MAT 540 : Problem Set 10

Due Saturday, December 7

1 The Dold-Kan correspondence

You need to look at the results of problems 1 and 2 of problem set 3 to do this problem.

Remember the simplicial category Δ and the category of simplicial sets **sSet** from problem 9 of problem set 1 and problem 2 of problem set 2. Let $\mathscr{C} = \mathrm{kar}((\mathbb{Z}[\Delta])^{\oplus})$ (see problems 1 and 2 of problem set 3), so that $\mathscr{C}$ is an additive pseudo-abelian category.

The category $\mathrm{Func}(\Delta^{\mathrm{op}}, \mathbf{Ab})$ is called the category of *simplicial abelian groups* and denoted by **sAb**; it is an abelian category, where kernel, cokernels and images are calculated in the obvious way (that is, $\mathrm{Ker}(X \to Y) = (\mathrm{Ker}(X_n \to Y_n))_{n\in\mathbb{N}}$ etc).

By the Yoneda lemma, the functor $h_{\mathscr{C}} : \mathscr{C} \to \mathrm{Func}(\mathscr{C}^{\mathrm{op}}, \mathbf{Ab})$ is fully faithful; by problems 1 and 2 of problem set 3, we have an equivalence $\mathrm{Func}_{\mathrm{add}}(\mathscr{C}^{\mathrm{op}}, \mathbf{Ab}) \simeq \mathrm{Func}(\Delta^{\mathrm{op}}, \mathbf{Ab}) = \mathbf{sAb}$. So we get a fully faithful functor $\mathscr{C} \to \mathbf{sAb}$, and we identify $\mathscr{C}$ with the essential image of this functor.

If X is a simplical set, we denote by $\mathbb{Z}^{(X)}$ the "free simplicial abelian group on X" : it is the simplicial abelian group sending $[n]$ to the free abelian group $\mathbb{Z}^{(X_n)}$ and $\alpha : [n] \to [m]$ to the unique group morphism from $\mathbb{Z}^{(X_m)}$ to $\mathbb{Z}^{(X_n)}$ extending $\alpha^* : X_m \to X_n$. If $u : X \to Y$ is a morphism of simplicial sets, we simply write $u : \mathbb{Z}^{(X)} \to \mathbb{Z}^{(Y)}$ for the morphism of simplicial abelian groups induced by u. If $\alpha : [n] \to [m]$, we also use α to denote the morphism $\Delta_n \to \Delta_m$ that is the image of α by the Yoneda embedding $h_\Delta : \Delta \to \mathbf{sSet}$.

(a). (1 point) For every $n \in \mathbb{N}$, show that the simplicial abelian group $\mathbb{Z}^{(\Delta_n)}$ is in $\mathscr{C}$. (Hint : It's the image of the object $[n]$ of Δ. Follow the identifications !)

Let $n \geq 1$. Remember from problem 9 of problem set 1 that we have defined morphisms $\delta_0, \delta_1, \ldots, \delta_n : [n-1] \to [n]$ in Δ by the condition that δ_i is the unique increasing map $[n-1] \to [n]$ such that $i \notin \mathrm{Im}(\delta_i)$. According to our previous conventions, we get morphisms $\delta_i : \Delta_{n-1} \to \Delta_n$ in **sSet** and $\delta_i : \mathbb{Z}^{(\Delta_{n-1})} \to \mathbb{Z}^{(\Delta_n)}$ in **sAb**. Remember also that, for $k \in [n]$, the *horn* Λ^n_k is the union of the images of the δ_i, for $i \in [n] - \{k\}$.

(b). (1 point) Show that $\mathbb{Z}^{(\Lambda^n_k)} = \sum_{i\in[n]-\{k\}} \mathrm{Im}(\delta_i)$, where the sum is by definition the image of the canonical morphism $\bigoplus_{i\in[n]-\{k\}} \mathrm{Im}(\delta_i) \to \mathbb{Z}^{(\Delta_n)}$ and we have identified $\mathbb{Z}^{(\Lambda^n_k)}$ to its image in $\mathbb{Z}^{(\Delta_n)}$.

If $f : [n] \to X$ is a map from $[n]$ to a set X, we also use the notation $(f(0) \to f(1) \to \ldots \to f(n))$ to represent f. Let $n \in \mathbb{N}$, and let S_n be the set of sequences $(a_1, \ldots, a_n) \in [n]$ such that $a_i \in \{i-1, i\}$ for every $i \in \{1, \ldots, n\}$; if $\underline{a} = (a_1, \ldots, a_n)$, we write $f_{\underline{a}} = (0 \to a_1 \to \ldots \to a_n) \in \mathrm{Hom}_{\mathbf{Set}}([n], [n])$ and $\varepsilon(\underline{a}) = (-1)^{\mathrm{card}(\{i|a_i\neq i\})}$.

(c). (1 point) For every $\underline{a} \in S_n$, show that $f_{\underline{a}} \in \mathrm{Hom}_\Delta([n], [n])$.

(d). (2 points) Let $p_n = \sum_{\underline{a}\in S_n} \varepsilon(\underline{a}) f_{\underline{a}} \in \mathrm{End}_{\mathscr{C}}(\mathbb{Z}^{(\Delta_n)})$. Show that p_n is a projector.

(e). (3 points) Show that $\mathbb{Z}^{(\Lambda_0^n)} = \mathrm{Im}(\mathrm{id}_{\mathbb{Z}^{(\Delta_n)}} - p_n) = \mathrm{Ker}(p_n)$. In particular, $\mathbb{Z}^{(\Lambda_0^n)}$ is an object of $\mathscr{C}$.

(f). (1 point) Let $I_n = \mathrm{Im}(p_n)$. This is also an object of $\mathscr{C}$. Show that we have an isomorphism $\mathbb{Z}^{(\Delta_n)} \simeq \mathbb{Z}^{(\Lambda_0^n)} \oplus I_n$ in $\mathscr{C}$.

(g). (2 points) If X is an object of **sAb** and $f : X \to I_n$ is a surjective morphism (that is, such that f_r is surjective for every $r \geq 0$), show that there exists a morphism $g : I_n \to X$ such that $f \circ g = \mathrm{id}_{I_n}$.

For every $k \in [n]$, define a simplicial subset $\Delta_n^{\leq k}$ of Δ_n by taking $\Delta_n^{\leq k}([m])$ equal to the set of nondecreasing $\alpha : [m] \to [n]$ such that either $\mathrm{card}(\mathrm{Im}(\alpha)) \leq k$, or $\mathrm{card}(\mathrm{Im}(\alpha)) = k+1$ and $0 \in \mathrm{Im}(\alpha)$. In particular, question (h)(i) says that $\Delta_n^{\leq n-1} = \Lambda_0^n$. (On the geometric realizations, $|\Delta_n|$ is a simplex of dimension n with vertices numbered by $0, 1, \ldots, n$, and $|\Delta_n^{\leq k}|$ is the union of its faces of dimension $\leq k$ that contain the vertex 0.)

(h). (i) (1 point) For every $k \in [n]$ and every $m \in \mathbb{N}$, show that

$$\Lambda_k^n([m]) = \{\alpha : [m] \to [n] \mid \text{either } \mathrm{card}(\mathrm{Im}(\alpha)) \leq n-1, \text{ or } \mathrm{card}(\mathrm{Im}(\alpha)) = n \text{ and } k \in \mathrm{Im}(\alpha)\}.$$

(ii) (1 point) For every $m \in \mathbb{N}$, show that the set

$$\{\alpha : [m] \to [n] \mid \mathrm{Im}(\alpha) \supset [n] - \{0\}\}$$

is a basis of the $\mathbb{Z}$-module $I_n([m])$.

(iii) (1 point) For every $k \in \{1, \ldots, n\}$, show that

$$\mathbb{Z}^{(\Delta_n^{\leq k})} / \mathbb{Z}^{(\Delta_n^{\leq k-1})} \simeq I_k^{\binom{n}{k}}.$$

(iv) (1 point) For every $k \in \{1, \ldots, n\}$, show that

$$\mathbb{Z}^{(\Delta_n^{\leq k})} \simeq \mathbb{Z}^{(\Delta_n^{\leq k-1})} \oplus I_k^{\binom{n}{k}}.$$

(i). (1 point) Show that there is an isomorphism $\mathbb{Z}^{(\Delta_n)} \simeq \bigoplus_{k=0}^{n} I_k^{\binom{n}{k}}$ in $\mathscr{C}$.

(j). (2 points) For all $n, m \in \mathbb{N}$, show that $\mathrm{Hom}_{\mathscr{C}}(I_n, I_m)$ is a free $\mathbb{Z}$-module of finite type. We denote its rank by $a_{n,m}$.

(k). (2 points) Show that $a_{n,n} \geq 1$ and $a_{n,n+1} \geq 1$ for every $n \in \mathbb{N}$. (Hint for the second: $\delta_0 : [n] \to [n+1]$.)

(l). (2 points) Show that, for all $n, m \in \mathbb{N}$, we have

$$\binom{n+m+1}{m} = \sum_{k=0}^{n} \sum_{l=0}^{m} a_{k,l} \binom{n}{k} \binom{m}{l}.$$

(m). (2 points) Show that, for all $n, m \in \mathbb{N}$, we have

$$\binom{n+m+1}{m} = \sum_{k=0}^{m} \binom{n+1}{k} \binom{m}{k}.$$

(n). (2 points) Show that $a_{n,n} = a_{n,n+1} = 1$ for every $n \in \mathbb{N}$ and $a_{n,m} = 0$ if $m \notin \{n, n+1\}$.

(o). (2 points) Let $\mathscr{I}$ be the full subcategory of $\mathscr{C}$ whose objects are the I_n for $n \in \mathbb{N}$. If $\mathscr{A}$ is an additive category, we consider the category $\mathscr{C}^{\leq 0}(\mathscr{A})$ of complexes of objects of $\mathscr{A}$ that are concentrated in degree ≤ 0 (that is, complexes $X \in \mathrm{Ob}(\mathscr{C}(\mathscr{A}))$ such that $X^n = 0$ for $n \geq 1$).

Give an equivalence of categories from $\mathrm{Func}_{\mathrm{add}}(\mathscr{I}^{\mathrm{op}}, \mathscr{A})$ to $\mathscr{C}^{\leq 0}(\mathscr{A})$.

(p). (2 points) Deduce an equivalence of categories from $\mathrm{Func}(\Delta^{\mathrm{op}}, \mathscr{A})$ to $\mathscr{C}^{\leq 0}(\mathscr{A})$, if $\mathscr{A}$ is a pseudo-abelian additive category. This is called the *Dold-Kan equivalence*.

(q). (2 points) Suppose that $\mathscr{A}$ is an abelian category, and let $X_\bullet$ be an object of $\mathrm{Func}(\Delta^{\mathrm{op}}, \mathscr{A})$. For $n \in \mathbb{N}$ and $i \in \{0, 1, \dots, n\}$, we denote the morphism $X_\bullet(\delta_i^n)$ by $d_i^n : X_n \to X_{n-1}$. The *normalized chain complex* of $X_\bullet$ is the complex $N(X_\bullet)$ in $\mathscr{C}^{\leq 0}(\mathscr{A})$ given by: for every $n \geq 0$,

$$N(X_\bullet)^{-n} = \bigcap_{1 \leq i \leq n} \mathrm{Ker}(d_i^n)$$

and $d_{N(X_\bullet)}^{-n}$ is the restriction of d_0^n. This defines a functor $N : \mathrm{Func}(\Delta^{\mathrm{op}}, \mathscr{A}) \to \mathscr{C}^{\leq 0}(\mathscr{A})$. Show that this functor is isomorphic to the equivalence of categories of the previous question.

Solution.

(a). We denote the faithful functor $\Delta \to \mathscr{C}$ by ι. Let $n \in \mathbb{N}$. If $m \in \mathbb{N}$, we have

$$h_{\mathscr{C}}(\iota([n]))(\iota([m])) = \mathrm{Hom}_{\mathscr{C}}(\iota([m]), \iota([n])) = \mathbb{Z}^{(\mathrm{Hom}_\Delta([m],[n]))} = \mathbb{Z}^{(\Delta_n([m]))} = \mathbb{Z}^{(\Delta_n)}([m]).$$

So the image of $\iota([n])$ by the fully faithful functor $\mathscr{C} \xrightarrow{h_{\mathscr{C}}} \mathrm{Func}(\mathscr{C}^{\mathrm{op}}, \mathbf{Ab}) \xrightarrow{\sim} \mathrm{Func}(\Delta^{\mathrm{op}}, \mathbf{Ab})$ is isomorphic to $\mathbb{Z}^{(\Delta_n)}$.

(b). If $X_\bullet$ is a simplicial set and if $m \in \mathbb{N}$, we denote by $(e_u)_{u \in X_m}$ the canonical basis of $\mathbb{Z}^{(X_\bullet)}([m]) = \mathbb{Z}^{(X_m)}$.

We need to show that, for every $m \in \mathbb{N}$, the subgroup $\mathbb{Z}^{(\Lambda_k^n)}([m])$ of $\mathbb{Z}^{(\Delta_n)}([m])$ is equal to $\sum_{i \in [n] - \{k\}} \mathrm{Im}(\delta_i([m]))$. Let $m \in \mathbb{N}$. For every $i \in [n]$, the morphism $\delta_i([m]) : \mathbb{Z}^{(\Delta_{n-1})}([m]) \to \mathbb{Z}^{(\Delta_n)}([m])$ is given on the canonical basis $(e_u)_{u \in \mathrm{Hom}_\Delta([m],[n])}$ of $\mathbb{Z}^{(\Delta_{n-1})}([m])$ by $\delta_i([m])(e_u) = e_{\delta_i \circ u}$. So $\sum_{i \in [n]-\{k\}} \mathrm{Im}(\delta_i([m]))$ is the $\mathbb{Z}$-submodule of $\mathbb{Z}^{(\Delta_n)}([m])$ generated by all the e_u for $u \in \mathrm{Hom}_\Delta([m],[n])$ factoring through some δ_i, $i \neq k$. This is the same as $\mathbb{Z}^{(\Lambda_k^n)}([m])$ by definition of the horn.

(c). We have to show that $f_{\underline{a}}$ is nondecreasing. Let $i \in \{0, \dots, n-1\}$. If $i = 0$, then $f_{\underline{a}}(i) = 0 \leq i$. Then $f_{\underline{a}}(i) = a_i \in \{i-1, i\}$, so $f_{\underline{a}}(i) \leq i$. On the other hand, we have $f_{\underline{a}}(i+1) = a_{i+1} \in \{i, i+1\}$, so $f_{\underline{a}}(i+1) \geq i \geq f_{\underline{a}}(i)$.

(d). Let $\underline{a} = (a_1, \dots, a_n) \in S_n$, and suppose that $\underline{a} \neq (1, \dots, n)$. Then $\mathrm{Im}(f_{\underline{a}})$ is strictly contained in $[n]$, and $0 \in \mathrm{Im}(f_{\underline{a}})$. This means that there exists $i_0 \in \{1, \dots, n\}$ such that $i_0 \notin \mathrm{Im}(f_{\underline{a}})$. Let $S_n' = \{(b_1, \dots, b_n) \in S_n \mid b_{i_0} = i_0\}$ and $S_n'' = \{(b_1, \dots, b_n) \in S_n \mid b_{i_0} = i_0 - 1\}$. We define a map $\iota : S_n' \to S_n''$ by sending $(b_1, \dots, b_n)$ to $(b_1, \dots, b_{i_0-1}, i_0 - 1, b_{i_0+1}, \dots, b_n)$. It is easy to see that ι is a bijection and that $\varepsilon(\iota(\underline{b})) = -\varepsilon(\underline{b})$ and that $f_{\underline{b}} \circ f_{\underline{a}} = f_{\varepsilon(\underline{b})} \circ f_{\underline{a}}$ for every $\underline{b} \in S_n'$. So

$$\begin{aligned} p_n \circ f_{\underline{a}} &= \sum_{\underline{b} \in S_n'} \varepsilon(\underline{b}) f_{\underline{b}} \circ f_{\underline{a}} + \sum_{\underline{b} \in S_n''} \varepsilon(\underline{b}) f_{\underline{b}} \circ f_{\underline{a}} \\ &= \sum_{\underline{b} \in S_n'} \varepsilon(\underline{b}) f_{\underline{b}} \circ f_{\underline{a}} - \sum_{\underline{b} \in S_n'} \varepsilon(\underline{b}) f_{\underline{b}} \circ f_{\underline{a}} \\ &= 0. \end{aligned}$$

On the other hand, if $\underline{a} = (1, \dots, n)$, then $\varepsilon(\underline{a}) = 1$ and $f_{\underline{a}} = \mathrm{id}_{[n]}$, so $p_n \circ f_{\underline{a}} = p_n$.

This shows that $p_n \circ p_n = p_n$.

(e). As p_n is a projector, we know that $\mathrm{Ker}(p_n)$ exists in $\mathscr{C}$ and that $\mathrm{Ker}(p_n) = \mathrm{Im}(\mathrm{id}_{\mathbb{Z}^{(\Delta_n)}} - p_n)$ by problem 2 of problem set 3.

For every $\underline{a} \in S_n$ such that $\underline{a} \neq (1, \dots, n)$, we have seen that there exists $i \in \{1, \dots, n\} - \mathrm{Im}(f_{\underline{a}})$, and then $f_{\underline{a}}$ factors through δ_i, so the image of $f_{\underline{a}}$ in the abelian category **sAb** is contained in $\mathbb{Z}^{(\Lambda_0^n)}$. As $\mathrm{id}_{\mathbb{Z}^{(\Delta_n)}} - p_n = -\sum_{\underline{a} \in S_n - \{(1,\dots,n)\}} \varepsilon(\underline{a}) f_{\underline{a}}$, this shows that $\mathrm{Im}(\mathrm{id}_{\mathbb{Z}^{(\Delta_n)}} - p_n) \subset \mathbb{Z}^{(\Lambda_0^n)}$.

If $i \in \{1, \dots, n\}$, then the same proof as in the solution of question (d) shows that $p_n \circ \delta_i = 0$, hence $(\mathrm{id}_{\mathbb{Z}^{(\Delta_n)}} - p_n) \circ \delta_i = \delta_i$. As $\mathbb{Z}^{(\Lambda_0^n)} = \sum_{i=1}^n \mathrm{Im}(\delta_i)$ by question (b), this implies that $\mathbb{Z}^{(\Lambda_0^n)} \subset \mathrm{Im}(\mathrm{id}_{\mathbb{Z}^{(\Delta_n)}} - p_n)$.

(f). This is question (b) of problem 2 of problem set 3.

(g). Let $i = \mathrm{id}_{[n]} \in \Delta_n([n])$, let e_i be the corresponding element of $\mathbb{Z}^{(\Delta_n)}([n])$, and let $\overline{e}_i$ be its image in $I_n([n])$. As f is surjective, we can find $x \in X_n$ such that $f_n(x) = \overline{e}_i$. Let $g' : \mathbb{Z}^{(\Delta_n)} \to X$ be the morphism corresponding to x by the bijection

$$\mathrm{Hom}_{\mathbf{sAb}}(\mathbb{Z}^{(\Delta_n)}, X) \simeq \mathrm{Hom}_{\mathbf{sSet}}(\Delta_n, X) \simeq X_n,$$

and let $g = q \circ g'$, where $q : \mathbb{Z}^{(\Delta_n)} \to I_n$ is the canonical projection. We want to show that $g \circ f = \mathrm{id}_{I_n}$. By the construction of g, we have $g \circ f(\overline{e}_i) = \overline{e}_i$. Let $m \in \mathbb{N}$. Remember that we denote by $(e_u)_{u \in \mathrm{Hom}_\Delta([m],[n])}$ the canonical basis of $\mathbb{Z}^{(\Delta_n)}([m])$. The family $(q(e_u))_{u \in \mathrm{Hom}_\Delta([m],[n])}$ spans $I_n([m])$, so it suffices to show that $g_m \circ f_m(q_m(e_u)) = q(e_u)$ for every u. Let $u \in \mathrm{Hom}_\Delta([m],[n])$. Then $i \circ u = u$, so $e_u = u^*(e_i)$, and

$$f_m \circ g_m(q_m(e_u)) = u^*(f_n \circ g_n(q_n(e_i))) = u^*(q(e_i)) = q(e_u).$$

(h). (i) The set $\Lambda_k^n([m])$ is the set of nondecreasing maps $\alpha : [m] \to [n]$ that factor through some δ_i, for $i \in [n] - \{k\}$. If $\alpha : [m] \to [n]$ is a nondecreasing map, then, by definition of δ_i, the map α factors through δ_i if and only if $i \notin \mathrm{Im}(\alpha)$. This shows that $\Lambda_k^n([m])$ does not contain any surjective α, contains all the α such that $|\mathrm{Im}(\alpha)| \leq n-1$, and contains an α such that $|\mathrm{Im}(\alpha)| = n$ if and only if $[n] - \mathrm{Im}(\alpha) \neq \{k\}$, i.e. $k \in \mathrm{Im}(\alpha)$. This is what we wanted to prove.

(ii) By (i), we have

$$\{\alpha \in \Delta_n([m]) \mid \mathrm{Im}(\alpha) \supset [n] - \{0\}\} = \Delta_n([m]) - \Lambda_0^n([m]),$$

so the family $(q_m(e_\alpha))_{\alpha \in \Delta_n([m]),\ \mathrm{Im}(\alpha) \supset [n]-\{0\}}$ is a basis of $I_n([m])$ (where q is as before the canonical projection $\mathbb{Z}^{(\Delta_n)} \to I_n$).

(iii) We fix $k \in \{1, \dots, n\}$. Let Ω be the set of $A \subset [n]$ such that $0 \in A$ and $|A| = k+1$. For every $A \in \Omega$, let $\beta_A : [k] \to [n]$ be the composition of the unique order-preserving bijection $[k] \overset{\sim}{\to} A$ and of the inclusion $A \subset [n]$; note that $\beta_A(0) = 0$. Consider the morphism $f_A : \mathbb{Z}^{(\Delta_k)} \to \mathbb{Z}^{(\Delta_n)}$ such that, for every $m \in \mathbb{N}$ and every $\alpha \in \mathrm{Hom}_\Delta([m],[k])$, we have $f_A(e_\alpha) = e_{\beta_A \circ \alpha}$. Note that $\mathrm{Im}(f_A) \subset \mathbb{Z}^{(\Delta_n^{\leq k})}$, so we can see f_A as a morphism from $\mathbb{Z}^{(\Delta_k)}$ to $\mathbb{Z}^{(\Delta_n^{\leq k})}$. Let $m \in \mathbb{N}$ and $\alpha \in \Lambda_0^k([m])$; if $|\mathrm{Im}(\alpha)| \leq k-1$, then $|\mathrm{Im}(\beta_A \circ \alpha)| \leq k-1$ and so $\beta_A \circ \alpha \in \Delta_n^{\leq k-1}([m])$; if $|\mathrm{Im}(\alpha)| = k$ and $0 \in \mathrm{Im}(\alpha)$, then $|\mathrm{Im}(\beta_A \circ \alpha)| = k$ and $0 \in \mathrm{Im}(\beta_A \circ \alpha)$, and so $\beta_A \circ \alpha \in \Delta_n^{\leq k-1}([m])$. This shows that $f_A(\mathbb{Z}^{(\Lambda_0^n)}) \subset \mathbb{Z}^{(\Delta_n^{\leq k-1})}$, hence that f_A induces a morphism $g_A : I_k \to \mathbb{Z}^{(\Delta_n^{\leq k})}/\mathbb{Z}^{(\Delta_n^{\leq k-1})}$.

Let $g = \sum_{A \in \Omega} g_A : I_k^\Omega \to \mathbb{Z}^{(\Delta_n^{\leq k})}/\mathbb{Z}^{(\Delta_n^{\leq k-1})}$. We claim that g is an isomorphism; this will finish the proof, because $|\Omega| = \binom{n}{k}$. Let $m \in \mathbb{N}$. For every $A \in \Omega$ and every $\alpha \in \operatorname{Hom}_\Delta([m],[k])$ such that either $|\operatorname{Im}(\alpha)| = k+1$, or $|\operatorname{Im}(\alpha)| = k$ and $0 \notin \operatorname{Im}(\alpha)$, we denote by $e_{A,\alpha} \in I_k^\Omega$ the basis element e_α of the copy of I_k corresponding to $A \in \Omega$. By (ii), this gives a basis of $(I_k^\Omega)([m])$. On the other, a basis of $(\mathbb{Z}^{(\Delta_n^{\leq k})}/\mathbb{Z}^{(\Delta_n^{\leq k-1})})([m])$ is given by the images of the basis elements $e_\beta \in \mathbb{Z}^{(\Delta_n^{\leq k})}([m])$ for $\beta \in \operatorname{Hom}_\Delta([m],[n])$ such that either $|\operatorname{Im}(\beta)| = k+1$ and $0 \in \operatorname{Im}(\beta)$, or $|\operatorname{Im}(\beta)| = k$ and $0 \notin \operatorname{Im}(\beta)$. To show that $g_m : (I_k^\Omega)([m]) \to (\mathbb{Z}^{(\Delta_n^{\leq k})}/\mathbb{Z}^{(\Delta_n^{\leq k-1})})([m])$ is an isomorphism, it suffices to notice that each $\beta \in \operatorname{Hom}_\Delta([m],[n])$ as in the previous sentence is equal to $\beta_A \circ \alpha$ for a unique $A \in \Omega$ and a unique $\alpha \in \operatorname{Hom}_\Delta([m],[k])$ (indeed, we must have $A = \operatorname{Im}(\beta)$ if $|\operatorname{Im}(\beta)| = k+1$ and $0 \in \operatorname{Im}(\beta)$, and $A = \{0\} \cup \operatorname{Im}(\beta)$ if $|\operatorname{Im}(\beta)| = k$ and $0 \notin \operatorname{Im}(\beta)$, and then A determines α because β_A is injective), and that we then have either $|\operatorname{Im}(\alpha)| = k+1$, or $|\operatorname{Im}(\alpha)| = k$ and $0 \notin \operatorname{Im}(\alpha)$.

(iv) This follows easily from (iii) and from question (g).

(i). By question (h)(iv) (and an easy induction), we have an isomorphism $\mathbb{Z}^{(\Delta_n)} \simeq \bigoplus_{k=0}^n I_k^{\binom{n}{k}}$ in **sAb**. As both sides are objects of $\mathscr{C}$ by question (f), and as $\mathscr{C}$ is a full subcategory of **sAb**, this isomorphism is an isomorphism in $\mathscr{C}$.

(j). As I_n (resp. I_m) is a direct factor of $\mathbb{Z}^{(\Delta_n)}$ (resp. $\mathbb{Z}^{(\Delta_m)}$) by question (f), the abelian group $\operatorname{Hom}_{\mathscr{C}}(I_n, I_m) = \operatorname{Hom}_{\mathbf{sAb}}(I_n, I_m)$ admits an injective morphism into

$$\operatorname{Hom}_{\mathbf{sAb}}(\mathbb{Z}^{(\Delta_n)}, \mathbb{Z}^{(\Delta_m)}) = \operatorname{Hom}_{\mathbf{sSet}}(\Delta_n, \mathbb{Z}^{(\Delta_m)}) = \mathbb{Z}^{(\Delta_m)}([n]) = \mathbb{Z}^{(\operatorname{Hom}_\Delta([n],[m]))}.$$

As the latter group is free and finitely generated, so is $\operatorname{Hom}_{\mathscr{C}}(I_n, I_m)$.

(k). We have $I_n \neq 0$ because $\Lambda_0^n \subsetneq \Delta_n$, so $0 \neq \mathrm{id}_{I_n} \in \operatorname{Hom}_{\mathscr{C}}(I_n, I_n)$, so $a_{n,n} \geq 1$.

Consider the unique nondecreasing injective map $\delta_0 : [n] \to [n+1]$ such that $0 \notin \operatorname{Im}(\delta_0)$. (In other words, we have $\delta_0(i) = i+1$ for every $i \in [n]$.) This induces a morphism $f : \mathbb{Z}^{(\Delta_n)} \to \mathbb{Z}^{(\Delta_{n+1})}$. If $m \in \mathbb{N}$ and $\alpha \in \Lambda_0^n([m])$, then $|\operatorname{Im}(\alpha)| \leq n$, so $|\operatorname{Im}(\delta_0 \circ \alpha)| \leq n$ and $\delta_0 \circ \alpha \in \Lambda_0^{n+1}([m])$. This shows that $f(\mathbb{Z}^{(\Lambda_0^n)}) \subset \mathbb{Z}^{(\Lambda_0^{n+1})}$, hence that f induces a morphism $g : I_n \to I_{n+1}$. Also, if $\alpha = \mathrm{id}_{[n]} \in \operatorname{Hom}_\Delta([n],[n])$, then $\delta_0 \circ \alpha \notin \Lambda_0^{n+1}([n])$, so the image by g of the class of e_α in $I_n([n])$ is not 0. This shows that $g \neq 0$, hence that $\operatorname{Hom}_{\mathscr{C}}(I_n, I_{n+1}) \neq 0$ and so $a_{n,n+1} \geq 1$.

(l). Let $n, m \in \mathbb{N}$. We have seen in the solution of question (j) that $\operatorname{Hom}_{\mathbf{sAb}}(\mathbb{Z}^{(\Delta_n)}, \mathbb{Z}^{(\Delta_m)})$ is a free $\mathbb{Z}$-module of rank $|\operatorname{Hom}_\Delta([n],[m])| = \binom{n+m+1}{m} = \binom{n+m+1}{n+1}$. On the other hand, by question (i), we have

$$\operatorname{Hom}_{\mathbf{sAb}}(\mathbb{Z}^{(\Delta_n)}, \mathbb{Z}^{(\Delta_m)}) \simeq \bigoplus_{k=0}^n \bigoplus_{l=0}^m (\operatorname{Hom}_{\mathbf{sAb}}(I_k, I_l))^{\binom{n}{k}\binom{m}{l}},$$

and the right hand side is a free $\mathbb{Z}$-module of rank $\sum_{k=0}^n \sum_{l=0}^m a_{k,l} \binom{n}{k}\binom{m}{l}$.

(m). Remember that Vandermonde's identity says that, for all $a, b, c \in \mathbb{N}$, we have

$$\binom{a+b}{c} = \sum_{j=0}^c \binom{b}{j}\binom{a}{c-j}.$$

Applying this to $a = n+1$ and $b = c = m$ and using the fact that $\binom{m}{k} = \binom{m}{m-k}$ for $0 \leq k \leq m$, we get

$$\binom{n+m+1}{m} = \sum_{k=0}^m \binom{m}{k}\binom{n+1}{k}.$$

To prove Vandermonde's identity, we consider an indeterminate t. By the binomial theorem, we have

$$(1+t)^a = \sum_{i=0}^{a} \binom{a}{i} t^i,$$

$$(1+t)^b = \sum_{j=0}^{b} \binom{b}{j} t^j$$

and

$$(1+t)^{a+b} = \sum_{c=0}^{a+b} \binom{a+b}{c} t^c.$$

As $(1+t)^{a+b} = (1+t)^a(1+t)^b$, if $c \in \mathbb{N}$, we get two formulas for the coefficient of t^c in this polynomial. The first formula is $\binom{a+b}{c}$, and the second formula is

$$\sum_{i,j\geq 0,\ i+j=c} \binom{a}{i}\binom{b}{j} = \sum_{j=0}^{c} \binom{a}{c-j}\binom{b}{j}.$$

(n). By questions (l) and (m), we have

$$\sum_{k=0}^{n}\sum_{l=0}^{m} a_{k,l}\binom{n}{k}\binom{m}{l} = \sum_{k=0}^{m}\binom{m}{k}\binom{n+1}{k} = \sum_{k=0}^{m}\binom{m}{k}\binom{n}{k} + \sum_{k=1}^{m}\binom{m}{k}\binom{n}{k-1},$$

where the second equality comes from Pascal's rule $\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$. By question (k) (and the obvious that all the $a_{k,l}$ are nonnegative), we have

$$\sum_{k=0}^{n}\sum_{l=0}^{m} a_{k,l}\binom{n}{k}\binom{m}{l} \geq \sum_{k=0}^{m}\binom{n}{k}\binom{m}{k} + \sum_{k=1}^{m}\binom{n}{k-1}\binom{m}{k}.$$

This implies that, for $k \in [n]$ and $l \in [m]$, we have $a_{k,l} = 0$ if $l \notin \{k, k+1\}$ and $a_{k,l} = 1$ if $l \in \{k, k+1\}$. As n and m were arbitrary, we get the conclusion.

(o). Let $F \in \mathrm{Func}(\mathscr{I}^{\mathrm{op}}, \mathscr{A})$. We define a complex $X \in \mathscr{C}^{\leq 0}(\mathscr{A})$ in the following way: For every $n \in \mathbb{N}$, we take $X^{-n} = F(I_n)$ and $d_X^{-n-1} : X^{-n-1} = F(I_{n+1}) \to X^{-n} = F(I_n)$ to the image by F of the element g_n of $\mathrm{Hom}_{\mathscr{C}}(I_n, I_{n+1})$ constructed in the solution of question (k). This defines a functor $\Phi : \mathrm{Func}_{\mathrm{add}}(\mathscr{I}^{\mathrm{op}}, \mathscr{A}) \to \mathscr{C}^{\leq 0}(\mathscr{A})$.

Conversely, let X be an object of $\mathscr{C}^{\leq 0}(\mathscr{A})$. We define a functor $F : \mathscr{I}^{\mathrm{op}} \to \mathscr{A}$ in the following way: For every $n \in \mathbb{N}$, we take $F(I_n) = X^{-n}$. Let $n, m \in \mathbb{N}$ and $f \in \mathrm{Hom}_{\mathscr{C}}(I_n, I_m)$. If $m \notin \{n, n+1\}$, then $f = 0$, so we must $F(f) = 0$. If $m = n$, then, by question (n), the morphism is of the form $a \cdot \mathrm{id}_{I_n}$, where $a \in \mathbb{Z}$, and we must set $F(f) = a\mathrm{id}_{X^{-n}}$. If $m = n+1$, then, by question (n), the morphism f is of the $a \cdot g_n$ with $a \in \mathbb{Z}$, and we set $F(f) = a \cdot d_X^{-n-1} : X^{-n-1} = F(I_{n+1}) \to X^{-n} = F(I_n)$. This defines a functor $\Psi : \mathscr{C}^{\leq 0}(\mathscr{A}) \to \mathrm{Func}_{\mathrm{add}}(\mathscr{I}^{\mathrm{op}}, \mathscr{A})$.

The fact that $\Phi \circ \Psi = \mathrm{id}_{\mathscr{C}^{\leq 0}(\mathscr{A})}$ follows immediately from the definitions of the functors Φ and Ψ, and the fact that $\Psi \circ \Phi = \mathrm{id}_{\mathrm{Func}_{\mathrm{add}}(\mathscr{I}^{\mathrm{op}}, \mathscr{A})}$ follows easily from the definition of these functors and from question (n).

(p). By problems 1 and 2 of problem set 3, we have an equivalence $\mathrm{Func}_{\mathrm{add}}(\mathscr{C}^{\mathrm{op}}, \mathscr{A}) \simeq \mathrm{Func}(\Delta^{\mathrm{op}}, \mathscr{A})$, so we can define a functor $\mathrm{Func}(\Delta^{\mathrm{op}}, \mathscr{A}) \to \mathscr{C}^{\leq 0}(\mathscr{A})$ by composing a quasi-inverse of this equivalence, the restriction functor $\mathrm{Func}_{\mathrm{add}}(\mathscr{C}^{\mathrm{op}}, \mathscr{A}) \to \mathrm{Func}_{\mathrm{add}}(\mathscr{I}^{\mathrm{op}}, \mathscr{A})$ and the equivalence $\mathrm{Func}_{\mathrm{add}}(\mathscr{I}^{\mathrm{op}}, \mathscr{A}) \xrightarrow{\sim} \mathscr{C}^{\leq 0}(\mathscr{A})$.

Showing that this is an equivalence of categories amounts to showing that the restriction functor $\mathrm{Func}_{\mathrm{add}}(\mathscr{C}^{\mathrm{op}}, \mathscr{A}) \to \mathrm{Func}_{\mathrm{add}}(\mathscr{I}^{\mathrm{op}}, \mathscr{A})$ is an equivalence of categories.

By the construction of the pseudo-abelian completion in problem 2 of problem set 3, every object of $\mathscr{C}$ is a direct summand of an object of $\mathbb{Z}[\Delta]^{\oplus}$, hence, by construction of the universal additive category in problem 1 of problem set 3, a direct summand of an object of the form $\bigoplus_{i\in I} \mathbb{Z}^{(\Delta_{n_i})}$, for $(n_i)_{i\in I}$ a finite family of nonnegative integers. By question (i), this implies that every object of $\mathscr{C}$ is a direct summand of an object of the form $\bigoplus_{i\in I} I_{n_i}$, for $(n_i)_{i\in I}$ a finite family of nonnegative integers.

Let $\mathscr{I}'$ be the full subcategory of $\mathscr{C}$ whose objects are finite direct sums of objects of $\mathscr{I}$; in other words, the category $\mathscr{I}'$ is the category $\mathscr{I}^{\oplus}$ defined in problem 2 of problem set 3. Then $\mathscr{I}'$ is an additive category and the preceding paragraph says that $\mathscr{C}$ is the pseudo-abelian completion of $\mathscr{I}'$. By problem 2 of problem set 3 (applied to the opposite categories), the restriction functor $\mathrm{Func}_{\mathrm{add}}(\mathscr{C}^{\mathrm{op}}, \mathscr{A}) \to \mathrm{Func}_{\mathrm{add}}(\mathscr{I}'^{\mathrm{op}}, \mathscr{A})$ is an an equivalence of categories. So it remains to show that the restriction functor $\mathrm{Func}_{\mathrm{add}}(\mathscr{I}'^{\mathrm{op}}, \mathscr{A}) \to \mathrm{Func}_{\mathrm{add}}(\mathscr{I}^{\mathrm{op}}, \mathscr{A})$ is an equivalence of categories. But this is proved in problem 1 of problem set 3.

(q). Let $DK : \mathrm{Func}(\Delta^{\mathrm{op}}, \mathscr{A}) \to \mathscr{C}^{\leq 0}(\mathscr{A})$ be the equivalence of categories of question (p).

Let $X_\bullet \in \mathrm{Func}(\Delta^{\mathrm{op}}, \mathscr{A})$. We still denote by $X_\bullet$ the corresponding functor $\mathscr{C}^{\mathrm{op}} \to \mathscr{A}$. Let $n \in \mathbb{N}$, and let $\delta = \sum_{i=1}^n \delta_i : \bigoplus_{i=1}^n \mathbb{Z}^{(\Delta_{n-1})} \to \mathbb{Z}^{(\Delta_n)}$, where we use the notation of question (b); by that question, we have $\mathbb{Z}^{(\Lambda_0^n)} = \mathrm{Im}(\delta)$, and by question (f), the canonical projection $\mathbb{Z}^{(\Delta_n)} \to I_n$ identifies I_n to $\mathrm{Coker}\,\delta$ and both $\mathrm{Im}\,\delta$ and $\mathrm{Coker}\,\delta$ are direct summands of $\mathbb{Z}^{(\Delta_n)}$. It is easy to deduce from this that, if $F : \mathscr{C}^{\mathrm{op}} \to \mathscr{C}'$ is any additive functor, then the morphism $F(I_n) \to F(\mathbb{Z}^{(\Delta_n)})$ is a kernel of $F(\delta)$. Applying this to $F = X_\bullet : \mathscr{C}^{\mathrm{op}} \to \mathscr{A}$, we see that $F(I_n) = DK(X_\bullet)^{-n}$ is canonically isomorphic to $\mathrm{Ker}(\bigoplus_{i=1}^n : d_i^n : X_n \to X_{n-1}^n) = \bigcap_{i=1}^n \mathrm{Ker}(d_i^n) = N(X^\bullet)^{-n}$. Also, as the nonzero morphism from I_{n-1} to I_n constructed in the solution of question (k) is the restriction of $\delta_0 : \mathbb{Z}^{(\Delta_{n-1})} \to \mathbb{Z}^{(\Delta_n)}$ (followed by the canonical projection $\mathbb{Z}^{(\Delta_n)} \to I_n$), its image by $X_\bullet$ is the restriction of d_0^n. So we get an isomorphism of complexes $DK(X_\bullet) \simeq N(X_\bullet)$, and this isomorphism is clearly functorial in $X_\bullet$.

□

2 The model structure on complexes

Let R be a ring, and let $\mathscr{A} = {}_R\mathbf{Mod}$. [1]

We denote by W the set of quasi-isomorphisms of $\mathscr{C}(\mathscr{A})$, by Fib the set of morphisms $f : X \to Y$ in $\mathscr{C}(\mathscr{A})$ such that $f^n : X^n \to Y^n$ is surjective for every $n \in \mathbb{Z}$ and by Cof the set of morphisms of $\mathscr{C}(\mathscr{A})$ that have the left lifting property relatively to every morphism of $W \cap \mathrm{Fib}$. We say that $X \in \mathrm{Ob}(\mathscr{C}(\mathscr{A}))$ is fibrant (resp. cofibrant) if the unique morphism $X \to 0$ (resp. $0 \to X$) is in Fib (resp. in Cof). The goal of this problem is to show that $(W, \mathrm{Fib}, \mathrm{Cof})$ is a model structure on $\mathscr{C}(\mathscr{A})$.

For every $M \in \mathrm{Ob}(\mathscr{A})$, let $K(M, n) = M[-n] \in \mathrm{Ob}(\mathscr{C}(\mathscr{A}))$, and let $D^n(M)$ be the complex X such that $X^n = X^{n+1} = M$, $d_X^n = \mathrm{id}_M$ and $X^i = 0$ if $i \notin \{n, n+1\}$. We also write $S^n = K(R, n)$ and $D^n = D^n(R)$. For every $M \in \mathrm{Ob}(\mathscr{A})$, the identity of M induces a morphism of complexes $K(M, n) \to D^{n-1}(M)$ (which is clearly functorial in M).

[1] We only need $\mathscr{A}$ to have all small limits and colimits and a nice enough projective generator, but we take $\mathscr{A} = {}_R\mathbf{Mod}$ to simplify the notation.

(a). (2 points) Show that the functor $D^n : {}_R\mathbf{Mod} \to \mathscr{C}(\mathscr{A})$ is left adjoint to the functor $\mathscr{C}(\mathscr{A}) \to \mathscr{A}$, $X \longmapsto X^n$, and that the functor $K(\cdot, n) : \mathscr{A} \to \mathscr{C}(\mathscr{A})$ is left adjoint to the functor Z^n.

(b). (1 point) Show that a morphism of $\mathscr{C}(\mathscr{A})$ is in Fib is and only if it has the right lifting property relatively to $0 \to D^n$ for every $n \in \mathbb{Z}$.

(c). (1 point) Show that D^n is cofibrant for every $n \in \mathbb{Z}$.

(d). (2 points) Show that S^n is cofibrant for every $n \in \mathbb{Z}$.

(e). Let $p : X \to Y$ be a morphism of $\mathscr{C}(\mathscr{A})$.

(i) (2 points) If p is in $W \cap \mathrm{Fib}$, show that it has the right lifting property relatively to the canonical morphism $S^n = K(R, n) \to D^{n-1}$ for every $n \in \mathbb{Z}$.

(ii) (3 points) If p has the right lifting property relatively to the canonical morphism $S^n \to D^{n-1}$ for every $n \in \mathbb{Z}$, show that it is in $W \cap \mathrm{Fib}$.

(f). (1 point) Show that the canonical morphism $S^n \to D^{n-1}$ is in Cof.

(g). Let $f : X \to Y$ be a morphism of $\mathscr{C}(\mathscr{A})$. Let $E = X \oplus \bigoplus_{n\in\mathbb{Z},\ y\in Y^n} D^n$, let $i : X \to E$ be the obvious inclusion and let $p : E \to Y$ be the morphism that is equal to f on the summand X and that, for every $n \in \mathbb{Z}$ and $y \in Y^n$, is equal on the corresponding summand D^n to the morphism $D^n \to Y$ corresponding to $y \in Y^n = \mathrm{Hom}_R(R, Y^n)$ by the adjunction of question (a). We clearly have $p \circ i = f$.

(i) (1 point) Show that i is in W.

(ii) (1 point) Show that i has the left lifting property relatively to any morphism of Fib.

(iii) (1 point) Show that p is in Fib.

(h). Let $f : X \to Y$ be a morphism of $\mathscr{C}(\mathscr{A})$. Let $X_0 = X$ and $f_0 = f$. For every $i \in \mathbb{N}$, we construct morphisms of complexes $j_i : X_i \to X_{i+1}$ and $f_{i+1} : X_{i+1} \to Y$ such that j_i is a monomorphism and in Cof and $f_{i+1} \circ j_i = f_i$ in the following way: Suppose that we already have $f_i : X_i \to Y$. Consider the set $\mathscr{D}_i$ of commutative squares

$$(D) \qquad \begin{array}{ccc} S^{n_D} & \xrightarrow{f_D} & X_i \\ \downarrow & & \downarrow{\scriptstyle f_i} \\ D^{n_D-1} & \xrightarrow[g_D]{} & Y \end{array}$$

(for some $n_D \in \mathbb{Z}$). Let $j_i : X_i \to X_{i+1}$ be defined by the cocartesian square

$$\begin{array}{ccc} \bigoplus_{D\in\mathscr{D}_i} S^{n_D} & \xrightarrow{\sum f_D} & X_i \\ \downarrow & & \downarrow{\scriptstyle j_i} \\ \bigoplus_{D\in\mathscr{D}_i} D^{n_D-1} & \longrightarrow & X_{i+1} \end{array}$$

The morphisms $f_i : X_i \to Y$ and $\sum g_D : \bigoplus_{D\in\mathscr{D}_i} D^{n_D-1} \to Y$ induce a morphism $f_{i+1} : X_{i+1} \to Y$, and we clearly have $f_{i+1} \circ j_i = f_i$.

Finally, let $F = \varinjlim_{i\in\mathbb{N}} X_i$ (where the transition morphisms are the j_i), let $j : X \to F$ be the morphism induced by j_0 and let $q : F \to Y$ be the morphism induced by the f_i.

(i) (1 points) Show that $q \circ j = f$.

(ii) (1 point) Show that j is a monomorphism.

(iii) (2 points) Show that j is in Cof.

(iv) (2 points) Show that q is in $W \cap \mathrm{Fib}$.

(i). (1 point) Show that every element of Cof is a monomorphism.

(j). (2 points) Show that every element of $W \cap \mathrm{Cof}$ has the left lifting property relatively to elements of Fib. (Hint: Use question (g).)

(k). (3 points) Show that $(W, \mathrm{Fib}, \mathrm{Cof})$ is a model structure on $\mathscr{C}(\mathscr{A})$.

(l). (2 points) Show that the intersections of $(W, \mathrm{Fib}, \mathrm{Cof})$ with $\mathscr{C}^-(\mathscr{A})$ also give a model structure on this category.

(m). (2 points) Let $f : A \to B$ be a morphism of $\mathscr{A}$. Show that f has the left lifting property relatively to epimorphisms of $\mathscr{A}$ if and only if it is injective with projective cokernel.

(n). (3 points) Let $i : X \to Y$ be a morphism of $\mathscr{C}^-(\mathscr{A})$. Show that i is in Cof if and only if, for every $n \in \mathbb{Z}$, the morphism i^n is injective with projective cokernel.

Solution.

(a). Let X be an object of $\mathscr{C}(\mathscr{A})$ and M be a left R-module. Giving a morphism of complexes from $D^n(M)$ to X amounts to giving R-linear maps $f : M \to X^n$ and $g : M \to X^{n+1}$ such that $g = d_X^n \circ f$; so there is no extra condition on f, and g is determined by f. In other words, we have constructed a bijection

$$\mathrm{Hom}_{\mathscr{C}(\mathscr{A})}(D^n(M), X) \xrightarrow{\sim} \mathrm{Hom}_R(M, X^n),$$

which is clearly functorial in M and X.

On the other hand, giving a morphism of complexes from $K(M, n)$ to X amounts to giving a R-linear map $f : M \to X^n$ such that $d_X^n \circ f = 0$; this is the same as giving a R-linear map $M \to \mathrm{Ker}(d_X^n) = Z^n(X)$. In other words, we have constructed a bijection

$$\mathrm{Hom}_{\mathscr{C}(\mathscr{A})}(K(M, n), X) \xrightarrow{\sim} \mathrm{Hom}_R(M, Z^n(X)),$$

which is clearly functorial in M and X.

Moreover, these adjunctions have the following property (which is clear on their construction): Let $u : K(M, n) \to D^{n-1}(M)$ be the morphism of complexes induced by id_M. If we have a morphism $f : D^{n-1}(M) \to X$ corresponding to $x \in X^{n-1}$, then the morphism $f \circ u : K(M, n) \to X$ corresponds to $d_X^{n-1}(x) \in Z^n(X)$.

(b). Let $f : X \to Y$ be a morphism of $\mathscr{C}(\mathscr{A})$. Saying that f has the right lifting property with respect to $0 \to D^n$ means that, for every morphism $g : D^n \to Y$, there exists $h : D^n \to X$ such that $f \circ h = g$. By question (a), this is equivalent to saying that the map $\mathrm{Hom}_R(R, X^n) \to \mathrm{Hom}_R(R, Y^n)$, $h \mapsto f \circ h$ is surjective, which is equivalent to the fact that $f^n : X^n \to Y^n$ is surjective. This proves the assertion.

(c). By question (b), the morphism $0 \to D^n$ has the left lifting property with respect to every fibration, so it is a cofibration.

(d). Let $f : X \to Y$ be a morphism in $W \cap \mathrm{Fib}$, and let $n \in \mathbb{Z}$. We want to show that $0 \to S^n$ has the left lifting property relatively to f. As $\mathrm{Hom}_{\mathscr{C}(\mathscr{A})}(S^n, C) = \mathrm{Hom}_R(R, Z^n(C)) = Z^n(C)$ for every object C of $\mathscr{C}(\mathscr{A})$ (by question (a)), this is equivalent to the fact that the map $Z^n(X) \to Z^n(Y)$ induced by f^n is surjective. So let $y \in Z^n(Y)$. As f is a quasi-isomorphism, there exists $x \in Z^n(X)$ such that $f^n(x) - y \in B^n(Y)$. Write $f^n(x) - y = d^Y_{n-1}(y')$, with $y' \in Y^{n-1}$. As f is in Fib, there exists $x' \in X^{n-1}$ such that $f^{n-1}(x') = y'$, and then we have

$$y = f^n(x) - d^Y_{n-1}(y') = f^n(x) - d^Y_{n-1}(f^{n-1}(x')) = f^n(x - d^X_{n-1}(x')).$$

Also, as $d^X_n \circ d^X_{n-1} = 0$, we still have $x - d^X_{n-1}(x') \in Z^n(X)$.

(e). By the solution of (a), saying that $p : X \to Y$ has the right lifting property relatively to $S^n \to D^{n-1}$ is equivalent to the following statement: For every $y' \in Y^{n-1}$, and for every $x \in Z^n(X)$ such that $d^{n-1}_Y(y') = p^n(x) \in Z^n(Y)$, there exists $x' \in X^{n-1}$ such that $d^{n-1}_X(x') = x$ and $p^{n-1}(x') = y'$.

(i) Suppose that $p \in W \cap \mathrm{Fib}$, and let $y' \in Y^{n-1}$ and $x \in Z^n(X)$ be such that $d^{n-1}_Y(y') = p^n(x)$. In particular, we have $p^n(X) \in B^n(Y)$; as p is a quasi-isomorphism, this implies that $x \in B^n(X)$, so there exists $x' \in X^{n-1}$ such that $d^{n-1}_X(x') = x$. We have

$$d^{n-1}_Y(p^{n-1}(x') - y') = p^n(d^{n-1}_X(x')) - p^n(x) = 0,$$

so $p^{n-1}(x') - y' \in Z^{n-1}(Y)$. By question (d), there exists $x'' \in Z^{n-1}(X)$ such that $p^{n-1}(x'') = p^{n-1}(x') - y'$, i.e. $y' - p^{n-1}(x' - x'')$. Moreover, as $x'' \in Z^{n-1}(X)$, we have $d^{n-1}_X(x' - x'') = d^{n-1}_X(x') = x$. So we are done.

(ii) Suppose that p has the right lifting property relatively to $S^n \to D^{n-1}$ for every $n \in \mathbb{Z}$.

We first show that p^n induces a surjective map $Z^n(X) \to Z^n(Y)$ for every $n \in \mathbb{Z}$. Indeed, let $n \in \mathbb{Z}$ and $y \in Z^n(Y)$. Then $d^n_Y(y) = 0 = p^{n+1}(0)$, so there exists $x \in X^n$ such that $d^n_X(x) = 0$, i.e. $x \in Z^n(X)$, and that $p^n(x) = y$.

Now we show that $p^n : X^n \to Y^n$ is surjective for every $n \in \mathbb{Z}$. Let $n \in \mathbb{Z}$ and $y \in Y^n$. Then $d^n_Y(y) \in Z^{n+1}(Y)$, so, by the previous paragraph, there exists $x' \in Z^{n+1}(X)$ such that $p^{n+1}(x') = d^n_Y(y)$. Then, by assumption, there exists $x \in X^n$ such that $d^n_X(x) = x'$ and $p^n(x) = y$.

We finally show that p is a quasi-isomorphism. Let $n \in \mathbb{Z}$. We already know that the map $\mathrm{H}^n(p) : \mathrm{H}^n(X) \to \mathrm{H}^n(Y)$ is surjective (because $Z^n(X) \to Z^n(Y)$ is surjective), so it remains to show that it is injective. Let $x \in Z^n(X)$, and suppose that $p^n(x) \in B^n(Y)$. Then there exists $y' \in Y^{n-1}$ such that $p^n(x) = d^{n-1}_Y(y')$, so we can also find $x' \in X^{n-1}$ such that $d^{n-1}_X(x') = x$ and $p^{n-1}(x') = y'$. In particular, we have $x \in B^n(X)$.

(f). This follows from question (e) and from the definition of Cof.

(g). (i) An easy calculation shows that the complex D^n has zero cohomology for every $n \in \mathbb{Z}$. As i is the direct sum of id_X and of morphisms $0 \to D^n$, this implies that i is a quasi-isomorphism.

(ii) The morphism id_X has the left lifting property relatively to any morphism of $\mathscr{C}(\mathscr{A})$, and morphisms $0 \to D^n$ have the left lifting property relatively to morphisms of Fib by question (b). Also, for every morphism of $\mathscr{C}(\mathscr{A})$, the set of morphisms that have the left lifting property relatively to f is stable by direct sums (this is easy, and it is also proved in Proposition VI.5.2.1 of the notes).

(iii) It is clear on the definition of p that every element of Y^n is in the image of p^n, for every $n \in \mathbb{Z}$. So p is in Fib.

(h). (i) For every $i \in \mathbb{N}$, the composition $X \to X_i \xrightarrow{f_i} Y$ is equal to

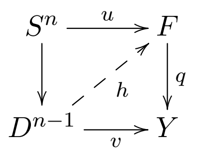
$$f_i \circ (j_{i-1} \circ j_{i-2} \circ \ldots \circ f_0) = f_{i-1} \circ (\circ j_{i-2} \circ \ldots \circ f_0) = \ldots = f_1 \circ j_0 = f.$$

So $q \circ j = f$.

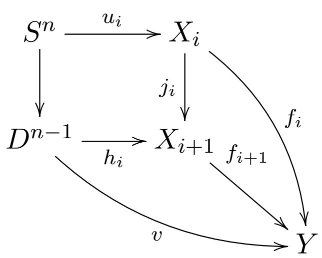
(ii) For every $i \in \mathbb{N}$, the morphism $X \to X_i$ (which is $j_{i-1} \circ j_{i-2} \circ \ldots \circ j_0$) is a monomorphism. As filering colimits are exact in $\mathscr{C}({}_R\mathbf{Mod})$ (because they are exact in ${}_R\mathbf{Mod}$), this implues that j is a monomorphism.

(iii) For every $i \in \mathbb{N}$, the morphism $\bigoplus_{D \in \mathscr{D}_i} S^{n_D} \to \bigoplus_{D \in \mathscr{D}_i} D^{n_D - 1}$ is in Cof by question (e). This easily implies that j_i is in Cof for every $i \in \mathbb{N}$, and then that j is is Cof (see Proposition VI.5.2.1 of the notes).

(iv) By question (e), it suppose to show that q has the right lifting property with respect to $S^n \to D^{n-1}$ for every $n \in \mathbb{Z}$. So fix $n \in \mathbb{Z}$, and consider a commutative square:

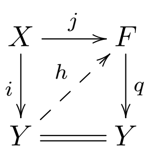


We want to find $h : D^{n-1} \to Y$ making the diagram commute. Remember that $\mathrm{Hom}_{\mathscr{C}(\mathscr{A})}(S^n, F) = Z^n(F)$ by (a). As $F^n = \varinjlim_{i \in \mathbb{N}} X_i^n$, there exists $i \in \mathbb{N}$ and $x \in X_i^n$ such that the element z of $Z^n(F)$ corresponding to u is the image of x_i in F^n. As $d_F^n(z) = 0$, the image in F^{n+1} of $d_{X_i}^n(x_i)$ is 0. But the morphism $X_i \to F$ is a monomorphism (for the same reason as in (i)), so $d_{x_i}^n(x_i) = 0$, i.e. $x_i \in Z^n(X_i)$. Let $u_i : S^n \to X_i$ be the morphism corresponding to x_i. By definition of X_{i+1}, there is a morphism $h_i : D^n \to Y$ making the following diagram commute:



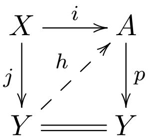
We get the desired morphism $h : D^n \to F$ by composing h_i with the canonical morphism $X_{i+1} \to F$.

(i). Let $i : X \to Y$ be an element of Cof. By question (h), we can write $i = q \circ j$, with $j : X \to F$ a monomorphism and $q \in W \cap \mathrm{Fib}$. In particular, we have a commutative square

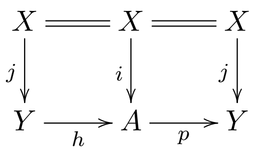


By definition of Cof, there exists $h : Y \to F$ such that $q \circ h = \mathrm{id}_Y$ and $h \circ i = j$. As j is a monomorphism, this implies that i is also a monomorphism.

(j). Let $j : X \to Y$ be an element of $W \cap \mathrm{Cof}$. By question (h), we can write $j = p \circ i$, where $i \in W$ has the left lifting property relatively to fibrations and $p \in \mathrm{Fib}$. As $j \in W$, we also have $p \in W$. Consider the commutative square

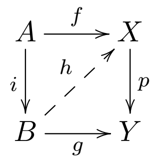


As $p \in W \cap \mathrm{Fib}$ and $j \in \mathrm{Cof}$, there exists $h : Y \to A$ such that $p \circ h = \mathrm{id}_Y$ and $h \circ i = j$. So we have a commutative diagram



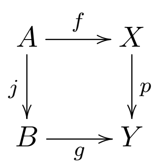
which shows that j is a retract of i. As i has the left lifting property relatively to fibrations, so does j. (This is easy, see Proposition VI.5.2.1 of the notes for a proof.)

(k). We check the axioms. First, the sets W, Fib and Cof clearly contain the identity morphisms and are stable by composition. Also, we know that $\mathscr{C}({}_R\mathbf{Mod})$ has all small limits and colimits, which is axiom (MC1). Axiom (MC2) (the fact that W satisfies the two out of three property) and the fact that W and Fib are stable by retracts are clear. The fact that Cof is stable by retract follows from its definition as the set of morphisms having the left lifting property relatively to elements of $W \cap \mathrm{Fib}$; this finishes the proof of (MC3). The existence of the two factorizations of axiom (MC5) is proved in questions (g) and (h). Finally, consider a commutative square



as in axiom (MC4). If $p \in W \cap \mathrm{Fib}$ and $i \in \mathrm{Cof}$, the existence of h follows from the definition of Cof. If $i \in W \cap \mathrm{Cof}$ and $p \in \mathrm{Fib}$, the existence of h follows from question (j).

(l). Let W^-, Fib^- and Cof^- be the intersections of W, Fib and Cof with $\mathscr{C}^-({}_R\mathbf{Mod})$. By the description of the functors $\mathrm{Hom}_{\mathscr{C}({}_R\mathbf{Mod})}(S^n, \cdot)$ and $\mathrm{Hom}_{\mathscr{C}({}_R\mathbf{Mod})}(D^n, \cdot)$ in question (a), if $f : X \to Y$ is a morphism of $\mathscr{C}^-({}_R\mathbf{Mod})$, then the algorithms of questions (g) and (h) produce factorizations of f in $\mathscr{C}^-({}_R\mathbf{Mod})$. So, to prove the statement, it suffices to check that Cof^- is the set of morphisms of $\mathscr{C}^-({}_R\mathbf{Mod})$ having the left lifting property relatively to the elements of $W^- \cap \mathrm{Fib}^-$. The fact that every morphism of Cof^- satisfies this property is clear. Conversely, let $j : A \to B$ be a morphism of $\mathscr{C}^-({}_R\mathbf{Mod})$ that has the left lifting property relatively to the elements of $W^- \cap \mathrm{Fib}^-$, and let $p : X \to Y$ be in $W \cap \mathrm{Fib}$. Consider a commutative square



As $A, B \in \mathrm{Ob}(\mathscr{C}^-({}_R\mathbf{Mod}))$, there exists $N \in \mathbb{Z}$ such that $A = \tau^{\leq N} A$ and $B = \tau^{\leq N} B$. Also, by question (d) and the properties of the truncation functors, the morphism

$\tau^{\leq N}p : \tau^{\leq N}X \to \tau^{\leq N}Y$ is still in $W \cap \mathrm{Fib}$, hence it is in $W^- \cap \mathrm{Fib}^-$. So we have a commutative square

$$\begin{array}{ccc} A & \xrightarrow{\tau^{\leq N}f} & \tau^{\leq N}X \\ {\scriptstyle j}\downarrow & \overset{h'}{\nearrow} & \downarrow{\scriptstyle \tau^{\leq N}p} \\ B & \xrightarrow[\tau^{\leq N}g]{} & \tau^{\leq N}Y \end{array}$$

with $\tau^{\leq N}p \in W^- \cap \mathrm{Fib}^-$. By the hypothesis on j, there exists $h'B \to \tau^{\leq N}X$ making the diagram commute. Composing h' with the canonical morphism $\tau^{\leq N}X \to X$, we get a morphism $h : B \to X$ such that $p \circ h = g$ and $h \circ j = f$.

(m). Let $f : A \to B$ be a morphism of left R-modules.

Suppose that f has the left lifting property with respect to every surjective morphism of left R-modules. Denote the canonical surjection $A \to \mathrm{Im}\, f$ by q. Applying the lifting property of f to the commutative square

$$\begin{array}{ccc} A & \xrightarrow{q} & \mathrm{Im}\, f \\ {\scriptstyle f}\downarrow & & \downarrow \\ B & \longrightarrow & 0 \end{array}$$

we get a morphism $h : B \to \mathrm{Im}\, f$ such that $h \circ f = q$. Applying the lifting property of f again, this time to the commutative square

$$\begin{array}{ccc} A & \xrightarrow{\mathrm{id}_A} & A \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle q} \\ B & \xrightarrow[h]{} & \mathrm{Im}\, f \end{array}$$

we get a morphism $s : B \to A$ such that $s \circ f = \mathrm{id}_A$. So f is injective and we have $B = \mathrm{Im}\, f \oplus P$, with $P = \mathrm{Ker}\, s$. It remains to show that P is projective. Let $u : M \to N$ be a surjective morphism of left R-modules, and let $g : P \to N$ be a R-linear map. We extend it to a R-linear map $g' : B \to N$ by taking $g' = 0$ on $\mathrm{Im}\, f$. Then we have a commutative square

$$\begin{array}{ccc} A & \xrightarrow{0} & M \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle u} \\ B & \xrightarrow[g']{} & N \end{array}$$

so there exists $h' : B \to M$ such that $u \circ h' = g'$. If $h = h'_{|P}$, we have $u \circ h = g$.

Conversely, suppose that f is injective with projective cokernel $P = \mathrm{Coker}\, f$. Let $p : B \to P$ be the canonical surjection. As P is projective, there exists $s : P \to B$ such that $p \circ s = \mathrm{id}_P$. Hence $B \simeq A \oplus P$, so we may assume that $B = A \oplus P$ and that $f = \begin{pmatrix} \mathrm{id}_A \\ 0 \end{pmatrix}$. Consider a commutative square

$$\begin{array}{ccc} A & \xrightarrow{u} & M \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle q} \\ B & \xrightarrow[v]{} & N \end{array}$$

with q a surjective map. As P is projective, there exists $h' : P \to M$ such that $q \circ h' = v_{|P}$. Let $h = \begin{pmatrix} u & h' \end{pmatrix} : B = A \oplus P \to N$. Then $h \circ f = u$ and $q \circ h = \begin{pmatrix} u & v_{|P} \end{pmatrix} = v$.

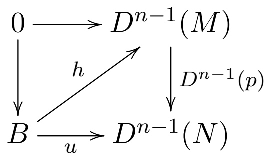
(n). We first prove that, for every $n \in \mathbb{Z}$, the functor $D^{n-1} : {}_R\mathbf{Mod} \to \mathscr{C}({}_R\mathbf{Mod})$ is right adjoint to the functor $\mathscr{C}({}_R\mathbf{Mod}) \to {}_R\mathbf{Mod}$, $X \mapsto X^n$. Let $n \in \mathbb{Z}$, let M be a left R-module and let X be an object of $\mathscr{C}({}_R\mathbf{Mod})$. Then giving a morphism of complexes $u : X \to D^{n-1}(M)$ is equivalent to giving two R-linear maps $u^{n-1} : X^{n-1} \to M$ and $u^n : X^n \to M$ such that $u^{n-1} \circ d_X^{n-2} = 0$ and $u^n \circ d_X^n = u^{n-1}$; as the second condition determines u^{n-1} and implies the first condition, this is equivalent to giving $u^n : X^n \to M$. So we have constructed a bijective map

$$\mathrm{Hom}_{\mathscr{C}({}_R\mathbf{Mod})}(X, D^{n-1}(M)) \to \mathrm{Hom}_R(X^n, M),$$

which is clearly functorial in X and M.

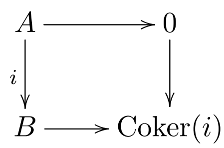
Let $i : A \to B$ be a morphism of $\mathscr{C}({}_R\mathbf{Mod})$. We suppose that i is in Cof, and we want to show that i^n is injective with projective kernel for every $n \in \mathbb{Z}$:

(1) Suppose first that $A = 0$, and let $n \in \mathbb{Z}$. We want to show that B^n is a projective R-module. Let $p : M \to N$ be a surjective map of left R-modules, and let $f : B^n \to N$ be a R-linear map. Then the morphism $D^{n-1}(p) : D^{n-1}(M) \to D^{n-1}(N)$ is a fibration, and it is acyclic because both $D^{n-1}(M)$ and $D^{n-1}(N)$ are acyclic complexes. Consider the morphism of complexes $u : B \to D^{n-1}(N)$ corresponding to $f : B^n \to N$ by the adjunction of the first paragraph. As B is cofibrant, there exists a morphism $h : B \to D^{n-1}(M)$ making the following diagram commute:



and then $h^n : B^n \to M$ satisfies the identity $p \circ h^n = f$. This shows that B^n is a projective R-module.

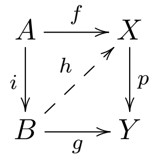
(2) Now we treat the general case. Note that we have a cocartesian diagram



By Corollary VI.1.2.4 of the notes, this implies that $0 \to \mathrm{Coker}(i)$ is a cofibration, i.e. that $\mathrm{Coker}(i)$ is cofibrant. By (1), this shows that i^n has projective cokernel for every $n \in \mathbb{Z}$. To show that i^n is injective, consider the morphism $u : A \to D^{n-1}(A^n)$ corresponding to id_{A^n} by the adjunction of the first paragraph. As $D^{n-1}(A^n)$ is an acyclic complex, the morphism $D^{n-1}(A^n)$ is an acyclic fibration, so there exists $h : B \to D^{n-1}A$ such that $h \circ i = u$, and in particular we have $h^n \circ i^n = \mathrm{id}_{A^n}$, which implies that i^n is injective.

Conversely, suppose that, for every $n \in \mathbb{Z}$, the morphism i^n is injective and has projective cokernel. We want to show that i is a cofibration. Let $P = \mathrm{Coker}(i)$. As each P^n is a projective, the morphisms $i^n : A^n \to B^n$ are split injections (i.e. there exists morphisms $a^n : B^n \to A^n$ such that $a^n \circ i^n = \mathrm{id}_{A^n}$), so, without loss of generality, we may assume that $B^n = A^n \oplus P^n$ and that $i^n = \begin{pmatrix} \mathrm{id}_{A^n} \\ 0 \end{pmatrix}$. As i is a morphism of complexes, we have $d_B^n = \begin{pmatrix} d_A^n & u^n \\ 0 & d_P^n \end{pmatrix}$, with $u^n : P^n \to A^{n+1}$.

Consider a commutative square (in $\mathscr{C}({}_R\mathbf{Mod})$)



with p an acyclic fibration. We want to show that there exists a morphism $h : B \to X$ making the diagram commute. As $g \circ i = p \circ f$, we have $g^n = \begin{pmatrix} p^n \circ f^n & v^n \end{pmatrix}$ with $v^n : P^n \to Y^n$, and the fact that g is a morphism of complexes is equivalent to the identities

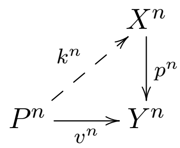
$$d_Y^n \circ v^n = p^{n+1} \circ f^{n+1} \circ u^n + v^{n+1} \circ d_P^n. \tag{1}$$

If $h : B \to X$ is a morphism such that $h \circ i = f$, then we must have $h^n = \begin{pmatrix} f^n & w^n \end{pmatrix}$, with $w^n : P^n \to X^n$. The fact that h is a morphism of complexes is equivalent to the identities

$$d_X^n \circ w^n = f^{n+1} \circ u^n + w^{n+1} \circ d_P^n, \tag{2}$$

and we have $p \circ h = g$ if and only $p^n \circ w^n = v^n$ for every $n \in \mathbb{Z}$.

Let $n \in \mathbb{Z}$. As $p^n : X^n \to Y^n$ is surjective and P^n is a projective R-module, there exists a R-linear map $k^n : P^n \to X^n$ such that $p^n \circ k^n = v^n$.



Let $r^n = d_X^n \circ k^n - k^{n+1} \circ d_P^n - f^{n+1} \circ u^n : P^n \to X^{n+1}$. We have

$$\begin{aligned} p^{n+1} \circ r^n &= d_Y^{n+1} \circ p^n \circ k^n - p^{n+1} \circ k^{n+1} \circ d_P^n - p^{n+1} \circ f^{n+1} \circ u^n \\ &= d_Y^{n+1} \circ v^n - v^{n+1} \circ d_P^n - p^{n+1} \circ f^{n+1} \circ u^n \\ &= 0 \quad \text{by (1)}. \end{aligned}$$

Let $K = \mathrm{Ker}(p)$. We just proved that $r^n : P^n \to X^{n+1}$ factors through a R-linear map $s^n : P^n \to K^{n+1}$. Also, we have

$$r^{n+1} \circ d_P^n = d_X^{n+1} \circ k^{n+1} \circ d_P^n - f^{n+2} \circ u^{n+1} \circ d_P^n$$

and

$$\begin{aligned} d_X^{n+1} \circ r^n &= -d_X^{n+1} \circ k^{n+1} \circ d_P^n - d_X^{n+1} \circ f^{n+1} \circ u^n \\ &= -d_X^{n+1} \circ k^{n+1} \circ d_P^n - f^{n+2} \circ d_A^{n+1} \circ u^n \\ &= -d_X^{n+1} \circ k^{n+1} \circ d_P^n - f^{n+2} \circ u^{n+1} \circ d_P^n, \end{aligned}$$

so $s^{n+1} \circ d_P^n = -d_K^{n+1} \circ s^n$. This means that the family $(s^n)_{n \in \mathbb{Z}}$ defines a morphism of complexes from P to $K[1]$. As P is a bounded above complex of projective R-modules and K is an acyclic complex, the dual of Theorem IV.3.2.1(i) of the notes says that s is homotopic to 0. This means that there exists a family of R-linear maps $(t^n : P^n \to K^n)_{n \in \mathbb{Z}}$ such that, for every $n \in \mathbb{Z}$, we have

$$s^n = t^{n+1} \circ d_P^n + d_{K[1]}^{n-1} \circ t^n = t^{n+1} \circ d_P^n - d_K^n \circ t^n.$$

For every $n \in \mathbb{Z}$, we set $w^n = k^n + t^n : P^n \to X^n$ and $h^n = \begin{pmatrix} f^n & w^n \end{pmatrix} : B^n \to X^n$. As $K^n = \mathrm{Ker}(p^n)$, we have

$$p^n \circ w^n = p^n \circ k^n = v^n,$$

so $p^n \circ h^n = g^n$. It remains to check that h is a morphism of complexes from B to X, so we check identity (2). Let $n \in \mathbb{Z}$. We have

$$\begin{aligned} d_X^n \circ w^n &= d_X^n \circ k^n + d_X^n \circ t^n \\ &= d_X^n \circ k^n + t^{n+1} \circ d_P^n - r^n \\ &= t^{n+1} \circ d_P^n + k^{n+1} \circ d_P^n + f^{n+1} \circ u^n \\ &= w^{n+1} \circ d_P^n + f^{n+1} \circ u^n, \end{aligned}$$

which is exactly (2).

□