MAT 540 : Problem Set 2

Due Thursday, September 26

1. Monoidal categories (extra credit)

A *monoidal category* is a category $\mathscr{C}$ equipped with a bifunctor $(\cdot) \otimes (\cdot) : \mathscr{C} \times \mathscr{C} \to \mathscr{C}$ (the tensor product or monoidal functor), with an identity (or unit) object $\mathbb{1}$ and with three natural isomorphisms $\alpha(A, B, C) : (A \otimes B) \otimes C \xrightarrow{\sim} A \otimes (B \otimes C)$, $\lambda(A) : \mathbb{1} \otimes A \xrightarrow{\sim} A$ and $\rho_A : A \otimes \mathbb{1} \xrightarrow{\sim} A$, satisfying the following conditions :

- for all $A, B, C, D \in \mathrm{Ob}(\mathscr{C})$, the following diagram commutes :

$$\begin{tikzcd}
((A \otimes B) \otimes C) \otimes D \arrow[r, "\alpha(A,B,C)\otimes \mathrm{id}_D"] \arrow[d, "\alpha(A\otimes B,C,D)"'] & (A \otimes (B \otimes C)) \otimes D \arrow[r, "\alpha(A,B\otimes C,D)"] & A \otimes ((B \otimes C) \otimes D) \arrow[d, "\mathrm{id}_A \otimes \alpha(B,C,D)"] \\
(A \otimes B) \otimes (C \otimes D) \arrow[rr, "\alpha(A,B,C\otimes D)"'] & & A \otimes (B \otimes (C \otimes D))
\end{tikzcd}$$

- for all $A, B \in \mathrm{Ob}(\mathscr{C})$, the following diagram commutes :

$$\begin{tikzcd}
(A \otimes \mathbb{1}) \otimes B \arrow[rr, "\alpha(A,\mathbb{1},B)"] \arrow[dr, "\rho(A)\otimes \mathrm{id}_B"'] & & A \otimes (\mathbb{1} \otimes B) \arrow[dl, "\mathrm{id}_A \otimes \lambda(B)"] \\
& A \otimes B &
\end{tikzcd}$$

Here are some examples :

- $\mathscr{C} = \mathbf{Set}$ or $\mathbf{Top}$, $\otimes = \times$, $\mathbb{1}$ is a singleton;
- $\mathscr{C} = \mathbf{Grp}$, $\otimes = \times$, $\mathbb{1} = \{1\}$;
- $\mathscr{C} = {}_R\mathbf{Mod}$ with R a commutative ring, $\otimes = \otimes_R$, $\mathbb{1} = R$;
- $\mathscr{C} = \mathrm{Func}(\mathscr{D}, \mathscr{D})$ with $\mathscr{D}$ a category, $\otimes = \circ$, $\mathbb{1} = \mathrm{id}_{\mathscr{D}}$.

A *monoid* in $\mathscr{C}$ is an object M of $\mathscr{C}$ together with two morphisms $\mu : M \otimes M \to M$ (multiplication) and $\eta : \mathbb{1} \to M$ (unit), such that the two following diagrams commute :

$$\begin{tikzcd}
M \otimes (M \otimes M) \arrow[r, "\mathrm{id}_M \otimes \mu"] & M \otimes M \arrow[r, "\mu"] & M \\
(M \otimes M) \otimes M \arrow[u, "\alpha(M,M,M)"] \arrow[r, "\mu \otimes \mathrm{id}_M"'] & M \otimes M \arrow[ur, "\mu"'] &
\end{tikzcd}$$

and

$$\begin{tikzcd}
M \otimes M & \mathbb{1} \otimes M \arrow[l, "\eta\otimes\mathrm{id}_M"'] \arrow[d, "\lambda(M)"] \\
M \otimes \mathbb{1} \arrow[u, "\mathrm{id}_M\otimes\eta"] \arrow[r, "\rho(M)"'] & M
\arrow[from=1-1, to=2-2, "\mu"]
\end{tikzcd}$$

(We can also define morphisms of monoids, and monoids in $\mathscr{C}$ form a category.)

Examples :

- A monoid in $(\mathbf{Set}, \times)$ is a monoid (in the usual sense).
- A monoid in $(\mathbf{Top}, \times)$ is a topological monoid.
- If R is a commutative ring, a monoid in $({}_R\mathbf{Mod}, \otimes)$ is a R-algebra. (In particular, a monoid in $(\mathbf{Ab}, \otimes_{\mathbb{Z}})$ is a ring.)
- A monoid in $(\mathrm{Func}(\mathscr{D}, \mathscr{D}), \circ)$ is called a *monad on* $\mathscr{D}$.

(a). (2 points) Let **Mon** be the category of (usual) monoids. It is a monoidal category, with the monoidal functor given by $\times$ and the unit object $\{1\}$. If (M, μ, η) is a monoid in **Mon**, show that M is a commutative monoid and μ is equal to the multiplication of M.

(b). (3 points) Let $F : \mathscr{C} \to \mathscr{D}$ and $G : \mathscr{D} \to \mathscr{C}$ be two functors such that (F, G) is a pair of adjoint functors, and let $\varepsilon : F \circ G \to \mathrm{id}_{\mathscr{D}}$ and $\eta : \mathrm{id}_{\mathscr{C}} \to G \circ F$ be the counit and unit of the adjunction. Define a morphism of functors $\mu : (G \circ F) \circ (G \circ F) \to G \circ F$ by $\mu(X) = G(\varepsilon(F(X))) : G(F \circ G(F(X))) \to G(F(X))$. Show that $(G \circ F, \mu, \eta)$ is a monad on $\mathscr{C}$.

Solution.

(a). We denote the monoid operation of M by $(a, b) \longmapsto a \cdot b$ and its unit element by 1. We also denote the map $\mu : M^2 \to M$ by $(a, b) \longmapsto a * b$. The fact that μ is a morphism of monoids says that, for all $a, b, c, d \in M$, we have

$$(*) \qquad (a * b) \cdot (c * d) = (a \cdot c) * (b \cdot d).$$

As $\eta : \{1\} \to M$ is a morphism of monoids, it sends 1 to $1 \in M$, so $\cdot$ and $*$ have the same unit. [1] So, if $a, d \in M$, we have

$$a \cdot d = (a * 1) \cdot (1 * d) = (a \cdot 1) * (1 \cdot d) = a * d,$$

and also

$$a \cdot d = (1 * a) \cdot (d * 1) = (1 \cdot d) * (a \cdot 1) = d * a.$$

This proves both statements. [2]

(b). Note that the operations $(\cdot) \circ \mathrm{id}_{\mathscr{C}}$ and $\mathrm{id}_{\mathscr{C}} \circ (\cdot)$ are the identity functor of the category $\mathrm{Func}(\mathscr{C}, \mathscr{C})$, so the functorial isomorphisms ρ and λ are just the identity in that case; similarly, as $(H \circ H') \circ H'' = H \circ (H' \circ H'')$ for any $H, H', H'' \in \mathrm{Func}(\mathscr{C}, \mathscr{C})$, the functorial isomorphism α is also the identity. So we have three things to prove :

(1) $\mu \circ (\mathrm{id}_{G\circ F} \otimes \eta) = \mathrm{id}_{G\circ F}$;

[1] This would be automatic even if we did not assume that η is a morphism of monoids : Let e be the unit of $*$. Then $1 = 1 \cdot 1 = (e * 1) \cdot (1 * e) = (e \cdot 1) * (1 \cdot e) = e * e = e$.

[2] Note that we did not use the associativity of $\cdot$ and $*$. In fact, we could deduce the associativity of $\cdot$ and $*$ from property (*).

(2) $\mu \circ (\eta \otimes \mathrm{id}_{G\circ F}) = \mathrm{id}_{G\circ F}$;

(3) $\mu \circ (\mu \otimes \mathrm{id}_{G\circ F}) = \mu \circ (\mathrm{id}_{G\circ F} \otimes \mu)$.

To prove (1), we note that, by definition of $\otimes$ and μ, for every $X \in \mathrm{Ob}(\mathscr{C})$, the left-hand side of (1) applied to X is the image by G of the composition

$$F(X) \xrightarrow{F(\eta(X))} F(G(F(X))) \xrightarrow{\varepsilon(F(X))} F(X) .$$

So (1) follows from the first statement of Proposition I.4.4 of the notes. The proof of (2) is similar : by definition of $\otimes$ and μ, for every $X \in \mathrm{Ob}(\mathscr{C})$, the left-hand side of (2) applied to X is the composition

$$G(F(X)) \xrightarrow{\eta(G(F(X)))} G(F(G(F(X)))) \xrightarrow{G(\varepsilon(F(X)))} G(F(X)) ,$$

and we can apply the second statement of Proposition I.4.4 of the notes.

It remains to prove (3). Let $X \in \mathrm{Ob}(\mathscr{C})$. Then, when applied to X, the square

$$\begin{tikzcd}
(G\circ F)\circ(G\circ F)\circ(G\circ F) \arrow[r, "\mu\otimes\mathrm{id}_{G\circ F}"] \arrow[d, "\mathrm{id}_{G\circ F}\otimes\mu"'] & (G\circ F)\circ(G\circ F) \arrow[d, "\mu"] \\
(G\circ F)\circ(G\circ F) \arrow[r, "\mu"'] & (G\circ F)
\end{tikzcd}$$

becomes

(*)
$$\begin{tikzcd}
G(F(G(F(G(F(X)))))) \arrow[r, "G(\varepsilon(F(G(F(X)))))"] \arrow[d, "G(F(G(\varepsilon(F(X)))))"'] & G(F(G(F(X)))) \arrow[d, "G(\varepsilon(F(X)))"] \\
G(F(G(F(X)))) \arrow[r, "G(\varepsilon(F(X)))"'] & G(F(X))
\end{tikzcd}$$

Let $Y = F(X)$ and $u = \varepsilon(Y) : F(G(Y)) \to Y$. As $\varepsilon : F \circ G \to \mathrm{id}_{\mathscr{D}}$ is a morphism of functors, the following square is commutative

$$\begin{tikzcd}
F(G(F(G(Y)))) \arrow[r, "\varepsilon(F(G(Y)))"] \arrow[d, "F(G(u))"'] & F(G(Y)) \arrow[d, "u"] \\
F(G(Y)) \arrow[r, "\varepsilon(Y)"'] & Y
\end{tikzcd}$$

Applying the functor F to this square, we recover the square (*), so (*) is also commutative.

$\square$

2. Geometric realization of a simplicial set Remember that the simplicial category Δ is the subcategory of **Set** whose objects are the sets $[n] = \{0, 1, \ldots, n\}$, for $n \in \mathbb{N}$, and whose morphisms are nondecreasing maps (where we put the usual order on $[n]$). The category of simplicial sets **sSet** is defined by **sSet** $= \mathrm{PSh}(\Delta) = \mathrm{Func}(\Delta^{\mathrm{op}}, \mathbf{Set})$; if X is a simplicial set, we write X_n for $X([n])$ and $\alpha^* : X_m \to X_n$ for $X(\alpha) : X([m]) \to X([n])$ (if $\alpha : [n] \to [m]$ is a nondecreasing map). The standard n-simplex Δ is the simplicial set represented by $[n]$, i.e. $\mathrm{Hom}_\Delta(\cdot, [n])$.

(a). Let $\mathscr{C}$ be a category and $F : \mathscr{C}^{\text{op}} \to \mathbf{Set}$ be a presheaf on $\mathscr{C}$. We consider the category $\mathscr{C}/F$ whose objects are pairs (X, x), with $X \in \text{Ob}(\mathscr{C})$ and $x \in F(X)$, and such that a morphism $(X, x) \to (Y, y)$ is a morphism $f : X \to Y$ in $\mathscr{C}$ with $F(f)(y) = x$. Note that we have an obvious faithful functor $G_F : \mathscr{C}/F \to \mathscr{C}$ (forgetting the second entry in a pair), so we get a functor $h_{\mathscr{C}} \circ G_F : \mathscr{C}/F \to \text{PSh}(\mathscr{C})$.

(i) (1 point) When does $\mathscr{C}/F$ have a terminal object ?

(ii) (2 points) Show that $\varinjlim(h_{\mathscr{C}} \circ G_F) = F$. (Hint : Use the second entries of the pairs to construct a morphism from $\varinjlim(h_{\mathscr{C}} \circ G_F)$ to F.) [3]

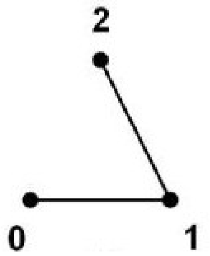
For every $n \in \mathbb{N}$, let $|\Delta_n| = \{(x_0, \ldots, x_n) \in [0,1]^{n+1} \mid x_0 + \ldots + x_n = 1\}$ with the subspace topology. If $f : [n] \to [m]$ is a map, we define $|f| : |\Delta_n| \to |\Delta_m|$ by $|f|(x_0, \ldots, x_n) = (\sum_{i \in f^{-1}(j)} x_i)_{0 \le j \le m}$. (With the convention that an empty sum is equal to 0.) Consider the functor $|.| : \Delta \to \mathbf{Top}$ sending $[n]$ to $|\Delta_n|$ and $f : [n] \to [m]$ to $|f|$.

Let X be a simplicial set, and consider the functor $G_X : \Delta/X \to \Delta$ of (a). The *geometric realization of* X is by definition the topological space $|X| = \varinjlim(|.| \circ G_X)$.

(b). (1 points) Show that this construction upgrades to a functor $|.| : \mathbf{sSet} \to \mathbf{Top}$. [4]

(c). (2 points) Show that, if X is Δ_n, then $|X| = |\Delta_n|$.

(d). (1 point) Give a simplicial set whose geometric realization is $\{(x_0, x_1, x_2) \in [0,1]^2 \mid x_0 = 0 \text{ or } x_2 = 0\}$. (Hint: why are the horns called horns ?)



(e). (2 points) Consider the functor $\text{Sing} : \mathbf{Top} \to \mathbf{sSet}$ given by $\text{Sing}(X) = \text{Hom}_{\mathbf{Top}}(|.|, X) : \Delta^{\text{op}} \to \mathbf{Set}$. (That is, if X is a topological space, then $\text{Sing}(X)$ is the simplicial set such that $\text{Sing}(X)_n$ is the set of continuous maps from $|\Delta_n|$ to X, and, if $f : [n] \to [m]$ is nondecreasing, then $f^* : \text{Sing}(X)_m \to \text{Sing}(X)_n$ sends a continuous map $u : |\Delta_m| \to X$ to $u \circ |f|$.) The simplicial set $\text{Sing}(X)$ is called the *singular simplicial complex of* X of X.

Show that $(|.|, \text{Sing})$ is a pair of adjoint functors.

Solution.

(a). (i) Suppose that (X, x) is a terminal object of $\mathscr{C}/F$. Let Y be an object of $\mathscr{C}$, and consider the map $\phi : \text{Hom}_{\mathscr{C}}(Y, X) \to F(Y)$ sending $f : Y \to X$ to $F(f)(x) \in F(Y)$. (Remember that F is a contravariant functor on $\mathscr{C}$.) We claim that ϕ is bijective. Indeed, if $f, g : Y \to X$ are two morphisms such that $F(f)(x) = F(g)(x)$, then they define morphisms from $(Y, F(f)(x))$ to (X, x) in the category $\mathscr{C}/F$, hence must be equal; so ϕ is injective. Also, if $y \in F(Y)$, then (Y, y) is an object of $\mathscr{C}/F$, so there exists a morphism $h : (Y, y) \to (X, x)$ in $\mathscr{C}/F$, that is, a morphism $h : Y \to X$ in $\mathscr{C}$ such that $F(h)(x) = y$; so ϕ is surjective.

[3]So every presheaf is a colimit of representable presheaves.

[4]This functor is called the *left Kan extension* of $|.| : \Delta \to \mathbf{Top}$ along the Yoneda embedding $\Delta \to \mathbf{sSet}$.

This proves that a terminal object in $\mathscr{C}/F$ is exactly a pair representing the functor F, so such a terminal object exists if and only if F is representable.

(ii) If $X \in \mathrm{Ob}(\mathscr{C})$ and $x \in F(X)$, then, by the Yoneda lemma, there is unique morphism $u_x : h_X \to F$ in $\mathrm{PSh}(\mathscr{C})$ such that $u_x(X)(\mathrm{id}_X) = x$. We claim that the family of these morphisms defines a cone under $h_{\mathscr{C}} \circ G_F$ with nadir F. This claim means that, for any two objects (X, x) and (Y, y) in $\mathscr{C}/F$ and any morphism $f : (X, x) \to (Y, y)$, the following diagram commutes :

$$\begin{tikzcd} h_X \arrow[r, "h_f"] \arrow[d, "u_x"'] & h_Y \arrow[dl, "u_y"] \\ F & \end{tikzcd}$$

As the morphism $u_y \circ h_f : h_X \to F$ sends $\mathrm{id}_X \in h_X(X)$ to $u_y(X)(f \circ \mathrm{id}_X) = F(f)(y) = F(x) = u_x(X)(\mathrm{id}_X)$, we have $u_y \circ h_f = u_x$ by the Yoneda lemma, so the diagram commutes, as desired.

By the universal property of the colimit, this gives a morphism $\phi : \varinjlim(h_{\mathscr{C}} \circ G_F) \to F$ in $\mathrm{PSh}(\mathscr{C})$.

Now we show that ϕ is an isomorphism. Let $F' = \varinjlim(h_{\mathscr{C}} \circ G_F)$. This is a colimit in the category of presheaves on $\mathscr{C}$, so we can use Proposition I.5.3.1 of the notes to compute it. Let Z be an object of $\mathscr{C}$. Then $F'(Z) = \varinjlim_{(X,x) \in \mathrm{Ob}(\mathscr{C}/F)} \mathrm{Hom}_{\mathscr{C}}(Z, X)$, and the map $\phi(Z) : F'(Z) \to F(Z)$ sends a morphism $f : Z \to X$ to $F(f)(x) \in F(Z)$. If $z \in F(Z)$, then (Z, z) is an object of $\mathscr{C}/F$, and $\phi(Z)(\mathrm{id}_Z) = z$; this shows that $\phi(Z)$ is surjective. Let (X, x) and (Y, y) be two objects of $\mathscr{C}/F$, let $f : Z \to X$ and $g : Z \to Y$ be morphisms of $\mathscr{C}$, and suppose that $F(f)(x) = F(g)(y)$. Let $z = F(f)(x)$. Then (Z, z) is an object of $\mathscr{C}/F$, the morphisms f and g induce morphisms $(Z, z) \to (X, x)$ and $(Z, z) \to (Y, y)$ in $\mathscr{C}/F$, and, in the square

$$\begin{tikzcd} \mathrm{Hom}_{\mathscr{C}}(Z, Z) \arrow[r, "\mathrm{Hom}_{\mathscr{C}}(Z,f)"] \arrow[d, "\mathrm{Hom}_{\mathscr{C}}(Z,g)"'] & \mathrm{Hom}_{\mathscr{C}}(Z, X) \arrow[d, "\phi(Z)"] \\ \mathrm{Hom}_{\mathscr{C}}(Z, Y) \arrow[r, "\phi(Z)"'] & F(Z) \end{tikzcd}$$

the element id_Z of $\mathrm{Hom}_{\mathscr{C}}(Z, Z)$ is sent to the same element z of Z by both paths. So the images of f and g in $F'(Z)$ are equal, which proves that $\phi(Z)$ is injective.

(b). For X a simplicial set, we set

$$L(X) = \coprod_{n \in \mathbb{N}} \coprod_{x \in X_n} |\Delta_n|,$$

so that $|X|$ is the quotient of $L(X)$ by the equivalence relation $\sim$ of Theorem I.5.2.1 of the notes, with the quotient topology. If $f : X \to Y$ is a morphism of simplicial sets, we denote by $L(f)$ a continuous map $L(X) \to L(Y)$ that, for each $n \in \mathbb{N}$ and each $x \in X_n$, sends the component $|\Delta_n|$ of $L(X)$ corresponding to (n, x) to the component $|\Delta_n|$ of $L(Y)$ corresponding to $(n, f_n(x))$ by $\mathrm{id}_{|\Delta_n|}$. This clearly defines a functor $L : \mathbf{sSet} \to \mathbf{Top}$. To show that $|.|$ upgrades to a functor, it suffices to show that, for every morphism $f : X \to Y$ in $\mathbf{sSet}$, the map $f' : L(X) \overset{L(f)}{\to} L(Y) \to |Y|$ factors through the quotient map $L(X) \to |X|$. Fix f, let $n, m \in \mathbb{N}$, $x \in X_n$, $y \in X_m$, $s \in |\Delta_n|$ and $t \in |\Delta_m|$ such that the images of $(n, x, s), (m, y, t) \in L(X)$ in $|X|$ are equal; we want to show that the images of

$(n, f_n(x), s), (m, f_m(y), t) \in L(Y)$ in $|Y|$ are also equal. We may assume that there exists $\alpha : [n] \to [m]$ such that $x = \alpha^*(y)$ and $t = |\alpha|(s)$. Then $f_n(x) = f_n(\alpha^*(y)) = \alpha^*(f_m(y))$, so $(n, f_n(x), s)$ and $(m, f_m(y), t)$ have the same image in $|Y|$.

(c). By (a)(i), the category Δ/Δ_n has a terminal object, which is $([n], \mathrm{id}_{[n]})$. It follows immediately from the definition of a cone under a functor that a cone $(S, (u_{m,x})_{m\in\mathbb{N}, x\in\Delta_n([m])})$ under $|.| \circ G_{\Delta_n}$ is uniquely determined by the continuous map $u_{n,\mathrm{id}_{[n]}} : |\Delta_n| \to S$, and that this map can be arbitrary. In other words, the functor sending a topological space S to the space of cones under $|.| \circ G_{\Delta_n}$ with nadir S is representable by $|\Delta_n|$. This means that $|\Delta_n| = \varinjlim(|.| \circ G_{\Delta_n}) = |\Delta_n|$.

(d). Let's take $X = \Lambda^2_1$ (see problem 9 of problem set 1). The geometric relaization $|X|$ is the quotient of $\coprod_{n\in\mathbb{N}} \coprod_{x\in X_n} |\Delta_n|$ by the equivalence relation $\sim$ of Theorem I.5.2.1 of the notes.

By definition, for every $n \in \mathbb{N}$, the set X_n is the set of nondecreasing maps $\alpha : [n] \to [2]$ such that $\{0, 2\} \not\subset \mathrm{Im}(\alpha)$. In particular, such a map always factors as $\alpha = \beta \circ \gamma$ with $\gamma : [n] \to [1]$ and $\beta : [1] \to [2]$ two nondecreasing maps such that $\beta \in X_1$, so $\alpha = \gamma^*(\beta)$, so, for every $s \in |\Delta_n|$, we have $(n, \alpha, s) \sim (1, \beta, |\gamma|(s))$. This means that $|X|$ is homeomorphic to the quotient of $\coprod_{n\in\{0,1\}} \coprod_{x\in X_n} |\Delta_n|$ by the relation of $\sim$.

For every $i \in [2]$, let $\alpha_i : [0] \to [2]$ be the map $0 \longmapsto i$, and $\delta_i : [1] \to [2]$ be the unique increasing map such that $\mathrm{Im}(\delta_i) = [2] - \{i\}$. Let β be the unique map from $[1]$ to $[0]$. Then $X_0 = \{\alpha_0, \alpha_1, \alpha_2\}$ and $X_1 = \{\delta_0, \delta_2, \alpha_0 \circ \beta, \alpha_1 \circ \beta, \alpha_2 \circ \beta\}$. Also, for every $i \in [2]$ and every $s \in |\Delta_1|$, we have $(1, \alpha_i \circ \beta, s) \sim (0, \alpha_i, |\beta|(s))$. So $|X|$ is the quotient of the disjoint union of three points corresponding to $\alpha_0, \alpha_1, \alpha_2$, say 0, 1 and 2, and of two line segments (homeomorphic to $[0, 1]$) corresponding to δ_0, δ_2, say I_0 and I_2, by the restriction of $\sim$. It is easy to see that this equivalence relation identifies the two extremities of I_0 (resp. I_2) with 1 and 2 (resp. 0 and 1), so $|X|$ is homeomorphic to the space of the figure.

(e). Let X be a simplicial set and Y be a topological space. By definition, we have $|X| = \varinjlim_{\Delta/X}(|.| \circ G_X)$, so, by Proposition I.5.3.4 of the notes, we have an isomorphism

$$\mathrm{Hom}_{\mathbf{Top}}(|X|, Y) \simeq \varprojlim_{(n,x)\in\mathrm{Ob}((\Delta/X)^{\mathrm{op}})} \mathrm{Hom}_{\mathbf{Top}}(|\Delta_n|, Y) = \varprojlim_{(n,x)\in\mathrm{Ob}((\Delta/X)^{\mathrm{op}})} \mathrm{Sing}(Y)_n.$$

Also, by question (a)(ii), we have $X = \varinjlim_{\Delta/X} G_X$, so, by the same proposition, we have

$$\mathrm{Hom}_{\mathbf{sSet}}(X, \mathrm{Sing}(Y)) \simeq \varprojlim_{(n,x)\in\mathrm{Ob}((\Delta/X)^{\mathrm{op}})} \mathrm{Hom}_{\mathbf{sSet}}(\Delta_n, \mathrm{Sing}(Y)) \simeq \varprojlim_{(n,x)\in\mathrm{Ob}((\Delta/X)^{\mathrm{op}})} \mathrm{Sing}(Y)_n$$

(the last isomorphism comes from the Yoneda lemma). So we get an isomorphism

$$\mathrm{Hom}_{\mathbf{Top}}(|X|, Y) \simeq \mathrm{Hom}_{\mathbf{sSet}}(X, \mathrm{Sing}(Y)),$$

and checking that it is an isomorphism of functors is straightforward.

□

3. Yoneda embedding and colimits Let k be a field, and let $\mathscr{C}$ be the category of k-vector spaces.

(a). (1 point) For every $n \in \mathbb{N}$, let $k[x]_{\leq n}$ be the vector space of polynomials of degree $\leq n$ in $k[x]$. Using the inclusions $k[x]_{\leq n} \subset k[x]_{\leq m}$ for $n \leq m$, we get a functor $F : \mathbb{N} \to \mathscr{C}$, $n \longmapsto k[x]_{\leq n}$. Show that $\varinjlim F = k[x]$.

(b). (2 points) Show that $h_{\mathscr{C}} : \mathscr{C} \to \mathrm{PSh}(\mathscr{C})$ does not commute with all colimits.

Solution.

(a). Note that the colimit is filtrant, because $\mathbb{N}$ is a directed poset. By an easy analogue Proposition I.5.6.3 of the notes to conclude that the $\varinjlim F$ is the quotient of $\bigoplus_{n\in\mathbb{N}} k[x]_{\leq n}$ by the subspace generated by the images of all the maps $u_{m,i} : k[x]_{\leq m} \to \bigoplus_{n\in\mathbb{N}} k[x]_{\leq n}$ sending $f \in k[x]_{\leq m}$ to $(f, -f)$, where the first entry is in the summand $k[x]_{\leq m}$ and the second entry is in the summand $k[x]_{\leq m+i}$, for every $m \in \mathbb{N}$ and every $i \geq 1$. [5] So the sum map from $\bigoplus_{n\in\mathbb{N}} k[x]_{\leq n} \to k[x]$ (sending a family $(f_0, f_1, \ldots)$ with finite support to $f_0 + f_1 + \ldots$) factors through $\varinjlim F$ and induces an isomorphism $\varinjlim F \xrightarrow{\sim} k[x]$.

(b). Let $V = k[x]$. We have seen in (a) that $V = \varinjlim_{n\in\mathbb{N}} k[x]_{\leq n}$, so we get a morphism of presheaves $u : \varinjlim_{n\in\mathbb{N}} h_{k[x]_{\leq n}} \to h_V$. If W is a k-vector space, $u(W)$ is the map from $(\varinjlim_{n\in\mathbb{N}} h_{k[x]_{\leq n}})(W) = \varinjlim_{n\in\mathbb{N}} \mathrm{Hom}_k(W, k[x]_{\leq n})$ to $\mathrm{Hom}_k(W, V)$ induced by the obvious injections $\mathrm{Hom}_k(W, k[x]_{\leq n}) \subset \mathrm{Hom}_k(W, V)$. So the image of $u(W)$ is the set of k-linear maps from W to V whose image is contained in one of the subspaces $k[x]_{\leq n}$ of V. In particular, $\mathrm{id}_V \in h_V(V)$ is not in the image of $u(V)$, so u is not an isomorphism.

$\square$

4. Filtrant colimits of modules (3 points)

Let R be a ring, let $\mathscr{I}$ be a filtrant category and let $F : \mathscr{I} \to {}_R\mathbf{Mod}$ be a functor. For every $i \in \mathrm{Ob}(\mathscr{I})$, we write $M_i = F(i)$. Let $\sim$ be the equivalence relation on $\coprod_{i\in\mathrm{Ob}(\mathscr{I})} M_i$ defined in Proposition I.5.6.2 of the notes; so $(i, x) \sim (j, y)$ if there exist morphisms $\alpha : i \to k$ and $\beta : j \to k$ in $\mathscr{I}$ such that $F(\alpha)(x) = F(\beta)(y)$). Let $M = \coprod_{i\in\mathrm{Ob}(\mathscr{I})} M_i/\sim$; this is the colimit of the composition $\mathscr{I} \xrightarrow{F} {}_R\mathbf{Mod} \xrightarrow{\mathrm{For}} \mathbf{Set}$. Denote by $q_i : M_i \to M$ the obvious maps.

Show that there exists a unique structure of left R-module on M such that all the q_i are R-linear maps, and that this structure makes $(M, (q_i))$ into a colimit of F.

Solution. Let $X = \coprod_{i\in I} M_i$. If (i, m) and (i, n) are elements of X such that $(i, m) \sim (j, n)$, and if $a \in R$, then $(i, m) \sim (j, n)$ (because the maps $F(\alpha)$ are all R-linear). So the action of R by left multiplication on X descends to an action on M. Now let (i_1, m_1) and (i_2, m_2) be elements of X. Choose morphisms $\alpha_1 : i_1 \to j$ and $\alpha_2 : i_2 \to i$ in $\mathscr{I}$. Then $(i_1, m_1) \sim (i, F(\alpha_1)(m_1))$ and $(i_2, m_2) \sim (i, F(\alpha_1)(m_2))$, so, if M has a structure of abelian group such that the map $M_i \to M$ is additive, this forces the image of $(i, F(\alpha_1)(m_1) + F(\alpha_2)(m_2))$ in M to be the sum of the images of (i_1, m_1) and (i_2, m_2) in M. We must check that this definition of addition does not depend on the choices, so we take $(j_1, n_1), (j_2, n_2) \in X$ such that $(j_1, n_1) \sim (i_1, m_1)$ and $(j_2, n_2) \sim (i_2, m_2)$. Choose morphisms $\alpha'_1 : j_1 \to j$ and $\alpha'_2 : j_2 \to j$. We want to check that $(i, F(\alpha_1)(m_1) + F(\alpha_2)(m_2)) \sim (j, F(\alpha'_1)(n_1) + F(\alpha'_2)(n_2))$. The hypothesis on (j_1, n_1) and (j_2, n_2) means that there exist morphisms $\beta_1 : i_1 \to k_1$, $\gamma_1 : j_1 \to k_1$, $\beta_2 : i_2 \to k_2$ and $\gamma_2 : j_2 \to k_2$ in $\mathscr{I}$ such that $F(\beta_1)(m_1) = F(\gamma_1)(n_1)$ and $F(\beta_2)(m_2) = F(\gamma_2)(n_2)$. As $\mathscr{I}$ is filtrant, we can find an object l of $\mathscr{I}$ and morphisms $\delta : i \to l$, $\delta_1 : k_1 \to l$, $\delta_2 : k_2 \to l$ and $\delta' : j \to l$, and then we can find a morphism $\epsilon : l \to l'$ such that

$$\epsilon \circ \delta \circ \alpha_1 = \epsilon \circ \delta_1 \circ \beta_1 : i_1 \to l',$$

$$\epsilon \circ \delta \circ \alpha_2 = \epsilon \circ \delta_2 \circ \beta_2 : i_2 \to l',$$

[5] We could also use problem 4 to calculate the colimit.

$$\epsilon \circ \delta' \circ \alpha'_1 = \epsilon \circ \delta_1 \circ \gamma_1 : j_1 \to l',$$

and

$$\epsilon \circ \delta' \circ \alpha'_2 = \epsilon \circ \delta_2 \circ \gamma_2 : i_1 \to l'.$$

Then

$$\begin{aligned}(l', F(\epsilon \circ \delta)(F(\alpha_1)(m_1) + F(\alpha_2)(m_2))) &= (l', F(\epsilon)(F(\delta_1 \circ \beta_1)(m_1) + F(\delta_2 \circ \beta_2)(m_2))) \\ &= (l', F(\epsilon)(F(\delta_1 \circ \gamma_1)(n_1) + F(\delta_2 \circ \gamma_2)(n_2))) \\ &= (l', F(\epsilon \circ \delta')(F(\alpha'_1)(n_1) + F(\alpha'_2)(n_2))),\end{aligned}$$

which implies that $(i, F(\alpha_1)(m_1) + F(\alpha_2)(m_2)) \sim (j, F(\alpha'_1)(n_1) + F(\alpha'_2)(n_2))$.

The fact that these two operations define a left R-module structure no M follows easily from their definition and from the fact that the M_i are left R-modules.

The obvious R-module maps $q_i : M_i \to M$ define a cone under F with apex M in the category ${}_R\mathbf{Mod}$. Let $(N, (v_i)_{I \in \mathrm{Ob}(\mathscr{I})})$ be another cone under F in ${}_R\mathbf{Mod}$. In particular, this defines a cone under $\mathrm{For} \circ F$ in $\mathbf{Set}$, where $\mathrm{For} : {}_R\mathbf{Mod} \to \mathbf{Set}$ is the forgetful functor. So there is a unique map $f : M \to N$ such that $f \circ q_i = v_i$ for every $i \in \mathrm{Ob}(\mathscr{I})$. We need to show that f is R-linear. Let $x_1, x_2 \in M$ and $a \in R$. We choose elements (i_1, m_1) and (i_2, m_2) of $\coprod_{i \in \mathrm{Ob}(\mathscr{I})} M_i$ representing x_1 and x_2; as we have seen in the definition of the addition on M, we may assume that $i_1 = i_2$. Then ax_1 is represented by (i_1, am_1), so $f(ax_1) = v_{i_1}(am_1) = av_{i_1}(m_1) = af(x_1)$, and $x_1 + x_2$ is represented by $(i_1, m_1 + m_2)$, so $f(x_1 + x_2) = v_{i_1}(m_1 + m_2) = v_{i_1}(m_1) + v_{i_1}(m_2) = f(x_1) + f(x_2)$.

$\square$

5. Filtrant colimits are exact (3 points)

Let R be a ring and $\mathscr{I}$ be a filtrant category. Show that the functor $\varinjlim : \mathrm{Func}(\mathscr{I}, {}_R\mathbf{Mod}) \to {}_R\mathbf{Mod}$ is exact, i.e. that if $u : F \to G$ and $v : G \to H$ are morphism of functors from $\mathscr{I}$ to ${}_R\mathbf{Mod}$ such that the sequence $0 \to F(i) \stackrel{u(i)}{\to} G(i) \stackrel{v(i)}{\to} H(i) \to 0$ is exact for every $i \in \mathrm{Ob}(\mathscr{I})$, then the sequence $0 \to \varinjlim F \stackrel{\varinjlim u}{\to} \varinjlim G \stackrel{\varinjlim v}{\to} \varinjlim H \to 0$ is exact. (Remember that we say that a sequence of R-modules $0 \to M \stackrel{f}{\to} N \stackrel{g}{\to} P \to 0$ is exact if $\mathrm{Ker}\, f = 0$, $\mathrm{Ker}\, g = \mathrm{Im}\, f$ and $\mathrm{Im}\, g = P$.)

Solution. First we note that, if $f : M \to N$ is a morphism of ${}_R\mathbf{Mod}$, then $\mathrm{Ker}(f) = \mathrm{Ker}(f, 0)$ is a finite limit in ${}_R\mathbf{Mod}$ and $\mathrm{Coker}(f) = \mathrm{Coker}(f, 0)$ is a (finite colimit). Also, we have $\mathrm{Im}(f) = \mathrm{Ker}(\mathrm{Coker}(f))$, and so $\mathrm{Im}(f) = N$ if and only if $\mathrm{Coker}(f) = 0$.

By Subsection I.5.4.1 of the notes and Corollary I.5.6.5 of the notes, we have (with the notation of the problem)

$$\mathrm{Ker}(\varinjlim u) = \varinjlim_{i \in \mathrm{Ob}(\mathscr{I})} \mathrm{Ker}(u(i)) = \varinjlim_{i \in \mathrm{Ob}(\mathscr{I})} 0 = 0$$

and

$$\mathrm{Coker}(\varinjlim v) = \varinjlim_{i \in \mathrm{Ob}(\mathscr{I})} \mathrm{Ker}(v(i)) = \varinjlim_{i \in \mathrm{Ob}(\mathscr{I})} 0 = 0.$$

Also,

$$\mathrm{Coker}(\varinjlim u) = \varinjlim_{i \in \mathrm{Ob}(\mathscr{I})} \mathrm{Coker}(u(i)),$$

so

$$\begin{aligned}\mathrm{Im}(\varinjlim u) = \mathrm{Ker}(\mathrm{Coker}(\varinjlim u)) &= \varinjlim_{i \in \mathrm{Ob}(\mathscr{I})} \mathrm{Ker}(\mathrm{Coker}(u(i))) \\ &= \varinjlim_{i \in \mathrm{Ob}(\mathscr{I})} \mathrm{Im}(u(i)) \\ &= \varinjlim_{i \in \mathrm{Ob}(\mathscr{I})} \mathrm{Ker}(v(i)) \\ &= \mathrm{Ker}(\varinjlim v).\end{aligned}$$

□

6. Objects of finite type and of finite presentation Let $\mathscr{C}$ a category that admits all filtrant colimits (indexed by small enough categories). An object X of $\mathscr{C}$ is called *of finite type* (resp. *of finite presentation* or *compact*) if, for every filtrant category $\mathscr{I}$ and every functor $F : \mathscr{I} \to \mathscr{C}$, the canonical map

$$\varinjlim_{i \in \mathrm{Ob}(\mathscr{I})} \mathrm{Hom}_{\mathscr{C}}(X, F(i)) \to \mathrm{Hom}_{\mathscr{C}}(X, \varinjlim F)$$

(see the beginning of Subsection I.5.4.2 of the notes) is injective (resp. bijective).

(a). Let R be a ring and M be a left R-module.

(i) (1 point) If M is free of finite type as a R-module, show that it is of finite presentation as an object of ${}_R\mathbf{Mod}$.

(ii) (2 points) If M is of finite type (resp. of finite presentation) as a R-module, show that it is of finite type (resp. of finite presentation) as an object of ${}_R\mathbf{Mod}$.

(iii) (1 point) Let $\mathscr{I}$ the poset of R-submodules of M that are of finite type, ordered by inclusion, and let $F : \mathscr{I} \to {}_R\mathbf{Mod}$ be the functor sending $N \subset M$ to M/N; if $N \subset N' \subset M$, we send the unique morphism $N \to N'$ in $\mathscr{I}$ to the canonical projection $M/N \to M/N'$. Show that $\varinjlim F = 0$.

(iv) (2 points) If M is of finite type (resp. of finite presentation) as an object of ${}_R\mathbf{Mod}$, show that it is of finite type (resp. of finite presentation) as an R-module.

(b). (6 points, extra credit) Let R be a commutative ring and S be a commutative R-algebra. Show that S is finitely presented as an R-algebra if and only if it is of finite presentation as an object of $R - \mathbf{CAlg}$.

(c). (i) (1 point) If X is a finite set with the discrete topology, show that X is of finite presentation as an object of **Top**.

(ii) (1 point) Let X be a topological space. Let $\mathscr{I}$ be the poset of finite sets of X ordered by inclusion; wee see $\mathscr{I}$ as a subcategory of **Top** (we use the subset topology on each finite $Y \subset X$), and we denote by $F : \mathscr{I} \to \mathbf{Top}$ the inclusion functor. Show that $X = \varinjlim F$ if the topology on X is the indiscrete (= coarse) topology.

(iii) (1 point) Let X be a topological space. If X is of finite presentation as an object of **Top**, show that it is finite.

(iv) (2 points) For $n \in \mathbb{N}$, let $X_n = \mathbb{N}_{\geq n} \times \{0, 1\}$, with the topology for which the open subsets are $\varnothing$ and $(\mathbb{N}_{\geq m} \times \{0\}) \cup (\mathbb{N}_{\geq n} \times \{1\})$, for $m \geq n$. Define $f_n : X_n \to X_{n+1}$

by $f_n(n,a) = (n+1,a)$ and $f_n(m,a) = (m,a)$ if $m > n$. Show that the X_n are topological spaces and that the maps f_n are continuous.

(v) (2 points) Show that $\varinjlim_{n\in\mathbb{N}} X_n$ is $\{0,1\}$ with the indiscrete topology. By $\varinjlim_{n\in\mathbb{N}} X_n$, we mean the colimit of the functor $F : \mathbb{N} \to \mathbf{Top}$ such that $F(n) = X_n$ and that, for each non-identity morphism $\alpha : n \to m$ in $\mathbb{N}$, that is, for $n < m$ in $\mathbb{N}$, $F(\alpha) = f_{m-1} \circ f_{m-2} \circ \ldots \circ f_n : X_n \to X_m$.

(vi) (2 points) Let X be a topological space. If X is of finite presentation as an object of **Top**, show that X is finite and has the discrete topology.

(d). (2 points) Let X be a topological space, and let $\mathrm{Open}(X)$ be the set of open subsets of X, ordered by inclusion. Show that X is compact if and only if X is of finite presentation as an object of $\mathrm{Open}(X)$.

Solution.

(a). (i) We can deduce this from the facts that :

- $\mathrm{Hom}_R(R, N) = N$ for every left R-module N (so R is of finite presentation as an object of ${}_R\mathbf{Mod}$);
- $\mathrm{Hom}_R(M_1 \oplus M_2, \cdot) = \mathrm{Hom}_R(M_1, \cdot) \oplus \mathrm{Hom}_R(M_2, \cdot)$ (so the direct sum of two objects of ${}_R\mathbf{Mod}$ of finite type (resp. of finite presentation) is also of finite type (resp. of finite presentation)).

Alternately, here is a very categorical way to answer the question. Let (F, G) be a pair of adjoint functors, with $F : \mathscr{C} \to \mathscr{D}$ and $G : \mathscr{D} \to \mathscr{C}$. Suppose that all filtrant colimits exist in $\mathscr{C}$ and $\mathscr{D}$ and that G commutes with filtrant colimits. Then we claim that F sends objects of finite presentation in $\mathscr{C}$ to objects of finite presentation in $\mathscr{D}$. Indeed, let $X \in \mathrm{Ob}(\mathscr{C})$. Then, for every functor $\alpha : \mathscr{I} \to \mathscr{D}$, with $\mathscr{I}$ filtrant, we have a commutative diagram :

$$\begin{tikzcd}
\varinjlim_{i\in\mathrm{Ob}(\mathscr{I})} \mathrm{Hom}_{\mathscr{D}}(F(X), \alpha(i)) \arrow[rr, "\sim"] \arrow[d] & & \varinjlim_{i\in\mathrm{Ob}(\mathscr{I})} \mathrm{Hom}_{\mathscr{C}}(X, G(\alpha(i))) \arrow[d] \\
\mathrm{Hom}_{\mathscr{D}}(F(X), \varinjlim \alpha) \arrow[r, "\sim"'] & \mathrm{Hom}_{\mathscr{C}}(X, G(\varinjlim \alpha)) \arrow[r, "\sim"'] & \mathrm{Hom}_{\mathscr{C}}(X, \varinjlim (G \circ \alpha))
\end{tikzcd}$$

If X is of finite presentation, then the rigth vertical morphism is an isomorphism, so the left vertical morphism also is.

We apply this to the pair of adjoint functors (Φ, For), where $\mathrm{For} : {}_R\mathbf{Mod} \to \mathbf{Set}$ is the forgetful functor and $\Phi : \mathbf{Set} \to {}_R\mathbf{Mod}$ sends a set X to the free left R-module on X. The fact that For commutes with filtrant colimits is Corollary I.5.6.3 of the notes. So it suffices to prove that finite sets are objects of finite presentation in **Set**. This follows from the fact that $\mathrm{Hom}_{\mathbf{Set}}(X, \cdot) = (\cdot)^X$ for every set X, and from Proposition I.5.6.4 of the notes. (It is also easy to see directly.)

(ii) Suppose that M is of finite type. Then we have an exact sequence $0 \to P \to N \to M \to 0$, with N free of finite type. Let $F : \mathscr{I} \to {}_R\mathbf{Mod}$ be a functor, with $\mathscr{I}$ filtrant. By problem 5 and the exactness properties of Hom_R, we have a commutative diagram with exact columns :

(*)
$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] \\
 & \varinjlim_{i\in\mathrm{Ob}(\mathscr{I})}\mathrm{Hom}_R(M,F(i)) \arrow[r, "(1)"] \arrow[d] & \mathrm{Hom}_R(M,\varinjlim F) \arrow[d] \\
 & \varinjlim_{i\in\mathrm{Ob}(\mathscr{I})}\mathrm{Hom}_R(N,F(i)) \arrow[r, "(2)"] \arrow[d] & \mathrm{Hom}_R(N,\varinjlim F) \arrow[d] \\
 & \varinjlim_{i\in\mathrm{Ob}(\mathscr{I})}\mathrm{Hom}_R(P,F(i)) \arrow[r, "(3)"] & \mathrm{Hom}_R(P,\varinjlim F)
\end{tikzcd}$$

By question (i), the arrow labeled (2) is an isomorphism, so the arrow labeled (1) is injective, which is what we wanted to prove.

Now assume that M is of finite presentation. Then we have an exact sequence $0\to P\to N\to M\to 0$, with N free of finite type and P of finite type. So, if we write the diagram (*) again, the arrow labeled (2) is an isomorphism by (i), and the arrow labeled (3) is injective by the previous paragraph. This implies that the arrow labeled (1) is an isomorphism,[6] which is what we wanted.

(iii) Note that $\mathscr{I}$ is a filtrant category, because it comes from a directed poset. (If N and N' are two submodules of finite type of M, then they are both contained in $N+N'$, which is also of finite type.) So we can use problem 4 to calculate $\varinjlim F$. Let $x\in\varinjlim F$, and let (N,m) be an element of $\coprod_{N\in\mathrm{Ob}(\mathscr{I})}(M/N)$ representing it (so N is a submodule of M of finite type, and $m\in M/N$). Then there exists a submodule N' of M of finite type such that $N\subset N'$ and that the image of m in M/N' is 0 (just take the submodule N' generated by N and by a preimage of m in M), so $(N,m)\sim(N',0)$ in $\coprod_{N\in\mathrm{Ob}(\mathscr{I})}(M/N)$, and so $x=0$. This shows that $\varinjlim F=0$.

(iv) Suppose that M is of finite type as an object of ${}_R\mathbf{Mod}$. Using the functor $F:\mathscr{I}\to{}_R\mathbf{Mod}$ of (iii), we see that the canonical morphism

$$\varinjlim_{N\in\mathrm{Ob}(\mathscr{I})}\mathrm{Hom}_R(M,M/N)\to\mathrm{Hom}_R(M,0)=0$$

is injective, which means that $\varinjlim_{N\in\mathrm{Ob}(\mathscr{I})}\mathrm{Hom}_R(M,M/N)=0$. Consider $\mathrm{id}_M\in\mathrm{Hom}_R(M,M/0)$. Its image in the filtrant colimit $\varinjlim_{N\in\mathrm{Ob}(\mathscr{I})}\mathrm{Hom}_R(M,M/N)$ is 0, so there exists a morphism $0\to N$ in $\mathscr{I}$ (that is, an object N of $\mathscr{I}$) such that the image of id_M in $\mathrm{Hom}_R(M,M/N)$ is 0. In other words, there exists a submodule N of M of finite type such that $M=N$, which means that M is of finite type.

Now suppose that M is of finite presentation as an object of ${}_R\mathbf{Mod}$. By the previous paragraph, M is a R-module of finite type, so there exists an exact sequence $0\to P\to N\to M\to 0$ with N a free R-module of finite type. We want to show that the R-module P is also of finite type. As in (iii), we consider the category $\mathscr{I}$ associated to the poset of finite type R-submodules of P, and the functor $F.G:\mathscr{I}\to{}_R\mathbf{Mod}$ defined by $F(Q)=P/Q$ and $G(Q)=N/Q$. For every $Q\in\mathrm{Ob}(\mathscr{I})$, we have an exact sequence $0\to F(Q)\to G(Q)\to N/P\to 0$. Using problem 5 and (iii), we get an exact sequence $0\to 0\to\varinjlim G\to N/P\to 0$. In other words, the canonical

[6] By the 4 lemma in the category $\mathbf{Ab}$, which I am assuming that you have seen in a previous class. This also follows from an easy diagram chase.

morphism $\varinjlim_{Q\in\mathrm{Ob}(\mathscr{I})} N/Q \to N/P$ (induced by the projections $N/Q \to N/P$, for $Q \subset P$) is an isomorphism. Using the isomorphism $N/P \overset{\sim}{\to} M$, we get an isomorphism $f : M \overset{\sim}{\to} \varinjlim_{Q\in\mathrm{Ob}(\mathscr{I})} N/Q$. As M is of finite presentation as an object of ${}_R\mathbf{Mod}$, there exists $Q \in \mathrm{Ob}(\mathscr{I})$ and a morphism $g : M \to N/Q$ such that f is the composition $M \overset{f}{\to} N/Q \to N/P$, where the second map is the canonical projection. This implies that the kernel of the morphism $N \to M$ is contained in Q, hence that $P = Q$ is of finite type.

(b). First we show that polynomial rings over R on finitely many indeterminates are of finite presentation as objects of $R-\mathbf{CAlg}$. For this, we apply the second proof of (a)(i) to the pair of adjoint functors (Φ, For), where $\mathrm{For} : R-\mathbf{CAlg} \to \mathbf{Set}$ is the forgetful functor and $\Phi : \mathbf{Set} \to R-\mathbf{CAlg}$ sends a set X to the free commutative R-algebra on X, that is, the polynomial ring $R[X]$. We already know that finite sets are objects of finite presentation in $\mathbf{Set}$. So it remains to check that $\mathrm{For} : R-\mathbf{CAlg} \to \mathbf{Set}$ commutes with filtrant colimits. The proof is exactly the same as for R-modules : using the procedure of problem 4, we show that, if $F : \mathscr{I} \to R-\mathbf{CAlg}$ is a functor with $\mathscr{I}$ filtrant, then there is a unique R-algebra structure on $\varinjlim(\mathrm{For}\circ F)$ that makes all the canonical morphisms $F(i) \to \varinjlim(\mathrm{For}\circ F)$ into R-algebra morphisms, and that $\varinjlim(\mathrm{For}\circ F)$ with this R-algebra structure satisfies the universal property characterizing the colimit of F. (We already know how to define the addition and the action of R, and we define the multiplication using the same trick as for the addition. See the solution of problem 4.)

Let S be a commutative finitely presented R-algebra. We show that S is of finite presentation as an object of $R-\mathbf{CAlg}$. Choose a surjective R-algebra morphism $f : S_0 := R[x_1, \dots, x_n] \to S$ whose kernel is finitely generated; write $\mathrm{Ker}(f) = (a_1, \dots, a_m)$ with $a_1, \dots, a_m \in S_0$, and let $g : S_1 := R[y_1, \dots, y_m] \to S_0$ be the unique R-algebra morphisms such that $g(y_j) = a_j$ for $1 \le j \le m$. For any commutative R-algebra T, we denote by $e_T : S_1 \to T$ the unique R-algebra morphism sending every y_j to 0. Then, if T is a commutative R-algebra, we have a sequence of maps

$$\mathrm{Hom}_{R-\mathbf{CAlg}}(S,T) \overset{u_T}{\to} \mathrm{Hom}_{R-\mathbf{CAlg}}(S_0,T) \overset{v_T}{\to} \mathrm{Hom}_{R-\mathbf{CAlg}}(S_1,T),$$

where $u_T(h) = h \circ f$ and $v_T(h') = h' \circ g$. As $f : S_0 \to S$ is surjective, the map u_T is injective. As the image of $g : S_1 \to S_0$ generates the ideal $\mathrm{Ker}(f)$, a morphism $h' : S_0 \to T$ factors as $S_0 \overset{f}{\to} S \overset{h}{\to} T$ if and only if it is zero on the image of g; in other words, the image of u_T is exactly the set of $h' \in \mathrm{Hom}_{R-\mathbf{CAlg}}(S_0,T)$ such that $v_T(h') = e_T$. In other words. we have just proved that the map u_T identifies the set $\mathrm{Hom}_{R-\mathbf{CAlg}}(S,T)$ with the fiber product of the diagram :

$$\begin{tikzcd} & \mathrm{Hom}_{R-\mathbf{CAlg}}(S_0,T) \arrow[d, "v_T"] \\ \{e_T\} \arrow[r] & \mathrm{Hom}_{R-\mathbf{CAlg}}(S_1,T) \end{tikzcd}$$

Let $* : R-\mathbf{CAlg} \to \mathbf{Set}$ be the functor sending T to the singleton $\{e_T\}$. The inclusion $\{e_T\} \subset \mathrm{Hom}_{R-\mathbf{CAlg}}(S_1,T)$ defines a morphism of functors $e : * \to \mathrm{Hom}_{R-\mathbf{CAlg}}(S_1,\cdot)$. Note also that u_T and v_T define morphisms of functors u and v. So u identifies the functor $\mathrm{Hom}_{R-\mathbf{CAlg}}(S_0,\cdot)$ with the fiber product of the diagram

$$\begin{tikzcd} & \mathrm{Hom}_{R-\mathbf{CAlg}}(S_0,\cdot) \arrow[d, "v"] \\ * \arrow[r, "e"'] & \mathrm{Hom}_{R-\mathbf{CAlg}}(S_1,\cdot) \end{tikzcd}$$

(in the category $\mathrm{Func}(R-\mathbf{CAlg},\mathbf{Set})$). As the three functors in this diagram commute with filtrant colimits by the first paragraph, as filtrant colimits commute with finite limits in **Set** (Proposition I.5.6.4 of the notes), the functor $\mathrm{Hom}_{R-\mathbf{CAlg}}(S,\cdot)$ also commutes with filtrant colimits.

It remains to show that a commutative R-algebra S that is of finite presentation as an object of $R-\mathbf{CAlg}$ is a finitely presented R-algebra. First, consider the poset $\mathscr{I}$ of finitely generated sub-R-algebras $S'\subset S$, seen as a category, and the obvious (inclusion) functor from $\mathscr{I}$ to $R-\mathbf{CAlg}$. The category $\mathscr{I}$ is clearly filtrant (because the union of two finitely generated subslalgebras of S is contained in a finitely generated subalgebra), and $\varinjlim F = S$ because we saw in the first paragraph that the forgetful functor $R-\mathbf{CAlg}\to\mathbf{Set}$ commutes with filtrant colimits. So the canonical map $\varinjlim_{S'\in\mathrm{Ob}(\mathscr{I})}\mathrm{Hom}_{R-\mathbf{CAlg}}(S,S')\to\mathrm{Hom}_{R-\mathbf{CAlg}}(S,S)$ is bijective, which implies that there exists a finitely generated subalgebra S' of S such that the identity of S factors through the inclusion $S'\subset S$, i.e. such that $S'=S$. So S is a finitely generated R-algebra. We write $S=R[x_1,\dots,x_n]/I$, with I an ideal of $R[x_1,\dots,x_n]$. Let $\mathscr{I}'$ be the poset of finite generated ideals $J\subset I$, seen as category; again, this is a clearly a filtrant category. Define a functor $G:\mathscr{I}'\to R-\mathbf{CAlg}$ by sending J to $R[x_1,\dots,x_n]/J$. For every $J\in\mathrm{Ob}(\mathscr{I}')$, let $u_J:G(J)=R[x_1,\dots,x_n]\to S$ be the quotient morphism. Then $(S,(u_J))$ is a cone under G, and we claim that it is a colimit of G. Indeed, let $(T,(v_J))$ be another cone under G. In particular, all the morphisms $R[x_1,\dots,x_n]\to R[x_1,\dots,x_n]/J\overset{v_J}{\to}T$ are equal, so we get a morphism $f:R[x_1,\dots,x_n]\to T$. Also, $\mathrm{Ker}(f)$ contains every finitely generated subideal of I, so it contains every element of I, so $I\subset\mathrm{Ker}(f)$, so f factors as $R[x_1,\dots,x_n]\to S\overset{g}{\to}T$. The morphism g is clearly a morphism of cones, and it is the only possible morphism of cones from $(S,(u_J))$ to $(T,(v_j))$ because all the maps u_J are surjective. As S is of finite presentation as an object of $R-\mathbf{CAlg}$, the canonical map

$$\varinjlim_{J\in\mathrm{Ob}(\mathscr{I}')}\mathrm{Hom}_{R-\mathbf{CAlg}}(S,R[x_1,\dots,x_n]/J)\to\mathrm{Hom}_{R-\mathbf{CAlg}}(S,S)$$

is bijective. In particular, there exists a finitely generated ideal $J\subset I$ such that the identity morphism of S factors as $S\to R[x_1,\dots,x_n]/J\to S$, where the second map is the quotient map; this forces J and I to be equal, so I is a finitely generated ideal, and so S is a finitely presented R-algebra.

(c). (i) As in (a)(i), we can do this directly or categorically. If we do it directly, we use the fact that a singleton is clearly of finite presentation in **Top**, and that a finite discrete set is a finite coproduct of singletons in **Top**. If we do it categorically, we apply the fact that we proved in (a)(i) to the pair of adjoitn functors (F,For), where $\mathrm{For}:\mathbf{Top}\to\mathbf{Set}$ is the forgetful functor (which preserves all colimits by Section I.5.5 of the notes) and F is its left adjoint, i.e. the functor that sends a set X to itself with the discrete topology (Example I.4.8 of the notes). Then the result follows from the fact that a finite set is of finite presentation as an object of **Set**, which we proved in (a)(i).

(ii) Let $\mathrm{For}:\mathbf{Top}\to\mathbf{Set}$ be the forgetful functor. It is easy to see that $\mathrm{For}(X)=\varinjlim(\mathrm{For}\circ F)$ (this just says that X is the union of all its finite subsets). We use this to identify X and $\varinjlim F$ as sets. Then X and $\varinjlim F$ are isomorphic as topological spaces if and only if the original topology on X coincides with the colimiit topology. Let U be a subset of X. It is open in the colimit topology if and only $U\cap Y$ is open in Y for every finite subset Y of X (using the subset topology on Y). This is certainly true if X has the coarse topology. [7]

[7] It is also true if X has the discrete topology...

(iii) Let X_0 be the underlying set of X with the coarse topology. Then the identity map $i : X \to X_0$ is continuous. As X is of finite presentation, question (ii) implies that i factors through a finite subset of X, hence that X is finite.

(iv) Let $n \in \mathbb{N}$, and let $(m_i)_{i\in I}$ be a family of integers $\geq n$. Then

$$\bigcup_{i\in I}(\mathbb{N}_{\geq m_i} \times \{0\}) \cup (\mathbb{N}_{\geq n} \times \{1\}) = (\mathbb{N}_{\geq \inf_{i\in I} m_i} \times \{0\}) \cup (\mathbb{N}_{\geq n} \times \{1\})$$

with $\inf_{i\in I} m_i \geq n$. Also, if I is finite, then

$$\bigcap_{i\in I}(\mathbb{N}_{\geq m_i} \times \{0\}) \cup (\mathbb{N}_{\geq n} \times \{1\}) = (\mathbb{N}_{\geq \sup_{i\in I} m_i} \times \{0\}) \cup (\mathbb{N}_{\geq n} \times \{1\})$$

So the family of "open sets" of the statement does define a topology on $\mathbb{N}_{\geq n} \times \{0, 1\}$.

Let $n \in \mathbb{N}$, and let $m \geq n + 1$. Then

$$f_n^{-1}((\mathbb{N}_{\geq m} \times \{0\}) \cup (\mathbb{N}_{\geq n+1} \times \{1\})) = \begin{cases} (\mathbb{N}_{\geq m} \times \{0\}) \cup (\mathbb{N}_{\geq n} \times \{1\}) & \text{if } m \geq n+2 \\ (\mathbb{N}_{\geq n} \times \{0\}) \cup (\mathbb{N}_{\geq n} \times \{0\}) & \text{if } m = n+1. \end{cases}$$

So f_n is continuous.

(v) We put the coarse topology on $\{0, 1\}$. Then the second projections maps $X_n \to \{0, 1\}$, hence define a cone under the functor F. So we get a continuous map $f : \varinjlim_{n\in\mathbb{N}} X_n \to \{0, 1\}$.

If $a \in \{0, 1\}$, then the image of $(0, a) \in X_0$, so its image by the obvious map $X_0 \to \coprod_{n\in\mathbb{N}} X_n \to \varinjlim_{n\in\mathbb{N}} X_n$ is a preimage of a by f. So f is surjective.

We prove that f is injective. Let $(m, a) \in X_n$ and $(m', b) \in X_{n'}$, and suppose that the images of (m, a) and (m', b) by the maps $X_n \to \coprod_{i\in\mathbb{N}} X_i \to \varinjlim_{i\in\mathbb{N}} X_i \xrightarrow{f} \{0, 1\}$ and $X_m \to \coprod_{i\in\mathbb{N}} X_i \to \varinjlim_{i\in\mathbb{N}} X_i \xrightarrow{f} \{0, 1\}$ are equal. We want to prove that (m, a) and (m', b) have the same image in $\varinjlim_{i\in\mathbb{N}} X_i$. As the f_i do not change the second coordinate of elements of X_i, the assumption implies that $a = b$. If $m > n$, then $f_{m-1} \circ \ldots \circ f_n(m, a) = (m, a) \in X_m$ has the same image as $(m, a) \in X_n$ in $\varinjlim_{i\in\mathbb{N}} X_i$; so we may assume that $n = m$. Similarly, we may assume that $n' = m'$. Up to switching n and n', we may assume that $n' \geq n$. If $n' = n$, we are done. Otherwise, we have $(n', a) = f_{n'-1} \circ \ldots \circ f_n(n, a)$, so $(n', a) \in X_{n'}$ and $(n, a) \in X_n$ have the same image in $\varinjlim_{i\in\mathbb{N}} X_i$.

It remains to prove that f^{-1} is continuous. If it were not, this would mean that $\{0\}$ or $\{1\}$ is open in $\varinjlim_{i\in\mathbb{N}} X_i$. But the preimages of $\{0\}$ and $\{1\}$ by the continuous map $X_n \to \varinjlim_{i\in\mathbb{N}} X_i$ are $\mathbb{N}_{\geq n} \times \{0\}$ and $\mathbb{N}_{\geq n} \times \{1\}$ respectively, and these are not open subsets of X_n. So neither $\{0\}$ nor $\{1\}$ is open in $\varinjlim_{i\in\mathbb{N}} X_i$.

(vi) We already know that X is finite by (iii). Let U be a subset of X, and let $f : X \to \{0, 1\}$ be the indicator map of U. Then f is continuous if we put the coarse topology on $\{0, 1\}$, so, by the hypothesis on X and question (v), there exists a continuous map $X \xrightarrow{g} X_n$ such that f is the composition of g and of the second projection $X_n \to \{0, 1\}$. As X is finite, there exists $m \geq n$ such that, for every $x \in X$, the first coordinate of $g(x) \in \mathbb{N}_{\geq n} \times \{0, 1\}$ is $< m$. Then

$$U = g^{-1}(\mathbb{N}_{\geq n} \times \{1\}) = g^{-1}((\mathbb{N}_{\geq m} \times \{0\}) \cup (\mathbb{N}_{\geq n} \times \{1\})).$$

As g is continuous, this proves that U is open in X. As U was an arbitrary subset of X, this shows that the topology of X is discrete.

(d). Let $\mathscr{U} = (U_i)_{i\in I}$ be a family of open subsets of X. Let $\mathscr{I}_I$ be the category associated to the poset of finite subsets of I, ordered by inclusions; then $\mathscr{I}_I$ is filtrant. Let $F_{\mathscr{U}} : \mathscr{I}_I \to \mathrm{Open}(X)$ be the functor sending a finite subset $J \subset I$ to $\bigcup_{j\in J} U_j$; if $J \subset J'$, then the image by $F_{\mathscr{U}}$ of the corresponding morphism of $\mathscr{I}_I$ is the inclusion $\bigcup_{j\in J} U_j \subset \bigcup_{j\in J'} U_j$. Then it is easy to see that $\varinjlim F_{\mathscr{U}} = \bigcup_{i\in I} U_i$.

Suppose that X is finite presentation as an object of $\mathrm{Open}(X)$, and let $\mathscr{U} = (U_i)_{i\in I}$ be an open covering of X. Then the identity morphism $X \to \varinjlim F_{\mathscr{U}}$ comes from a morphism $X \to \bigcup_{j\in J} U_j$ with $J \in \mathrm{Ob}(\mathscr{I}_I)$, or, in other words, there exists a finite subset J of I such that $X \subset \bigcup_{j\in J} U_j$. This means that X is compact.

Conversely, suppose that X is compact, and let $F : \mathscr{I} \to \mathrm{Open}(X)$ be a functor, with $\mathscr{I}$ filtrant. Let $U = \varinjlim F$. We claim that $U = \bigcup_{i\in \mathrm{Ob}(\mathscr{I})} F(i)$. For every $i \in \mathrm{Ob}(\mathscr{I})$, we have a morphism $F(i) \to U$ in $\mathrm{Open}(X)$, so $F(i) \subset U$. Conversely, let $U' = \bigcup_{i\in \mathrm{Ob}(\mathscr{I})} F(i)$. Then we have a morphism $F(i) \to U'$ in $\mathrm{Open}(X)$ for every $i \in \mathrm{Ob}(\mathscr{I})$, and this defines a cone under F with apex U', so the universal property of the colimit implies that we have a morphism $U \to U'$ in $\mathrm{Open}(X)$, that is, that $U \subset U'$.

Now we show that the map $\alpha : \varinjlim_{i\in \mathrm{Ob}(\mathscr{I})} \mathrm{Hom}_{\mathrm{Open}(X)}(X, F(i)) \to \mathrm{Hom}_{\mathrm{Open}(X)}(X, U)$ is bijective. Note that, as all Hom sets in $\mathrm{Open}(X)$ are empty sets or singletons, and as $\mathscr{I}$ is filtrant, the source of α has at most one element. If $U \neq X$, then $\mathrm{Hom}_{\mathrm{Open}(X)}(X, U) = \varnothing$ and $\mathrm{Hom}_{\mathrm{Open}(X)}(X, F(i)) = \varnothing$ for every $i \in \mathrm{Ob}(\mathscr{I})$, so α is bijective. Suppose that $U = X$; then $\mathrm{Hom}_{\mathrm{Open}(X)}(X, U) = \{\mathrm{id}_X\}$, and we want to show that id_X has a preimage by α. This is equivalent to the fact that $X = F(i)$ for some $i \in \mathrm{Ob}(\mathscr{I})$. As X is compact and as $X = U = \bigcup_{i\in \mathrm{Ob}(\mathscr{I})} F(i)$, we know that there exist $i_1, \ldots, i_n \in \mathrm{Ob}(\mathscr{I})$ such that $X = F(i_1) \cup \ldots \cup F(i_n)$. As $\mathscr{I}$ is filtrant, there exists $j \in \mathrm{Ob}(\mathscr{I})$ and morphisms $i_1 \to j$, $\ldots$, $i_n \to j$. So we have morphisms $F(i_r) \to F(j)$ in $\mathrm{Open}(X)$ for $1 \leq r \leq n$, that is, $F(j)$ contains $F(i_1), \ldots, F(i_n)$; this implies that $F(j) = X$.

$\square$