_n \otimes_R \mathbb{C} \).
vi. (1) Show that \( \det(L_X'(0)) = \det(I_n - iX)^{n-1} \).

e) (3) Show that the linear functional sending \( f \in C_c(\text{SO}(n)) \) to
\[
\int_{A_n} f(\Phi(X)) \frac{1}{|\det L_X'(0)|} dX
\]
defines a left Haar measure on \( \text{SO}(n) \). (Hint : Note that \( (L_X \circ L_Y)(0) = L_X(Y) \), and use the chain rule.)

Solution.

a) If \( X \in M_n(\mathbb{R}) \), then \(-1\) is not an eigenvalue of \( X \) if and only if \( I_n + X \) is invertible, i.e. if and only if the formula defining \( \Phi(X) \) makes sense. So the set of definition of \( \Phi \) is the open set defined by the equation \( \det(I_n + X) \neq 0 \). Note also that \( I_n - X \) and \( I_n + X \) commute, so \( I_n - X \) and \((I_n + X)^{-1} \) commute (if the second is defined), so we also have \( \Phi(X) = (I_n + X)^{-1}(I_n - X) \).

Let \( X \in M_n(\mathbb{R}) \) such that \( \Phi(X) \) is defined. Then we have
\[
I_n + \Phi(X) = ((I_n + X) + (I_n - X))(I_n + X)^{-1} = 2(I_n + X)^{-1}
\]
and
\[
I_n - \Phi(X) = ((I_n + X) - (I_n - X))(I_n + X)^{-1} = 2X(I_n + X)^{-1}.
\]
In particular, \( I_n + \Phi(X) \) is invertible, so \( \Phi(\Phi(X)) \) makes sense, and we have
\[
\Phi(\Phi(X)) = (I_n - \Phi(X))(I_n + \Phi(X))^{-1} = 2X(I_n + X)^{-1}(2(I_n + X)^{-1})^{-1} = X.
\]

b) Let \( g \in \text{SO}(n) \). Then we can find \( P \in \text{GL}_n(\mathbb{R}) \) such that
\[
P g P^{-1} = \begin{pmatrix}
  r_1 & 0 & \ldots & 0 \\
  0 & r_2 & 0 & 0 \\
  \vdots & \ddots & \ddots & 0 \\
  0 & \ldots & 0 & r_m
\end{pmatrix},
\]
where:
• if \( n \) is even, then \( m = n/2 \) and \( r_1, \ldots, r_m \) are \( 2 \times 2 \) matrices of the form
\[
\begin{pmatrix}
\cos \theta_i & \sin \theta_i \\
-\sin \theta_i & \cos \theta_i
\end{pmatrix},
\]
with \( \theta_i \in [0, 2\pi) \);

• if \( n \) is odd, then \( m = (n + 1)/2 \), the matrix \( r_m \) is the \( 1 \times 1 \) matrix \( 1 \) and \( r_1, \ldots, r_{m-1} \) are \( 2 \times 2 \) matrices of the form
\[
\begin{pmatrix}
\cos \theta_i & \sin \theta_i \\
-\sin \theta_i & \cos \theta_i
\end{pmatrix},
\]
with \( \theta_i \in [0, 2\pi) \).

In both cases, \(-1\) is an eigenvalue of \( g \) if and only if one at least one of the \( \theta_i \) is equal to \( \pi \). So, by varying the \( \theta_i \), we can find a sequence of elements of \( \text{SO}(n) \) that converge to \( g \) and don’t have \(-1\) as an eigenvalue. This proves that \( U \) is dense in \( \text{SO}(n) \).

Next, as antisymmetric matrices have only imaginary eigenvalues, the function \( \Phi \) is defined on \( A_n \). Note also that it is clear on the definition of \( \Phi \) that \( \Phi \) is continuous on its open set of definition. By the second part of question (a), \( \Phi \) is injective and, to see that it suffices to prove that \( \Phi(U) \) is dense in \( Y \) (because then its inverse will be \( \Phi \)). So we just need to show that \( \Phi(A_n) = U \). Using again the fact that \( \Phi(\Phi(X)) = X \) whenever this makes sense, we see that it suffices to prove that \( \Phi(A_n) \subset U \) and \( \Phi(U) \subset A_n \).

Let \( X \in A_n \). Then \( X^T = -X \), so \( \Phi(X)^T = (I_n + X^T)^{-1}(I_n - X^T) = (I_n - X)^{-1}(I_n + X) \), and hence \( \Phi(X)^T \Phi(X) = I_n \), which means that \( \Phi(X) \in \text{O}(n) \). As \( \Phi \) is continuous and \( A_n \) is connected, \( \Phi(A_n) \) is connected. But \( I_n = \Phi(0) \in \Phi(A_n) \), so \( \Phi(A_n) \) is contained in \( \text{SO}(n) \).

Let \( X \in \text{SO}(n) \) such that \(-1\) is not an eigenvalue of \( X \). Then \( X^T = X^{-1} \), so
\[
\Phi(X)^T = (I_n - X^T)(I_n + X^T)^{-1} = (I_n - X^{-1})(I_n + X^{-1})^{-1} = (X - I_n)(X + I_n)^{-1} = -\Phi(X).
\]

So we have \( \Phi(U) \subset A_n \).

c) Fix \( X \in A_n \). Note that the formula \( \Phi(L_X Y) = \Phi(X) \Phi(Y) \) can also be written
\[
L_X Y = \Phi(\Phi(X) \Phi(Y)), \text{ by (a)}.
\]
For \( Y \in A_n \), \( \Phi(X) \Phi(Y) \) has an image by \( \Phi \) (which will automatically be in \( A_n \) by (b)) if and only if \( \Phi(Y) \in \Phi(X)^{-1} U \). So we can take \( V = \Phi(U \cap (\Phi(X)^{-1} U)) \); this is dense in \( A_n \) because \( U \cap (\Phi(X)^{-1} U) \) is dense in \( \text{SO}(n) \) by (b). Then the image of \( V \) by the map \( L_X : Y \mapsto \Phi(\Phi(X) \Phi(Y)) \) is \( W := \Phi((\Phi(X)U) \cap U) \).

The map \( L_X : V \to W \) is continuous and surjective. In fact, as \( \Phi \) is infinitely differentiable (it is given by rational functions in the entries of its arguments, by the formula saying that \( A^{-1} \) is \( \det(A)^{-1} \) times the transpose of its cofactor matrix, for every \( A \in \text{GL}_n(\mathbb{R}) \)), the map \( L_X \) is also infinitely differentiable.

Let \( X' = \Phi(\Phi(X)^{-1}) \in A_n \). Then we get as above a continuous and surjective map
\[
L_{X'} : W \to V, \text{ defined by the formula } L_{X'}(Y) = \Phi(\Phi(X)^{-1} \Phi(Y)).
\]
The maps \( L_X \) and \( L_{X'} \) are inverses of each other, and in particular they are both diffeomorphisms.

Finally, if \( Y = 0 \), then \( \Phi(Y) = I_n \). So \( \Phi(Y) \in U \), and we also have \( \Phi(Y) \in \Phi(X)^{-1} U \), because \( \Phi(X) \Phi(Y) = \Phi(X) \in U \). This shows that \( 0 \in V \).

d) i. Let \( X \in A_n \). As \( L_X \) is defined at the point \( 0 \), the differential \( L_X'(0) \) makes sense; also, as \( L_X \) is a diffeomorphism, \( \det(L_X'(0)) \neq 0 \).

ii. Note that, for \( Y \in A_n \),
\[
(I_n + X + Y)^{-1} = (I_n + X)^{-1}(I_n + (I_n + X)^{-1} Y) = (I_n + X)^{-1}(I_n - Y(I_n + X)^{-1} + o(Y)),
\]
hence
\[
\Phi(X + Y) = (I_n - X - Y)(I_n + X + Y)^{-1}
\]
\[
= ((I_n - X) - Y)(I_n + X)^{-1}((I_n - Y)(I_n + X)^{-1} + o(Y))
\]
\[
= \Phi(X) - \Phi(X)Y(I_n + X)^{-1} - Y(I_n + X)^{-1} + o(Y).
\]

In particular (taking \(X = 0\)), we have
\[
\Phi(Y) = I_n - 2Y + o(Y).
\]

So
\[
\Phi(X)\Phi(Y) = \Phi(X) - 2\Phi(X)Y + o(Y),
\]

and
\[
L_X(Y) = \Phi(\Phi(X)\Phi(Y)) = \Phi(\Phi(X) - 2\Phi(X)Y + o(Y)) = \Phi(\Phi(X) - \Phi(X)y(-2\Phi(X)Y)(I_n + \Phi(X))^{-1} - (-2\Phi(X)Y)(1_n + \Phi(X))^{-1} + o(Y).
\]

Using \(\Phi(\Phi(X)) = X\) and \(I_n + \Phi(X) = 2(I_n + X)^{-1}\) (see (a)), we can simplify this last expression to
\[
X + X\Phi(X)Y(I_n + X)^{-1} + \Phi(X)Y(I_n + X) + o(Y) = X + (I_n + X)\Phi(X)Y(I_n + Y) + o(Y)
\]
\[
= X + (I_n - X)Y(I_n + X) + o(Y).
\]

But then the conclusion that \(L'_X(0)(Y) = (I_n - X)Y(I_n + X)\) follows immediately from the definition of the differential.

iii. As \(X\) is antisymmetric and has real entries, it is normal, so the spectral theorem says that \(X\) is diagonalizable in an orthonormal basis of \(\mathbb{C}^n\); in other words, there exists a unitary matrix \(P\) such that \(PXP^{-1}\) is diagonal. We have already used the fact that the eigenvalues of \(X\) are imaginary, but it is easy to recheck it quickly: we have \(X^* = -X\) and \(P^* = P^{-1}\), and \((PXP^{-1})^* = (P^*)^{-1}X^*P^* = -PXP^{-1}\). As \(PXP^{-1}\) is diagonal, this means that its diagonal entries (which are the eigenvalues of \(X\)) are all imaginary.

iv. It follows directly from the definition of \(Y_{jk}\) that \(Y_{jk}^T = -Y_{jk}\), so \(Y_{jk} \in A_n \otimes \mathbb{R} \mathbb{C}\). Furthermore, by (ii), we have
\[
L'_X(0)(Y_{ij}) = (I_n - X)Y_{ij}(I_n + X)
\]
\[
= (I_n - X)(v_j v_k^T)(I_n - X^T) - (I_n - X)(v_k v_j^T)(I_n - X^T)
\]
\[
= (1 - i\lambda_j)(v_j v_k^T)(1 - i\lambda_k) - (1 - i\lambda_k)(v_k v_j^T)(1 - i\lambda_j)
\]
\[
= (1 - i\lambda_j)(1 - i\lambda_k)Y_{ij}.
\]

v. As \((v_1, \ldots, v_n)\) is a basis of \(\mathbb{C}^n\), the matrices \(v_j v_k^T\), for \(1 \leq j, k \leq n\), form a basis of \(M_n(\mathbb{C})\). So the matrices \(Y_{jk} = (v_j v_k^T) - (v_j v_k^T)^T\), for \(1 \leq j, k \leq n\), generate \(A_n \otimes \mathbb{R} \mathbb{C}\). Note that \(Y_{jj} = 0\) and \(Y_{kj} = -Y_{jk}\), so \(A_n \otimes \mathbb{R} \mathbb{C}\) is actually spanned by the matrices \(Y_{jk}\), for \(1 \leq j < k \leq n\). As there are \(n(n-1)/2\) such matrices and \(\dim_{\mathbb{C}}(A_n \otimes \mathbb{R} \mathbb{C}) = \dim_{\mathbb{C}}(A_n) = n(n-1)/2\), they form a basis of \(A_n \otimes \mathbb{R} \mathbb{C}\).

vi. By (iv) and (v), we have
\[
\det(L'_X(0)) = \prod_{1 \leq j < k \leq n} (1 - i\lambda_j)(1 - i\lambda_k) = \prod_{r=1}^{n} (1 - i\lambda_r)^{n-1}
\]
(because each \(1 - i\lambda_r\) appears \(n - 1\) times in the first big product: \((n - r)\) times as the first factor \((1 - i\lambda_j)\), and \((r - 1)\) times as the second factor \((1 - i\lambda_k))\).
To get the result, we just need to note that the eigenvalues of \( I_n - iX \) are \( 1 - i\lambda_1, \ldots, 1 - i\lambda_n \), so that
\[
\det(I_n - X) = \prod_{r=1}^{n} (1 - i\lambda_r).
\]
e) Let us denote this functional by \( \Lambda \). First, by question (e), the function \( X \mapsto \frac{1}{\det(L_X'(0))} \) is defined everywhere on \( A_n \) and continuous, so the integral defining \( \Lambda \) makes sense.

We need to check that \( \Lambda \) is positive and invariant by left translations. We first check the positivity. Let \( f \in C_c^+ (\text{SO}(n)) \). Then we can find \( \varepsilon > 0 \) and a nonempty open subset \( \Omega \) of \( \text{SO}(n) \) such that \( f_{|\Omega} \geq \varepsilon \). As \( U \) is open dense in \( \text{SO}(n) \), its intersection with \( \Omega \) is open and nonempty, so \( \Phi(U \cap \Omega) \) is open and nonempty in \( A_n \), and we have
\[
\Lambda(f) \geq \varepsilon \int_{\Phi(U \cap \Omega)} \frac{1}{|\det(L_X'(0))|} dX > 0
\]
(because the function \( X \mapsto \frac{1}{|\det(L_X'(0))|} \) is continuous and positive on \( \Phi(U \cap \Omega) \)).

Now we check the left invariance. Fix \( f \in C_c(\text{SO}(n)) \). Let \( g \in U \). Then \( \Lambda(L_g f) = \int_{A_n} f(g^{-1}\Phi(Y)) \frac{1}{|\det(L_Y'(0))|} dY \). Choose \( X, X' \in A_n \) such that \( \Phi(X) = g^{-1} \) and \( \Phi(X') = g \). Then
\[
\Lambda(L_g f) = \int_{A_n} f(\Phi(X)\Phi(Y)) \frac{1}{|\det(L_Y'(0))|} dY
\]
\[
= \int_V f(\Phi(X)\Phi(Y)) \frac{1}{|\det(L_Y'(0))|} dY \quad \text{(because } \text{vol}(A_n - V) = 0 \text{)}
\]
\[
= \int_V f(\Phi(L_X Y)) \frac{1}{|\det(L_Y'(0))|} dY.
\]
Now note that, if \( Y \in V \), then so does \( L_Y(0) = Y \), so \( L_X(Y) = L_X \circ L_Y(0) = L_X Y (0) \) makes sense, and we have by the chain rule
\[
L_{L_X Y}(0) = L_X(0) \circ L_Y(0),
\]
hence in particular
\[
\frac{1}{|\det(L_Y'(0))|} = \frac{|\det(L_X'(0))|}{|\det(L_{L_X Y}'(0))|}.
\]
This implies that
\[
\Lambda(g)(f) = \int_V f(\Phi(L_X Y)) \frac{|\det(L_X'(0))|}{|\det(L_{L_X Y}'(0))|} dY.
\]
Using the substitution \( Z = L_X Y \), we see that this is equal to
\[
\int_W f(\Phi(Z)) \frac{1}{|\det(L_Z'(0))|} dZ.
\]
As \( \text{vol}(A_N - W) = 0 \), the last integral is equal to \( \int_{A_n} f(\Phi(Z)) \frac{1}{|\det(L_Z'(0))|} dZ \), i.e. to \( \Lambda(f) \).

So we have shown that the function \( \text{SO}(n) \to \mathbb{C}, \; g \mapsto \Lambda(L_g f) \) is constant on the open dense subset \( U \). As this function is continuous (it is the composition of the continuous function \( \text{SO}(n) \to C_c(\text{SO}(n)) \), \( g \mapsto L_g f \) and of the continuous linear function \( \Lambda : C_c(\text{SO}(n)) \to \mathbb{C} \)), it is constant on the whole \( \text{SO}(n) \), which means that \( \Lambda(L_g f) = \Lambda(f) \) for every \( g \in \text{SO}(n) \).

\[\Box\]