MAT 540 : Problem Set 1

Due Thursday, September 19

1.

(a). (2 points) In the category **Set**, show that a morphism is a monomorphism (resp. an epimorphism) if and only it is injective (resp. surjective).

(b). (2 points) Let $\mathscr{C}$ be a category and $F : \mathscr{C} \to$ **Set** be a *faithful* functor, show that any morphism f of $\mathscr{C}$ whose such that $F(f)$ is injective (resp. surjective) is a monomorphism (resp. an epimorphism).

(c). (2 points) What are the monomorphisms and epimorphisms in ${}_R\mathbf{Mod}$?

(d). (2 points) What are the monomorphisms in **Top** ? Give an example of a continuous morphism with dense image that is not an epimorphism in **Top**. [1]

(e). (2 points) Find a category $\mathscr{C}$, a faithful $F : \mathscr{C} \to$ **Set** and a monomorphism f in $\mathscr{C}$ such that $F(f)$ is not injective.

(f). (1 point) Find an epimorphism in **Ring** that is not surjective.

(g). The goal of this question is to show that any epimorphism in **Grp** is a surjective map. Let $\phi : G \to H$ be a morphism of groups, and suppose that it is an epimorphism in **Grp**. Let $A = \mathrm{Im}(\phi)$. Let $S = \{*\} \sqcup (H/A)$, where $\{*\}$ is a singleton, and let $\mathfrak{S}$ be the group of permutations of S. We denote by σ the element of $\mathfrak{S}$ that switches $*$ and A and leaves the other elements of H/A fixed. For every $h \in H$, we denote by $\psi_1(h)$ the element of $\mathfrak{S}$ that leaves $*$ fixed and acts on H/A by left translation by H; this defines a morphism of groups $\psi_1 : H \to \mathfrak{S}$. We denote by $\psi_2 : H \to \mathfrak{S}$ the morphism $\sigma\psi_1\sigma^{-1}$.

(i) (2 points) Show that $\psi_1 = \psi_2$.

(ii) (1 point) Show that $A = H$.

Solution.

(a). Let X, Y be sets and $f : X \to Y$ be a map.

Suppose that f is injective. If $g_1, g_2 : Z \to X$ are maps such that $f \circ g_1 = f \circ g_2$, then, for every $z \in Z$, we have $f(g_1(z)) = f(g_2(z))$, hence $g_1(z) = g_2(z)$; so $g_1 = g_2$. This shows that f is a monomorphism.

Conversely, suppose that f is a monomorphism. Let $x, x' \in X$ such that $x \neq x'$. Let $\{*\}$ be a singleton, and consider the maps $g_1, g_2 : \{*\} \to X$ defined by $g_1(*) = x$ and $g_2(*) = x'$. As $g_1 \neq g_2$, we have $f \circ g_1 \neq f \circ g_2$, so $f(x) \neq f(x')$. This shows that f is injective.

[1] In fact, the epimorphisms in **Top** are the surjective continuous maps.

Suppose that f is surjective. If $h_1, h_2 : Y \to Z$ are maps such that $h_1 \circ f = h_2 \circ f$, then, for every $y \in Y$, there exists $x \in X$ such that $f(x) = y$, and then $h_1(y) = h_1(f(x)) = h_2(f(x)) = h_2(y)$; so $h_1 = h_2$. This shows that f is a monomorphism.

Conversely, suppose that f is an epimorphism. Let $y_0 \in Y$, let $Z = \{a, b\}$ be a set with two distinct elements, and define $h_1, h_2 : Y \to Z$ by $h_1(y) = a$ for every $y \in Y$, $h_2(y) = a$ for every $y \in Y - \{y_0\}$ and $h_2(y_0) = b$. We have $h_1 \neq h_2$, so $h_1 \circ f \neq h_2 \circ f$. As h_1 and h_2 coincide on $Y - \{y_0\}$, this implies that $y_0 \in \mathrm{Im}(f)$. So f is surjective.

(b). Let $f : X \to Y$ be a morphism of $\mathscr{C}$. Suppose that $F(f)$ is injective. Let $g_1, g_2 : Z \to X$ be morphisms of $\mathscr{C}$ such that $f \circ g_1 = f \circ g_2$. Then $F(f) \circ F(g_1) = F(f) \circ F(g_2)$, so $F(g_1) = F(g_2)$ by a). As F is faithful, this implies that $g_1 = g_2$. So f is a monomorphism.

Suppose that $F(f)$ is surjective. Let $h_1, h_2 : Y \to Z$ be morphisms of $\mathscr{C}$ such that $h_1 \circ f = h_2 \circ f$. Then $F(h_1) \circ F(f) = F(h_2) \circ F(f)$, so $F(h_1) = F(h_2)$ by a). As F is faithful, this implies that $h_1 = h_2$. So f is an epimorphism.

(c). By b), any R-linear that is injective (resp. surjective) is a monomorphism (resp. epimorphism) in ${}_R\mathbf{Mod}$.

Conversely, let $f : M \to N$ be a monomorphism in ${}_R\mathbf{Mod}$. Consider the inclusion map $g_1 : \mathrm{Ker}(f) \to M$ and the map $g_2 = 0 : \mathrm{Ker}(f) \to M$. By definition of the kernel, we have $f \circ g_1 = f \circ g_2 = 0$, so $g_1 = g_2$, so $\mathrm{Ker}(f) = 0$, so f is injective.

Now let $f : M \to N$ be an epimorphism in ${}_R\mathbf{Mod}$. Consider the obvious surjection $h_1 : N \to \mathrm{Coker}(f)$ and the zero map $h_2 : N \to \mathrm{Coker}(f)$. By definition of the cokernel, we have $h_1 \circ f = h_2 \circ f = 0$, so $h_1 = h_2$, so $\mathrm{Coker}(f) = 0$, so f is surjective.

(d). By b), we know that any (continuous) injection is a monomorphism in **Top**. Conversely, let $f : X \to Y$ be a monomorphism in **Top**. Let $x, x' \in X$ such that $x \neq x'$. Let $\{*\}$ be a singleton with the discrete topology, and consider the maps $g_1, g_2 : \{*\} \to X$ defined by $g_1(*) = x$ and $g_2(*) = x'$; these maps are continuous, hence morphisms in **Top**. As $g_1 \neq g_2$, we have $f \circ g_1 \neq f \circ g_2$, so $f(x) \neq f(x')$. This shows that f is injective.

Let $X = \{s, \eta\}$ be a set with two distinct points. We put the topology on X for which the open sets are $\varnothing$, X and $\{\eta\}$. Note that $\{\eta\}$ is dense in X. Let $f : X \to X$ be the map sending every point of X to η. Then f has dense image, but f is not an epimorphism, because $\mathrm{id}_X \circ f = f \circ f$, while $f \neq \mathrm{id}_X$.

(e). Let $\mathscr{C}$ be the subcategory of **Set** whose objects are $\{0\}$ and $\{0, 1\}$, and whose morphisms are the identities and the unique map f from $\{0, 1\}$ to $\{0\}$. Then f is a monomorphism in $\mathscr{C}$, but it is not injective. (And the inclusion is a faithful functor from $\mathscr{C}$ to **Set**.)

(f). Consider the inclusion $f : \mathbb{Z} \to \mathbb{Q}$. It is an epimorphism in **Ring**. Indeed, let R be a ring and let $h_1, h_2 : \mathbb{Q} \to R$ are morphisms of rings such that $h_1 \circ f = h_2 \circ f$. For every $m \in \mathbb{Z} - \{0\}$, the image of m in $\mathbb{Q}$ is invertible, so $h_1(m), h_2(m) \in R^\times$. For every $x \in \mathbb{Q}$, we can write $x = nm^{-1}$ with $n \in \mathbb{N}$ and $m \in \mathbb{Z} - \{0\}$, and then $h_1(x) = h_1(n)h_1(m)^{-1} = h_2(n)h_2(m)^{-1} = h_2(x)$.

More generally, if A is a commutative ring and S is a multiplicative subset of A, then the canonical map $A \to S^{-1}A$ is an epimorphism in **Ring**.

(g). [2]

(i) Note that $\psi_1(h)_{|S-\{*,A\}} = \psi_2(h)_{S-\{*,A\}}$ for every $h \in H$.

[2]This proof comes from [**?**].

Let $h \in A = \mathrm{Im}(\phi)$. We have $\psi_1(h)(*) = *$. On the other hand, the action of h on H/A by left translation fixes A, so $\psi_1(h)(A) = A$. So $\psi_1(h)_{\{*,A\}}$ is the identity morphism of $\{*, A\}$. This implies that $\psi_2(h)_{\{*,A\}}$ is also the identity morphism of $\{*, A\}$, hence that $\psi_1(h) = \psi_2(h)$. So ψ_1 and ψ_2 are equal on the image of ϕ, which implies that $\psi_1 \circ \phi \psi_2 \circ \phi$. As ϕ is an epimorphism, we deduce that $\psi_1 = \psi_2$.

(ii) Let $h \in A$. Then $\psi_1(h)(*) = *$, and $\psi_2(h)(*) = \sigma \circ \psi_1(h)(A) = \sigma(hA)$. By (i), we know that $\psi_1(h) = \psi_2(h)$, so $* = \sigma(hA)$. This is only possible if $hA = A$, i.e. if $h \in A$. So $H = A = \mathrm{Im}(\phi)$, and ϕ is surjective.

□

2. Let $F : \mathscr{C} \to \mathscr{C}'$ be a functor.

(a). (3 points) If F has a quasi-inverse, show that it is fully faithful and essentially surjective.

(b). (4 points) If F is fully faithful and essentially surjective, construct a functor $G : \mathscr{C}' \to \mathscr{C}$ and isomorphisms of functors $F \circ G \simeq \mathrm{id}_{\mathscr{C}}$ and $G \circ F \simeq \mathrm{id}_{\mathscr{C}'}$.

Solution.

(a). Let $G : \mathscr{C}' \to \mathscr{C}$ be a quasi-inverse of F, and let $u : G \circ F \xrightarrow{\sim} \mathrm{id}_{\mathscr{C}}$ and $v : F \circ G \xrightarrow{\sim} \mathrm{id}_{\mathscr{C}'}$ be isomorphisms of functors.

Let $X, Y \in \mathrm{Ob}(\mathbb{C})$. We denote by β the map $\mathrm{Hom}_{\mathscr{C}}(X, Y) \to \mathrm{Hom}_{\mathscr{C}'}(F(X), F(Y))$ given by F. Consider the map $\alpha : \mathrm{Hom}_{\mathscr{C}'}(F(X), F(Y)) \to \mathrm{Hom}_{\mathscr{C}}(X, Y)$ that we get by composing $G : \mathrm{Hom}_{\mathscr{C}'}(F(X), F(Y)) \to \mathrm{Hom}_{\mathscr{C}}(G \circ F(X), G \circ F(Y))$ and the map $\mathrm{Hom}_{\mathscr{C}}(G \circ F(X), G \circ F(Y)) \to \mathrm{Hom}_{\mathscr{C}}(X, Y)$, $g \longmapsto u(Y) \circ g \circ u(X)^{-1}$. We claim that $\alpha \circ \beta$ is the identity on $\mathrm{Hom}_{\mathscr{C}}(X, Y)$. Indeed, let $f \in \mathrm{Hom}_{\mathscr{C}}(X, Y)$. As u is a morphism of functors, the following diagram is commutative :

$$\begin{tikzcd}
G \circ F(X) \arrow[r, "G\circ F(f)"] \arrow[d, "u(X)"'] & G \circ F(Y) \arrow[d, "u(Y)"] \\
X \arrow[r, "f"'] & Y
\end{tikzcd}$$

This shows that $u(Y) \circ G \circ F(f) \circ u(X)^{-1} = f$, i.e. that $\alpha \circ \beta(f) = f$. In particular, the map β is injective and the map α is surjective. This shows that F is faithful. Applying this result to G (which is also an equivalence of categories, with quasi-inverse F), we see that the map α is also injective, hence it is bijective, hence β is also bijective. This shows that F is fully faithful.

Let $X' \in \mathrm{Ob}(\mathscr{C}')$. Then $v : F(G(X')) \xrightarrow{\sim} X'$ is an isomorphism, and $G(X') \in \mathrm{Ob}(\mathscr{C})$. This shows that F is essentially surjective.

(b). We construct the functor G. Let $X' \in \mathrm{Ob}(\mathscr{C}')$; we choose an object X of $\mathscr{C}$ and an isomorphism $u(X') : F(X) \xrightarrow{\sim} X'$, and we set $G(X) = X'$. Let $X', Y' \in \mathrm{Ob}(\mathscr{C}')$, and let $X = G(X')$ and $Y = G(Y')$. We define a map $\mathrm{Hom}_{\mathscr{C}'}(X', Y') \to \mathrm{Hom}_{\mathscr{C}'}(F(X), F(Y))$ by $f' \longmapsto u(Y')^{-1} \circ f' \circ u(X')$. Composing this with the inverse of the bijection $F : \mathrm{Hom}_{\mathscr{C}}(X, Y) \xrightarrow{\sim} \mathrm{Hom}_{\mathscr{C}'}(F(X), F(Y))$, we get a map $\mathrm{Hom}_{\mathscr{C}'}(X', Y') \to \mathrm{Hom}_{\mathscr{C}}(X, Y)$, which we denote by G.

Next we show that G is a functor. If $X' \in \mathrm{Ob}(\mathscr{C}')$, then $u(X')^{-1} \circ \mathrm{id}_{X'} \circ u(X') = \mathrm{id}_{F(G(X'))}$, so $G(\mathrm{id}_{X'}) = \mathrm{id}_{G(X')}$. Let $f' : X' \to Y'$ and $g' : Y' \to Z'$ be two morphisms of $\mathscr{C}'$, and let $f : X \to Y$ and $g : Y \to Z$ be their images by G. By definition of G on morphisms, we have $F(f) = u(Y')^{-1} \circ f' \circ u(X')$ and $F(g) = u(Z')^{-1} \circ g' \circ u(Y')$, so $F(g \circ f) = F(g) \circ F(f) = u(Z')^{-1} \circ (g' \circ f') \circ u(X') = F(G(g' \circ f'))$. As F is faithful, this implies that $g \circ f = G(g' \circ f')$, i.e. that $G(g') \circ G(f') = G(g' \circ f')$. So G is a functor.

Finally, we show that G is a quasi-inverse of F. For every $X' \in \mathrm{Ob}(\mathscr{C}')$, we have by definition of $G(X')$ an isomorphism $u(X') : F(G(X')) \xrightarrow{\sim} X'$. We need to show that this defines an isomorphism of functors $F \circ G \xrightarrow{\sim} \mathrm{id}_{\mathscr{C}'}$. So let $f' : X' \to Y'$ be a morphism of $\mathscr{C}'$. By definition of $G(f')$, we have $u(Y') \circ F(G(f')) = f' \circ u(X')$, which is what we wanted. We still need to define an isomorphism of functors $v : G \circ F \xrightarrow{\sim} \mathrm{id}_{\mathscr{C}}$. Let $X \in \mathrm{Ob}(\mathscr{C})$. By definition of G, we have an isomorphism $u(F(X)) : F(G(F(X))) \xrightarrow{\sim} F(X)$. As F is fully faithful, there is a unique $v(X) \in \mathrm{Hom}_{\mathscr{C}}(G(F(X)), X)$ such that $F(v(X)) = u(F(X))$, and $v(X)$ is an isomorphism because a fully faithful functor is conservative. It remains to show that this defines a morphism of functors. So let $f : X \to Y$ be a morphism of $\mathscr{C}$. Then $F(G(F(f))) = u(F(Y))^{-1} \circ F(f) \circ u(F(X))$, so

$$F(f) \circ F(v(X)) = F(f) \circ u(F(X)) = u(F(Y)) \circ F(G(F(f))) = F(v(Y)) \circ F(G(F(f))).$$

Using the fact that F is faithful (and is a functor), we get $f \circ v(X) = v(Y) \circ G(F(f))$, which is what we wanted.

□

3. Let $\mathscr{C}$ be the full subcategory of **Ab** whose objects are finitely generated abelian groups.

(a). (2 points) Show that every natural endomorphism of $\mathrm{id}_{\mathscr{C}}$ is multiplication by some $n \in \mathbb{Z}$.

(b). (3 points) Consider the functor $F : \mathscr{C} \to \mathscr{C}$ that sends an abelian group A to $A_{\mathrm{tor}} \oplus (A/A_{\mathrm{tor}})$ (and acts in the obvious way on morphisms), where A_{tor} is the torsion subgroup of A. Show that there is no natural isomorphism $F \xrightarrow{\sim} \mathrm{id}_{\mathscr{C}}$.

Solution.

(a). Let $u : \mathrm{id}_{\mathscr{C}} \to \mathrm{id}_{\mathscr{C}}$ be a morphism of functors. Then $u(\mathbb{Z}) \in \mathrm{End}_{\mathbf{Ab}}(\mathbb{Z})$, so $u(\mathbb{Z})$ is of the form $n\mathrm{id}_{\mathbb{Z}}$ for some $n \in \mathbb{Z}$. Let A be an arbitrary abelian group. We want to show that $u(A) = n\mathrm{id}_A$. Let $a \in A$. We consider the morphism of groups $f : \mathbb{Z} \to A$ sending 1 to a. As u is a morphism of functors, we have a commutative diagram :

$$\begin{tikzcd} \mathbb{Z} \arrow[r, "u(\mathbb{Z})"] \arrow[d, "f"'] & \mathbb{Z} \arrow[d, "f"] \\ A \arrow[r, "u(A)"'] & A \end{tikzcd}$$

In particular, $u(A)(a) = u(A)(f(1)) = f(u(\mathbb{Z})(1)) = f(n) = na$. So $u(A) = n\mathrm{id}_A$.

(b). Suppose that $u : F \xrightarrow{\sim} \mathrm{id}_{\mathscr{C}}$ is a natural isomorphism. For every abelian groups A, consider the morphism $v(A) : A \to A/A_{\mathrm{tor}} \oplus A_{\mathrm{tor}}$ that is the composition of the canonical surjection $A \to A/A_{\mathrm{tor}}$ and of the injection $A/A_{\mathrm{tor}} \to A/A_{\mathrm{tor}} \oplus A_{\mathrm{tor}}$. It is easy to see that this defines a morphism of functors $v : \mathrm{id}_{\mathscr{C}} \to F$. So $u \circ v$ is an endomorphism of $\mathrm{id}_{\mathscr{C}}$, and, by a), there exists $n \in \mathbb{Z}$ such that $u \circ v$ is the multiplication by n. As $v(\mathbb{Z}) = \mathrm{id}_{\mathbb{Z}}$ by definition of v and $u(\mathbb{Z})$ is an isomorphism, we must have $n = \pm 1$. Now take $A = \mathbb{Z}/2\mathbb{Z}$. Then $v(A) = 0$, so $u \circ v(A) = 0$, so n is divisible by 2. This is a contradiction.

□

4. (2 points, extra credit) Let k be a field, and let $F : \mathbf{Mod}_k \to \mathbf{Mod}_k$ be the functor sending a k-vector space V to $V \otimes_k V$ and a k-linear transformation f to $f \otimes f$. Show that the only morphism of functors from $\mathrm{id}_{\mathbf{Mod}_k}$ to F is the zero one, i.e. the morphism $u : \mathrm{id}_{\mathbf{Mod}_k} \to F$ such that $u(V) = 0$ for every k-vector space V.

Solution. Let $u : \mathrm{id}_{\mathbf{Mod}_k} \to F$ be a morphism of functors. Then $u(k)$ is a k-linear map from k to $k \otimes_k k$, so there exists a unique $\lambda \in k$ such that $u(k)(1) = \lambda(1 \otimes 1)$.

Let V be a k-vector space, and let $v \in V$. We denote by $f; k \to V$ the unique k-linear map such that $f(1) = v$. As u is a morphism of functors, we have $u(V) \circ f = (f \otimes f) \circ u(k)$, and in particular $u(V)(v) = u(V)(f(1)) = (f \otimes f)(\lambda(1 \otimes 1)) = \lambda(v \otimes v)$.

Take $V = k^2$, and let (e_1, e_2) be the canonical basis of V. We know that $(e_1 \otimes e_1, e_1 \otimes e_2, e_2 \otimes e_1, e_2 \otimes e_2)$ is a basis of $V \otimes_k V$. Using the previous paragraph, we see that

$$u(V)(e_1 + e_2) = \lambda(e_1 + e_2) \otimes (e_1 + e_2) = \lambda(e_1 \otimes e_1) + \lambda(e_1 \otimes e_2) + \lambda(e_2 \otimes e_1) + \lambda(e_2 \otimes e_2).$$

On the other hand, as $u(V)$ is k-linear, we have

$$u(V)(e_1 + e_2) = u(V)(e_1) + u(V)(e_2) = \lambda(e_1 \otimes e_1) + \lambda(e_2 \otimes e_2).$$

This is only possible if $\lambda = 0$. But then, by the calculation of the previous paragraph, we have $u(W) = 0$ for every k-vector space W.

Note that we did not use the fact that k is a field, so the result is also true for the catgeory of modules over a commutative ring.

□

5. (4 points) Let $\mathscr{C}$ be a category. Remember that the category $\mathrm{PSh}(\mathscr{C})$ of presheaves on $\mathscr{C}$ is the category $\mathrm{Func}(\mathscr{C}^{\mathrm{op}}, \mathbf{Set})$.

Let F be a presheaf on $\mathscr{C}$ and X be an object of $\mathscr{C}$. Let $\Phi : \mathrm{Hom}_{\mathrm{PSh}(\mathscr{C})}(h_X, F) \to F(X)$ be the map defined by $\Phi(u) = u(X)(\mathrm{id}_X)$. Let $\Psi : F(X) \to \mathrm{Hom}_{\mathrm{PSh}(\mathscr{C})}(h_X, F)$ be the map sending $x \in F(X)$ to the morphism of functors $\Psi(x) : h_X \to F$ such that $\Psi(x)(Y) : h_X(Y) = \mathrm{Hom}_{\mathscr{C}}(Y, X) \to F(Y)$ sends $f : Y \to X$ to $F(f)(x) \in F(Y)$. Show that Φ and Ψ are bijections that are inverses of each other.

Solution. We show that $\Psi \circ \Phi$ is the identity of $\mathrm{Hom}_{\mathrm{PSh}(\mathscr{C})}(h_X, F)$. Let $u \in \mathrm{Hom}_{\mathrm{PSh}(\mathscr{C})}(h_X, F)$. Let Y be an object of $\mathscr{C}$. As u is a morphism of functors, we have a commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathscr{C}}(X, X) \arrow[r, "u(X)"] \arrow[d, "h_X(f)"'] & F(X) \arrow[d, "F(f)"] \\
\mathrm{Hom}_{\mathscr{C}}(Y, X) \arrow[r, "u(Y)"'] & F(Y)
\end{tikzcd}$$

In particular, we have

$$F(f)(\Phi(u)) = F(f)(u(X)(\mathrm{id}_X)) = u(Y)(h_X(f)(\mathrm{id}_X)) = u(Y)(f).$$

As $F(f)(\Phi(u)) = \Psi(\Phi(u))(Y)(f)$ by definition of Ψ, this shows that $\Psi(\Phi(u))(Y) = u(Y)$, hence that $\Psi(\Phi(u)) = u$.

Now we show that $\Phi \circ \Psi$ is the identity of $F(X)$. Let $x \in F(X)$. Then $\Phi(\Psi(x)) = \Psi(x)(X)(\mathrm{id}_X) = F(\mathrm{id}_X)(x) = \mathrm{id}_{F(X)}(x) = x$.

$\square$

6.

(a). (2 points) Show that the categories **Set** and $\mathbf{Set}^{\mathrm{op}}$ are not equivalent. (Hint : If $F : \mathbf{Set}^{\mathrm{op}} \to \mathbf{Set}$ is an equivalence of categories, show that $F(\varnothing)$ is a singleton and that $F(X) = \varnothing$ for X a singleton.)

(b). (1 point) Let $\mathscr{C}$ be the full subcategory of **Set** whose objects are finite sets. Show that $\mathscr{C}$ and $\mathscr{C}^{\mathrm{op}}$ are not equivalent.

(c). (1 point) Show that **Rel** and $\mathbf{Rel}^{\mathrm{op}}$ are equivalent.

(d). (2 points) Let $\mathscr{D}$ be the full subcategory of **Ab** whose objects are finite abelian groups. Show that $\mathscr{D}$ and $\mathscr{D}^{\mathrm{op}}$ are equivalent.

Solution.

(a). Suppose that there exists an equivalence of categories $F : \mathbf{Set} \to \mathbf{Set}^{\mathrm{op}}$. For every set X, the set

$$\mathrm{Hom}_{\mathbf{Set}^{\mathrm{op}}}(F(X), F(\varnothing)) = \mathrm{Hom}_{\mathbf{Set}}(F(\varnothing), F(X)) \simeq \mathrm{Hom}_{\mathbf{Set}}(\varnothing, X)$$

is a singleton (because there is a unique map from the empty set into X). So $F(\varnothing)$ is a singleton.

Similarly, if X is a singleton, then, for every set Y, the set

$$\mathrm{Hom}_{\mathbf{Set}^{\mathrm{op}}}(F(X), F(Y)) = \mathrm{Hom}_{\mathbf{Set}}(F(Y), F(X)) \simeq \mathrm{Hom}_{\mathbf{Set}}(Y, X)$$

is a singleton. So $F(X)$ is the empty set.

Now let X be a singleton and Y be a set with two elements. Then $\mathrm{Hom}_{\mathbf{Set}}(X, Y)$ is a set with two elements. But on the other hand, we have

$$\mathrm{Hom}_{\mathbf{Set}}(X, Y) \simeq \mathrm{Hom}_{\mathbf{Set}^{\mathrm{op}}}(F(X), F(Y)) = \mathrm{Hom}_{\mathbf{Set}}(F(Y), \varnothing),$$

and $\mathrm{Hom}_{\mathbf{Set}}(F(Y), \varnothing)$ has at most one element (it is empty if $F(Y) \neq \varnothing$, and it only contains $\mathrm{id}_\varnothing$ if $F(Y) = \varnothing$). This is a contradiction.

(b). The proof of a) works just as well.

(c). Let $F : \mathbf{Rel} \to \mathbf{Rel}^{\mathrm{op}}$ be defined by $F(X) = X$ for every set X and, for all sets X, Y and every subset f of $X \times Y$, $F(f) = \{(y, x) \mid (x, y) \in f\}$. We want to show that F is a functor. (Then it will clearly be an equivalence, and even an isomorphism of categories.) Let X, Y, Z be sets and $f : X \to Y$, $g : Y \to Z$ be morphisms in **Rel**; that is, f is a subset of $X \times Y$ and g is a subset of $Y \times Z$. Then, in **Rel**, we have $g \circ f = \{(x, z) \mid \exists y \in Y,\ (x, y) \in f \text{ and } (y, z) \in g\}$. On the other hand, in $\mathbf{Rel}^{\mathrm{op}}$, we have $F(f) \circ F(g) = \{(z, x) \in Z \times X \mid \exists y \in Y,\ (y, x) \in F(f) \text{ and } (z, y) \in F(g)\}$. This is clearly equal to $F(g \circ f)$.

(d). Consider the functor $F = \mathrm{Hom}_{\mathbf{Ab}}(\cdot, \mathbb{Q}/\mathbb{Z}) : \mathbf{Ab}^{\mathrm{op}} \to \mathbf{Ab}$. If A is a finite abelian group, then so is $F(A)$. So F induces a functor $\mathscr{D}^{\mathrm{op}} \to \mathscr{D}$, which we still denote by F. We can also see F as a functor from $\mathscr{D}$ to $\mathscr{D}^{\mathrm{op}}$. We claim that F is an equivalence of categories, and in fact that it is its own quasi-inverse. To show this, it suffices to construct an functorial isomorphism $\mathrm{id}_{\mathscr{D}} \xrightarrow{\sim} F \circ F$. For every finite abelian group A, we consider the map $u : A \to F(F(A)) = \mathrm{Hom}_{\mathbf{Ab}}(\mathrm{Hom}_{\mathbf{Ab}}(A, \mathbb{Q}/\mathbb{Z}), \mathbb{Q}/\mathbb{Z})$, $a \longmapsto (f \longmapsto f(a))$. The fact that this defines a morphism of functors is a straightforward verification. The fact that is an isomorphism if Pontrjagin duality for finite abelian groups. (By the structure theorem for finite abelian groups, it suffices to check that $u(A)$ is an isomorphism for A of the form $\mathbb{Z}/n\mathbb{Z}$, which is easy.)

$\square$

7. (4 points) Let $\mathscr{C}$ and $\mathscr{C}'$ and $F : \mathscr{C} \to \mathscr{C}'$, $G : \mathscr{C}' \to \mathscr{C}$ be two functors. We consider the two bifunctions $H_1, H_2 : \mathscr{C}^{\mathrm{op}} \times \mathscr{C} \to \mathbf{Set}$ defined by $H_1 = \mathrm{Hom}_{\mathscr{C}'}(F(\cdot), \cdot)$ and $H_2 = \mathrm{Hom}_{\mathscr{C}}(\cdot, G(\cdot))$. Suppose that we are given, for every $X \in \mathrm{Ob}(\mathscr{C})$ and every $Y \in \mathrm{Ob}(\mathscr{C}')$, a bijection $\alpha(X,Y) : H_1(X,Y) \xrightarrow{\sim} H_2(X,Y)$. Show that the two following statements are equivalent :

(i) The family of bijections $(\alpha(X,Y))_{X \in \mathrm{Ob}(\mathscr{C}), Y \in \mathrm{Ob}(\mathscr{C}')}$ defines an isomorphism of functors $H_1 \xrightarrow{\sim} H_2$.

(ii) For every morphism $f : X_1 \to X_2$ in $\mathscr{C}$, every morphism $g : Y_1 \to Y_2$ in $\mathscr{C}'$, and for all $u \in \mathrm{Hom}_{\mathscr{C}'}(F(X_1), Y_1)$ and $v \in \mathrm{Hom}_{\mathscr{C}'}(F(X_2), Y_2)$, the square

$$\begin{tikzcd} F(X_1) \arrow[r, "u"] \arrow[d, "F(f)"'] & Y_1 \arrow[d, "g"] \\ F(X_2) \arrow[r, "v"'] & Y_2 \end{tikzcd}$$

is commutative if and only if the square

$$\begin{tikzcd} X_1 \arrow[r, "\alpha(X_1,Y_1)(u)"] \arrow[d, "f"'] & G(Y_1) \arrow[d, "G(g)"] \\ X_2 \arrow[r, "\alpha(X_2,Y_2)(v)"'] & G(Y_2) \end{tikzcd}$$

is commutative.

Solution. The key is to write explicitly what it means for the $(\alpha(X,Y))$ to define a morphism of functors. It means that, for every morphism $f : X_1 \to X_2$ in $\mathscr{C}$ (that is, a morphism $X_2 \to X_1$ in $\mathscr{C}^{\mathrm{op}}$) and for every morphism $g : Y_1 \to Y_2$ in $\mathscr{C}'$, the following square commutes :

$$\begin{tikzcd} \mathrm{Hom}_{\mathscr{C}'}(F(X_2), Y_1) \arrow[r, "H_1(f,g)"] \arrow[d, "\alpha(X_2,Y_1)"'] & \mathrm{Hom}_{\mathscr{C}'}(F(X_1), Y_2) \arrow[d, "\alpha(X_1,Y_2)"] \\ \mathrm{Hom}_{\mathscr{C}}(X_2, G(Y_1)) \arrow[r, "H_2(f,g)"'] & \mathrm{Hom}_{\mathscr{C}}(X_1, G(Y_2)) \end{tikzcd}$$

The fact that the square commutes says exactly that, for every morphism $w : F(X_2) \to Y_1$ in $\mathscr{C}'$, we have

$$\alpha(X_1, Y_2)(g \circ w \circ F(f)) = G(g) \circ \alpha(X_2, Y_1)(w) \circ f.$$

Suppose that (i) holds. Using the calculation of the previous, we get :

(a) Taking $X_1 = X_2$, $f = \mathrm{id}_{X_1}$ and $g : Y_1 \to Y_2$ arbitrary : for every $u : F(X_1) \to Y_1$, we have

$$\alpha(X_1, Y_2)(g \circ u) = G(g) \circ \alpha(X_1, Y_1)(u).$$

(b) Taking $f : X_1 \to X_2$ arbitrary, $Y_1 = Y_2$ and $g = \mathrm{id}_{Y_2}$: for every $v : F(X_2) \to Y_2$, we have

$$\alpha(X_1, Y_2)(v \circ F(f)) = \alpha(X_2, Y_2)(v) \circ f.$$

Suppose that we are in the situation of (ii), that is, we are given morphisms $f : X_1 \to X_2$ in $\mathscr{C}$, $g : Y_1 \to Y_2$ in $\mathscr{C}'$, and $u \in \mathrm{Hom}_{\mathscr{C}'}(F(X_1), Y_1)$ and $v \in \mathrm{Hom}_{\mathscr{C}'}(F(X_2), Y_2)$. We want to show that the top square of (ii) commutes if and only if the bottom squqre commutes.

Suppose that the top square commutes, that is, that $v \circ F(f) = g \circ u$. Applying (a) and (b), we get

$$G(g) \circ \alpha(X_1, Y_1)(u) = \alpha(X_1, Y_2)(g \circ u) = \alpha(X_1, Y_2)(v \circ F(f)) = \alpha(X_2, Y_2)(v) \circ f.$$

This shows that the bottom square commutes.

Conversely, suppose that the bottom square commutes, that is, that $G(g) \circ \alpha(X_1, Y_1)(u) = \alpha(X_2, Y_2)(v) \circ f$. Again, applying (a) and (b), we get

$$\alpha(X_1, Y_2)(g \circ u) = G(g) \circ \alpha(X_1, Y_1)(u) = \alpha(X_2, Y_2)(v) \circ f = \alpha(X_1, Y_2)(v \circ F(f)).$$

As $\alpha(X_1, Y_2)$ is bijective, this implies that $g \circ u = v \circ F(f)$, which means that the top square commutes.

Now we assume that (ii) holds, and we want to show that (i) also holds. Let $f : X_1 \to X_2$ be a morphism in $\mathscr{C}$, $g : Y_1 \to Y_2$ be a morphism in $\mathscr{C}'$, and $w : F(X_2) \to Y_1$ be a morphism in $\mathscr{C}'$. We want to show that $\alpha(X_1, Y_2)(g \circ w \circ F(f)) = G(g) \circ \alpha(X_2, Y_1)(w) \circ f$. We apply (i) to $u = w \circ F(f) : F(X_1) \to Y_1$ and $v = g \circ w : F(X_2) \to Y_2$. We obviously have $g \circ u = v \circ F(f)$, so, by (i), this implies that

$$\alpha(X_2, Y_2)(g \circ w) \circ f = G(g) \circ \alpha(X_1, Y_1)(w \circ F(f)). \tag{*}$$

Applying (*) to the particular case where $Y_1 = Y_2$ and $g = \mathrm{id}_{Y_1}$, we get:

$$\alpha(X_2, Y_1)(w) \circ f = \alpha(X_1, Y_1)(w \circ F(f)). \tag{**}$$

Applying (**) with w replaced by $g \circ w : F(X_2) \to Y_2$, we get

$$\alpha(X_2, Y_2)(g \circ w) \circ f = \alpha(X_1, Y_2)(g \circ w \circ F(f)). \tag{***}$$

Putting (*), (**) and (***) together gives

$$\begin{aligned}\alpha(X_1, Y_2)(g \circ w \circ F(f)) = \alpha(X_2, Y_2)(g \circ w) \circ f &= G(g) \circ \alpha(X_1, Y_1)(w \circ F(f)) \\ &= G(g) \circ \alpha(X_1, Y_1)(w \circ F(f)),\end{aligned}$$

which is what we wanted to prove.

□

8. Remember that a functor $F : \mathscr{C} \to \mathbf{Set}$ is called representable if there exists an object X of $\mathscr{C}$ and an element x of $F(X)$ such that the morphism of functors $u : \mathrm{Hom}_{\mathscr{C}}(X, \cdot) \to F$ defined by $u(Y) : \mathrm{Hom}_{\mathscr{C}}(X, Y) \to F(Y)$, $(f : X \to Y) \longmapsto F(f)(x)$ is an isomorphism. The couple (X, x) is then said to represent F.

The following functors are representable. For each of them, give a couple representing the functor. (If the functor is only defined on objects, it is assumed to act on morphisms in the obvious way.) (1 point per functor) (a). The identity endofunctor of **Set**.

(b). The functor $F : \mathbf{Grp} \to \mathbf{Set}$, $G \longmapsto G^n$, where $n \in \mathbb{N}$.

(c). The forgetful functor $\mathbf{Mod}_R \to \mathbf{Set}$, where R is a ring.

(d). The forgetgul functor $\mathbf{Ring} \to \mathbf{Set}$.

(e). The functor $\mathbf{Ring} \to \mathbf{Set}$, $R \longmapsto R^\times$.

(f). The functor $F : \mathbf{Cat} \to \mathbf{Set}$ that takes a category to its set of objects.

(g). The functor $F : \mathbf{Cat} \to \mathbf{Set}$ that takes a category to its set of morphisms (i.e. $\bigcup_{X,Y \in \mathrm{Ob}(\mathscr{C})} \mathrm{Hom}_{\mathscr{C}}(X, Y)$).

(h). The functor $F : \mathbf{Cat} \to \mathbf{Set}$ that takes a category to its set of isomorphisms.

(i). The functor $F : \mathbf{Top}_* \to \mathbf{Set}$ that takes a pointed topological space (X, x) to the set of continuous loops on X with base point x.

(j). The functor $F : \mathbf{Set}^{\mathrm{op}} \to \mathbf{Set}$ such that $F(X) = \mathfrak{P}(X)$ and, for every map $f : X \to Y$, $F(f) : \mathfrak{P}(Y) \to \mathfrak{P}(X)$ is the map $A \longmapsto f^{-1}(A)$.

(k). The functor $F : \mathbf{Top}^{\mathrm{op}} \to \mathbf{Set}$ that sends a topological space to its set of open subsets. (If $f : X \to Y$ is a continuous map, $F(f) : F(Y) \to F(X)$ is the map $U \longmapsto f^{-1}(U)$.)

(l). If k is a field, the functor $F : \mathbf{Mod}_k^{\mathrm{op}} \to \mathbf{Set}$ that sends a k-vector space to the underlying set of V^* (so F is the composition of the duality functor $\mathbf{Mod}_k^{\mathrm{op}} \to \mathbf{Mod}_k$ and of the forgetful functor from $\mathbf{Mod}_k$ to **Set**.)

Solution.

(a). Take $X = \{x\}$ to be a singleton and x to be the unique element of $F(X) = X$. Then, for every set Y, $u(Y) : \mathrm{Hom}_{\mathbf{Set}}(X, Y) \to F(Y) = Y$ sends $f : X \to Y$ to $f(x) \in F(Y) = Y$; it is clearly bijective.

(b). Let $X = F_n$ be the free group on n generators $(x_1, \ldots, x_n)$, and $x = (x_1, \ldots, x_n) \in F(F_n) = (F_n)^n$. For every group G, the map $u(G) : \mathrm{Hom}_{\mathbf{Grp}}(F_n, G) \to G^n$ sends $f : F_n \to G$ to $(f(x_1), \ldots, f(x_n)) \in G^n$. The fact that this is bijective is the universal property of the free group F_n.

(c). Take $X = R$ with the obvious right R-action, and $x = 1 \in F(R) = R$. Then, for every right R-module M, the map $u(M) : \mathrm{Hom}_R(R, M) \to F(M) = M$ sends $f : R \to M$ to $f(1)$. This is bijective because R is a free R-module with base $\{1\}$.

(d). Take X equal to the polynomial ring $\mathbb{Z}[x]$ and $x \in F(X) = X$ to be the indeterminate. For every ring R, the map $u(R) : \mathrm{Hom}_{\mathbf{Ring}}(\mathbb{Z}[x], R) \to F(R) = R$ sends $f : \mathbb{Z}[x] \to R$ to $f(x) \in R$. The fact that this is bijective is the universal property of the polynomial ring.

(e). Take $X = \mathbb{Z}[x, x^{-1}]$ (the polynomial ring $\mathbb{Z}[x]$ localized at the indeterminaye x) and x to be the indeterminate in $F(X) = \mathbb{Z}[x, x^{-1}]^\times$. For every ring R, the map $u(R) : \mathrm{Hom}_{\mathbf{Ring}}(\mathbb{Z}[x], R) \to F(R) = R^\times$ sends $f : \mathbb{Z}[x] \to R$ to $f(x) \in R^\times$. The fact that this is bijective follows from the universal properties of the polynomial ring of the localization.

(f). Let X be the category with only one object $*$ and such that $\mathrm{End}_X(*) = \{\mathrm{id}_*\}$, and let $x \in F(X) = \{*\}$ be the unique object. (Note that X is the category corresponding to the poset $[0]$.) If $\mathscr{C}$ is a category, the map $u(\mathscr{C}) : \mathrm{Func}(X, \mathscr{C}) \to F(\mathscr{C}) = \mathrm{Ob}(\mathscr{C})$ takes a functor $G : X \to \mathscr{C}$ to $G(*) \in \mathrm{Ob}(\mathscr{C})$. This map is bijective, with inverse the map

$v(\mathscr{C}) : \mathrm{Ob}(\mathscr{C}) \to \mathrm{Func}(X, \mathscr{C})$ sending $c \in \mathrm{Ob}(\mathscr{C})$ to the functor $G : X \to \mathscr{C}$ defined by $G(*) = c$ and $G(\mathrm{id}_*) = \mathrm{id}_c$.

(g). Let X be the category corresponding to the poset $[1]$, that is, X has two objects 0 and 1, and a unique non-identity morphism $\alpha : 0 \to 1$. Let $x \in F(X)$ be the morphism α. If $\mathscr{C}$ is a category, the map $u(\mathscr{C}) : \mathrm{Func}(X, \mathscr{C}) \to F(X)$ sends a functor $G : X \to \mathscr{C}$ to $G(\alpha) \in \mathrm{Hom}_{\mathscr{C}}(F(0), F(1))$. Let $v(\mathscr{C}) : F(X) \to \mathrm{Func}(X, \mathscr{C})$ be defined as follows : if $f : c_0 \to c_1$ is a morphism of $\mathscr{C}$, that is, an element of $F(\mathscr{C})$, we defined a functor $G : X \to \mathscr{C}$ by $G(0) = c_0$, $G(1) = c_1$ and $G(\alpha) = f$. Then $v(\mathscr{C})$ is an inverse of $u(\mathscr{C})$, so $u(\mathscr{C})$ is bijective.

Let X be the category such that $\mathrm{Ob}(X) = \{0, 1\}$, and such that the only two non-identity morphisms of X are morphisms $\alpha : 0 \to 1$ and $\beta : 1 \to 0$ such that $\alpha \circ \beta = \mathrm{id}_1$ and $\beta \circ \alpha = \mathrm{id}_0$. If $\mathscr{C}$ is a category, the map $u(\mathscr{C}) : \mathrm{Func}(X, \mathscr{C}) \to F(X)$ sends a functor $G : X \to \mathscr{C}$ to $G(\alpha) \in \mathrm{Hom}_{\mathscr{C}}(F(0), F(1))$, which is an isomorphism with inverse $G(\beta)$. Let $v(\mathscr{C}) : F(X) \to \mathrm{Func}(X, \mathscr{C})$ be defined as follows : if $f : c_0 \to c_1$ is an isomorphism of $\mathscr{C}$, that is, an element of $F(\mathscr{C})$, we defined a functor $G : X \to \mathscr{C}$ by $G(0) = c_0$, $G(1) = c_1$, $G(\alpha) = f$ and $G(\beta) = f^{-1}$. Then $v(\mathscr{C})$ is an inverse of $u(\mathscr{C})$, so $u(\mathscr{C})$ is bijective.

(h). Remember that a loop on a topological space Y with base point y is just a continuous map γ from S^1 (the unit circle in $\mathbb{C}$) to Y such that $\gamma(1) = y$. In other words, it is a morphism from $(S^1, 1)$ to (Y, y) in the category $\mathbf{Top}_*$. So we can take $X = (S^1, 1)$ and $x = \mathrm{id}_{S^1} \in F(X)$.

(i). For every set Y, we have a bijection $v(Y) : \mathfrak{P}(Y) \overset{\sim}{\to} \mathrm{Hom}_{\mathbf{Set}}(Y, \{0, 1\})$ sending a subset A of Y to its characteristic function. So we can take $X = \{0, 1\}$ and $x = \{1\} \in \mathfrak{P}(X)$. Indeed, if Y is a set, then the map $u(Y) : \mathrm{Hom}_{\mathbf{Set}^{\mathrm{op}}}(X, Y) = \mathrm{Hom}_{\mathbf{Set}}(Y, X) \to \mathfrak{P}(Y)$ sends $f : Y \to \{0, 1\}$ to $f^{-1}(\{1\})$, which is the inverse of the bijection $v(Y)$.

(j). Let X be the Sierpinski space, that is, the topological space $\{s, \eta\}$ where the open subsets are $\varnothing$, $\{\eta\}$ and $\{s, \eta\}$, and let $x = \{\eta\} \in F(X)$. Then, if Y is a topological space, the map $u(Y) : \mathrm{Hom}_{\mathbf{Set}^{\mathrm{op}}}(X, Y) = \mathrm{Hom}_{\mathbf{Set}}(Y, X) \to \mathfrak{P}(Y)$ sends $f : Y \to \{s, \eta\}$ to the open subset $f^{-1}(\{\eta\})$ of Y. Conversely, if U is an open subset of Y, then the map $f : Y \to \{s, \eta\}$ such that $f(y) = \eta$ for $y \in Y$ and $f(y) = s$ for $y \in Y - U$ is continuous. So $u(Y)$ is bijective.

(k). For every k-vector space V, we have $F(V) = \mathrm{Hom}_k(V, k)$. So we can take $X = k$ (with the obvious action of k) and $x = \mathrm{id}_k \in \mathrm{Hom}_k(k, k)$. Indeed, for every k-vector space V, the map $u(V) : \mathrm{Hom}_{\mathbf{Mod}_k^{\mathrm{op}}}(k, V) = \mathrm{Hom}_k(V, k) \to F(V) = \mathrm{Hom}_k(V, k)$ sends $f : V \to k$ to $\mathrm{id}_k \circ f = f$. This is the identity of $F(V)$, so it is obviously bijective.

$\square$

9. (extra credit) The simplicial category Δ is defined in Example I.2.1.8(5) of the notes. It is the category whose objects are the finite sets $[n] = \{0, 1, \ldots, n\}$ with their usual order and whose morphisms are the nondecreasing maps between these sets.

The category **sSet** of *simplicial sets* if $\mathrm{Func}(\Delta^{\mathrm{op}}, \mathbf{Set})$. So a simplicial set is by definition a functor $X : \Delta^{\mathrm{op}} \to \mathbf{Set}$; in that case, we write X_n for $X([n])$ and, if $f : [n] \to [m]$, we often write $f^* : X_m \to X_n$ for $X(f)$. For example, for each $n \in \mathbb{N}$, the *standard simplex* of dimension n is the simplicial set $\mathrm{Hom}_{\Delta}(\cdot, [n])$.

If X is a simplicial set, a simplicial subset Y of X is the data of a subset Y_n of X_n, for every $n \in \mathbb{N}$, such that $\alpha^*(Y_m) \subset Y_n$ for every morphism $\alpha : [n] \to [m]$ in Δ. We can form images of morphisms of simplicial sets, and unions and intersections of simplicial subsets, in the obvious way.

If we see each poset $[n]$ as a category in the usual way, then the morphisms of Δ become functors, so this allows us to see Δ as a subcategory of **Cat**.

Let $\mathscr{C}$ be a category. Its *nerve* $N(\mathscr{C})$ is the restriction to Δ^{op} of the functor $\mathrm{Hom}_{\mathbf{Cat}}(\cdot,\mathscr{C})$ on $\mathbf{Cat}^{\mathrm{op}}$; it is a functor from Δ^{op} to **Set**, i.e. a simplicial set. As $\mathrm{Hom}_{\mathbf{Cat}}$ is a bifunctor, this construction is functorial in $\mathscr{C}$, and we get a nerve functor $N : \mathbf{Cat} \to \mathbf{sSet}$.

(a). (3 points) If $\mathscr{C}$ is a category, show that $N(\mathscr{C})_0 \simeq \mathrm{Ob}(\mathscr{C})$ and $N(\mathscr{C})_1 \simeq \coprod_{X,Y\in\mathrm{Ob}(\mathscr{C})} \mathrm{Hom}_{\mathscr{C}}(X,Y)$. Can you give a similar description of $N(\mathscr{C})_n$ for $n \geq 2$?

(b). (1 point) Let $n \in \mathbb{N}$. Show that the nerve of $[n]$ is isomorphic to Δ_n.

(c). (1 point) Let $n \in \mathbb{N}$. Show that there exists $e_n \in \Delta_n([n])$ such that, for every simplicial set X, the map $\mathrm{Hom}_{\mathbf{sSet}}(\Delta_n, X) \xrightarrow{\sim} X_n$ sending u to $u_n(e_n)$ is bijective.

(d). (1 point) For every category $\mathscr{C}$ and every simplicial set X, if $u, v : X \to N(\mathscr{C})$ are two morphisms of simplicial sets such that $u_i, v_i : X_i \to N(\mathscr{C})_i$ are equal for $i \in \{0,1\}$, show that $u = v$.

(e). (1 point) We denote by $\Delta_{\leq 2}$ the full subcategory of Δ whose objects are [0], [1] and [2]; if X is a simplicial set, we denote by $X_{\leq 2}$ its restriction to $\Delta_{\leq 2}$ (which is a functor $\Delta^{\mathrm{op}}_{\leq 2} \to \mathbf{Set}$).

Let X be a simplicial set and $\mathscr{C}$ be a category. Show that every morphism $X_{\leq 2} \to N(\mathscr{C})_{\leq 2}$ extends to a morphism $X \to N(\mathscr{C})$.

(f). (2 points) Show that the functor $N : \mathbf{Cat} \to \mathbf{sSet}$ is fully faithful.

Let $n \in \mathbb{N}$ For every $k \in [n]$, we denote by δ_k the unique injective increasing map $[n-1] \to [n]$ such that $k \notin \mathrm{Im}(\delta_k)$. This induces a map $\Delta_{n-1} \to \Delta_n$, that we also denote by δ_k; the image of this map is called the kth facet of Δ_n.

If $k \in [n]$, the *horn* Λ^n_k is the union of all the facets of Δ_n except for the kth one; in other words, it is the simplicial subset of Δ_n defined by

$$\Lambda^n_k([m]) = \{f \in \mathrm{Hom}_\Delta([m],[n]) \mid \exists l \in [n]-\{k\} \text{ and } g \in \mathrm{Hom}_\Delta([m],[n-1]) \text{ with } f = \delta_l \circ g\}.$$

(g). (1 point) Let $\mathscr{C}$ be a category. If $n \geq 3$ and $k \in [n] - \{0, n\}$, show that every morphism of simplicial sets $\Lambda^n_k \to X$ extends uniquely to a morphism $\Delta_n \to X$.

(h). (1 point) Let $\mathscr{C}$ be a category. Show that every morphism of simplicial sets $\Lambda^2_1 \to X$ extends uniquely to a morphism $\Delta_2 \to X$.

(i). (2 points) Show that a simplicial set X is the nerve of a category if and only if, for every $n \in \mathbb{N}$, every $0 < k < n$ and every morphism of simplicial sets $u : \Lambda^n_k \to X$, the morphism u extends uniquely to a morphism $\Delta_n \to X$.

Solution.

(a). By problem 8(f), the functor $\mathbf{Cat} \to \mathbf{Set}$, $\mathscr{C} \longmapsto \mathrm{Ob}(\mathscr{C})$ is represented by [0]. As $N(\mathscr{C})_0 = \mathrm{Hom}_{\mathbf{Cat}}([0],\mathscr{C})$, this gives an isomorphism $N(\mathscr{C})_0 \simeq \mathrm{Ob}(\mathscr{C})$, natural in $\mathscr{C}$. Similarly, by 8(g), the functor $\mathbf{Cat} \to \mathbf{Set}$, $\mathscr{C} \longmapsto \coprod_{X,Y\in\mathrm{Ob}(\mathscr{C})} \mathrm{Hom}_{\mathscr{C}}(X,Y)$ is represented by [1]. As $N(\mathscr{C})_1 = \mathrm{Hom}_{\mathbf{Cat}}([1],\mathscr{C})$, this gives an isomorphism $N(\mathscr{C})_1 \simeq \coprod_{X,Y\in\mathrm{Ob}(\mathscr{C})} \mathrm{Hom}_{\mathscr{C}}(X,Y)$, also natural in $\mathscr{C}$. Note that, if $\delta_0, \delta_1 : [0] \to [1]$ are the two maps defined by $\delta_0(0) = 1$ and $\delta_1(0) = 0$, then $\delta_1^* : N(\mathscr{C})_1 \to N(\mathscr{C})_0$ sends a morphism to its source and $\delta_0^* : N(\mathscr{C})_1 \to N(\mathscr{C})_0$ sends a morphism to its target.

Let $\mathscr{C}$ be category. For $n \geq 1$, consider the set M_n of sequences of n composable morphisms $c_0 \overset{f_1}{\to} c_1 \overset{f_2}{\to} \ldots \overset{f_n}{\to} c_n$ of $\mathscr{C}$, which we will also write as $(f_1, \ldots, f_n)$. We have a map $\alpha : N(\mathscr{C})_n \to M_n$ sending a functor $F : [n] \to \mathscr{C}$ to the sequence $F(0) \to F(1) \to \ldots \to F(n)$, where the morphism $F(i) \to F(i+1)$ is the image by F of the unique morphism $i \to i+1$ in $[n]$. This uniquely determines the functor F, because, for $i \leq j$ in $[n]$, the unique morphism $i \to j$ is the composition of $i \to i+1 \to i+2 \to \ldots \to j$. For the same reason, every element of M_n comes from a functor $F : [n] \to \mathscr{C}$. So we get a bijection $N(\mathscr{C})_n \overset{\sim}{\to} M_n$. (We can easily make M_n into a functor **Cat** $\to$ **Set**, and then this bijection is an isomorphism of functors.)

We will identify $N(\mathscr{C})_n$ with M_n in the rest of this solution. We also write $M_0 = \mathrm{Ob}(\mathscr{C})$. (we can think of $c \in \mathrm{Ob}(\mathscr{C})$ as a length 0 sequence of composable morphisms (c).)

Let $\alpha : [m] \to [n]$ be a nondecreasing map. We can give an explicit description of the map $\alpha^* : N(\mathscr{C})_n \to N(\mathscr{C})_m$ by chasing through the identifications. If $n = 0$ and $m \geq 1$, then α^* sends $c \in \mathrm{Ob}(\mathscr{C})$ to the sequence $(\mathrm{id}_c, \ldots, \mathrm{id}_c) \in M_m$. If $n \geq 1$ and $m = 0$, let $i = \alpha(0)$; then α^* sends the sequence $c_0 \overset{f_1}{\to} c_1 \overset{f_2}{\to} \ldots \overset{f_n}{\to} c_n$ to c_i. Suppose that $n, m \geq 1$, let $c_0 \overset{f_1}{\to} c_1 \overset{f_2}{\to} \ldots \overset{f_n}{\to} c_n$ be an element of M_n, and let $d_0 \overset{g_1}{\to} d_1 \overset{g_2}{\to} \ldots \overset{g_m}{\to} d_m$ be its image by α^*. For $i \in \{1, \ldots, m\}$, we have :

- if $\alpha(i-1) = \alpha(i)$, then $d_{i-1} = d_i = c_{\alpha(i)}$ and $g_i = \mathrm{id}_{c_{\alpha(i)}}$;
- if $\alpha(i-1) < \alpha(i)$, then $g_i = f_{\alpha(i)} \circ \ldots \circ f_{\alpha(i-1)+1}$.

(b). As we have identified Δ to a subcategory of **Cat**, this is just the definition of Δ_n.

(c). Let $e_n = \mathrm{id}_{[n]} \in \Delta_n([n]) = \mathrm{Hom}_\Delta([n],[n])$. The fact that the map of the statement is bijective is exactly the Yoneda lemma (Theorem I.3.2.2 of the notes).

(d). Suppose that $u, v : X \to N(\mathscr{C})$ satisfy the condition of the question. Let $n \geq 2$, and let $x \in X_n$. Write $u(x) = (c_0 \overset{f_1}{\to} c_1 \overset{f_2}{\to} \ldots \overset{f_n}{\to} c_n)$ and $v(x) = (d_0 \overset{g_1}{\to} d_1 \overset{g_2}{\to} \ldots \overset{g_n}{\to} d_n)$. We want to show that $u(x) = v(x)$, that is, that $f_i = g_i$ for every $i \in \{1, \ldots, n\}$. Fix $i \in \{1, \ldots, n\}$, and consider the map $\alpha : [1] \to [n]$ sending 0 to $i-1$ and 1 to i. Then α is a morphism in Δ, so we have a commutative diagram

$$\begin{tikzcd} X_n \arrow[r, "u_n"] \arrow[d, "\alpha^*"'] & N(\mathscr{C})_n \arrow[d, "\alpha^*"] \\ X_1 \arrow[r, "u_1"'] & N(\mathscr{C})_1 \end{tikzcd}$$

and a similar commutative diagram for v. By definition of the bijection $N(\mathscr{C})_n \overset{\sim}{\to} M_n$, the map α^* sends a sequence $e_0 \overset{h_1}{\to} e_1 \overset{h_2}{\to} \ldots \overset{h_n}{\to} e_n$ to $h_i : e_{i-1} \to e_i$. So we get $f_i = \alpha^*(u_n(x)) = u_1(\alpha^*(x)) = v_1(\alpha^*(x)) = \alpha^*(v_n(x)) = g_i$.

(e). Let $u : X_{\leq 2} \to N(\mathscr{C})_{\leq 2}$. We want to show that u extends to a morphism of simplicial sets $v : X \to N(\mathscr{C})$. The solution of question (d) tells us how we must extend u: Let $n \geq 2$, and, for $i \in \{1, \ldots, n\}$, let $\alpha_i^n = \alpha_i : [1] \to [n]$ be the map $m \longmapsto m + i - 1$. Then, for every $x \in X_n$, $v_n(x)$ must be the sequence $(u_1(\alpha_1^*(x)), \ldots, u_1(\alpha_n^*(x)))$ of morphisms of $\mathscr{C}$. These morphisms are composable : indeed, if we denote by $\delta_0, \delta_1 : [0] \to [1]$ the two maps defined by $\delta_0(0) = 1$ and $\delta_1(0) = 0$, then $\alpha_i \circ \delta_0 = \alpha_{i+1} \circ \delta_1$ for $1 \leq i \leq n-1$, so the target $u_0(\delta_0^* \alpha_i^*(x))$ of $u_1(\alpha_i^*(x))$ is equal to the source $u_0(\delta_1^* \alpha_{i+1}^*(x))$ of $u_1(\alpha_{i+1}^*(x))$.

We have to check that $v_2 = u_2$ and that v is a morphism of simplicial sets. The proof that $v_2 = u_2$ is exactly as in the solution of (d). To show that v is a morphism of simplicial sets, we take a nondecreasing map $\alpha : [m] \to [n]$ and we show that $v_m \circ \alpha^* = \alpha^* \circ v_n$.

We can write $\alpha = \alpha' \circ \alpha''$ with α', α'' both nondecreasing, α' injective and α'' surjective, and it suffices to show the statement for α' and α''. Moreover, we can write α' (resp. α'') as a composition of injective (resp. surjective) nondecreasing maps $[p] \to [p+1]$ (resp. $[p+1] \to [p]$). So we may assume that α is injective or surjective and that $n = m \pm 1$.

Suppose first that $\alpha : [n+1] \to [n]$ is a surjective nondecreasing map. Then there is a unique $i \in [n]$ such that $\alpha(i) = \alpha(i+1) = i$, $\alpha(j)$ for $0 \leq j < i$ and $\alpha(j) = j-1$ for $i+1 < j \leq n+1$. Let $x \in X_n$, and let $(f_1, \dots, f_n) = v_n(x)$. The map $\alpha^* : N(\mathscr{C})_n \to N(\mathscr{C})_{n+1}$ sends the sequence of composable morphisms $(f_1, \dots, f_n)$ to $(f_1, \dots, f_i, \mathrm{id}_c, f_{i+1}, \dots, f_n)$, where c is the target of f_i. By definition, $v_{n+1}(\alpha^*(x)) = (g_1, \dots, g_{n+1})$, with $g_j = u_1({\alpha_j^{n+1}}^*\alpha^*(x))$. If $1 \leq j \leq i$, then $\alpha \circ \alpha_j^{n+1} = \alpha_j^n$, so $g_j = f_j$. If $i+2 \leq j \leq n+1$, then $\alpha \circ \alpha_j^{n+1} = \alpha_{j-1}^n$, so $g_j = f_{j-1}$. Finally, $\alpha_{i+1}^n \circ \alpha : [1] \to [n]$ is the map sending every element of $[1]$ to i, so it is equal to $\alpha' \circ \alpha''$, where $\alpha' : [0] \to [n]$ sends 0 to i and $\alpha'' : [1] \to [0]$ is the unique map; so $g_{i+1} = u_1(\alpha''^* \circ \alpha'^*(x)) = \alpha''^* u_0(\alpha'^*(x))$ is $\mathrm{id}_{c'}$, where $c' = u_0(\alpha'^*(x))$; as $\alpha' = \alpha_i^n \circ \delta_0$, we have $c' = \delta_0^*(f_i)$, that is, c' is the target c of f_i, as we wanted.

Now we take $\alpha : [n-1] \to [n]$ injective and increasing; we may also assume $n \geq 3$, as we already the result for $n \leq 2$. There exists $i \in [n]$ such that $\mathrm{Im}(\alpha) = [n] - \{i\}$, that is, such that α is the map δ_i defined before (g). Let $x \in X_n$, and let $c_0 \xrightarrow{f_1} \dots \xrightarrow{f_n} c_n$ be $v_n(x)$ and $(g_1, \dots, g_{n-1}$ be $v_{n-1}(\alpha^*(x))$. As we saw in the solution of (a), the map $\alpha^* : M_n \to M_{n-1}$ sends the sequence of composable morphisms $c_0 \xrightarrow{f_1} \dots \xrightarrow{f_n} c_n$ to the sequence :

- $c_0 \xrightarrow{f_1} \dots \xrightarrow{f_{n-1}} c_{n-1}$ if $i = n$;
- $c_1 \xrightarrow{f_2} \dots \xrightarrow{f_n} c_n$ if $i = 0$;
- $c_0 \xrightarrow{f_1} \dots c_{i-1} \xrightarrow{f_{i+1} \circ f_i} c_{i+1} \dots \xrightarrow{f_n} c_n$ if $1 \leq i \leq n-1$.

If $1 \leq j \leq i-1$, we have $\alpha \circ \alpha_j^{n-1} = \alpha_j^n$, which implies that $g_j = f_j$. If $i+1 \leq j \leq n-1$, we have $\alpha \circ \alpha_j^{n-1} = \alpha_{j+1}^n$, which implies that $g_j = f_{j+1}$. To finish the proof that $\alpha^*(v_n(x)) = v_{n-1}(\alpha^*(x))$, it remains to consider the case $j = i \in \{1, \dots, n-1\}$. Then $\alpha \circ \alpha_j^{n-1} = \alpha' \circ \delta_1$, where $\alpha' : [2] \to [n]$ is the map $x \longmapsto x + i - 1$ and $\delta_1 : [1] \to [2]$ is the map sending 0 to 0 and 1 to 2. Hence $g_j = u_1(\delta_1^* \alpha'^*(x)) = \delta_1^* u_2(\alpha'^* x)$, so if $u_2(\alpha'^*(x)) = (h_1, h_2)$ then $g_j = h_2 \circ h_1$; but it is easy to see that $u_2(\alpha'^*(x)) = (f_i, f_{i+1})$ (by looking at the composition of α' with $\alpha_1^2, \alpha_2^2 : [1] \to [2]$), so we are done.

(f). Let $\mathscr{C}$ and $\mathscr{C}'$ be categories. We want to show that the map $N : \mathrm{Func}(\mathscr{C}, \mathscr{C}') \to \mathrm{Hom}_{\mathbf{sSet}}(N(\mathscr{C}), N(\mathscr{C}'))$ is bijective, so we try to construct an inverse of this map.

Let $u : N(\mathscr{C}) \to N(\mathscr{C}')$ be a morphism of simplicial sets. We denote by F the map $\mathrm{Ob}(\mathscr{C}) \simeq N(\mathscr{C})_0 \xrightarrow{u_0} N(\mathscr{C}')_0 \simeq \mathrm{Ob}(\mathscr{C}')$. Let $f : c_0 \to c_1$ be a morphism of $\mathscr{C}$. We saw in (a) that this morphism corresponds to a functor $T : [1] \to \mathscr{C}$, that is, an element of $N(\mathscr{C})_1$. We denote by $F(f) : d_0 \to d_1$ the morphism of $\mathscr{C}'$ corresponding to $u_1(T) \in N(\mathscr{C}')_1$. We want to show that $d_0 = F(c_0)$ and $d_1 = F(c_1)$. Let $i \in \{0, 1\}$, and consider the map $\alpha : [0] \to [1]$ sending 0 to i. This is a morphism of Δ, and $\alpha^* : N(\mathscr{C})_1 \to N(\mathscr{C})_0$ sends a morphism of $\mathscr{C}$ to its source if $i = 0$ and its target if $i = 1$. Using the commutativity of the diagram

$$\begin{tikzcd} N(\mathscr{C})_1 \arrow[r, "u_1"] \arrow[d, "\alpha^*"'] & N(\mathscr{C}'_1) \arrow[d, "\alpha^*"] \\ N(\mathscr{C}_0) \arrow[r, "u_0"'] & N(\mathscr{C}'_0) \end{tikzcd}$$

we see that $d_0 = F(c_0)$ and $d_1 = F(c_1)$. Now we show that F is a functor. There are two conditions to check :

(1) Consider the unique map $\alpha : [1] \to [0]$. This is a morphism of Δ, and $\alpha^* : N(\mathscr{C})_0 \to N(\mathscr{C})_1$ sends the element of $N(\mathscr{C})_0$ corresponding to an object c of $\mathscr{C}$ to the element of $N(\mathscr{C})_1$ corresponding to id_c. As $u_1 \circ \alpha^* = \alpha^* \circ u_0$, we get that, for every $c \in \mathrm{Ob}(\mathscr{C})$, $F(\mathrm{id}_c) = \mathrm{id}_{F(c)}$.

(2) Consider the map $\alpha : [1] \to [2]$ sending 0 to 0 and 1 to 2, and the map $\sigma_i : [1] \to [2]$, $m \longmapsto m+i$, for $i \in \{0,1\}$. Then α^* (resp. σ_0^*, resp. σ_1^*) sends the element of $N(\mathscr{C})_2$ corresponding to a sequence $c_0 \stackrel{f_1}{\to} c_1 \stackrel{f_2}{\to} c_2$ to the element of $N(\mathscr{C})_1$ corresponding to $f_2 \circ f_1$ (resp. f_1, resp. f_2). (This is clear on the identifications of (a).)

   Let $c_0 \stackrel{f_1}{\to} c_1 \stackrel{f_2}{\to} c_2$ be a sequence of composable morphisms of $\mathscr{C}$. Using the previous paragraph and the fact that u is a morphism of functors, we see that the image by u of the element of $N(\mathscr{C})_2$ corresponding to this sequence is the sequence $F(c_0) \stackrel{F(f_1)}{\to} F(c_1) \stackrel{F(f_2)}{\to} F(c_2)$, and using this and the fact that $\alpha^* \circ u_2 = u_1 \circ \alpha^*$, we finally get $F(f_2) \circ F(f_1) = F(f_2 \circ f_1)$.

So we have constructed a map $\Phi : \mathrm{Hom}_{\mathbf{sSet}}(N(\mathscr{C}), N(\mathscr{C}')) \to \mathrm{Func}(\mathscr{C}, \mathscr{C}')$, and it is clear on the construction that, for every functor $F : \mathscr{C} \to \mathscr{C}'$, we have $\Phi(N(F)) = F$. Now let $u : N(\mathscr{C}) \to N(\mathscr{C}')$ be a morphism of simplicial sets, and let $F = \Phi(u)$. We want to show that $N(F) = u$. Again, it is clear from the construction of Φ that $N(F)_0 = u_0$ and $N(F)_1 = u_1$. But then the fact that $N(F) = u$ follows from (c).

(g). If $n \geq 4$, then, for every $k \in [n]$, the morphism $\Lambda^n_{k,\leq 2} \to \Delta_{n,\leq 2}$ induced by the inclusion $\Lambda^n_k \subset \Delta_n$ is the identity morphism. So, by (d) and (e), every morphism $\Lambda^n_k \to N(\mathscr{C})$ extends uniquely to a morphism $\Delta_n \to N(\mathscr{C})$.

We still need to treat the case $n = 3$. Note that the uniqueness of the extension will follow from the fact that $\Lambda^3_{k,\leq 1} = \Delta_{3,\leq 1}$.

Let $\partial\Delta_3$ be the union of all the faces of Δ_3. Then the inclusion $\partial\Delta_3 \subset \Delta_3$ induces an equality $\partial\Delta_{3,\leq 2} = \Delta_{3,\leq 2}$, so it suffices to show that the morphism $u : \Lambda^3_k \to N(\mathscr{C})$ extends to $\partial\Delta_3$. As $\partial\Delta_3 = \Lambda^3_k \cup \delta_{k*}(\Delta_2)$ and $\Lambda^3_k \cap \delta_{k*}(\Delta_2) = \delta_{k*}(\partial\Delta_2)$, it suffices to extend u from $\delta_{k*}(\partial\Delta_2)$ to $\delta_{k*}(\Delta_2)$. For $0 \leq i < j \leq 3$, let $\alpha_{i,j} : [1] \to [3]$ be the map sending 0 to i and 1 to j; note that $\alpha_{i,j} \in \Lambda^3_k([3])$. Let $f_i = u_3(\alpha_{i-1,i})$, for $1 \leq i \leq 3$. We treat the case $k = 1$, the case $k = 2$ is similar. Factoring both $\alpha_{1,2}$ and $\alpha_{2,3}$ through the morphism $\delta_0 : [2] \to [3]$, we see that f_3 and f_2 are composable, and that $f_3 \circ f_2 = u_3(\alpha_{1,3})$. Factoring both $\alpha_{0,1} = f_1$ and $\alpha_{1,3}$ through the morphism $\delta_2 : [2] \to [3]$, we see that $f_3 \circ f_2 = u_3(\alpha_{1,3})$ and f_1 are composable, and that $f_3 \circ f_2 \circ f_1 = u_3(\alpha_{0,3})$. Similary, using $\delta_3 : [2] \to [3]$, we show that $f_2 \circ f_1 = u_3(\alpha_{0,2})$.

In particular, we see that $u_3(\alpha_{0,3}) = f_3 \circ f_2 \circ f_1 = u_3(\alpha_{3,2}) \circ u_3(\alpha_{0,2})$. This is exactly the condition that we need to extend u from $\delta_{1*}(\partial\Delta_2)$ to $\delta_{1*}(\Delta_2)$. (See the solution of the next question.)

(h). Let $u : \Lambda^2_1 \to N(\mathscr{C})$. We want to extend u to a morphism $v : \Delta_2 \to N(\mathscr{C})$. Remember that, by the Yoneda lemma, giving v is the same as giving an element e of $N(\mathscr{C})_2$; the fact that v extends u then says that, for every $\alpha : [m] \to [2]$ such that $\alpha \in \Lambda^2_1([n])$, we have $\alpha^*(e) = u_m(\alpha)$.

Note that the maps δ_2 and δ_0 from [1] to [2] are in $\Lambda^2_1([1])$ by definition of the horn Λ^2_1. We set $f_1 = u_1(\delta_2)$ and $f_2 = u_1(\delta_0)$. Comparing the compositions of δ_0 and δ_2 with the two maps $[0] \to [1]$, we see that (f_1, f_2) is a sequence of composable morphisms of $\mathscr{C}$, hence an element of e of $N(\mathscr{C})_2$; we denote by $v : \Delta_2 \to N(\mathscr{C})$ the corresponding morphism, that is, the unique morphism such that $v_2(e_2) = (f_1, f_2)$. Using the method of the solution of (d), we see that this is the only possibility for a morphism extending u (such a morphism must send $e_2 \in \Delta_2([2])$ to (f_1, f_2)).

It remains to show that v does extend u. Let $\alpha : [m] \to [2]$ be an element of $\Lambda_1^2([2])$; by definition of the horn, this means that we can write $\alpha = \delta_i \circ \beta$, with $\beta : [m] \to [1]$ nondecreasing and $i \in \{0, 2\}$. Then $v_2(\alpha) = \alpha^*(e) = \beta^*(\delta_i^*(e))$ and $u_2(\alpha) = \beta^* u_1(\delta_i)$, so it suffices to show that $\delta_i^*(e) = u_1(\delta_i)$; but this follows from the definition of f_1 and f_2 and the description of $\delta_i^* : N(\mathscr{C})_2 \to N(\mathscr{C})_1$ in (a).

(i). Let X be a simplicial set, and suppose that every morphism $u : \Lambda_k^n \to X$ with $0 < k < n$ extends uniquely to Δ_n. We denote by $d_0, d_1 : [0] \to [1]$ the two maps sending 0 to 0 and 1 respectively, and by s the unique map from [1] to [0]. We construct a category $\mathscr{C}$ in the following way :

(1) We take $\mathrm{Ob}(\mathscr{C}) = X_0$.

(2) If $c, d \in X_0$, we have $\mathrm{Hom}_{\mathscr{C}}(c, d) = \{f \in X_1 \mid d_0^*(f) = c \text{ and } d_1^*(f) = d\}$.

(3) For every $c \in X_0$, we denote by id_c the element $s^*(c)$ of X_1.

(4) Let $c, d, e \in X_0$ and $f \in \mathrm{Hom}_{\mathscr{C}}(c, d)$, $g \in \mathrm{Hom}_{\mathscr{C}}(d, e)$. We want to construct a morphism $u : \Lambda_1^2 \to X$. Let $\alpha : [m] \to [2]$ be an element of $\Lambda_1^2([m])$. By definition of Λ_1^2, there exists $\beta : [m] \to [1]$ and $j \in \{0, 2\}$ such that $\alpha = \delta_j \circ \beta$. We set $u_m(\alpha) = \beta^*(f_j)$, with $f_j = f$ if $j = 2$ and $f_j = g$ if $j = 0$. We must check that this is well-defined; if α can be written as $\beta \circ \delta_0$ and $\beta' \circ \delta_2$, with $\beta : [m] \to [1]$, this means that $\mathrm{Im}(\alpha) = \{1\}$, so $\mathrm{Im}(\beta) = \{0\}$ and $\mathrm{Im}(\beta') = \{1\}$, so there exists $\gamma : [m] \to [0]$ such that $\beta = d_0 \circ \gamma$ and $\beta' = d_1 \circ \gamma$, hence $\beta^*(g) = \gamma^*(d) = \beta'^*(f)$. We now check that u is a morphism of simplicial sets. If $\alpha : [m] \to [2]$ is an element of $\Lambda_1^2([m])$, write $\alpha = \delta_j \circ \beta$, with $\beta : [m] \to [1]$ and $j \in \{0, 2\}$; then, for every $\gamma : [m'] \to [m]$, we have $\alpha \circ \gamma = \delta_j \circ (\beta \circ \gamma)$, so $u_{m'}(\gamma) = (\beta \circ \gamma)^*(f_j) = \gamma^*(\beta^*(f_j)) = \gamma^*(u_m(\beta))$. So u is a morphism of simplicial sets, and, by assumption, it extends uniquely to a morphism $v : \Delta_2 \to X$. We take $g \circ f = v_1(\delta_1)$. It is easy to check that $g \circ f \in \mathrm{Hom}_{\mathscr{C}}(c, e)$.

It is easy to check that the identity morphisms are unit elements for the composition.

We check that the composition law of $\mathscr{C}$ is associative. Let (f_1, f_2, f_3) be a sequence of composable morphisms in $\mathscr{C}$. Remember that we have maps $\delta_i : [2] \to [3]$, inducing morphisms of simplicial sets $\delta_{i*} : \Delta_2 \to \Delta_3$. As in the construction of the composition in (4), we use the pair (f_1, f_2) to construct a morphism $\delta_{3*}(\Delta_2) \to X$, the pair (f_2, f_3) to construct a morphism $\delta_{1*}(\Delta_2) \to X$, and the pair $(f_1, f_3 \circ f_2)$ to construct a morphism $\delta_{2*}(\Delta_2) \to X$. These three morphisms glue to a morphism $\Lambda_1^3 \to X$, which extends uniquely to $v : \Delta_3 \to X$. In particular, if we define maps $\alpha_{i,j} : [1] \to [3]$ as in (g), we see as in that question that

$$v_1(\alpha_{0,3}) = v_1(\alpha_{3,2}) \circ v_1(\alpha_{2,0}) = f_3 \circ (f_2 \circ f_1) = v_1(\alpha_{3,1} \circ \alpha_{1,0}) = (f_3 \circ f_2) \circ f_1.$$

For $n \geq 1$ and $1 \leq i \leq n$, let the $\alpha_i^n : [1] \to [n]$ be as in the solution of (e).

Let $n \geq 2$. If $1 \leq m \leq n$ $0 \leq i_0 < i_1 < \ldots < i_m \leq n$, we denote by $\alpha_{i_0,\ldots,i_m} : [1] \to [n]$ the map sending $r \in [m]$ to $i_r \in [n]$. If $x \in X_n$, we define morphisms $f_{1,x}, \ldots, f_{n,x}$ in $\mathscr{C}$ by $f_i = \alpha_i^{n*}(x)$. As $\alpha_i^n \circ d_0 = \alpha_{i+1}^n \circ d_1$ for $1 \leq i \leq n-1$, the f_i form a sequence of composable morphisms, so $(f_1, \ldots, f_n) \in N(\mathscr{C})_n$. We claim that :

(A) For every $(f_1, \ldots, f_n)$ ni $N(\mathscr{C})_n$, there exists $x \in X$ such that $(f_{1,x}, \ldots, f_{n,x}) = (f_1, \ldots, f_n)$.

(B) If $x, y \in X_n$ are such that $f_{i,x} = f_{i,y}$ for $1 \leq i \leq n$, then $x = y$.

We prove (A). Let $(f_1, \ldots, f_n)$ be a sequence of composable morphisms in $\mathscr{C}$. For $0 \leq i_0 < i_1 \leq n$, we define a morphism $u_{i_0,i_1} : \alpha_{i_0,i_1,*}(\Delta_1) \to X$ by sending $\alpha_{i_0,i_1,*}(e_1)$ to $f_{i_1} \circ \ldots f_{i_0+1} \in X_1$, where $e_1 \in \Delta_1([1])$ is the element defined in (c). Suppose that $0 \leq i_0 < i_1 < i_2 \leq n$. Then the morphisms u_{i_0,i_1}, u_{i_1,i_2} and u_{i_0,i_2} agree on the intersections of their domains (because the f_i are composable), so they glue to a morphism $u'_{i_0,i_1,i_2} : \alpha_{i_0,i_1,i_2,*}(\partial\Delta_2) \to X$; by the property of X, this morphism extends uniquely to a morphism $u_{i_0,i_1,i_2} : \alpha_{i_0,i_1,i_2,*}(\Delta_2) \to X$. Now take $0 \leq i_0 < i_1 < i_2 < i_3 \leq n$. Then the morphisms u_{i_0,i_1,i_2}, u_{i_0,i_1,i_3}, u_{i_0,i_2,i_3} and u_{i_1,i_2,i_3} agree on the intersections of their domains (we just recover one of the u_{i_r,i_s} on such an intersection), so they glue to a morphism $u'_{i_0,i_1,i_2,i_3} : \alpha_{i_0,i_1,i_2,i_3,*}(\partial\Delta_3) \to X$; by the property of X, this morphism extends uniquely to a morphism $u_{i_0,i_1,i_2,i_3} : \alpha_{i_0,i_1,i_2,i_3,*}(\Delta_3) \to X$. We continue in this way until we get a morphism $u_{0,1,\ldots,n} : \Delta_n \to X$ extending the original u_{i_0,i_1}; the corresponding element of X_n has the required property.

Now we prove (B). By the Yoneda lemma, the elements $u, v \in X_n$ correspond to two morphisms $u_x, u_y : \Delta_n \to X$, and the condition of (B) says that u_x and u_y agree on $\alpha_{i_0,i_1,*}(\Delta_1)$ for all $i_0, i_1 \in [n]$ such that $i_0 < i_1$. But we saw in the proof of (A) that there is a unique way to extend a family of morphisms $\alpha_{i_0,i_1,*}(\Delta_1) \to X$ (agreeing on the intersections of the $\alpha_{i_0,i_1,*}(\Delta_1)$) to a morphism $\Delta_n \to X$. So $u_x = u_y$, that is, $x = y$.

Finally, we define $u : X \to N(\mathscr{C})$ by taking u_1 and u_0 to be the obvious bijections and by sending $x \in X_n$ to the sequence of maps $(f_{1,x}, \ldots, f_{n,x})$, for every $n \geq 2$. This induces a morphism of functors from $X_{\leq 2}$ to $N(\mathscr{C})_{\leq 2}$ by the definition of the composition and the description of the maps between the $N(\mathscr{C})_n$ in (a). Then the solution of (e) shows that u is a morphism of simplicial sets. Points (A) and (B) imply that u is an isomorphism.

$\square$