MAT 540 : Problem Set 2

Due Thursday, September 26

1. Monoidal categories (extra credit)

A *monoidal category* is a category $\mathscr{C}$ equipped with a bifunctor $(\cdot)\otimes(\cdot) : \mathscr{C}\times\mathscr{C}\to\mathscr{C}$ (the tensor product or monoidal functor), with an identity (or unit) object $\mathbb{1}$ and with three natural isomorphisms $\alpha(A,B,C) : (A\otimes B)\otimes C \xrightarrow{\sim} A\otimes(B\otimes C)$, $\lambda(A) : \mathbb{1}\otimes A \xrightarrow{\sim} A$ and $\rho_A : A\otimes\mathbb{1}\xrightarrow{\sim} A$, satisfying the following conditions :

- for all $A,B,C,D\in\mathrm{Ob}(\mathscr{C})$, the following diagram commutes :

$$\begin{tikzcd}
((A\otimes B)\otimes C)\otimes D \arrow[r, "\alpha(A,B,C)\otimes\mathrm{id}_D"] \arrow[d, "\alpha(A\otimes B,C,D)"'] & (A\otimes(B\otimes C))\otimes D \arrow[r, "\alpha(A,B\otimes C,D)"] & A\otimes((B\otimes C)\otimes D) \arrow[d, "\mathrm{id}_A\otimes\alpha(B,C,D)"] \\
(A\otimes B)\otimes(C\otimes D) \arrow[rr, "\alpha(A,B,C\otimes D)"'] & & A\otimes(B\otimes(C\otimes D))
\end{tikzcd}$$

- for all $A,B\in\mathrm{Ob}(\mathscr{C})$, the following diagram commutes :

$$\begin{tikzcd}
(A\otimes\mathbb{1})\otimes B \arrow[rr, "\alpha(A,\mathbb{1},B)"] \arrow[dr, "\rho(A)\otimes\mathrm{id}_B"'] & & A\otimes(\mathbb{1}\otimes B) \arrow[dl, "\mathrm{id}_A\otimes\lambda(B)"] \\
& A\otimes B &
\end{tikzcd}$$

Here are some examples :

- $\mathscr{C}=$ **Set** or **Top**, $\otimes=\times$, $\mathbb{1}$ is a singleton;
- $\mathscr{C}=$ **Grp**, $\otimes=\times$, $\mathbb{1}=\{1\}$;
- $\mathscr{C}={}_R\mathbf{Mod}$ with R a commutative ring, $\otimes=\otimes_R$, $\mathbb{1}=R$;
- $\mathscr{C}=\mathrm{Func}(\mathscr{D},\mathscr{D})$ with $\mathscr{D}$ a category, $\otimes=\circ$, $\mathbb{1}=\mathrm{id}_{\mathscr{D}}$.

A *monoid* in $\mathscr{C}$ is an object M of $\mathscr{C}$ together with two morphisms $\mu : M\otimes M\to M$ (multiplication) and $\eta : \mathbb{1}\to M$ (unit), such that the two following diagrams commute :

$$\begin{tikzcd}
M\otimes(M\otimes M) \arrow[r, "\mathrm{id}_M\otimes\mu"] & M\otimes M \arrow[r, "\mu"] & M \\
(M\otimes M)\otimes M \arrow[u, "\alpha(M,M,M)"] \arrow[r, "\mu\otimes\mathrm{id}_M"'] & M\otimes M \arrow[ur, "\mu"'] &
\end{tikzcd}$$

and

$$\begin{tikzcd}
M \otimes M \arrow[dr, "\mu"] & \mathbb{1} \otimes M \arrow[l, "\eta\otimes\mathrm{id}_M"'] \arrow[d, "\lambda(M)"] \\
M \otimes \mathbb{1} \arrow[u, "\mathrm{id}_M\otimes\eta"] \arrow[r, "\rho(M)"'] & M
\end{tikzcd}$$

(We can also define morphisms of monoids, and monoids in $\mathscr{C}$ form a category.)

Examples :

- A monoid in $(\mathbf{Set}, \times)$ is a monoid (in the usual sense).
- A monoid in $(\mathbf{Top}, \times)$ is a topological monoid.
- If R is a commutative ring, a monoid in $({}_R\mathbf{Mod}, \otimes)$ is a R-algebra. (In particular, a monoid in $(\mathbf{Ab}, \otimes_{\mathbb{Z}})$ is a ring.)
- A monoid in $(\mathrm{Func}(\mathscr{D}, \mathscr{D}), \circ)$ is called a *monad on* $\mathscr{D}$.

(a). (2 points) Let **Mon** be the category of (usual) monoids. It is a monoidal category, with the monoidal functor given by $\times$ and the unit object $\{1\}$. If (M, μ, η) is a monoid in **Mon**, show that M is a commutative monoid and μ is equal to the multiplication of M.

(b). (3 points) Let $F : \mathscr{C} \to \mathscr{D}$ and $G : \mathscr{D} \to \mathscr{C}$ be two functors such that (F, G) is a pair of adjoint functors, and let $\varepsilon : F \circ G \to \mathrm{id}_{\mathscr{D}}$ and $\eta : \mathrm{id}_{\mathscr{C}} \to G \circ F$ be the counit and unit of the adjunction. Define a morphism of functors $\mu : (G \circ F) \circ (G \circ F) \to G \circ F$ by $\mu(X) = G(\varepsilon(F(X))) : G(F \circ G(F(X))) \to G(F(X))$. Show that $(G \circ F, \mu, \eta)$ is a monad on $\mathscr{C}$.

2. Geometric realization of a simplicial set Remember that the simplicial category Δ is the subcategory of **Set** whose objects are the sets $[n] = \{0, 1, \dots, n\}$, for $n \in \mathbb{N}$, and whose morphisms are nondecreasing maps (where we put the usual order on $[n]$). The category of simplicial sets **sSet** is defined by $\mathbf{sSet} = \mathrm{PSh}(\Delta) = \mathrm{Func}(\Delta^{\mathrm{op}}, \mathbf{Set})$; if X is a simplicial set, we write X_n for $X([n])$ and $\alpha^* : X_m \to X_n$ for $X(\alpha) : X([m]) \to X([n])$ (if $\alpha : [n] \to [m]$ is a nondecreasing map). The standard n-simplex Δ is the simplicial set represented by $[n]$, i.e. $\mathrm{Hom}_{\Delta}(\cdot, [n])$.

(a). Let $\mathscr{C}$ be a category and $F : \mathscr{C}^{\mathrm{op}} \to \mathbf{Set}$ be a presheaf on $\mathscr{C}$. We consider the category $\mathscr{C}/F$ whose objects are pairs (X, x), with $X \in \mathrm{Ob}(\mathscr{C})$ and $x \in F(X)$, and such that a morphism $(X, x) \to (Y, y)$ is a morphism $f : X \to Y$ in $\mathscr{C}$ with $F(f)(y) = x$. Note that we have an obvious faithful functor $G_F : \mathscr{C}/F \to \mathscr{C}$ (forgetting the second entry in a pair), so we get a functor $h_{\mathscr{C}} \circ G_F : \mathscr{C}/F \to \mathrm{PSh}(\mathscr{C})$.

   (i) (1 point) When does $\mathscr{C}/F$ have a terminal object ?

   (ii) (2 points) Show that $\varinjlim(h_{\mathscr{C}} \circ G_F) = F$. (Hint : Use the second entries of the pairs to construct a morphism from $\varinjlim(h_{\mathscr{C}} \circ G_F)$ to F.) [1]

For every $n \in \mathbb{N}$, let $|\Delta_n| = \{(x_0, \dots, x_n) \in [0,1]^{n+1} \mid x_0 + \dots + x_n = 1\}$ with the subspace topology. If $f : [n] \to [m]$ is a map, we define $|f| : |\Delta_n| \to |\Delta_m|$ by $|f|(x_0, \dots, x_n) = (\sum_{i \in f^{-1}(j)} x_i)_{0 \le j \le m}$. (With the convention that an empty sum is equal to 0.) Consider the functor $|.| : \Delta \to \mathbf{Top}$ sending $[n]$ to $|\Delta_n|$ and $f : [n] \to [m]$ to $|f|$.

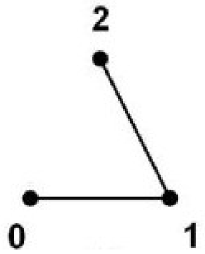
[1] So every presheaf is a colimit of representable presheaves.

Let X be a simplicial set, and consider the functor $G_X : \Delta/X \to \Delta$ of (a). The *geometric realization of* X is by definition the topological space $|X| = \varinjlim(|.| \circ G_X)$.

(b). (1 points) Show that this construction upgrades to a functor $|.| : \mathbf{sSet} \to \mathbf{Top}$. [2]

(c). (2 points) Show that, if X is Δ_n, then $|X| = |\Delta_n|$.

(d). (1 point) Give a simplicial set whose geometric realization is $\{(x_0, x_1, x_2) \in [0,1]^2 \mid x_0 = 0 \text{ or } x_2 = 0\}$. (Hint: why are the horns called horns ?)



(e). (2 points) Consider the functor $\mathrm{Sing} : \mathbf{Top} \to \mathbf{sSet}$ given by $\mathrm{Sing}(X) = \mathrm{Hom}_{\mathbf{Top}}(|.|, X) : \Delta^{\mathrm{op}} \to \mathbf{Set}$. (That is, if X is a topological space, then $\mathrm{Sing}(X)$ is the simplicial set such that $\mathrm{Sing}(X)_n$ is the set of continuous maps from $|\Delta_n|$ to X, and, if $f : [n] \to [m]$ is nondecreasing, then $f^* : \mathrm{Sing}(X)_m \to \mathrm{Sing}(X)_n$ sends a continuous map $u : |\Delta_m| \to X$ to $u \circ |f|$.) The simplicial set $\mathrm{Sing}(X)$ is called the *singular simplicial complex of* X of X.

Show that $(|.|, \mathrm{Sing})$ is a pair of adjoint functors.

3. Yoneda embedding and colimits Let k be a field, and let $\mathscr{C}$ be the category of k-vector spaces.

(a). (1 point) For every $n \in \mathbb{N}$, let $k[x]_{\leq n}$ be the vector space of polynomials of degree $\leq n$ in $k[x]$. Using the inclusions $k[x]_{\leq n} \subset k[x]_{\leq m}$ for $n \leq m$, we get a functor $F : \mathbb{N} \to \mathscr{C}$, $n \longmapsto k[x]_{\leq n}$. Show that $\varinjlim F = k[x]$.

(b). (2 points) Show that $h_{\mathscr{C}} : \mathscr{C} \to \mathrm{PSh}(\mathscr{C})$ does not commute with all colimits.

4. Filtrant colimits of modules (3 points)

Let R be a ring, let $\mathscr{I}$ be a filtrant category and let $F : \mathscr{I} \to {}_R\mathbf{Mod}$ be a functor. For every $i \in \mathrm{Ob}(\mathscr{I})$, we write $M_i = F(i)$. Let $\sim$ be the equivalence relation on $\coprod_{i \in \mathrm{Ob}(\mathscr{I})} M_i$ defined in Proposition I.5.6.2 of the notes; so $(i, x) \sim (j, y)$ if there exist morphisms $\alpha : i \to k$ and $\beta : j \to k$ in $\mathscr{I}$ such that $F(\alpha)(x) = F(\beta)(y)$. Let $M = \coprod_{i \in \mathrm{Ob}(\mathscr{I})} M_i / \sim$; this is the colimit of the composition $\mathscr{I} \xrightarrow{F} {}_R\mathbf{Mod} \xrightarrow{\mathrm{For}} \mathbf{Set}$. Denote by $q_i : M_i \to M$ the obvious maps.

Show that there exists a unique structure of left R-module on M such that all the q_i are R-linear maps, and that this structure makes $(M, (q_i))$ into a colimit of F.

5. Filtrant colimits are exact (3 points)

Let R be a ring and $\mathscr{I}$ be a filtrant category. Show that the functor $\varinjlim : \mathrm{Func}(\mathscr{I}, {}_R\mathbf{Mod}) \to {}_R\mathbf{Mod}$ is exact, i.e. that if $u : F \to G$ and $v : G \to H$ are mor-

[2]This functor is called the *left Kan extension* of $|.| : \Delta \to \mathbf{Top}$ along the Yoneda embedding $\Delta \to \mathbf{sSet}$.

phism of functors from $\mathscr{I}$ to ${}_R\mathbf{Mod}$ such that the sequence $0 \to F(i) \overset{u(i)}{\to} G(i) \overset{v(i)}{\to} H(i) \to 0$ is exact for every $i \in \mathrm{Ob}(\mathscr{I})$, then the sequence $0 \to \varinjlim F \overset{\varinjlim u}{\to} \varinjlim G \overset{\varinjlim v}{\to} \varinjlim H \to 0$ is exact. (Remember that we say that a sequence of R-modules $0 \to M \overset{f}{\to} N \overset{g}{\to} P \to 0$ is exact if $\mathrm{Ker}\, f = 0$, $\mathrm{Ker}\, g = \mathrm{Im}\, f$ and $\mathrm{Im}\, g = P$.)

6. Objects of finite type and of finite presentation Let $\mathscr{C}$ a category that admits all filtrant colimits (indexed by small enough categories). An object X of $\mathscr{C}$ is called *of finite type* (resp. *of finite presentation* or *compact*) if, for every filtrant category $\mathscr{I}$ and every functor $F : \mathscr{I} \to \mathscr{C}$, the canonical map

$$\varinjlim_{i \in \mathrm{Ob}(\mathscr{I})} \mathrm{Hom}_{\mathscr{C}}(X, F(i)) \to \mathrm{Hom}_{\mathscr{C}}(X, \varinjlim F)$$

(see the beginning of Subsection I.5.4.2 of the notes) is injective (resp. bijective).

(a). Let R be a ring and M be a left R-module.

- (i) (1 point) If M is free of finite type as a R-module, show that it is of finite presentation as an object of ${}_R\mathbf{Mod}$.
- (ii) (2 points) If M is of finite type (resp. of finite presentation) as a R-module, show that it is of finite type (resp. of finite presentation) as an object of ${}_R\mathbf{Mod}$.
- (iii) (1 point) Let $\mathscr{I}$ the poset of R-submodules of M that are of finite type, ordered by inclusion, and let $F : \mathscr{I} \to {}_R\mathbf{Mod}$ be the functor sending $N \subset M$ to M/N; if $N \subset N' \subset M$, we send the unique morphism $N \to N'$ in $\mathscr{I}$ to the canonical projection $M/N' \to M/N$. Show that $\varinjlim F = 0$.
- (iv) (2 points) If M is of finite type (resp. of finite presentation) as an object of ${}_R\mathbf{Mod}$, show that it is of finite type (resp. of finite presentation) as an R-module.

(b). (4 points, extra credit) Let R be a commutative ring and S be a commutative R-algebra. Show that S is finitely presented as an R-algebra if and only if it is of finite presentation as an object of $R - \mathbf{CAlg}$.

(c).
- (i) (1 point) If X is a finite set with the discrete topology, show that X is of finite presentation as an object of $\mathbf{Top}$.
- (ii) (1 point) Let X be a topological space. Let $\mathscr{I}$ be the poset of finite sets of X ordered by inclusion; wee see $\mathscr{I}$ as a subcategory of $\mathbf{Top}$ (we use the subset topology on each finite $Y \subset X$), and we denote by $F : \mathscr{I} \to \mathbf{Top}$ the inclusion functor. Show that $X = \varinjlim F$ if the topology on X is the indiscrete (= coarse) topology.
- (iii) (1 point) Let X be a topological space. If X is of finite presentation as an object of $\mathbf{Top}$, show that it is finite.
- (iv) (2 points) For $n \in \mathbb{N}$, let $X_n = \mathbb{N}_{\geq n} \times \{0, 1\}$, with the topology for which the open subsets are $\varnothing$ and $(\mathbb{N}_{\geq m} \times \{0\}) \cup (\mathbb{N}_{\geq n} \times \{1\})$, for $m \geq n$. Define $f_n : X_n \to X_{n+1}$ by $f_n(n, a) = (n + 1, a)$ and $f_n(m, a) = (m, a)$ if $m > n$. Show that the X_n are topological spaces and that the maps f_n are continuous.
- (v) (2 points) Show that $\varinjlim_{n \in \mathbb{N}} X_n$ is $\{0, 1\}$ with the indiscrete topology. By $\varinjlim_{n \in \mathbb{N}} X_n$, we mean the colimit of the functor $F : \mathbb{N} \to \mathbf{Top}$ such that $F(n) = X_n$ and that, for each non-identity morphism $\alpha : n \to m$ in $\mathbb{N}$, that is, for $n < m$ in $\mathbb{N}$, $F(\alpha) = f_{m-1} \circ f_{m-2} \circ \ldots \circ f_n : X_n \to X_m$.

(vi) (2 points) Let X be a topological space. If X is of finite presentation as an object of **Top**, show that X is finite and has the discrete topology.

(d). (2 points) Let X be a topological space, and let $\mathrm{Open}(X)$ be the set of open subsets of X, ordered by inclusion. Show that X is compact if and only if X is of finite presentation as an object of $\mathrm{Open}(X)$.