

MAT 449 : Representation theory

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I Representations of topological groups

I.1 Topological groups

Definition I.1.1. A *topological group* is a topological set G with the structure of a group such that the multiplication map $G \times G \rightarrow G, (x, y) \mapsto xy$ and the inversion map $G \rightarrow G, x \mapsto x^{-1}$ are continuous.

We usually will denote the unit of G by 1 or e .

Example I.1.2. - Any group with the discrete topology is a topology group. Frequently used examples include finite groups, free groups (both commutative and noncommutative) and “arithmetic” matrix groups such as $GL_n(\mathbb{Z})$ and $SL_n(\mathbb{Z})$.

- The additive groups of \mathbb{R} and \mathbb{C} are topological groups.
- The group $GL_n(\mathbb{C})$, with the topology given by any norm on the \mathbb{C} -vector space $M_n(\mathbb{C})$, is a topological group,¹ hence so are all its subgroups if we put the induced topology on them. For example $S^1 := \{z \in \mathbb{C} \mid |z| = 1\}$, $GL_n(\mathbb{R})$, $SU(n)$, $SO(n)$ etc.
- (See problem set 1.) The additive group of \mathbb{Q}_p and the group $GL_n(\mathbb{Q}_p)$ are topological groups.

Definition I.1.3. We say that a topological space X is *locally compact* if every point of X has a compact neighborhood.

Remark I.1.4. If X is Hausdorff, this is equivalent to the fact that every point of X has a basis of compact neighborhoods.

Note that we do not assume that neighborhoods of points in topological spaces are open.

Notation I.1.5. Let G be a group, and let $A, B \subset G$, $x \in G$ and $n \geq 1$. We use the following notation :

$$xA = \{xy, y \in A\} \quad \text{and} \quad Ax = \{yx, y \in A\}$$
$$AB = \{yz, y \in A, z \in B\}$$

¹See problem set 1 for a proof.

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$$A^n = AA \dots A \quad (n \text{ factors})$$

$$A^{-1} = \{y^{-1}, y \in A\}$$

Definition I.1.6. We say that a subset A of G is *symmetric* if $A = A^{-1}$.

Proposition I.1.7. Let G be a topological group.

1. If U is an open subset of G and A is any subset of G , then the sets UA , AU and U^{-1} are open.
2. If U is a neighborhood of 1 in G , then there is an open symmetric neighborhood V of 1 such that $V^2 \subset U$.
3. If H is a subgroup of G , then its closure \overline{H} is also a subgroup of G .
4. If H is an open subgroup of G , then it is also closed.
5. If A and B are compact subsets of G , then the set AB is also compact.
6. Let H be a subgroup of G . Then the quotient G/H (with the quotient topology) is :
 - a) Hausdorff if H is closed;
 - b) Locally compact if G is locally compact;
 - c) A topological group if H is normal.

Proof. 1. For $x \in G$, we denote by $l_x : G \rightarrow G$ (resp. $r_x : G \rightarrow G$) left (resp. right) multiplication by x . We also denote by $\iota : G \rightarrow G$ the map $x \mapsto x^{-1}$. By the axioms for topological groups, all these maps are continuous.

Now note that $U^{-1} = \iota^{-1}(U)$, $AU = \bigcup_{x \in A} l_x^{-1}(U)$ and $UA = \bigcup_{x \in A} r_x^{-1}(U)$. So U^{-1} , AU and UA are open.

2. We may assume that U is open. Let $m : G \times G \rightarrow G$, $(x, y) \mapsto xy$. Then m is continuous, so $W := m^{-1}(U)$ is open. We have $(1, 1) \in W$ because $1^2 = 1 \in U$. By definition of the product topology on $G \times G$, there exists an open subset $\Omega \ni 1$ of G such that $\Omega \times \Omega \subset W$. We have $\Omega^2 \subset U$ by definition of W . Let $V = \Omega \cap \Omega^{-1}$. We know that Ω^{-1} is open by (a), so V is open, and it is symmetric by definition. We clearly have $1 \in V$ and $V^2 \subset \Omega^2 \subset U$.
3. Consider the map $u : G \times G \rightarrow G$, $(x, y) \mapsto xy^{-1}$; then a nonempty subset A of G is a subgroup if and only if $u(A \times A) \subset A$. Also, by the axioms of topological groups, the map u is continuous. Hence, for every $Z \subset G \times G$, $u(\overline{Z}) \subset \overline{u(Z)}$. Applying this to $H \times H$ (whose closure is $\overline{H} \times \overline{H}$), we see that \overline{H} is a subgroup of G .
4. We have $G = H \sqcup ((G - H)H)$. If H is open, then $(G - H)H$ is also open by (a), hence H is closed.
5. The multiplication map $m : G \times G \rightarrow G$ is continuous by hypothesis. As $AB = m(A \times B)$ and $A \times B$ is compact, the set AB is also compact.

6. a) Let $x, y \in G$ be such that $xH \neq yH$. By question (a), $x(G - H)y^{-1}$ is open, so its complement xHy^{-1} is closed. Also, by the assumption that $xH \neq yH$, the unit 1 is not in xHy^{-1} . By (b), there exists a symmetric open set U such that $U^2 \subset G - xHy^{-1}$. Let's show that $UxH \cap UyH = \emptyset$, which will prove the result because UxH (resp. UyH) is an open neighborhood of xH (resp. yH) in G/H . If $UxH \cap UyH \neq \emptyset$, then we can find $u_1, u_2 \in U$ and $h_1, h_2 \in H$ such that $u_1xh_1 = u_2yh_2$. But then $xh_1h_2^{-1}y^{-1} = u_1^{-1}u_2 \in xHy^{-1} \cap U^2$, which is not possible.
- b) Let $xH \in G/H$. If K is a compact neighborhood of x in G , then its image in G/H is a compact neighborhood of xH in G/H .
- c) If H is normal, then G/H is a group. Let's show that its multiplication is continuous. Let $x, y \in G$. Any open neighborhood of xyH in G/H is of the form $UxyH$, with U an open neighborhood of xy in G . By the continuity of multiplication on G , there exists open neighborhoods V and W of x and y in G such that $VW \subset U$. Then VH and WH are open neighborhoods of xH and yH in G/H , and we have $(VH)(WH) \subset UH$. (Remember that, as H is normal, $AH = HA$ for every subset A of G .) Let's show that inversion is continuous on G/H . Let $x \in G$. Any open neighborhood of $x^{-1}H$ in G/H is of the form UH , with U an open neighborhood of x^{-1} in G . By question (a), the set U^{-1} is open, so $U^{-1}H$ is an open neighborhood of xH in G/H , and we have $(U^{-1}H)^{-1} = HU = UH$.

□

Remark I.1.8. In particular, if G is a topological group, then $G/\overline{\{1\}}$ is a Hausdorff topological group. We are interested in continuous group actions of G on vector spaces, so we could replace G by $G/\overline{\{1\}}$ to study them. Hence, **in what follows, we will only Hausdorff topological groups (unless otherwise specified).**

Definition I.1.9. A *compact group* (resp. a *locally compact group*) is a Hausdorff and compact (resp. locally compact) topological group.

Example I.1.10. Among the groups of example I.1.2, finit discrete groups and the groups S^1 , $SU(n)$ and $SO(n)$ are compact. All the other groups are locally compact. We get a non-locally compact group by considering the group of invertible bounded linear endomorphism of an infinite-dimensional Banach space (see problem set 1).

Translation operators : Let G be a group, $x \in G$ and $f : G \rightarrow \mathbb{C}$ be a function. We define two functions $L_x f, R_x f : G \rightarrow \mathbb{C}$ by :

$$L_x f(y) = f(x^{-1}y) \quad \text{and} \quad R_x f(y) = f(yx).$$

We chose the convention so that $L_{xy} = L_x \circ L_y$ and $R_{xy} = R_x \circ R_y$. Note that, if G is a topological group and f is continuous, then $L_x f$ and $R_x f$ are also continuous.

Function spaces : Let X be a topological set. If $f : X \rightarrow \mathbb{C}$ is a function, we write

$$\|f\|_\infty = \sup_{x \in X} |f(x)| \in [0, +\infty].$$

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We also use the following notation :

- $\mathcal{C}(X)$ for the set of continuous functions $f : X \rightarrow \mathbb{C}$;
- $\mathcal{C}_b(X)$ for the set of bounded continuous functions $f : X \rightarrow \mathbb{C}$ (i.e. elements f of $\mathcal{C}(X)$ such that $\|f\|_\infty < +\infty$);
- $\mathcal{C}_0(X)$ for the set of continuous functions $X \rightarrow \mathbb{C}$ that vanish at infinity (i.e. such that, for every $\varepsilon > 0$, there exists a compact subset K of X such that $|f(x)| < \varepsilon$ for every $x \notin K$);
- $\mathcal{C}_c(X)$ for the set of continuous functions with compact support from X to \mathbb{C} .

Note that we have $\mathcal{C}(X) \supset \mathcal{C}_b(X) \supset \mathcal{C}_0(X) \supset \mathcal{C}_c(X)$, with equality if X is compact. The function $\|\cdot\|_\infty$ is a norm on $\mathcal{C}_b(X)$ and its subspaces, and $\mathcal{C}_b(X)$ and $\mathcal{C}_0(X)$ are complete for this norm (but not $\mathcal{C}_c(X)$, unless X is compact).

Definition I.1.11. Let G be a topological group. A function $f : G \rightarrow \mathbb{C}$ is called *left (resp. right) uniformly continuous* if $\|L_x f - f\|_\infty \rightarrow 0$ as $x \rightarrow 1$ (resp. $\|R_x f - f\|_\infty \rightarrow 0$ as $x \rightarrow 1$).

Proposition I.1.12. If $f \in \mathcal{C}_c(G)$, then f is both left and right uniformly continuous.

Proof. We prove that f is right uniformly continuous (the proof that it is left uniformly continuous is similar). Let K be the support of f . Let $\varepsilon > 0$. For every $x \in K$, we choose a neighborhood U_x of 1 such that $|f(xy) - f(x)| < \frac{\varepsilon}{2}$ for every $y \in U_x$; by proposition I.1.7, we can find a symmetric open neighborhood V_x of 1 such that $V_x^2 \subset U_x$. We have $K \subset \bigcup_{x \in K} xV_x$. As K is compact, we can find $x_1, \dots, x_n \in K$ such that $K \subset \bigcup_{i=1}^n x_i V_{x_i}$. Let $V = \bigcup_{i=1}^n V_{x_i}$, this is a symmetric open neighborhood of 1.

We claim that, if $y \in V$, then $\|R_y f - f\|_\infty < \varepsilon$. Indeed, let $y \in V$, and let $x \in G$. First assume that $x \in K$. Then there exists $i \in \{1, \dots, n\}$ such that $x \in x_i V_{x_i}$. Then we have $xy \in x_i V_{x_i} V_{x_i} \subset x_i U_{x_i}$, hence

$$|f(xy) - f(x)| \leq |f(xy) - f(x_i)| + |f(x_i) - f(x)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Now assume that $xy \in K$. Then there exists $i \in \{1, \dots, n\}$ such that $xy \in x_i V_{x_i}$, and we have $x = xyy^{-1} \in x_i V_{x_i} V_{x_i} \subset x_i U_{x_i}$. Hence

$$|f(xy) - f(x)| \leq |f(xy) - f(x_i)| + |f(x_i) - f(x)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Finally, if $x, xy \notin K$, then $f(x) = f(xy) = 0$, and of course $|f(xy) - f(x)| < \varepsilon$.

□

Remark I.1.13. We put the topology given by $\|\cdot\|_\infty$ on $\mathcal{C}_b(G)$. Then a function $f \in \mathcal{C}_b(G)$ is left (resp. right) uniformly continuous if and only if the map $G \rightarrow \mathcal{C}_b(G)$, $x \mapsto L_x f$ (resp. $x \mapsto R_x f$) is continuous at the unit of G .

Using the fact that $L_{xy} = L_x \circ L_y$ and $R_{xy} = R_x \circ R_y$ and the operators L_x and R_x preserve $\mathcal{C}_c(G)$, we see that the proposition above implies that, if $f \in \mathcal{C}_c(G)$, then the two maps $G \rightarrow \mathcal{C}_c(G)$ sending $x \in G$ to $L_x f$ and to $R_x f$ are continuous.

I.2 Haar measures

Definition I.2.1. Let X be a topological space.

1. The σ -algebra of *Borel sets* on X is the σ -algebra on X generated by the open subsets of X . A *Borel measure* on X is a measure on this σ -algebra.
2. A *regular Borel measure* on X is a measure μ on the σ -algebra of Borel sets of X satisfying the following properties :
 - a) For every compact subset K of X , $\mu(K) < +\infty$;
 - b) μ is *outer regular* : for every Borel subset E of X , we have $\mu(E) = \inf\{\mu(U), U \supset E \text{ open}\}$;
 - c) μ is *inner regular* : for every $E \subset X$ that is either Borel of finite measure or open, we have $\mu(E) = \sup\{\mu(K), K \subset E \text{ compact}\}$.

Notation I.2.2. We denote by $\mathcal{C}_c^+(X)$ the subset of nonzero $f \in \mathcal{C}_c(X)$ such that $f(X) \subset \mathbb{R}_{\geq 0}$.

Theorem I.2.3 (Riesz representation theorem). *Let X be a locally compact Hausdorff space, and let $\Lambda : \mathcal{C}_c(X) \rightarrow \mathbb{C}$ be a linear functional such that $\Lambda(f) \geq 0$ for every $f \in \mathcal{C}_c^+(X)$.² Then there exists a unique regular Borel measure μ on X such that, for every $f \in \mathcal{C}_c(X)$,*

$$\Lambda(f) = \int_X f d\mu.$$

Definition I.2.4. Let G be a locally compact group. A *left (resp. right) Haar measure* on G is a nonzero regular Borel measure μ on G such that, for every Borel set E of G and every $x \in G$, we have $\mu(xE) = \mu(E)$ (resp. $\mu(Ex) = \mu(E)$).

Example I.2.5. 1. If G is a discrete group, then the counting measure is a left and right Haar measure on G .

2. Lebesgue measure is a left and right Haar measure on the additive group of \mathbb{R} .

Proposition I.2.6. *Let G be a locally compact group and μ be a regular Borel measure on G .*

1. *Let $\tilde{\mu}$ be the Borel measure on G defined by $\tilde{\mu}(E) = \mu(E^{-1})$. Then μ is a left Haar measure if and only if $\tilde{\mu}$ is a right Haar measure.*
2. *The measure μ is a left Haar measure on G if and only if we have : for every $f \in \mathcal{C}_c(G)$, for every $y \in G$, $\int_G L_y f d\mu = \int_G f d\mu$.*
3. *If μ is a left Haar measure on G , then $\mu(U) > 0$ for every nonempty open subset of G and $\int_G f d\mu > 0$ for every $f \in \mathcal{C}_c^+(G)$.*

²Such a linear functional is called *positive*.

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Proof. 1. First, note that $\tilde{\mu}$ is a regular Borel measure on G because $x \mapsto x^{-1}$ is a homeomorphism from G to itself.

If $E \subset G$ is a Borel set and $x \in E$, then $\tilde{\mu}(Ex) = \mu(x^{-1}E^{-1})$. This implies the statement.

2. Let $x \in G$, and let μ_x be the Borel measure on G defined by $\mu_x(E) = \mu(xE)$. (This is indeed a regular Borel measure on G , because $y \mapsto xy$ is a homeomorphism from G to itself.) Then, for every measurable function $f : G \rightarrow \mathbb{C}$, we have $\int_G f d\mu_x = \int_G L_x f d\mu$. (This is obvious for characteristic functions of Borel subsets, and we get the general case by approximating f by linear combinations of characteristic functions.)

On the one hand, the measure μ is a left Haar measure if and only if $\mu = \mu_x$ for every $x \in G$. On the other hand, by the uniqueness in the Riesz representation theorem (and the paragraph above), for $x \in G$, we have $\mu = \mu_x$ if and only if $\int_G f d\mu = \int_G L_x f d\mu$ for every $f \in \mathcal{C}_c(G)$. The statement follows.

3. Suppose that there exists a nonempty open subset U of G such that $\mu(U) = 0$. Then $\mu(xU) = 0$ for every $x \in G$, so we may assume that $1 \in U$. Let K be a compact subset of G . Then $K \subset \bigcup_{x \in K} xU$, so there exist $x_1, \dots, x_n \in K$ such that $K \subset \bigcup_{i=1}^n x_i U$. As $\mu(x_i U) = 0$ for every i , this implies that $\mu(K) = 0$. But then, by inner regularity of μ , we get $\mu(G) = 0$, which contradicts the fact that μ is nonzero.

Let $f \in \mathcal{C}_c^+(G)$. Then $U := \{x \in G \mid f(x) > \frac{1}{2}\|f\|_\infty\}$ is a nonempty open subset of G , so $\mu(U) > 0$. But we have $f \geq \frac{1}{2}\|f\|_\infty \mathbf{1}_U$, hence $\int_G f d\mu \geq \frac{1}{2}\|f\|_\infty \mu(U) > 0$.

□

Theorem I.2.7. *Let G be a locally compact group. Then :*

1. *There exists a left Haar measure on G ;*
2. *If μ_1 and μ_2 are two left Haar measures on G , then there exists $c \in \mathbb{R}_{>0}$ such that $\mu_2 = c\mu_1$.*

By proposition I.2.6, this theorem implies the similar result for right Haar measures.

Proof. We first prove existence. The idea is very similar to the construction of Lebesgue measure on \mathbb{R} . Suppose that $c > 0$, and that $\varphi \in C_c^+(\mathbb{R})$ is bounded by 1 and very close to the characteristic function of the interval $[0, c]$. If $f \in \mathcal{C}_c(\mathbb{R})$ does not vary too quickly on intervals of length c , then we can approximate f by a linear combination of left translates of φ : $f \simeq \sum c_i L_{x_i} \varphi$, and then $\int f d\mu \simeq \sum c_i L_{x_i} \int \varphi d\mu$. As $c \rightarrow 0$, we will be able to approximate every $f \in C_c(\mathbb{R})$ (because we know that these functions are uniformly continuous), and we'll be able to define $\int f d\mu$ by going to the limit. On a general locally compact group, we replace the intervals by smaller and smaller compact neighborhoods of 1.

Now here is the rigorous proof. Let $f, \varphi \in C_c^+(G)$. Then $U := \{x \in G \mid \varphi(x) > \frac{1}{2}\|\varphi\|_\infty\}$ is a nonempty open subset of G and we have $\varphi \geq \frac{1}{2}\|\varphi\|_\infty \mathbf{1}_U$. As the support of f is compact, it can be covered by a finite number of translates of U , so there exist $x_1, \dots, x_n \in G$ and $c_1, \dots, c_n \in \mathbb{R}_{\geq 0}$

such that $f \leq \sum_{i=1}^n c_i L_{x_i} \varphi$. Hence, if we define $(f : \varphi)$ to be the infimum of all finite sums $\sum_{i=1}^n c_i$ with $c_1, \dots, c_n \in \mathbb{R}_{\geq 0}$ and such that there exist $x_1, \dots, x_n \in G$ with $f \leq \sum_{i=1}^n c_i L_{x_i} \varphi$, we have $(f : \varphi) < +\infty$. We claim that :

- (I.2.0.0.1) $(f : \varphi) = (L_x f : \varphi) \quad \forall x \in G$
- (I.2.0.0.2) $(f_1 + f_2 : \varphi) \leq (f_1 : \varphi) + (f_2 : \varphi)$
- (I.2.0.0.3) $(cf : \varphi) = c(f : \varphi) \quad \forall c \geq 0$
- (I.2.0.0.4) $(f_1 : \varphi) \leq (f_2 : \varphi) \quad \text{if } f_1 \leq f_2$
- (I.2.0.0.5) $(f : \varphi) \geq \frac{\|f\|_\infty}{\|\varphi\|_\infty}$
- (I.2.0.0.6) $(f : \varphi) \leq (f : \psi)(\psi : \varphi) \quad \forall \psi \in \mathcal{C}_c^+(G) - \{0\}$

The first four properties are easy. For the fifth property, note that, if $f \leq \sum_{i=1}^n c_i L_{x_i} \varphi$, then

$$\|f\|_\infty \leq \sum_{i=1}^n c_i \|L_{x_i} \varphi\|_\infty = \left(\sum_{i=1}^n c_i \right) \|\varphi\|_\infty.$$

Finally, the last property is a consequence of the following fact : Let $\psi \in \mathcal{C}_c^+(G)$. If we have $f \leq \sum_{i=1}^n c_i L_{x_i} \psi$ and $\psi \leq \sum_{j=1}^m d_j L_{y_j} \varphi$, then $f \leq \sum_{i=1}^n \sum_{j=1}^m c_i d_j L_{x_i y_j} \varphi$.

Now we fix $f_0 \in \mathcal{C}_c^+(G)$. By I.2.0.0.5, we know that $(f_0 : \varphi) > 0$. We define $I_\varphi : \mathcal{C}_c^+(G) \rightarrow \mathbb{R}_{\geq 0}$ by

$$I_\varphi(f) = \frac{(f : \varphi)}{(f_0 : \varphi)}.$$

By I.2.0.0.1-I.2.0.0.4, we have

$$\begin{aligned} I_\varphi(f) &= I_\varphi(L_x f) \quad \forall x \in G \\ I_\varphi(f_1 + f_2) &\leq I_\varphi(f_1) + I_\varphi(f_2) \\ I_\varphi(cf) &= c I_\varphi(f) \quad \forall c \geq 0 \\ I_\varphi(f_1) &\leq I_\varphi(f_2) \quad \text{if } f_1 \leq f_2 \end{aligned}$$

If the second inequality was an equality (that is, if I_φ were additive), we could extend I_φ to a positive linear functional on $\mathcal{C}_c(G)$ and apply the Riesz representation theorem. This is not quite true, but we have the following result :

Claim : For all $f_1, f_2 \in \mathcal{C}_c^+(G)$ and $\varepsilon > 0$, there exists a neighborhood V of 1 in G such that we have $I_\varphi(f_1) + I_\varphi(f_2) \leq I_\varphi(f_1 + f_2) + \varepsilon$ whenever $\text{supp}(\varphi) \subset V$.

Let's first prove the claim. Choose a function $g \in \mathcal{C}_c^+(G)$ such that $g(x) = 1$ for every $x \in \text{supp}(f_1 + f_2)$, and let δ be a positive real number. Let $h = f_1 + f_2 + \delta g$. We define functions $h_1, h_2 : G \rightarrow \mathbb{R}_{\geq 0}$ by

$$h_i(x) = \begin{cases} \frac{f_i(x)}{h(x)} & \text{if } f_i(x) \neq 0 \\ 0 & \text{if } f_i(x) = 0. \end{cases}$$

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Note that h_i is equal to $\frac{f_i}{h}$, hence continuous on the open subset $\{x \in G \mid h(x) \neq 0\}$. As G is the union of this open subset and of the open subset $G - \text{supp}(f_i)$ (on which h_i is also continuous), this shows that h_i is continuous, hence $h_i \in \mathcal{C}_c^+(G)$. Note also that we have $f_i = h_i h$.

By proposition I.1.12, there exists a neighborhood V of 1 such that, for $i \in \{1, 2\}$ and $x, y \in G$ with $y^{-1}x \in V$, we have $|h_i(x) - h_i(y)| < \delta$. Let $\varphi \in \mathcal{C}_c^+(G)$ be such that $\text{supp}(\varphi) \subset V$. If $c_1, \dots, c_n \in \mathbb{R}_{\geq 0}$ and $x_1, \dots, x_n \in G$ are such that $h \leq \sum_{j=1}^n c_j L_{x_j} \varphi$, then, for every $x \in G$ and $i \in \{1, 2\}$,

$$f_i(x) = h(x)h_i(x) \leq \sum_{j=1}^n c_j \varphi(x_j^{-1}x) h_i(x) \leq \sum_{j=1}^n c_j \varphi(x_j^{-1}x) (h_i(x_j) + \delta),$$

because $\varphi(x_j^{-1}x) = 0$ unless $x_j^{-1}x \in V$. Hence

$$(f_1 : \varphi) + (f_2 : \varphi) \leq \sum_{j=1}^n c_j (h_1(x_j) + h_2(x_j) + 2\delta).$$

Since $h_1 + h_2 \leq 1$, we get

$$(f_1 : \varphi) + (f_2 : \varphi) \leq (1 + 2\delta) \sum_{j=1}^n c_j,$$

hence, taking the infimum over the family (c_1, \dots, c_n) and dividing by $(f_0 : \varphi)$, we get

$$I_\varphi(f_1) + I_\varphi(f_2) \leq (1 + 2\delta) I_\varphi(h) \leq (1 + 2\delta) (I_\varphi(f_1 + f_2) + \delta I_\varphi(g)).$$

The right-hand side of this tends to $I_\varphi(f_1 + f_2)$ as δ tends to 0, so we get the desired inequality by taking δ small enough. This finishes the proof of the claim.

We come back to the construction of a left Haar measure on G . For every $f \in \mathcal{C}_c^+(G)$, let $X_f = [(f_0 : f)^{-1}, (f : f_0)] \subset \mathbb{R}$. Let $X = \prod_{f \in \mathcal{C}_c^+(G)} X_f$, endowed with the product topology. Then, by Tychonoff's theorem, X is a compact Hausdorff space. It is the space of functions $I : \mathcal{C}_c^+(G) \rightarrow \mathbb{R}$ such that $I(f) \in X_f$ for every f (with the topology of pointwise convergence). Also, by I.2.0.6, we have $I_\varphi \in I$ for every $\varphi \in \mathcal{C}_c^+(G)$. For every neighborhood V of 1 in G , let $K(V)$ be the closure of $\{I_\varphi \mid \text{supp}(\varphi) \subset V\}$ in X . We have $K(V) \neq \emptyset$ for every V , so $K(V_1) \cap \dots \cap K(V_n) \supset K(\bigcap_{i=1}^n V_i) \neq \emptyset$ for every finite family V_1, \dots, V_n of neighborhoods of 1 in G . As X is compact, this implies that the intersection of all the sets $K(V)$ is nonempty. We choose an element I of this intersection.

Let's show that I is invariant by left translations, additive and homogenous of degree 1. (That is, it has the same properties as I_φ , but is also additive instead of just subadditive.) Let $f_1, f_2 \in \mathcal{C}_c^+(G)$, $c \in \mathbb{R}_{\geq 0}$, $x \in G$ and $\varepsilon > 0$. Choose a neighborhood V of 1 in G such that $I_\varphi(f_1) + I_\varphi(f_2) \leq I_\varphi(f_1 + f_2) + \varepsilon$ whenever $\text{supp}(\varphi) \subset V$; this exists by the claim. By definition of I , it is in the closure $\{I_\varphi \mid \text{supp}(\varphi) \subset V\}$, which means that there exists $\varphi \in \mathcal{C}_c^+(G)$ such

that $\text{supp}(\varphi) \subset V$ and $|I(aL_y g) - I_\varphi(aL_y g)| < \varepsilon$ for $g \in \{f_1, f_2, f_1 + f_2\}$, $y \in \{1, x\}$ and $a \in \{1, c\}$. Then we get :

$$|I(L_x f_1) - I(f_1)| \leq |I(L_x f_1) - I_\varphi(L_x f_1)| + |I_\varphi(L_x f_1) - I_\varphi(f_1)| + |I_\varphi(f_1) - I(f_1)| < 2\varepsilon,$$

$$|I(cf_1) - cI(f_1)| \leq |I(cf_1) - I_\varphi(cf_1)| + |I_\varphi(cf_1) - cI_\varphi(f_1)| + |cI_\varphi(f_1) - cI(f_1)| < \varepsilon(1 + c)$$

and

$$\begin{aligned} |I(f_1 + f_2) - I(f_1) - I(f_2)| &\leq |I_\varphi(f_1 + f_2) - I_\varphi(f_1) - I_\varphi(f_2)| \\ &+ |I(f_1 + f_2) - I_\varphi(f_1 + f_2)| + |I(f_1) - I_\varphi(f_1)| + |I(f_2) - I_\varphi(f_2)| < 4\varepsilon. \end{aligned}$$

As ε is arbitrary, this implies that $I(L_x f_1) = I(f_1)$, $I(cf_1) = cI(f_1)$ and $I(f_1 + f_2) = I(f_1) + I(f_2)$.

Now we extend I to a linear functional $\mathcal{C}_c(G) \rightarrow \mathbb{C}$, that we will still denote by I . Let $f \in \mathcal{C}_c(G)$. Then we can write $f = (f_1 - f_2) + i(g_1 - g_2)$, with $f_1, f_2, g_1, g_2 \in \mathcal{C}_c^+(G) \cup \{0\}$ (for example, take $f_1 = \max(0, \text{Re}(f))$, $f_2 = \max(0, -\text{Re}(f))$, $g_1 = \max(0, \text{Im}(f))$ and $g_2 = \max(0, -\text{Im}(f))$). We set $I(f) = I(f_1) - I(f_2) + i(I(g_1) - I(g_2))$ (with the convention that $I(0) = 0$). If $f = (F_1 - F_2) + i(G_1 - G_2)$, with $F_1, F_2, G_1, G_2 \in \mathcal{C}_c^+(G) \cup \{0\}$, then $F_1 + f_2 = F_2 + f_1$ and $G_1 + g_2 = G_2 + g_1$, so we get the same result for $I(f)$. Also, it is easy to check that I is a linear functional from $\mathcal{C}_c(G)$ to \mathbb{C} , and it is positive by construction. By the Riesz representation theorem, there exists a regular Borel measure μ on G such that $I(f) = \int_G f d\mu$. By proposition I.2.6, this measure is a left Haar measure.

We now prove the second statement of the theorem (uniqueness of left Haar measure up to a constant). Let μ_1, μ_2 be two left Haar measures on G . By the uniqueness in the Riesz representation theorem (and the fact that $\mathcal{C}_c^+(G)$ generates $\mathcal{C}_c(G)$) it suffices to find a positive real number c such that $\int f d\mu_1 = c \int f d\mu_2$ for every $f \in \mathcal{C}_c^+(G)$. By proposition I.2.6, we have $\int_G f d\mu_2 > 0$ for every $f \in \mathcal{C}_c^+(G)$. So it suffices to show that, if $f, g \in \mathcal{C}_c^+(G)$, we have

$$\frac{\int f d\mu_1}{\int f d\mu_2} = \frac{\int g d\mu_1}{\int g d\mu_2} \quad (*).$$

Let $f, g \in \mathcal{C}_c^+(G)$. Let V_0 be a symmetric compact neighborhood of 1, and set

$$A = (\text{supp}(f))V_0 \cup V_0(\text{supp}(f))$$

and

$$B = (\text{supp}(g))V_0 \cup V_0(\text{supp}(g)).$$

Then A and B are compact by proposition I.1.7. If $y \in V_0$, the functions $x \mapsto f(xy) - f(yx)$ and $x \mapsto g(xy) - g(yx)$ are supported on A and B respectively.

Let $\varepsilon > 0$. By proposition I.1.12, there exists a symmetric neighborhood $V \subset V_0$ of 1 such that, for every $x \in G$ and every $y \in V$, we have $|f(xy) - f(yx)| < \varepsilon$ and $|g(xy) - g(yx)| < \varepsilon$.

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Let $h \in \mathcal{C}_c^+(G)$ be such that $\text{supp}(h) \subset V$ and $h(x) = h(x^{-1})$ for every $x \in G$. Then

$$\begin{aligned} \left(\int_G h d\mu_2 \right) \left(\int_G f d\mu_1 \right) &= \int_{G \times G} h(y) f(x) d\mu_1(x) d\mu_2(y) \\ &= \int_{G \times G} h(y) f(yx) d\mu_1(x) d\mu_2(y). \end{aligned}$$

(We use the left invariance of μ_1 . Also, we can apply Fubini's theorem, because all the functions are supported on compact sets, and compact sets have finite measure.) Similarly, we have

$$\begin{aligned} \left(\int_G h d\mu_1 \right) \left(\int_G f d\mu_2 \right) &= \int_{G \times G} h(x) f(y) d\mu_1(x) d\mu_2(y) \\ &= \int_{G \times G} h(y^{-1}x) f(y) d\mu_1(x) d\mu_2(y) \\ &= \int_{G \times G} h(x^{-1}y) f(y) d\mu_1(x) d\mu_2(y) \\ &= \int_{G \times G} h(y) f(xy) d\mu_1(x) d\mu_2(y). \end{aligned}$$

Hence

$$\begin{aligned} \left| \left(\int_G h d\mu_1 \right) \left(\int_G f d\mu_2 \right) - \left(\int_G h d\mu_2 \right) \left(\int_G f d\mu_1 \right) \right| &= \left| \int_{G \times G} h(y) (f(xy) - f(yx)) d\mu_1(x) d\mu_2(y) \right| \\ &\leq \varepsilon \mu_1(A) \int_G h d\mu_2, \end{aligned}$$

as $\text{supp}(h) \subset V$. Dividing by $(\int_G f d\mu_2)(\int_G h d\mu_2)$, we get

$$\left| \left(\int_G h d\mu_1 \right) \left(\int_G h d\mu_2 \right)^{-1} - \left(\int_G f d\mu_1 \right) \left(\int_G f d\mu_2 \right)^{-1} \right| \leq \varepsilon \mu_1(A) \left(\int_G f d\mu_2 \right)^{-1}.$$

Similarly, we have

$$\left| \left(\int_G h d\mu_1 \right) \left(\int_G h d\mu_2 \right)^{-1} - \left(\int_G g d\mu_1 \right) \left(\int_G g d\mu_2 \right)^{-1} \right| \leq \varepsilon \mu_1(B) \left(\int_G g d\mu_2 \right)^{-1}.$$

Taking the sum gives

$$\left| \frac{\int_G f d\mu_1}{\int_G f d\mu_2} - \frac{\int_G g d\mu_1}{\int_G g d\mu_2} \right| \leq \varepsilon \left(\frac{\mu_1(A)}{\int_G f d\mu_2} + \frac{\mu_1(B)}{\int_G g d\mu_2} \right).$$

As ε is arbitrary, this gives the desired equality (*).

□

We now want to compare left and right Haar measures.

Proposition I.2.8. *Let G be a locally compact group. Let $x \in G$. Then there exists $\Delta(x) \in \mathbb{R}_{>0}$ such that, for every left Haar measure μ on G , we have $\mu(Ex) = \Delta(x)\mu(E)$. Moreover, $\Delta : G \rightarrow \mathbb{R}_{>0}$ is a continuous group homomorphism (where the group structure on $\mathbb{R}_{>0}$ is given by multiplication) and, for every left Haar measure μ on G , every $x \in G$ and every $f \in L^1(\mu)$, we have*

$$\int_G R_x f d\mu = \Delta(x^{-1}) \int_G f d\mu.$$

Proof. Let $x \in G$, and μ be a left Haar measure on G . Then the measure μ_x defined by $\mu_x(E) = \mu(Ex)$ is also a left Haar measure on G , so, by the uniqueness statement in theorem I.2.7, there exists $\Delta(x) \in \mathbb{R}_{>0}$ such that $\mu_x = \Delta(x)\mu$, that is, $\mu(Ex) = \Delta(x)\mu(E)$ for every Borel subset E of G . Suppose that λ is another left Haar measure on G . Then, again by theorem I.2.7, there exists $c > 0$ such that $\lambda = c\mu$, and so we get, for every Borel subset E of G ,

$$\lambda(Ex) = c\mu(Ex) = c\Delta(x)\mu(E) = \Delta(x)\lambda(E).$$

This proves the first statement.

We prove that Δ is a morphism of groups. Let $x, y \in G$, and let E be a Borel subset of G such that $\mu(E) \neq 0$. Then

$$\Delta(xy)\mu(E) = \mu(Exy) = \Delta(y)\mu(Ey) = \Delta(y)\Delta(x)\mu(E),$$

hence $\Delta(xy) = \Delta(x)\Delta(y)$.

We now prove the last statement. If E is a Borel subset of G and $x \in G$, then $R_x \mathbf{1}_E = \mathbf{1}_{Ex^{-1}}$, so we get

$$\int_G R_x \mathbf{1}_E d\mu = \mu(Ex^{-1}) = \Delta(x^{-1})\mu(E) = \Delta(x)^{-1} \int_G \chi_E d\mu$$

by definition of Δ . This proves the result for $f = \chi_E$. The general case follows by approximating f by linear combinations of functions $\mathbf{1}_E$.

Finally, we prove that Δ is continuous. Let $f \in \mathcal{C}_c^+(G)$. We know that the function $G \rightarrow \mathbb{C}$, $x \mapsto R_{x^{-1}}f$ is continuous (see remark I.1.13), so the function $G \rightarrow \mathbb{C}$, $x \mapsto \int_G R_{x^{-1}}f d\mu$ is also continuous. But we have just seen that $\int_G R_{x^{-1}}f d\mu = \Delta(x)^{-1} \int_G f d\mu$, and we know that $\int_G f d\mu > 0$ by proposition I.2.6. Hence Δ is continuous. □

Definition I.2.9. The function Δ of the previous proposition is called the *modular function* of G . We say that the group G is *unimodular* if $\Delta = 1$ (that is, if some (or any) left Haar measure on G is also a right Haar measure).

Remark I.2.10. Suppose that $\alpha : G \rightarrow G$ is a homeomorphism such that for every $x \in G$, we have $\beta(x) \in G$ satisfying : for every $y \in G$, $\alpha(xy) = \beta(x)\alpha(y)$. (For example, α could be right translation by a fixed element of G , or a continuous group isomorphism with continuous inverse.)

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Then we can generalize the construction of proposition I.2.8 to get a $\Delta(\alpha) \in \mathbb{R}_{>0}$ satisfying : for every $f \in \mathcal{C}_c(G)$, for every left Haar measure μ on G ,

$$\Delta(\alpha) \int_G f(\alpha(x)) d\mu(x) = \int_G f(x) d\mu(x)$$

(or equivalently $\mu(\alpha(E)) = \Delta(\alpha)\mu(E)$ for every Borel subset E of G). Moreover, if $\beta : G \rightarrow G$ satisfies the same conditions as α , then so does $\alpha \circ \beta$ and we have $\Delta(\alpha \circ \beta) = \Delta(\alpha)\Delta(\beta)$.

Example I.2.11. 1. Any compact group is unimodular. Indeed, if G is compact, then $\Delta(G)$ is a compact subgroup of $\mathbb{R}_{>0}$, but the only compact subgroup of $\mathbb{R}_{>0}$ is $\{1\}$. In particular, a compact group G has a unique left and right Haar measure μ such that $\mu(G) = 1$; we call this measure the *normalized Haar measure* of G .

2. Any discrete group is unimodular. Indeed, if G is discrete, then $\Delta(G)$ is a discrete subgroup of $\mathbb{R}_{>0}$, but the only discrete subgroup of $\mathbb{R}_{>0}$ is $\{1\}$.

Of course, in this case, we already knew the result, because we have a left Haar measure on G that is also a right Haar measure : the counting measure.

3. If G is commutative, then left and right translations are equal on G , so G is unimodular.

4. The groups $\mathrm{GL}_n(\mathbb{R})$ and $\mathrm{GL}_n(\mathbb{C})$ are unimodular.

5. The group of invertible upper triangular matrices in $M_2(\mathbb{R})$ is not unimodular (see problem set 1). In fact, its modular function is

$$\Delta : \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \longmapsto |ac^{-1}|.$$

6. Remember the commutator subgroup $[G, G]$ is the subgroup generated by all the $xyx^{-1}y^{-1}$, for $x, y \in G$. It is a normal subgroup of G , and every group morphism from G to a commutative group is trivial on $[G, G]$. In particular, the modular function Δ is trivial on $[G, G]$, so G is unimodular if $G = [G, G]$. More generally, using the first example, we see that G is unimodular if the quotient group $G/[G, G]$ is compact.

Proposition I.2.12. Let G be a locally compact group, and let μ be a left Haar measure on G . We define a right Haar measure ν on G by $\nu(E) = \mu(E^{-1})$ (see proposition I.2.6).

Then, for every $f \in \mathcal{C}_c(G)$, we have

$$\int_G f(x^{-1}) d\mu_G(x) = \int_G f(x) d\nu(x) = \int_G \Delta(x^{-1}) f(x) d\mu(x).$$

We also write this property as $d\nu(x) = \Delta(x^{-1})d\mu(x)$, or $d\mu_G(x^{-1}) = \Delta(x^{-1})d\mu_G(x)$.

Proof. We prove the first equality. It is actually true for every $f \in L^1(G)$. If f is characteristic function of a Borel subset E , then $x \mapsto f(x^{-1})$ is the characteristic function of E^{-1} , so $\int f(x^{-1})d\mu(x) = \int f d\nu$ by definition of ν . We get the general result by approximation f by linear combination of characteristic functions of Borel subsets.

We prove the second equality. Consider the linear function $\Lambda : \mathcal{C}_c(G) \rightarrow \mathbb{C}$, $f \mapsto \int_G \Delta(x^{-1})f(x)d\mu(x)$. As Δ takes its values in $\mathbb{R}_{>0}$, Λ is positive. Also, for every $y \in G$, we have

$$\Lambda(R_y f) = \int_G f(xy)\Delta(x^{-1})d\mu(x) = \Delta(y) \int_G f(xy)\Delta((xy)^{-1})d\mu(x) = \int_G f(x)\Delta(x^{-1})d\mu(x) = \Lambda(f)$$

(using the left invariance of μ and the fact that Δ is morphism of groups). So the unique regular Borel measure ρ that corresponds to Λ by the Riesz representation theorem is a right Haar measure (see proposition I.2.6). By theorem I.2.7, there exists $c > 0$ such that $\rho = c\nu$. To finish the proof, it suffices to show that $c = 1$. Suppose that $c \neq 1$. Then we can find a compact symmetric neighborhood U of 1 such that, for every $x \in U$, we have $|\Delta(x^{-1}) - 1| \leq \frac{1}{2}|c - 1|$. As U is symmetric, we have $\mu(U) = \nu(U)$, hence

$$|c - 1|\mu(U) = |c\nu(U) - \mu(U)| = \left| \int_U (\Delta(x^{-1}) - 1)d\mu(x) \right| \leq \frac{1}{2}|c - 1|\mu(U),$$

which contradicts the fact that $\mu(U) \neq 0$ (by proposition I.2.6).

□

I.3 Representations

In this section, G is a topological group.

I.3.1 Continuous representations

Definition I.3.1.1. If V and W are normed \mathbb{C} -vector spaces, we denote by $\text{Hom}(V, W)$ the \mathbb{C} -vector space of bounded linear operators from V to W , and we put on it the topology given by the operator norm $\|\cdot\|_{op}$. We also write $\text{End}(V)$ for $\text{Hom}(V, V)$, and $\text{GL}(V)$ for $\text{End}(V)^\times$, with the topology induced by that of $\text{End}(V)$.

Definition I.3.1.2. Let V be a normed \mathbb{C} -vector space. Then a (*continuous*) *representation* of G on V is a group morphism ρ from G to the group of \mathbb{C} -linear automorphisms of V such that the action map $G \times V \rightarrow V$, $(g, v) \mapsto \rho(g)(v)$, is continuous.

We refer to the representation by (ρ, V) , ρ or often simply by V . Sometimes, we don't explicitly name the map ρ and write the action of G on V as $(g, v) \mapsto gv$.

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Remark I.3.1.3. - The definition makes sense if V is any topological vector space (over a topological field).

- If (ρ, V) is a continuous representation of G , then the action of every $g \in G$ on V is a continuous endomorphism of V , so we get a group morphism $\rho : G \rightarrow \text{GL}(V)$. But this morphism is not necessarily continuous, unless V is finite-dimensional (see proposition I.3.5.1). An example of this is given by the representations of G on $L^p(G)$ defined below.
- If $\rho : G \rightarrow \text{GL}(V)$ is a morphism of groups that is continuous for the weak* topology on $\text{End}(V)$, then it is not necessarily a continuous representation. (For example, take $G = \text{GL}(V)$, with the topology induced by the weak* topology on $\text{End}(V)$, and $\rho = \text{id}$. This is not a continuous representation of G on V .)

Example I.3.1.4. - The *trivial representation* of G on V is the representation given by $\rho(x) = \text{id}_V$ for every $x \in G$. (It is a continuous representation.)

- The identity map of $\text{GL}(V)$ is a continuous representation of $\text{GL}(V)$ on V .
- If $G = S^1$ and $n \in \mathbb{Z}$, the map $G \rightarrow \mathbb{C}, z \mapsto z^n$ is a continuous representation of G on \mathbb{C} .
- The map $\rho : \mathbb{R} \rightarrow \text{GL}_2(\mathbb{C}), x \mapsto \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$ is a continuous representation of \mathbb{R} on \mathbb{C}^2 .
- See example I.3.1.11 for the representations of G on its function spaces.

Definition I.3.1.5. Let (ρ_1, V_1) and (ρ_2, V_2) be two representations of G . An *intertwining operator* (or *G -equivariant map*) from V_1 to V_2 is a bounded \mathbb{C} -linear map $T : V_1 \rightarrow V_2$ such that, for every $g \in G$ and every $v \in V_1$, we have $T(\rho_1(g)v) = \rho_2(g)T(v)$.

We write $\text{Hom}_G(V_1, V_2)$ for the space of intertwining operators from V_1 to V_2 , and $\text{End}_G(V_1)$ for the space of intertwining operators from V_1 to itself.

We say that the representations (ρ_1, V_1) and (ρ_2, V_2) are *isomorphic* (or *equivalent*) if there exists intertwining operators $T : V_1 \rightarrow V_2$ and $T' : V_2 \rightarrow V_1$ such that $T' \circ T = \text{id}_{V_1}$ and $T \circ T' = \text{id}_{V_2}$.

Definition I.3.1.6. Let (ρ, V) be a representation of V .

1. A *subrepresentation* of V (or *G -invariant subspace*) is a linear subspace W such that, for every $g \in G$, we have $\rho(g)(W) \subset W$.
2. The representation (ρ, V) is called *irreducible* if $V \neq 0$ and if its only closed G -invariant subspaces are 0 and V . Otherwise, the representation is called *reducible*.
3. The representation (ρ, V) is called *indecomposable* if, whenever $V = W_1 \oplus W_2$ with W_1 and W_2 two closed G -invariant subspaces of V , we have $W_1 = 0$ or $W_2 = 0$.
4. The representation (ρ, V) is called *semisimple* if there exists a family $(W_i)_{i \in I}$ of closed G -invariant subspaces of V that are in direct sum and such that $\bigoplus_{i \in I} W_i$ is dense in V . (If I is finite, the direct sum is also closed in V , so this implies that $V = \bigoplus_{i \in I} W_i$.)

Remark I.3.1.7. If (ρ, V) is a representation of G and $W \subset V$ is a G -stable subspace, then its closure \overline{W} is also stable by G .

Example I.3.1.8. The representation ρ of \mathbb{R} on \mathbb{C}^2 given by $\rho(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$ is indecomposable but not irreducible.

Lemma I.3.1.9. Let (ρ_1, V_1) and (ρ_2, V_2) be two representations of G , and let $T : V_1 \rightarrow V_2$ be an intertwining operator. Then $\text{Ker}(T)$ is a subrepresentation of V_1 , and $\text{Im}(T)$ is a subrepresentation of V_2 .

Proof. Let $v \in \text{Ker}(T)$ and $g \in G$. Then $T(\rho_1(g)(v)) = \rho_2(g)(T(v)) = 0$, so $\rho_1(g)(v) \in \text{Ker}(T)$.

Now let $w \in \text{Im}(T)$, and choose $v \in V_1$ such that $w = T(v)$. Then $\rho_2(g)(w) = T(\rho_1(g)(v)) \in \text{Im}(T)$.

□

Proposition I.3.1.10. Let V be a normed vector space and $\rho : G \rightarrow \text{End}(V)$ be a multiplicative map. We denote by $\|\cdot\|_{op}$ the operator norm on $\text{End}(V)$. Suppose that :

- (a) For every $g \in G$, we have $\|\rho(g)\|_{op} \leq 1$;
- (b) For every $v \in V$, the map $G \rightarrow V, g \mapsto \rho(g)(v)$ is continuous.

Then (ρ, V) is a continuous representation of G .

Proof. Let $g_0 \in G, v_0 \in V$, and $\varepsilon > 0$. We want to find a neighborhood U of g in G and a $\delta > 0$ such that : $g \in U$ and $\|v - v_0\| < \delta \Rightarrow \|\rho(g)(v) - \rho(g_0)(v_0)\| < \varepsilon$.

Choose a neighborhood U of g in G such that : $g \in U \Rightarrow \|\rho(g)(v_0) - \rho(g_0)(v_0)\| < \varepsilon/2$, and take $\delta = \varepsilon/2$. Then, if $g \in U$ and $\|v - v_0\| < \delta$, we have

$$\begin{aligned} \|\rho(g)(v) - \rho(g_0)(v_0)\| &\leq \|\rho(g)(v) - \rho(g)(v_0)\| + \|\rho(g)(v_0) - \rho(g_0)(v_0)\| \\ &< \|\rho(g)\| \|v - v_0\| + \varepsilon/2 \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon, \end{aligned}$$

because $\|\rho(g)\| \geq 1$.

□

Example I.3.1.11. 1. We have defined, for every $x \in G$, two endomorphisms L_x and R_x of the space of functions on G , and these endomorphisms preserve $\|\cdot\|_\infty$. So, by proposition I.3.1.10 and remark I.1.13, they define two representations of G on $\mathcal{C}_c(G)$.

2. Suppose that G is locally compact Hausdorff. We fix a left Haar measure dx on G , and we denote $L^p(G)$ the L^p spaces for this measure, for $1 \leq p \leq \infty$. The left invariance of the measure implies that the operators L_x preserve the L^p norm, so we get a \mathbb{C} -linear left

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action of G on $L^p(G)$, and, by proposition I.3.1.10, to show that it is a representation, we just need to show that, if $f \in L^p(G)$, the map $G \rightarrow L^p(G)$, $x \mapsto L_x f$ is continuous. This is not necessarily true if $p = \infty$, but it is for $1 \leq p < \infty$, by proposition I.3.1.13 below. So we get a representation of G on $L^p(G)$ for $1 \leq p < \infty$.

If we chose instead a right Haar measure on G , then the operators R_x would define a representation of G on $L^p(G)$ for $1 \leq p < \infty$. So, if G is unimodular, we get two commuting representations of G on $L^p(G)$.

Definition I.3.1.12. Let G be a locally compact group with a left (resp. right) Haar measure dx , and let $L^2(G)$ be the corresponding L^2 space. The representation of G on $L^2(G)$ given by the operators L_x (resp. R_x) is called the *left* (resp. *right*) regular representation of G .

Proposition I.3.1.13. Let G be a locally compact group, let μ be a left Haar measure on G , and let $L^p(G)$ be the corresponding L^p space. Suppose that $1 \leq p < \infty$.

Then, for every $f \in L^p(G)$, we have $\|L_x f - f\|_p \rightarrow 0$ and $\|R_x f - f\|_p \rightarrow 0$ as $x \rightarrow 1$.

Proof. Suppose first that $f \in \mathcal{C}_c(G)$, and fix a compact neighborhood V of 1. Then $K := V(\text{supp } f) \cup (\text{supp } f)V$ is compact by proposition I.1.7, so $\mu(K) < +\infty$. For every $x \in V$, we have $\text{supp}(f), \text{supp}(L_x f), \text{supp}(R_x f) \subset K$, so $\|L_x f - f\|_p \leq \mu(K)^{1/p} \|L_x f - f\|_\infty$ and $\|R_x f - f\|_p \leq \mu(K)^{1/p} \|R_x f - f\|_\infty$. The result then follows from proposition I.1.12.

Now let f be any element of $L^p(G)$. We still fix a compact neighborhood V of 1, and we set $C = \sup_{x \in V} \Delta(x)^{-1/p}$. Let $\varepsilon > 0$. There exists $g \in \mathcal{C}_c(G)$ such that $\|f - g\|_p < \varepsilon$. Then we have, for $x \in V$,

$$\|L_x f - f\|_p \leq \|L_x(f - g)\|_p + \|L_x g - g\|_p + \|g - f\|_p \leq 2\varepsilon + \|L_x g - g\|_p$$

(as $\|L_x(f - g)\|_p = \|f - g\|_p$) and

$$\|R_x f - f\|_p \leq \|R_x(f - g)\|_p + \|R_x g - g\|_p + \|g - f\|_p \leq (1 + C)\varepsilon + \|R_x g - g\|_p$$

(as $\|R_x(f - g)\|_p = \Delta(x)^{-1/p} \|f - g\|_p$). We have seen in the first part of the proof that $\|L_x g - g\|_p$ and $\|R_x g - g\|_p$ tend to 0 as x tends to 1, so we can find a neighborhood $U \subset V$ of 1 such that $\|L_x f - f\|_p \leq 3\varepsilon$ and $\|R_x f - f\|_p \leq (2 + C)\varepsilon$ for $x \in U$.

□

I.3.2 Unitary representations

Remember that a (*complex*) *Hilbert space* is a \mathbb{C} -vector space V with a Hermitian inner product³ such that V is complete for the corresponding norm. If V is a finite-dimensional \mathbb{C} -vector space with a Hermitian inner product, then it is automatically complete, hence a Hilbert space. We will usually denote the inner product on all Hermitian inner product spaces by $\langle \cdot, \cdot \rangle$.

³We will always assume Hermitian inner products to be \mathbb{C} -linear in the first variable.

Notation I.3.2.1. Let V and W be Hermitian inner product spaces. For every continuous \mathbb{C} -linear map $T : V \rightarrow W$, we write $T^* : W \rightarrow V$ for the adjoint of T , if it exists. Remember that we have $\langle T(v), w \rangle = \langle v, T^*(w) \rangle$ for every $v \in V$ and $w \in W$, and that T^* always exists if V and W are Hilbert spaces.

If V' is a subspace of V , we write V'^\perp for the orthogonal of V' ; it is defined by

$$(V')^\perp = \{v \in V \mid \forall v' \in V', \langle v, v' \rangle = 0\}.$$

Finally, we write $U(V)$ for the group of unitary endomorphisms of V , that is, of endomorphisms T of V that preserve the inner product ($\langle T(v), T(w) \rangle = \langle v, w \rangle$ for all $v, w \in V$). A unitary endomorphism T is automatically bounded and invertible (with inverse equal to T^*).

The following result is an immediate corollary of proposition I.3.1.10 (and of the fact that unitary operators have norm 1).

Corollary I.3.2.2. *If V is a Hilbert space and $\rho : G \rightarrow U(V)$ is a morphism of groups, then the following are equivalent :*

1. *The map $G \times V \rightarrow V$, $(g, v) \mapsto \rho(g)(v)$, is continuous.*
2. *For every $v \in V$, the map $G \rightarrow V$, $g \mapsto \rho(g)(v)$, is continuous.*

Definition I.3.2.3. If V is a Hilbert space, a *unitary representation* of G on V is a morphism of groups $\rho : G \rightarrow U(V)$ satisfying the conditions of the proposition above.

These representations are our main object of study.

Example I.3.2.4. If (X, μ) is any measure space, then $L^2(X)$ is a Hilbert space, with the following inner product :

$$\langle f, g \rangle = \int_X f(x) \overline{g(x)} d\mu(x).$$

So if G is a locally compact group, then the left regular representation and right regular representations of G are unitary representations of G (on the same space if G is unimodular).

Remark I.3.2.5. Note that ρ is still not necessarily a continuous map in general. (Unless $\dim_{\mathbb{C}} V < +\infty$.) For example, it is not continuous for the left regular representation of S^1 .

Also, note that we don't need the completeness of V in the proof, so the proposition is actually true for any Hermitian inner product space.

Lemma I.3.2.6. *Let (ρ, V) be a unitary representation of G . Then, for every G -invariant subspace W of V , the subspace W^\perp is also G -invariant.*

In particular, if W is a *closed* G -invariant subspace of V , then we have $V = W \oplus W^\perp$ with W^\perp a closed G -invariant subspace.

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Proof. Let $v \in W^\perp$ and $g \in G$. Then, for every $w \in W$, we have

$$\langle \rho(g)(v), w \rangle = \langle v, \rho(g)^{-1}w \rangle = 0$$

(the last equality comes from the fact that $\rho(g)^{-1}w \in W$), hence $\rho(g)(v) \in W^\perp$. □

Lemma I.3.2.7. *Let (ρ_1, V_1) and (ρ_2, V_2) be two unitary representations of G , and let $T : V_1 \rightarrow V_2$ be an intertwining operator. Then $T^* : V_2 \rightarrow V_1$ is also an intertwining operator.*

Proof. Let $w \in V_2$ and $g \in G$. Then, for every $v \in V_1$, we have

$$\begin{aligned} \langle v, T^*(\rho_2(g)(w)) \rangle &= \langle T(v), \rho_2(g)(w) \rangle = \langle \rho_2(g)^{-1}T(v), w \rangle = \langle T(\rho_1(g)^{-1}(v)), w \rangle = \\ &= \langle \rho_1(g)^{-1}(v), T^*(w) \rangle = \langle v, \rho_1(g)T^*(w) \rangle. \end{aligned}$$

So $T^*(\rho_2(g)(w)) = \rho_1(g)(T^*(w))$. □

Theorem I.3.2.8. *Assume that the group G is compact Hausdorff. Let $(V, \langle \cdot, \cdot \rangle_0)$ be a Hilbert space and $\rho : G \rightarrow \text{GL}(V)$ be a continuous representation of G on V . Then there exists a Hermitian inner product $\langle \cdot, \cdot \rangle$ on V satisfying the following properties :*

1. *There exist real numbers $c, C > 0$ such that, for every $v \in V$, we have $c|\langle v, v \rangle_0| \leq |\langle v, v \rangle| \leq C|\langle v, v \rangle_0|$. In other words, the norms coming from the two inner products are equivalent, and so V is still a Hilbert space for the inner product $\langle \cdot, \cdot \rangle$.*
2. *The representation ρ is unitary for the inner product $\langle \cdot, \cdot \rangle$.*

Remark I.3.1. (a) If V is irreducible, it follows from Schur's lemma (see theorem I.3.4.1) that this inner product is unique up to a constant.

- (b) This is false for noncompact groups. For example, consider the representation ρ of \mathbb{R} on \mathbb{C}^2 given by $\rho(t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$. There is no inner product on \mathbb{C}^2 that makes this representation unitary (otherwise $\rho(\mathbb{R})$ would be a closed subgroup of the unitary group of this inner product, hence compact, but this impossible because $\rho(\mathbb{R}) \simeq \mathbb{R}$).

Proof of the theorem. We define $\langle \cdot, \cdot \rangle : V \times V \rightarrow \mathbb{C}$ by the following formula : for all $v, w \in V$,

$$\langle v, w \rangle = \int_G \langle \rho(g)v, \rho(g)w \rangle_0 dg,$$

where dg is a normalized Haar measure on G . This defines a Hermitian form on V , and we have $\langle \rho(g)v, \rho(g)w \rangle = \langle v, w \rangle$ for every $v, w \in V$ and $g \in G$ by left invariance of the measure.

If we prove property (1), it will also imply that $\langle \cdot, \cdot \rangle$ is definite (hence an inner product), and so we will be done. Let $v \in V$. Then the two maps $G \rightarrow V$ sending v to $\rho(g)(v)$ and to $\rho(g)^{-1}(v)$ are continuous. As G is compact, they are both bounded. By the uniform boundedness principle (theorem I.3.2.11), there exist $A, B \in \mathbb{R}_{>0}$ such that $\|\rho(g)^{-1}\|_{op} \leq A$ and $\|\rho(g)\|_{op} \leq B$ for every $g \in G$. By the submultiplicativity of the operator norm, the first inequality implies that $\|\rho(g)\| \leq A^{-1}$ for every $g \in G$. So the definition of $\langle \cdot, \cdot \rangle$ (and the fact that G has volume 1) gives property (1), with $c = A^{-2}$ and $C = B^2$.

□

Corollary I.3.2.9. *If G is compact Hausdorff, then every nonzero finite-dimensional continuous representation of G is semisimple.*

Proof. We may assume that the representation is unitary by the theorem. We prove the corollary by induction on $\dim V$. The result is obvious if $\dim V \leq 1$, so assume that $\dim V \geq 2$ and that we know the result for all spaces of strictly smaller dimension. If V is irreducible, we are done. Otherwise, there is a G -invariant subspace $W \subsetneq V$ such that $W \neq 0$. This subspace is closed because it is finite-dimensional, and we have $V = W \oplus W^\perp$ with W^\perp invariant by lemma I.3.2.6. As $\dim(W), \dim(W^\perp) < \dim(V)$, we can apply the induction hypothesis to W and W^\perp and conclude that they are semisimple. But then their direct sum V is also semisimple.

□

Remark I.3.2.10. This is still true (but harder to prove) for infinite-dimensional unitary representations of compact groups, but it is false for infinite-dimensional unitary representations of noncompact groups, or for finite-dimensional (non-unitary) representations of noncompact groups.

Theorem I.3.2.11 (Uniform boundedness principle or Banach-Steinhaus theorem). *Let V and W be normed vector spaces, and suppose that V is a Banach space (i.e. that it is complete for the metric induced by its norm). Let $(T_i)_{i \in I}$ be a family of bounded linear operators from V to W .*

If the family $(T_i)_{i \in I}$ is pointwise bounded (that is, if $\sup_{i \in I} \|T_i(v)\| < +\infty$ for every $v \in V$), then it is bounded (that is, $\sup_{i \in I} \|T_i\|_{op} < +\infty$).

Proof. ⁴ Suppose that $\sup_{i \in I} \|T_i\|_{op} = +\infty$, and choose a sequence $(i_n)_{n \geq 0}$ of elements of I such that $\|T_{i_n}\|_{op} \geq 4^n$. We define a sequence $(v_n)_{n \geq 0}$ of elements of V in the following way :

- $v_0 = 0$;
- For $n \geq 1$, we can find, thanks to the lemma below, an element v_n of V such that $\|v_n - v_{n-1}\| \leq 3^{-n}$ and $\|T_{i_n}(v_n)\| \geq \frac{2}{3} 3^{-n} \|T_{i_n}\|_{op}$.

⁴Taken from a paper of Alan Sokal.

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We have $\|v_n - v_m\| \leq \frac{1}{2}3^{-n}$ for $m \geq n$, so the sequence $(v_n)_{n \geq 0}$ is a Cauchy sequence; as V is complete, it has a limit v , and we have $\|v_n - v\| \leq \frac{1}{2}3^{-n}$ for every $n \geq 0$. The inequality $\|T_{i_n}(v_n)\| \geq \frac{2}{3}3^{-n}\|T_{i_n}\|_{op}$ and the triangle inequality now imply that $\|T_{i_n}(x)\| \geq \frac{1}{6}3^{-n}\|T_{i_n}\|_{op} \geq \frac{1}{6}(\frac{4}{3})^n$, and so the sequence $(\|T_{i_n}(x)\|)_{n \geq 0}$ is unbounded, which contradicts the hypothesis.

□

Lemma I.3.2.12. *Let V and W be two normed vector spaces, and let $T : V \rightarrow W$ be a bounded linear operator. Then for any $v \in V$ and $r > 0$, we have*

$$\sup_{v' \in B(v, r)} \|T(v')\| \geq r\|T\|_{op},$$

where $B(v, r) = \{v' \in V \mid \|v - v'\| < r\}$.

Proof. For every $x \in V$, we have

$$\|T(x)\| \leq \frac{1}{2}(\|T(v+x)\| + \|T(v-x)\|) \leq \max(\|T(v+x)\|, \|T(v-x)\|).$$

Taking the supremum over $x \in B(0, r)$ gives the inequality of the lemma.

□

Finally, we have the following result, whose proof uses the spectral theorem.

Theorem I.3.2.13. *If G is a compact group, then every irreducible unitary representation is finite-dimensional.*

Proof. See problem set 5.

□

I.3.3 Cyclic representations

Definition I.3.3.1. Let (ρ, V) be a continuous representation of G , and let $v \in V$. Then the closure W of $\text{Span}\{\rho(g)(v), g \in G\}$ is a subrepresentation of V , called the *cyclic subspace* generated by v .

If $V = W$, we say that V is a *cyclic representation* and that v is a *cyclic vector* for V .

Example I.3.3.2. An irreducible representation is cyclic, and every nonzero vector is a cyclic vector for it.

The converse is not true. For example, consider the representation ρ of the symmetric group \mathfrak{S}_n on \mathbb{C}^n defined by $\rho(\sigma)(x_1, \dots, x_n) = (x_{\sigma^{-1}(1)}, \dots, x_{\sigma^{-1}(n)})$, and let $v = (1, 0, \dots, 0) \in \mathbb{C}^n$. Then the set $\rho(\mathfrak{S}_n)(v)$ is the canonical basis of \mathbb{C}^n , hence it generates \mathbb{C}^n , and so v is a cyclic vector for ρ . But ρ is not irreducible, because $\mathbb{C}(1, 1, \dots, 1)$ is a subrepresentation.

Proposition I.3.3.3. *Every unitary representation of G is a direct sum of cyclic representations.*

If the indexing set is infinite, we understand the direct sum to be the closed direct sum (that is, the closure of the algebraic direct sum).

Proof. Let (π, V) be a unitary representation of G . By Zorn's lemma, we can find a maximal collection $(W_i)_{i \in I}$ of pairwise orthogonal cyclic subspaces of V . Suppose that V is not the direct sum of the W_i , then there exists a nonzero vector $v \in (\bigoplus_{i \in I} W_i)^\perp$. By lemma I.3.2.6, the cyclic subspace generated by v is included in $(\bigoplus_{i \in I} W_i)^\perp$, which contradicts the maximality of the family $(W_i)_{i \in I}$. Hence $V = \overline{\bigoplus_{i \in I} W_i}$. □

I.3.4 Schur's lemma

The following theorem is fundamental. We will not be able to prove it totally until we have the spectral theorem for normal endomorphisms of Hilbert spaces.

Theorem I.3.4.1 (Schur's lemma). *Let (ρ_1, V_1) and (ρ_2, V_2) be two representations of G , and let $T : V_1 \rightarrow V_2$ be an intertwining operator.*

1. *If V_1 is irreducible, then T is either zero or injective.*
2. *If V_2 is irreducible, then T is zero or has dense image.*
3. *Suppose that V_1 is unitary. Then it is irreducible if and only if $\text{End}_G(V_1) = \mathbb{C} \cdot \text{id}_{V_1}$.*
4. *Suppose that V_1 and V_2 are unitary and irreducible. Then $\text{Hom}_G(V_1, V_2)$ is of dimension zero (if V_1 and V_2 are not isomorphic) or 1 (if V_1 and V_2 are isomorphic).*

Proof. We prove the first two points. By lemma I.3.1.9, $\text{Ker}(T)$ and $\text{Im}(T)$ are G -invariant subspaces of V_1 and V_2 . Moreover, $\text{Ker}(T)$ is a closed subspace of V_1 . If V_1 is irreducible, then its only closed invariant subspaces are 0 and V_1 ; this gives the first point. If V_2 is irreducible, then its only closed invariant subspaces are 0 and V_2 ; this gives the second point.

We prove the third point. Suppose first that V_1 is not irreducible. Then it has a closed invariant subspace W such that $0 \neq W \neq V_1$, and orthogonal projection on W is a G -equivariant endomorphism by lemma I.3.4.3. So $\text{End}_G(V_1)$ strictly contains $\mathbb{C} \cdot \text{id}_{V_1}$.

Now suppose that V_1 is irreducible, and let $T \in \text{End}_G(V_1)$. We want to show that $T \in \mathbb{C} \text{id}_{V_1}$. If V_1 is finite-dimensional, then T has an eigenvalue λ , and then $\text{Ker}(T - \lambda \text{id}_{V_1})$ is a nonzero G -invariant subspace of V_1 , hence equal to V_1 , and we get $T = \lambda \text{id}_{V_1}$. In general, we still know that every $T \in \text{End}(V)$ has a nonzero spectrum (by theorem II.1.1.3), but, if λ is in the spectrum of T , we only know that $T - \lambda \text{id}_V$ is not invertible, not that $\text{Ker}(T - \lambda \text{id}_V) \neq 0$. So we cannot apply the same strategy. Instead, we will use a corollary of the spectral theorem (theorem II.4.1).

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Note that the subgroup $\rho_1(G)$ of $\text{End}(V_1)$ satisfies the hypothesis of corollary II.4.4 because V_1 is irreducible, so its centralizer in $\text{End}(V_1)$ is equal to $\mathbb{C}\text{id}_{V_1}$; but this centralizer is exactly $\text{End}_G(V_1)$, so we are done.

We prove the fourth point. Let $T : V_1 \rightarrow V_2$ be an intertwining operator. Then $T^* : V_2 \rightarrow V_1$ is also an intertwining operator by lemma I.3.2.7, so $T^*T \in \text{End}_G(V_1)$ and $TT^* \in \text{End}_G(V_2)$. By the third point, there exists $c \in \mathbb{C}$ such that $T^*T = c\text{id}_{V_1}$. If $c \neq 0$, then T is injective and $\text{Im}(T)$ is closed (because $\|T(v)\| \geq \frac{|c|}{\|T^*\|_{op}}\|v\|$ for every $v \in V_1$, see lemma I.3.4.2), so T is an isomorphism by the second point, and its inverse $c^{-1}T^*$; hence V_1 and V_2 are isomorphic, and $\text{Hom}_G(V_1, V_2) \simeq \text{End}_G(V_1)$ is 1-dimensional. Suppose that $c = 0$. If $T \neq 0$, then it has dense image by the second point, but then $T^* = 0$ by the first point, hence $T = (T^*)^* = 0$, which is absurd; so $T = 0$. So we have proved that, if $\text{Hom}_G(V_1, V_2) \neq 0$, then V_1 and V_2 must be isomorphic; this finishes the proof of the fourth point. □

Lemma I.3.4.2. *Let V, W be two normed vector spaces, and let $T : V \rightarrow W$ be a bounded linear operator. Suppose that V is complete. If there exists $c > 0$ such that $\|T(v)\| \geq c\|v\|$ for every $v \in V$, then $\text{Im}(T)$ is closed.*

Proof. Let $(v_n)_{n \in \mathbb{N}}$ be a sequence of elements of V such that the sequence $(T(v_n))_{n \in \mathbb{N}}$ converges to a $w \in W$. We want to show that $w \in \text{Im}(T)$. Note that, for all $n, m \in \mathbb{N}$, we have $\|v_n - v_m\| \leq c^{-1}\|T(v_n) - T(v_m)\|$. This implies that $(v_n)_{n \in \mathbb{N}}$ is a Cauchy sequence, so it has a limit $v \in V$ because V is complete. As T is continuous, we have $w = \lim_{n \rightarrow +\infty} T(v_n) = T(v)$, so $w \in \text{Im}(T)$. □

Lemma I.3.4.3. *Let (ρ, V) be a unitary representation of G , let W be a closed subspace of V , and let π be the orthogonal projection on W , seen as a linear endomorphism of V .*

Then W is G -invariant if and only if π is G -equivariant.

Proof. Suppose that π is G -equivariant. Let $w \in W$ and $g \in G$. Then $\rho(g)(w) = \rho(g)(\pi(w)) = \pi(\rho(g)(w)) \in W$. So W is invariant by G .

Conversely, suppose that W is G -invariant. By lemma I.3.2.6, its orthogonal W^\perp is also invariant by G . Let $v \in V$ and $g \in G$. We write $w = \pi(g)$ and $w' = g - \pi(g)$. Then $\rho(g)(v) = \rho(g)(w) + \rho(g)(w')$ with $\rho(g)(w) \in W$ and $\rho(g)(w') \in W^\perp$, so $\pi(\rho(g)(v)) = \rho(g)(w)$. □

Corollary I.3.4.4. *If G is commutative, then every irreducible unitary representation of G is 1-dimensional.*

So each unitary irreducible representation of G is equivalent to one (and only one) continuous group morphism $G \rightarrow S^1$.

Proof. Let (ρ, V) be an irreducible unitary representation. As G is commutative, the operators $\rho(x)$ and $\rho(y)$ commute for all $x, y \in G$, so we have $\rho(x) \in \text{End}_G(V)$ for every $x \in G$. By Schur's lemma, this implies that $\rho(x) \in \mathbb{C} \cdot \text{id}_V$ for every $x \in G$. In particular, every linear subspace of V is invariant by G . As V is irreducible, it has no nontrivial closed invariant subspaces, so it must be 1-dimensional. □

Example I.3.4.5. Let $G = \mathbb{R}$. Then every irreducible unitary representation of G is of the form $\rho_y : x \mapsto e^{ixy}$, for $y \in \mathbb{R}$. The representation ρ_y factors through $S^1 \simeq \mathbb{R}/\mathbb{Z}$ if and only if $y \in 2\pi\mathbb{Z}$.

I.3.5 Finite-dimensional representations

Remember that, if V is a finite-dimensional \mathbb{C} -vector space, then all norms on V are equivalent. So V has a canonical topology, and so does $\text{End}(V)$ (as another finite-dimensional vector space).

Proposition I.3.5.1. *Let V be a normed \mathbb{C} -vector space and $\rho : G \rightarrow \text{GL}(V)$ be a morphism of groups. Consider the following conditions.*

- (i) *The map $G \times V \rightarrow V$, $(g, v) \mapsto \rho(g)(v)$, is continuous (i.e. ρ is a continuous representation of G on V).*
- (ii) *For every $v \in V$, the map $G \rightarrow V$, $g \mapsto \rho(g)(v)$, is continuous.*
- (iii) *The map $\rho : G \rightarrow \text{GL}(V)$ is continuous.*

Then we have (iii) \Rightarrow (i) \Rightarrow (ii). If moreover V is finite-dimensional, then all three conditions are equivalent.

Proof.

(i) \Rightarrow (ii) is obvious.

(ii) \Rightarrow (iii) : Suppose that V is finite-dimensional, and let (e_1, \dots, e_n) be a basis of V , and let $\|\cdot\|$ be the norm on V defined by $\|\sum_{i=1}^n x_i e_i\| = \sup_{1 \leq i \leq n} |x_i|$. We use the corresponding operator norm on $\text{End}(V)$ and still denote it by $\|\cdot\|$. Let $g_0 \in G$ and let $\varepsilon > 0$; we are looking for a neighborhood U of $g_0 \in G$ such that : $g \in U \Rightarrow \|\rho(g) - \rho(g_0)\| \leq \varepsilon$.

For every $i \in \{1, \dots, n\}$, the function $G \rightarrow V$, $g \mapsto \rho(g)(e_i)$, is continuous by assumption, so there exists a neighborhood U_i of g_0 in G such that :

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$g \in U \Rightarrow \|\rho(g)(e_i) - \rho(g_0)(e_i)\| \leq \varepsilon/n$. Let $U = \bigcap_{i=1}^n U_i$. Then if $g \in U$, for every $v = \sum_{i=1}^n x_i e_i \in V$, we have

$$\|\rho(g)(v) - \rho(g_0)(v)\| \leq \sum_{i=1}^n \|x_i\| \|\rho(g)(e_i) - \rho(g_0)(e_i)\| < \sum_{i=1}^n |x_i| \varepsilon/n \leq \varepsilon \|v\|,$$

which means that $\|\rho(g) - \rho(g_0)\| \leq \varepsilon$.

(iii) \Rightarrow (i) : Let $g_0 \in G$, $v_0 \in V$, and $\varepsilon > 0$. We want to find a neighborhood U of g and G and a $\delta > 0$ such that : $g \in U$ and $\|v - v_0\| < \delta \Rightarrow \|\rho(g)(v) - \rho(g_0)(v_0)\| < \varepsilon$.

Choose a δ such that $0 < \delta \leq \frac{\varepsilon}{2\|\rho(g_0)\|}$, and let U be a neighborhood of g_0 in G such that : $g \in G \Rightarrow \|\rho(g) - \rho(g_0)\| < \frac{\varepsilon}{2(\|v_0\| + \delta)}$. Then, if $g \in U$ and $\|v - v_0\| < \delta$, we have $\|v\| \leq \|v_0\| + \delta$, and hence

$$\begin{aligned} \|\rho(g)(v) - \rho(g_0)(v_0)\| &\leq \|\rho(g)(v) - \rho(g_0)(v)\| + \|\rho(g_0)(v) - \rho(g_0)(v_0)\| \\ &\leq \|\rho(g) - \rho(g_0)\| \|v\| + \|\rho(g_0)\| \|v - v_0\| \\ &< \frac{\varepsilon}{2(\|v_0\| + \delta)} (\|v_0\| + \delta) + \|\rho(g_0)\| \delta \\ &\leq \varepsilon/2 + \varepsilon/2 = \varepsilon. \end{aligned}$$

□

I.4 The convolution product and the group algebra

Let G be a locally compact group, and let dx be a left Haar measure on G . We denote by $L^p(G)$ the L^p spaces for this measure. We also denote by Δ the modular function of G .

I.4.1 Convolution on $L^1(G)$ and the group algebra of G

Definition I.4.1.1. Let f and g be functions from G to \mathbb{C} . The *convolution* of f and g , denoted by $f * g$, is the function $x \mapsto \int_G f(y)g(y^{-1}x)dy$ (if it makes sense).

Proposition I.4.1.2. Let $f, g \in L^1(G)$. Then the integral $\int_G f(y)g(y^{-1}x)dy$ is absolutely convergent for almost every x in G , so $f * g$ is defined almost everywhere, and we have $f * g \in L^1(G)$ and

$$\|f * g\|_1 \leq \|f\|_1 \|g\|_1.$$

Proof. By the Fubini-Tonelli theorem and the left invariance of the measure on G , the function $G \times G \rightarrow \mathbb{C}$, $(x, y) \mapsto f(y)g(y^{-1}x)$ is integrable and we have

$$\int_{G \times G} |f(y)g(y^{-1}x)| dx dy = \int_{G \times G} |f(y)||g(x)| dx dy = \|f\|_1 \|g\|_1.$$

I.4 The convolution product and the group algebra

So the first statement also follows from Fubini's theorem, and the second statement is obvious. □

Note that the convolution product is clearly linear in both arguments.

Proposition I.4.1.3. *Let $f, g \in L^1(G)$.*

1. *For almost every $x \in G$, we have*

$$\begin{aligned} f * g(x) &= \int_G f(y)g(y^{-1}x)dy \\ &= \int_G f(xy)g(y^{-1})dy \\ &= \int_G f(y^{-1})g(yx)\Delta(y^{-1})dy \\ &= \int_G f(xy^{-1})g(y)\Delta(y^{-1})dy \\ &= \int_G f(y)L_yg(x)dy \\ &= \int_G g(y^{-1})R_yf(x)dy. \end{aligned}$$

2. *For every $h \in L^1(G)$, we have*

$$(f * g) * h = f * (g * h).$$

(In other words, the convolution product is associative.)

3. *For every $x \in G$, we have*

$$L_x(f * g) = (L_xf) * g$$

and

$$R_x(f * g) = f * (R_xg).$$

4. *If G is abelian, then $f * g = g * f$.*

Proof. 1. We get the equalities of the first four lines by using the substitutions $y \mapsto xy$ and $y \mapsto y^{-1}$, the left invariance of dy and proposition I.2.12. The last two lines are just reformulations of the first two.

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2. For almost every $x \in G$, we have

$$\begin{aligned}
 ((f * g) * h)(x) &= \int_G (f * g)(y) h(y^{-1}x) dy \\
 &= \int_{G \times G} f(z) g(z^{-1}y) h(y^{-1}x) dz dy \\
 &= \int_G f(z) \left(\int_G g(z^{-1}y) h(y^{-1}x) dy \right) dz \\
 &= \int_G f(z) \left(\int_G g(y) h(y^{-1}z^{-1}x) dy \right) dz \\
 &= \int_G f(z) (g * h)(z^{-1}x) dz \\
 &= (f * (g * h))(x).
 \end{aligned}$$

3. This follows immediately from the definition and the equality of the first two lines in point (1).

4. This follows from (1) and from the fact that $\Delta = 1$.

□

Definition I.4.1.4. A *Banach algebra* (over \mathbb{C}) is an associative \mathbb{C} -algebra A with a norm $\|\cdot\|$ making A a Banach space (i.e. a complete normed vector space) and such that, for every $x, y \in A$, we have $\|xy\| \leq \|x\|\|y\|$ (i.e. the norm is submultiplicative). If A has a unit e , we also require that $\|e\| = 1$.

Note that we do not assume that A has a unit. If it does, we say that A is *unital*.

Example I.4.1.5. (a) If V is a Banach space, then $\text{End}(V)$ is a Banach algebra.

(b) By propositions I.4.1.2 and I.4.1.3, the space $L^1(G)$ with the convolution product is a Banach algebra. We call it the (L^1) *group algebra* of G .

Remark I.4.1.6. If the group G is discrete and dx is the counting measure, then $\delta_1 := \mathbf{1}_{\{1\}}$ is a unit for the convolution product. In general, $L^1(G)$ does not always have a unit. (It does if and only if G is discrete.) We can actually see it as a subalgebra of a bigger Banach algebra which does have a unit, the *measure algebra* $\mathcal{M}(G)$ of G (see for example section 2.5 of [8]) :

Remember that a (*complex*) *Radon measure* on G is a bounded linear functional on $\mathcal{C}_0(G)$ (with the norm $\|\cdot\|_\infty$). We denote by $\mathcal{M}(G)$ the space of Radon measures and by $\|\cdot\|$ its norm (which is the operator norm); this is a Banach space. If μ is a Radon measure, we write $f \mapsto \int_G f(x) d\mu(x)$ for the corresponding linear functional on $\mathcal{C}_0(G)$. We define the convolution product $\mu * \nu$ of two Radon measures μ and ν to be the linear functional

$$f \mapsto \int_{G \times G} f(xy) d\mu(x) d\nu(y).$$

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Then it is not very hard to check that $\|\mu * \nu\| \leq \|\mu\| \|\nu\|$ and that the convolution product is associative on $\mathcal{M}(G)$. This makes $\mathcal{M}(G)$ into a Banach algebra, and the Dirac measure at 1 is a unit element of $\mathcal{M}(G)$.

Note also that $\mathcal{M}(G)$ is commutative if and only if G is abelian. Indeed, it is obvious on the definition of $*$ that $\mathcal{M}(G)$ is commutative if G is abelian. To show the converse, we denote by δ_x the Dirac measure at x (so $\int_G f d\delta_x = f(x)$). Then we clearly have $\delta_x * \delta_y = \delta_{xy}$ for every $x, y \in G$. So, if $\mathcal{M}(G)$ is commutative, then $\delta_{xy} = \delta_{yx}$ for every $x, y \in G$, and this implies that G is abelian.

Even though $L^1(G)$ does not contain the unit of $\mathcal{M}(G)$, we have families of functions called “approximate identities” that will be almost as good as δ_1 in practice. In particular, we will be able to prove that $L^1(G)$ is commutative if and only if G is abelian.

Definition I.4.1.7. A (symmetric, continuous) approximate identity with supports in a basis of neighborhoods \mathcal{U} of 1 in G is a family of functions $(\psi_U)_{U \in \mathcal{U}}$ in $\mathcal{C}_c^+(G)$ such that, for every $U \in \mathcal{U}$, we have

- $\text{supp}(\psi_U) \subset U$;
- $\psi_U(x^{-1}) = \psi_U(x), \forall x \in G$;
- $\int_G \psi_U(x) dx = 1$.

For some results, we don’t need the continuity of the ψ_U or the fact that $\psi_U(x^{-1}) = \psi_U(x)$.

Proposition I.4.1.8. For every basis of neighborhoods \mathcal{U} of 1 in G , there exists an approximate identity with supports in \mathcal{U} .

Proof. Let $U \in \mathcal{U}$. Then U contains a symmetric neighborhood $V \subset U$ of 1 and a compact neighborhood $K \subset V$ of 1, and, by corollary A.3.1, there exists a continuous function $f : X \rightarrow [0, 1]$ with compact support contained in V such that $f|_K = 1$. In particular, $f \neq 0$, so $f \in \mathcal{C}_c^+(X)$. Define $g : X \rightarrow [0, 2]$ by $g(x) = f(x) + f(x^{-1})$. Then $g \in \mathcal{C}_c^+(X)$ (because $g|_K = 2$) and $\text{supp}(g) \subset V \subset U$. Now take $\psi_U = \frac{1}{\int_G g(x) dx} g$.

□

Proposition I.4.1.9. Let \mathcal{U} be a basis of neighborhoods of 1 in G , and let $(\psi_U)_{U \in \mathcal{U}}$ be an approximate identity with supports in \mathcal{U} .

1. For every $f \in L^1(G)$, we have $\|\psi_U * f - f\|_1 \rightarrow 0$ and $\|f * \psi_U - f\|_1 \rightarrow 0$ as $U \rightarrow \{1\}$. In fact, we have :

$$\|\psi_U * f - f\|_1 \leq \sup_{y \in U} \|L_Y f - f\|_1$$

and

$$\|f * \psi_U - f\|_1 \leq \sup_{y \in U} \|R_Y f - f\|_1.$$

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2. If $f \in L^\infty(G)$ and f is left (resp. right) uniformly continuous, then $\|\psi_U * f - f\|_\infty \rightarrow 0$ (resp. $\|f * \psi_U - f\|_\infty \rightarrow 0$) as $U \rightarrow \{1\}$. In fact, we have :

$$\|\psi_U * f - f\|_\infty \leq \sup_{y \in U} \|L_y f - f\|_\infty$$

and

$$\|f * \psi_U - f\|_\infty \leq \sup_{y \in U} \|R_y f - f\|_\infty.$$

In point (2), note that if $f : G \rightarrow \mathbb{C}$ is bounded and $g \in \mathcal{C}_c(G)$, then the integral defining $(f * g)(x)$ converges absolutely for every $x \in G$.

Proof. 1. Let $U \in \mathcal{U}$. For every $x \in G$, we have

$$(\psi_U * f)(x) - f(x) = \int_G \psi_U(y)(L_y f(x) - f(x))dy$$

(because $\int_G \psi_U(y)dy = 1$). So

$$\begin{aligned} \|\psi_U * f - f\|_1 &= \int_G \left| \int_G \psi_U(y)(L_y f(x) - f(x))dy \right| dx \\ &\leq \int_{G \times G} \psi_U(y) |L_y f(x) - f(x)| dy dx \\ &\leq \int_G \psi_U(y) \|L_y f - f\|_1 dy \\ &\leq \sup_{y \in U} \|L_y f - f\|_1. \end{aligned}$$

The first convergence result then follows from the fact that $\|L_y f - f\|_1 \rightarrow 0$ as $y \rightarrow 1$, which is proposition I.3.1.13.

The proof of the second convergence result is similar (we get that $\|f * \psi_U - f\|_1 \leq \sup_{y \in U} \|R_y f - f\|_1$ and apply proposition I.3.1.13).

2. Let $U \in \mathcal{U}$. Then for every $x \in G$,

$$|(\psi_U * f)(x) - f(x)| \leq \int_G \psi_U(y) |L_y f(x) - f(x)| dy.$$

As $\psi_U(y) = 0$ for $y \notin U$, this implies that

$$|(\psi_U * f)(x) - f(x)| \leq \left(\sup_{y \in U} |L_y f(x) - f(x)| \right) \left(\int_G \psi_U(y) dy \right) = \sup_{y \in U} |L_y f(x) - f(x)|.$$

Taking the supremum over $x \in G$ gives

$$\|\psi_U * f - f\|_\infty \leq \sup_{y \in U} \|L_y f - f\|_\infty.$$

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So the first statement follows immediately from the definition of left uniform continuity. The proof of the second statement is similar. □

Corollary I.4.1.10. 1. The Banach algebra $L^1(G)$ is commutative if and only if the group G is abelian.

2. Let \mathcal{I} be a closed linear subspace of $L^1(G)$. Then \mathcal{I} is a left (resp. right) ideal if and only if it is stable under the operators L_x (resp. R_x), $x \in G$.

Proof. 1. If G is abelian, then we have already seen that $L^1(G)$ is commutative. Conversely, suppose that $L^1(G)$ is commutative. Let $x, y \in G$. Let $f \in \mathcal{C}_c(G)$, and choose an approximate identity $(\psi_U)_{U \in \mathcal{U}}$. By proposition I.4.1.3, we have, for every $U \in \mathcal{U}$,

$$(R_x f) * (R_y \psi_U) = R_y((R_x f) * \psi_U) = R_y(\psi_U * (R_x f)) = R_y R_x(\psi_U * f) = R_{yx}(f * \psi_U)$$

and

$$(R_x f) * (R_y \psi_U) = (R_y \psi_U) * (R_x f) = R_x(f * (R_y \psi_U)) = R_x R_y(f * \psi_U) = R_{xy}(f * \psi_U).$$

Evaluating at 1 gives $(f * \psi_U)(xy) = (f * \psi_U)(yx)$. But proposition I.4.1.9 (and proposition I.1.12) implies that $\|f * \psi_U - f\|_\infty \rightarrow 0$ as $U \rightarrow \{1\}$, so we get

$$f(xy) = \lim_{U \rightarrow \{1\}} (f * \psi_U)(xy) = \lim_{U \rightarrow \{1\}} (f * \psi_U)(yx) = f(yx).$$

As this is true for every $f \in \mathcal{C}_c(G)$, we must have $xy = yx$ (this follows from local compactness and Urysohn's lemma).

2. We prove the result for left ideals (the proof for right ideals is similar). Suppose that \mathcal{I} is a left ideal, and let $x \in G$. Choose an approximate identity $(\psi_U)_{U \in \mathcal{U}}$. We know that $\psi_U * f \rightarrow f$ in $L^1(G)$ as $U \rightarrow \{1\}$, and so $L_x(\psi_U * f) \rightarrow L_x f$ as $U \rightarrow \{1\}$ (because L_x preserves the L^1 norm). But $L_x(\psi_U * f) = (L_x \psi_U) * f$ by proposition I.4.1.3; as \mathcal{I} is a left ideal, we have $(L_x \psi_U) * f \in \mathcal{I}$ for every $U \in \mathcal{U}$, and as \mathcal{I} is closed, this finally implies that $L_x f \in \mathcal{I}$.

Conversely, suppose that \mathcal{I} is stable by all the operators L_x , $x \in G$. Let $f \in L^1(G)$ and $g \in \mathcal{I}$. By proposition I.4.1.3, we have $f * g = \int_G f(y) L_y g dy$. By the definition of the integral, the function $f * g$ is in the closure of the span of the $L_y g$, $y \in G$, and so it is in \mathcal{I} by hypothesis (and because \mathcal{I} is closed). □

I.4.2 Representations of G vs representations of $L^1(G)$

Definition I.4.2.1. A Banach $*$ -algebra is a Banach algebra A with an involutive anti-automorphism $*$. (That, for every $x, y \in A$ and $\lambda \in \mathbb{C}$, we have $(x+y)^* = x^* + y^*$, $(\lambda x)^* = \bar{\lambda} x^*$, $(xy)^* = y^* x^*$ and $(x^*)^* = x$.)

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The anti-automorphism $*$ is called an *involution* on the Banach algebra A .

Example I.4.2.2. (a) \mathbb{C} , with the involution $z^* = \bar{z}$.

- (b) If G is a locally compact group with a left Haar measure, then $L^1(G)$ with the convolution product and the involution $*$ defined by $f^*(x) = \Delta(x)^{-1} \overline{f(x^{-1})}$ is a Banach $*$ -algebra (note that f^* is in $L^1(G)$ and that we have $\int_G f^*(x) dx = \int_G \overline{f(x)} dx$ and $\int_G |f^*(x)| dx = \int_G |f(x)| dx$ by proposition I.2.12; so $\|f^*\|_1 = \|f\|_1$). It is commutative if and only if G is abelian, and it has a unit if and only if G is discrete.
- (c) If X is a locally compact Hausdorff space, the space $\mathcal{C}_0(X)$ with the norm $\|\cdot\|_\infty$, the usual (pointwise) multiplication and the involution $*$ defined by $f^*(x) = \overline{f(x)}$ is a commutative Banach $*$ -algebra. It has a unit if and only if X is compact (and the unit is the constant function 1).
- (d) Let H be a Hilbert space. Then $\text{End}(H)$, with the operator norm and the involution $T \mapsto T^*$ (where T^* is the adjoint of T as above) is a unital Banach $*$ -algebra. It is commutative if and only if $\dim_{\mathbb{C}}(H) = 1$.

Definition I.4.2.3. (i) If A and B are two Banach $*$ -algebras, a $*$ -homomorphism from A to B is a morphism of \mathbb{C} -algebras $u : A \rightarrow B$ that is bounded as a linear operator and such that $u(x^*) = u(x)^*$, for every $x \in A$.

- (ii) A *representation* of a Banach $*$ -algebra A on a Hilbert space H is a $*$ -homomorphism π from A to $\text{End}(H)$. We say that the representation is *nondegenerate* if, for every $v \in H - \{0\}$, there exists $x \in A$ such that $\pi(x)(v) \neq 0$.

We will need the following result, which we will prove in the next section. (See corollary II.3.9.)

Proposition I.4.2.4. Let V be a Hilbert space. Then, for every $T \in \text{End}(H)$ such that $TT^* = T^*T$, we have

$$\|T\|_{op} = \lim_{n \rightarrow \infty} \|T^n\|_{op}^{1/n}.$$

Corollary I.4.2.5. Let A be a Banach $*$ -algebra such that $\|x^*\| = \|x\|$ for every $x \in A$, and let π be a representation of A on a Hilbert space V . Then $\|\pi\|_{op} \leq 1$.

Proof. By definition, the operator π is bounded; let $C = \|\pi\|_{op}$. Let $x \in A$, and let $T = \pi(x^*x) \in \text{End}(H)$. Note that $T = T^*$. For every $n \geq 1$, we have

$$\|T^n\| \leq C \|(x^*x)\|_1^n \leq C \|x\|^{2n}$$

(because $\|x^*\| = \|x\|$). On the other hand, we have

$$\|T\|_{op} = \lim_{n \rightarrow +\infty} \|T^n\|^{1/n}$$

by proposition I.4.2.4, hence

$$\|\pi(x)\|_{op} = \|\pi(x)^* \pi(x)\|_{op}^{1/2} = \|T\|_{op}^{1/2} \leq \left(\lim_{n \rightarrow +\infty} C^{1/n} \|x\|^{2n/n} \right)^{1/2} = \|x\|.$$

In other words, $\|\pi\|_{op} \leq 1$.

□

We now fix a locally compact group G as before.

Theorem I.4.2.6. 1. Let (π, V) be a unitary representation of G . We define a map from $L^1(G)$ to the space of linear endomorphisms of V , still denoted by π , in the following way : if $f \in L^1(G)$, we set

$$\pi(f) = \int_G f(x) \pi(x) dx,$$

by which we mean that

$$\pi(f)(v) = \int_G f(x) \pi(x)(v) dx$$

for every $v \in V$ (the integral converges by problem set 4).

Then this is a nondegenerate representation of the Banach $*$ -algebra $L^1(G)$ on V , and moreover we have, for every $x \in G$ and every $f \in L^1(G)$,

$$\pi(L_x f) = \pi(x) \pi(f) \quad \text{and} \quad \pi(R_x f) = \Delta(x)^{-1} \pi(f) \pi(x)^{-1}.$$

2. Every nondegenerate representation π of the Banach $*$ -algebra $L^1(G)$ on a Hilbert space V comes from a unitary representation π of the group G as in point (1).

Moreover, if $(\psi_U)_{U \in \mathcal{U}}$ is an approximate identity, then, for every $x \in G$ and every $v \in V$, we have

$$\pi(x)(v) = \lim_{U \rightarrow \{1\}} \pi(L_x \psi_U)(v).$$

3. Let (π, V) be a unitary representation of G , and $\pi : L^1(G) \rightarrow \text{End}(V)$ be the associated $*$ -homomorphism. Then a closed subspace W of V is G -invariant if and only if $\pi(f)(W) \subset W$ for every $f \in L^1(G)$.

4. Let (π_1, V_1) and (π_2, V_2) be unitary representations of G , and $\pi_i : L^1(G) \rightarrow \text{End}(V_i)$, $i = 1, 2$, be the associated $*$ -homomorphisms. Then a bounded linear map $T : V_1 \rightarrow V_2$ is G -equivariant if and only if $T \circ \pi_1(f) = \pi_2(f) \circ T$ for every $f \in L^1(G)$.

Proof. 1. If $f \in L^1(G)$, then the map $\pi(f) : V \rightarrow V$ is clearly \mathbb{C} -linear, and we have for every $v \in V$:

$$\|\pi(f)(v)\| = \left\| \int_G f(x) \pi(x)(v) dx \right\| \leq \int_G |f(x)| \|v\| dx \leq \|v\| \|f\|_1,$$

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so the endomorphism $\pi(f)$ of V is bounded and $\|\pi(f)\|_{op} \leq \|f\|_1$. Also, it is easy to see that the map $\pi : L^1(G) \rightarrow \text{End}(H)$ sending f to $\pi(f)$ is \mathbb{C} -linear, and the equality $\|\pi(f)\|_{op} \leq \|f\|_1$ implies that it is also bounded (we also see that $\|\pi\|_{op}$ is bounded by 1, as it should according to corollary I.4.2.5).

Let $f, g \in L^1(G)$. Then, for every $v \in V$,

$$\begin{aligned}\pi(f * g)(v) &= \int_{G \times G} f(y)g(y^{-1}x)\pi(x)(v)dx dy \\ &= \int_G f(y) \left(\int_G g(y^{-1}x)\pi(x)(v)dx \right) dy \\ &= \int_G f(y) \left(\int_G g(x)\pi(yx)(v)dx \right) dy \\ &= \int_G f(y)\pi(y)(\pi(g)(v))dy \\ &= \pi(f)(\pi(g)(v)).\end{aligned}$$

So $\pi(f * g) = \pi(f) \circ \pi(g)$. Also,

$$\begin{aligned}\pi(f^*)(v) &= \int_G \Delta(x)^{-1} \overline{f(x^{-1})} \pi(x)(v) dx \\ &= \int_G \overline{f(x)} \pi(x^{-1})(v) dx \quad \text{by proposition I.2.12} \\ &= \int_G \overline{f(x)} \pi(x)^*(v) dx,\end{aligned}$$

so that, if $w \in V$,

$$\langle \pi(f^*)(v), w \rangle = \int_G \langle \overline{f(x)} \pi(x)^*(v), w \rangle dx = \int_G \langle v, f(x) \pi(x)(w) \rangle = \langle v, \pi(f)(w) \rangle.$$

This means that $\pi(f^*) = \pi(f)^*$. So we have proved that π is a $*$ -homomorphism.

Now we show the last statement. Let $f \in L^1(G)$ and $x \in G$. Then, for every $v \in V$,

$$\begin{aligned}\pi(x)(\pi(f)(v)) &= \int_G f(y)\pi(x)(\pi(y)(v))dy \\ &= \int_G f(x^{-1}y)\pi(y)(v)dy \\ &= \pi(L_x f)(v)\end{aligned}$$

and

$$\begin{aligned}\pi(f)(\pi(x)^{-1}(v)) &= \int_G f(y)\pi(y)(\pi(x^{-1})(v))dy \\ &= \Delta(x) \int_G f(yx)\pi(y)(v)dy \\ &= \Delta(x)\pi(R_x f)(v).\end{aligned}$$

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Finally, we show that the representation $\pi : L^1(G) \rightarrow \text{End}(V)$ is nondegenerate. Let $v \in V - \{0\}$, and choose a compact neighborhood K of 1 in G such that $\|\pi(x)(v) - v\| \leq \frac{1}{2}\|v\|$ for every $x \in K$. Let $f = \text{vol}(K)^{-1}\mathbf{1}_K$. Then

$$\|\pi(f)(v) - v\| = \frac{1}{\text{vol}(K)} \left\| \int_K (\pi(x)(v) - v) dx \right\| \leq \frac{1}{2}\|v\|,$$

and in particular $\pi(f)(v) \neq 0$.

Finally, we show the last statement. Let $(\psi_U)_{U \in \mathcal{U}}$ be an approximate identity, and let $x \in G$ and $v \in V$.

2. Let π be a nondegenerate representation of the Banach $*$ -algebra $L^1(G)$ on a Hilbert space V . Choose an approximate identity $(\psi_U)_{U \in \mathcal{U}}$ of G . The idea of the proof is that $\pi(x)$ should be the limit of the $\pi(L_x \psi_U)$ as U tends to $\{1\}$.

We now make the idea of proof above more rigorous. Note that, by corollary I.4.2.5, we have $\|\pi\|_{op} \leq 1$. Let W be the span of the $\pi(f)(v)$, for $f \in L^1(G)$ and v . I claim that W is dense in V . Indeed, let $v \in W^\perp$. Then, for every $f \in L^1(G)$, we have $\langle \pi(f)(v), v' \rangle = \langle v, \pi(f^*)(v') \rangle = 0$ for all $v' \in V$, hence $\pi(f)(v) = 0$. As π is nondegenerate, this is only possible if $v = 0$. Hence $W^\perp = 0$, which means that W is dense in V .

Let $x \in G$. We want to define an element $\tilde{\pi}(x) \in \text{End}(V)$ such that, for every $f \in L^1(G)$, we have $\tilde{\pi}(x)\pi(f) = \pi(L_x f)$. This forces us to define $\tilde{\pi}(x)$ on an element $w = \sum_{j=1}^n \pi(f_j)(v_j)$ of W ($f_j \in L^1(G)$, $v_j \in V$) as

$$\tilde{\pi}(x)(w) = \sum_{j=1}^n \pi(L_x f_j)(v_j).$$

This is well-defined because, for every $n \geq 1$, and for all $f_1, \dots, f_n \in L^1(G)$ and $v_1, \dots, v_n \in V$, we have

$$\begin{aligned} \sum_{j=1}^n \pi(L_x f_j)(v_j) &= \lim_{U \rightarrow \{1\}} \sum_{j=1}^n \pi(L_x (\psi_U * f_j))(v_j) \\ &= \lim_{U \rightarrow \{1\}} \sum_{j=1}^n \pi((L_x \psi_U) * f_j)(v_j) \\ &= \lim_{U \rightarrow \{1\}} \pi(L_x \psi_U) \left(\sum_{j=1}^n \pi(f_j)(v_j) \right) \end{aligned}$$

so $\sum_{j=1}^n \pi(L_x f_j)(v_j) = 0$ if $\sum_{j=1}^n \pi(f_j)(v_j) = 0$.

Moreover, as $\|\pi(L_x \psi_U)\|_{op} \leq \|\pi\|_{op} \|\psi_U\|_1 \leq 1$ for every $U \in \mathcal{U}$, we have $\|\tilde{\pi}(x)(w)\| \leq \|w\|$ for every $w \in W$, so $\tilde{\pi}(x)$ is a bounded linear operator of norm ≤ 1

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on W , hence extends by continuity to a bounded linear operator $\tilde{\pi}(x) \in \text{End}(V)$ of norm ≤ 1 .

Next, using the fact that $L_{xy} = L_x \circ L_y$, we see that, for all $x, y \in G$, $\tilde{\pi}(xy) = \tilde{\pi}(x)\tilde{\pi}(y)$ on W , hence on all of V . Similarly, the fact that $L_1 = \text{id}_{L^1(G)}$ implies that $\tilde{\pi}(1) = \text{id}_V$. Also, for every $x \in G$, we have, if $v \in V$,

$$\|v\| = \|\tilde{\pi}(x^{-1})\tilde{\pi}(x)(v)\| \leq \|\tilde{\pi}(x^{-1})\|_{op} \|\tilde{\pi}(x)(v)\| \leq \|\tilde{\pi}(x)(v)\| \leq \|v\|,$$

so $\|\tilde{\pi}(x)(v)\| = \|v\|$, i.e., $\tilde{\pi}(x)$ is a unitary operator.

Let $v \in V$. We want to show that the map $G \rightarrow V$, $x \mapsto \tilde{\pi}(x)(v)$ is continuous. By proposition I.3.1.10, this will imply that $\tilde{\pi} : G \rightarrow \text{End}(V)$ is a unitary representation of G on V . We first suppose that $v = \pi(f)(v')$, with $f \in L^1(G)$ and $v' \in V$. Then $\tilde{\pi}(x)(v) = \pi(L_x f)(v')$, so the result follows from the continuity of the map $G \rightarrow L^1(G)$, $x \mapsto L_x f$ (see proposition I.3.1.13), of π and of the evaluation map $\text{End}(V) \rightarrow V$, $T \mapsto T(v')$. As finite sums of continuous functions $G \rightarrow V$ are continuous, we get the result for every $v \in W$. Now we treat the general case. Let $x \in G$ and $\varepsilon > 0$. We must find a neighborhood U of x in G such that, for every $y \in U$, we have $\|\tilde{\pi}(y)(v) - \tilde{\pi}(x)(v)\| < \varepsilon$. Choose $w \in W$ such that $\|v - w\| < \varepsilon/3$, and a neighborhood U of x in G such that, for every $y \in U$, we have $\|\tilde{\pi}(y)(w) - \tilde{\pi}(x)(w)\| < \varepsilon/3$ (this is possible by the first part of this paragraph). Then, for every $y \in U$, we have

$$\begin{aligned} \|\tilde{\pi}(y)(v) - \tilde{\pi}(x)(v)\| &\leq \|\tilde{\pi}(y)(v) - \tilde{\pi}(y)(w)\| + \|\tilde{\pi}(y)(w) - \tilde{\pi}(x)(w)\| + \|\tilde{\pi}(x)(w) - \tilde{\pi}(x)(v)\| \\ &< \|v - w\| + \varepsilon/3 + \|v - w\| \text{ (because } \tilde{\pi}(x) \text{ and } \tilde{\pi}(y) \text{ are unitary)} \\ &< \varepsilon, \end{aligned}$$

as desired.

We show that the representation $\tilde{\pi}$ of $L^1(G)$ induced by $\tilde{\pi}$ is the representation π that we started from. Let $f, g \in L^1(G)$. Then, for every $v \in V$,

$$\begin{aligned} \tilde{\pi}(f)\pi(g)(v) &= \int_G f(x)\tilde{\pi}(x)\pi(g)(v)dx \\ &= \int_G f(x)\pi(L_x g)(v)dx \\ &= \int_G \pi(f(x)L_x g)(v)dx \\ &= \pi\left(\int_G f(x)L_x g dx\right)(v) \\ &= \pi(f * g)(v) \\ &= \pi(f)\pi(g)(v). \end{aligned}$$

So, if $f \in L^1(G)$, then $\tilde{\pi}(f)$ and $\pi(f)$ are equal on W . As W is dense in V , this implies that $\tilde{\pi}(f) = \pi(f)$.

I.4 The convolution product and the group algebra

Finally, we show the last statement. Let $(\psi_U)_{U \in \mathcal{U}}$ be an approximate identity as above. Let $x \in G$. We have already seen that, for every $v \in W$, we have

$$\tilde{\pi}(x)(v) = \lim_{U \rightarrow \{1\}} \pi(L_x \psi_U)(v).$$

As both sides are continuous functions of $v \in V$ (for the right hand side, we use the fact that $\|\pi(L_x \psi_U)\|_{op} = 1$, this identity extends to all $v \in V$.

3. Suppose that W is G -invariant. Let $f \in L^1(G)$ and $w \in W$. As $\pi(f)(w) = \int_G f(x) \pi(x)(w) dx$ is a limit of linear combinations of elements of the form $\pi(x)(w)$, $x \in G$, it is still in W .

Conversely, suppose that $\pi(f)(W) \subset W$ for every $f \in L^1(G)$. Let $x \in G$, and let $(\psi_U)_{U \in \mathcal{U}}$ be an approximate identity. Then, by the last statement of (2), for every $w \in W$, we have

$$\pi(x)(w) = \lim_{U \rightarrow \{1\}} \pi(L_x \psi_U)(w) \in W.$$

So W is G -invariant.

4. Let $T : V_1 \rightarrow V_2$ be a bounded linear map, and let $W \subset V_1 \times V_2$ be the graph of T ; this is a closed linear subspace of $V_1 \times V_2$. Then T is G -equivariant if and only W is G -invariant, and T is $L^1(G)$ -equivariant if and only W is stable by all the $\pi_1(f) \times \pi_2(f)$, $f \in L^1(G)$. So the conclusion follows from point (3).

□

Example I.4.2.7. Let π be the representation of G given by $\pi(x)(f) = L_x f$ (see example I.3.1.11). Then, for every $f, g \in L^1(G)$, we have $\pi(f)(g) = f * g$. Indeed, we have $\pi(f)(g) = \int_G f(x) L_x g dx$ by definition of $\pi(f)$, so the statement follows from problem 4 of problem set 4.

I.4.3 Convolution on other L^p spaces

We will only see a few results that we'll need later to prove the Peter-Weyl theorem for compact groups. The most important case is that of $L^2(G)$.

Most of the results are based on Minkowski's inequality, which is proved in problem set 4. Here, we only state it for functions on G .

Proposition I.4.3.1 (Minkowski's inequality). *Let $p \in [1, +\infty)$, and let φ be a function from $G \times G$ to \mathbb{C} . Then*

$$\left(\int_G \left| \int_G \varphi(x, y) d\mu(y) \right|^p d\mu(x) \right)^{1/p} \leq \int_G \left(\int_G |\varphi(x, y)|^p d\mu(x) \right)^{1/p} d\mu(y),$$

in the sense that if the right hand side is finite, then $\int_G \varphi(x, y) d\mu(y)$ converges absolutely for almost all $x \in G$, the left hand side is finite and the inequality holds.

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Corollary I.4.3.2. *Let $p \in [1, +\infty)$, and let $f \in L^1(G)$ and $g \in L^p(G)$.*

1. *The integral defining $f * g(x)$ converges absolutely for almost every $x \in G$, and we have $f * g \in L^p(G)$ and $\|f * g\|_p \leq \|f\|_1 \|g\|_p$.*
2. *If G is unimodular, then the same conclusions hold with $f * g$ replaced by $g * f$.*

Proof. 1. we apply Minkowski's inequality to the function $\varphi(x, y) = f(y)g(y^{-1}x)$. For every $y \in G$, we have

$$\int_G |\varphi(x, y)|^p d\mu(x) = |f(y)|^p \int_G |g(x)|^p d\mu(x) = |f(y)|^p \|g\|_p^p$$

by left invariance of μ , so

$$\int_G \left(\int_G |\varphi(x, y)|^p d\mu(x) \right)^{1/p} d\mu(y) = \|g\|_p \int_G |f(y)| d\mu(y) = \|f\|_1 \|g\|_p.$$

Minkowski's inequality first says that $\int_G \varphi(x, y) d\mu(y) = f * g(x)$ converges absolutely for almost all $x \in G$, which is the first statement. The rest of Minkowski's inequality is exactly the fact that $\|f * g\|_p \leq \|f\|_1 \|g\|_p$.

2. Suppose that G is unimodular. Then

$$g * f(x) = \int_G g(y) f(y^{-1}x) d\mu(x) = \int_G g(xy^{-1}) f(y) d\mu(y).$$

So the proof is the same as in (1), by applying Minkowski's inequality to the function $\varphi(x, y) = g(xy^{-1}) f(y)$.

□

Now we generalize proposition I.4.1.9 to other L^p spaces.

Corollary I.4.3.3. *Let \mathcal{U} be a basis of neighborhoods of 1 in G , and let $(\psi_U)_{U \in \mathcal{U}}$ be an approximate identity with supports in \mathcal{U} . Then, for every $1 \leq p < +\infty$, if $f \in L^p(G)$, we have $\|\psi_U * f - f\|_p \rightarrow 0$ and $\|f * \psi_U - f\|_p \rightarrow 0$ as $U \rightarrow \{1\}$.*

Proof. Let $U \in \mathcal{U}$ and $f \in L^p(G)$. Then we have, for every $x \in G$,

$$\psi_U * f(x) - f(x) = \int_G \psi_U(y) (L_y f(x) - f(x)) d\mu(y)$$

(because $\int_G \psi_U d\mu = 1$). Applying Minkowski's inequality to the function $\varphi(x, y) = \psi_U(y) (L_y f(x) - f(x))$, we get

$$\|\psi_U * f - f\|_p \leq \int_G \|L_y f - f\|_p \psi_U(y) d\mu(y) \leq \sup_{y \in U} \|L_y f - f\|_p.$$

Similarly, we have

$$\begin{aligned} f * \psi_U(x) - f(x) &= \int_G f(xy) \psi_U(y^{-1}) d\mu(y) - f(x) \int_G \psi_U(y) d\mu(y) \\ &= \int_G (R_y f(x) - f(x)) \psi_U(y) d\mu(y). \end{aligned}$$

So applying Minkowski's inequality to the function $\varphi(x, y) = (R_y f(x) - f(x)) \psi_U(y)$ gives

$$\|f * \psi_U - f\|_p \leq \int_G \|R_y f - f\|_p \psi_U(y) d\mu(y) \leq \sup_{y \in U} \|R_y f - f\|_p.$$

Hence both statements follow from proposition I.3.1.13. □

Finally, we prove that the convolution products makes functions more regular in some cases. The most important case (for us) in the following proposition is when G is compact and $p = q = 2$.

Proposition I.4.3.4. *Suppose that G is unimodular. Let $p, q \in (1, +\infty)$ such that $p^{-1} + q^{-1} = 1$ and let $f \in L^p(G)$, $g \in L^q(G)$.*

*Then $f * g$ exists, $f * g \in \mathcal{C}_0(G)$ and $\|f * g\|_\infty \leq \|f\|_p \|g\|_q$.*

Proof. Let $x \in G$. We have

$$f * g(x) = \int_G f(y) R_x g(y^{-1}) d\mu(y).$$

As G is unimodular, the function $y \mapsto R_x g(y^{-1})$ is still in $L^q(G)$ and has the same L^q norm as g . So, by Hölder's inequality,⁵ the integral above converges absolutely and we have $|f * g(x)| \leq \|f\|_p \|g\|_q$. This proves the existence of $f * g$ and the result about its norm. It also shows that the bilinear map $L^p(G) \times L^q(G) \rightarrow L^\infty(G)$, $(f, g) \mapsto f * g$ is continuous. As $\mathcal{C}_0(G)$ is closed in $L^\infty(G)$ and $\mathcal{C}_c(G)$ is dense in both $L^p(G)$ and $L^q(G)$, it suffices to prove that $f * g \in \mathcal{C}_0(G)$ if $f, g \in \mathcal{C}_c(G)$.

So let $f, g \in \mathcal{C}_c(G)$. Let $x \in G$, let $\varepsilon > 0$, and choose a neighborhood U of x such that, for every $y \in G$ and $x' \in U$, we have $|g(yx) - g(yx')| \leq \varepsilon$. Then, if $x' \in U$,

$$|f * g(x) - f * g(x')| \leq \int_G |f(y)| |g(y^{-1}x) - g(y^{-1}x')| d\mu(y) \leq \varepsilon \int_G |f(y)| d\mu(y).$$

This shows that $f * g$ is continuous. Let $K = (\text{supp } g)(\text{supp } f)$; this is a compact subset of G . We want to show that $\text{supp}(f * g) \subset K$, which will finish the proof. Let $x \in G$, and suppose that $f * g(x) \neq 0$. Then there exists $y \in G$ such that $f(y)g(y^{-1}x) \neq 0$. We must have $y \in \text{supp } f$ and $y^{-1}x \in \text{supp } g$, so $x \in y(\text{supp } g) \subset K$. □

⁵Which reduces to the Cauchy-Schwarz inequality when $p = q = 2$.

II Some Gelfand theory

II.1 Banach algebras

In this section, A will be a Banach algebra. (See definition I.4.1.4.) Note that the submultiplicativity of the norm implies that the multiplication is a continuous map from $A \times A$ to A .

We suppose for now that A has a unit e and denote by A^\times the group of invertible elements of A .

II.1.1 Spectrum of an element

Definition II.1.1.1. Let $x \in A$.

(i) The *spectrum* of x in A is

$$\sigma(x) = \sigma_A(x) = \{\lambda \in \mathbb{C} \mid \lambda e - x \notin A^\times\}.$$

(ii) The *spectral radius* of x is

$$\rho(x) = \inf_{n \geq 1} \|x^n\|^{1/n}.$$

We will see below how that $\rho(x)$ is equal to $\sup\{|\lambda|, \lambda \in \sigma(x)\}$ (which justifies the name “spectral radius”).

We start by proving some basic properties of invertible elements and the spectral radius. (Note that point (i) does not use the completeness of A , so it stays true in any normed algebra.)

Proposition II.1.1.2. (i) If $x, y \in A^\times$ are such that $\|x - y\| \leq \frac{1}{2}\|x^{-1}\|^{-1}$, then we have

$$\|x^{-1} - y^{-1}\| \leq 2\|x^{-1}\|^2\|y - x\|.$$

In particular, the map $x \mapsto x^{-1}$ is a homeomorphism from A^\times onto itself.

(ii) For every $x \in A$, we have

$$\rho(x) = \lim_{n \rightarrow +\infty} \|x^n\|^{1/n}$$

(Gelfand's formula).

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(iii) Let $x \in A$. If $\rho(x) < 1$ (for example if $\|x\| < 1$), then $e - x \in A^\times$ and $(e - x)^{-1} = \sum_{n \geq 0} x^n$, with the convention that $x^0 = e$. (In particular, the series converges.)

(iv) The group A^\times is open in A .

Proof. (i) We have

$$\|y^{-1}\| - \|x^{-1}\| \leq \|y^{-1} - x^{-1}\| = \|y^{-1}(x - y)x^{-1}\| \leq \|y^{-1}\| \|x - y\| \|x^{-1}\| \leq \frac{1}{2} \|y^{-1}\|.$$

In particular, $\|y^{-1}\| \leq 2\|x^{-1}\|$. Combining this with the inequality above gives

$$\|y^{-1} - x^{-1}\| \leq \|y^{-1}\| \|x - y\| \|x^{-1}\| \leq 2\|x^{-1}\|^2 \|x - y\|,$$

which is the first statement. This also show that the map $x \mapsto x^{-1}$ is continuous. As this map is equal to its own inverse, it is a homeomorphism.

(ii) Let $\varepsilon > 0$. We want to find $N \in \mathbb{Z}_{\geq 1}$ such that $\|x^n\|^{1/n} \leq \rho(x) + \varepsilon$ for $n \geq N$. (We already know that $\|x^n\|^{1/n} \geq \rho(x)$ by definition of $\rho(x)$, so this is enough to establish the result.) By definition of $\rho(x)$, we can find $m \geq 1$ such that $\|x^m\|^{1/m} \leq \rho(x) + \frac{1}{2}\varepsilon$. For every integer $n \geq 1$, we can write $n = mq(n) + r(n)$, with $q(n), r(n) \in \mathbb{N}$ and $0 \leq r(n) \leq m - 1$. Note that

$$\frac{q(n)}{n} = \frac{1}{m} \left(1 - \frac{r(n)}{n}\right) \xrightarrow{n \rightarrow +\infty} \frac{1}{m},$$

hence

$$\|x^m\|^{q(n)/n} \|x\|^{r(n)/n} \xrightarrow{n \rightarrow +\infty} \|x^m\|^{1/m}.$$

Choose $N \geq 1$ such that, for $n \geq 1$, we have

$$\|x^m\|^{q(n)/n} \|x\|^{r(n)/n} \leq \|x^m\|^{1/m} + \frac{\varepsilon}{2} \leq \rho(x) + \varepsilon.$$

Then, if $n \geq N$, we have

$$\|x^n\|^{1/n} = \|x^{mq(n)} x^{r(n)}\|^{1/n} \leq \|x^m\|^{q(n)/n} \|x\|^{r(n)/n} \leq \rho(x) + \varepsilon,$$

as desired. ¹

(iii) Fix $r \in \mathbb{R}$ such that $\rho(x) < r < 1$. Then, by (ii), we have $\|x^n\| \leq r^n$ for n big enough. For every $n \in \mathbb{N}$, we write $S_n = \sum_{k=0}^n x^k$. Then, if $m \geq n$ are big enough, we have

$$\|S_m - S_n\| = \left\| \sum_{k=n+1}^m x^k \right\| \leq r^{n+1} \sum_{k=0}^m r^k = r^{n+1} \frac{1}{1-r}.$$

¹The reasoning used in this proof is sometimes called *Fekete's lemma*. See <https://en.wikipedia.org/wiki/Subadditivity>.

So the sequence $(S_n)_{n \geq 0}$ is a Cauchy sequence, and it converges because A is complete. This means that the series $\sum_{n \geq 0} x^n$ converges. Moreover, for every $n \geq 0$, we have

$$(e - x)S_n = S_n(e - x) = \sum_{k=0}^n x^k - \sum_{k=1}^{n+1} x^k = e - x^{n+1}.$$

This tends to e as $n \rightarrow +\infty$, so $\sum_{n \geq 0} x^n$ is the inverse of $e - x$.

(iv) Let $x \in A^\times$. If $y \in A$ is such that $\|y - x\| < \|x^{-1}\|^{-1}$, then we have $\|e - x^{-1}y\| \leq \|x^{-1}\| \|x - y\| < 1$. So, by (iii), $x^{-1}y \in A^\times$, hence $y \in A^\times$.

□

Theorem II.1.1.3. *For every $x \in A$, the spectrum $\sigma_A(x)$ is a nonempty compact subset of \mathbb{C} , and we have*

$$\rho(x) = \max\{|\lambda|, \lambda \in \sigma_A(x)\}.$$

This explains the name “spectral radius” for $\rho(x)$. Note in particular that, although the spectrum of x depends on A (for example, if we consider a Banach subalgebra B of A containing x , then we have $\sigma_B(x) \supset \sigma_A(x)$, but this may not be an equality), the spectral radius of x does not.

Proof. Consider the map $F : \mathbb{C} \rightarrow A$ sending $\lambda \in \mathbb{C}$ to $\lambda e - x$. Then F is continuous, and $\sigma_A(x)$ is the inverse of the closed subset $A - A^\times$ of A , so $\sigma_A(x)$ is closed in \mathbb{C} .

Next, let $\lambda \in \mathbb{C}$ such that $|\lambda| > \rho(x)$. Then $\rho(\lambda^{-1}x) < 1$, so, by (iii) of proposition II.1.1.2, we have $e - \lambda^{-1}x = \lambda^{-1}(\lambda e - x) \in A^\times$, which immediately implies that $\lambda \notin \sigma_A(x)$. So we have shown that

$$\rho(x) \geq \sup\{|\lambda|, \lambda \in \sigma_A(x)\}.$$

In particular, $\sigma_A(x)$ is a closed and bounded subset of \mathbb{C} , so it is compact.

Let's show that $\sigma_A(x)$ is not empty. Let $T : A \rightarrow \mathbb{C}$ be a bounded linear functional, and define $f : \mathbb{C} - \sigma_A(x) \rightarrow \mathbb{C}$ by $f(\lambda) = T((\lambda e - x)^{-1})$. If $\lambda, \mu \in \mathbb{C} - \sigma_A(x)$, then

$$(\lambda e - x)^{-1} - (\mu e - x)^{-1} = (\lambda e - x)^{-1}((\mu e - x) - (\lambda e - x))(\mu e - x)^{-1} = -(\lambda - \mu)(\lambda e - x)^{-1}(\mu e - x)^{-1},$$

so, if $\lambda \neq \mu$, we get

$$\frac{f(\lambda) - f(\mu)}{\lambda - \mu} = -T((\lambda e - x)^{-1}(\mu e - x)^{-1}).$$

Using the continuity of the function $y \mapsto y^{-1}$ (see (i) of proposition II.1.1.2), we get, for every $\lambda \in \mathbb{C} - \sigma_A(x)$,

$$\lim_{\mu \rightarrow \lambda} \frac{f(\lambda) - f(\mu)}{\lambda - \mu} = -T((\lambda e - x)^{-2}).$$

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In particular, the function f is holomorphic on $\mathbb{C} - \sigma_A(x)$. Let's prove that f vanishes at ∞ , i.e. that $f(\lambda)$ tends to 0 when $|\lambda| \rightarrow +\infty$. Let $\lambda \in \mathbb{C}$ such that $|\lambda| > \rho(x)$. Then, by (iii) of proposition II.1.1.2,

$$(\lambda e - x)^{-1} = \lambda^{-1}(e - \lambda^{-1}x)^{-1} = \frac{1}{\lambda} \sum_{n \geq 0} \frac{1}{\lambda^n} x^n,$$

so

$$\|(\lambda e - x)^{-1}\| \leq \frac{1}{|\lambda|} \sum_{n \geq 0} \frac{\|x\|^n}{|\lambda|^n} = \frac{1}{|\lambda|} \frac{1}{1 - |\lambda|^{-1}\|x\|}.$$

This tends to 0 as $|\lambda| \rightarrow +\infty$; as T is continuous, so does $f(\lambda)$.

Now suppose that $\sigma_A(x) = \emptyset$. Then f is an entire function, and $f(\lambda) \rightarrow 0$ as $|\lambda| \rightarrow +\infty$. By Liouville's theorem, this implies that $f = 0$, i.e. that $T((\lambda e - x)^{-1}) = 0$ for every $\lambda \in \mathbb{C}$. But this is true for every $T \in \text{Hom}(A, \mathbb{C})$ and bounded linear functionals on A separate points by the Hahn-Banach theorem, so we get that $(\lambda e - x)^{-1} = 0$ for every $\lambda \in \mathbb{C}$. This is impossible, because $(\lambda e - x)^{-1} \in A^\times$. So $\sigma_A(x) \neq \emptyset$.

Finally, we prove that

$$\rho(x) \leq \max\{|\lambda|, \lambda \in \sigma_A(x)\}.$$

Let $r = \max\{|\lambda|, \lambda \in \sigma_A(x)\}$. We already know that $r \leq \rho(x)$. Assume that $r < \rho(x)$, and pick r' such that $r < r' < \rho(x)$. Let $T \in \text{Hom}(A, \mathbb{C})$ and define $f : \mathbb{C} - \sigma_A(x) \rightarrow \mathbb{C}$ as before. Then we have seen that f is holomorphic on $\mathbb{C} - \sigma_A(x) \supset \{\lambda \in \mathbb{C} \mid |\lambda| > r\}$. We have also seen that, if $|\lambda| > \rho(x)$, then

$$(\lambda e - x)^{-1} = \sum_{n \geq 0} \frac{1}{\lambda^{n+1}} x^n,$$

hence

$$f(\lambda) = \sum_{n \geq 0} \frac{T(x^n)}{\lambda^{n+1}}.$$

By uniqueness of the power series expansion, this is still valid for $|\lambda| > r$. In particular, the series $\sum_{n \geq 0} \frac{T(x^n)}{(r')^{n+1}}$ converges, so the sequence $(\frac{T(x^n)}{(r')^{n+1}})_{n \geq 0}$ converges to 0, and in particular it is bounded. Consider the sequence $(\alpha_n)_{n \geq 0}$ of bounded linear functionals on $\text{Hom}(A, \mathbb{C})$ defined by $\alpha_n(T) = \frac{T(x^n)}{(r')^{n+1}}$. We just saw that, for every $T \in \text{Hom}(A, \mathbb{C})$, the sequence $(\alpha_n(T))_{n \geq 0}$ is bounded. By the uniform boundedness principle (theorem I.3.2.11), this implies that the sequence $(\|\alpha_n\|_{op})_{n \geq 0}$ is bounded. But note that, by the Hahn-Banach theorem, we have $\|\alpha_n\|_{op} = \left\| \frac{x^n}{(r')^{n+1}} \right\|$. So the sequence $(\frac{\|x^n\|}{(r')^{n+1}})_{n \geq 0}$ is bounded. Choose a real number C bounding it. Then we get

$$\rho(x) = \lim_{n \rightarrow +\infty} \|x^n\|^{1/n} \leq \lim_{n \rightarrow +\infty} C^{1/n} (r')^{(n+1)/n} = r',$$

a contradiction. So $r \geq \rho(x)$.

□

II.1.2 The Gelfand-Mazur theorem

It is a well-known fact that every finite-dimensional \mathbb{C} -algebra that is a field is isomorphic to \mathbb{C} . This is the Banach algebra analogue.

Corollary II.1.2.1 (Gelfand-Mazur theorem). *Let A be a Banach algebra in which every nonzero element is invertible. Then A is isomorphic to \mathbb{C} (i.e. $A = \mathbb{C}e$).*

Proof. Let $x \in A$. By theorem II.1.1.3, $\sigma_A(x) \neq \emptyset$. Let $\lambda \in \sigma_A(x)$, then $\lambda e - x$ is not invertible, so $x = \lambda e$ by hypothesis. □

Definition II.1.2.2. We say that a subset I of A is an *ideal* if it is an ideal in the usual algebraic sense, i.e. if I is a \mathbb{C} -subspace of A that is stable by left and right multiplication by every element of A . We say that I is a *proper ideal* of A if I is an ideal of A and $I \neq A$.

If I is an ideal of A , then it is easy to see that \bar{I} is also an ideal.

Remember also the definition of the quotient norm.

Definition II.1.2.3. Let V be a normed vector space and $W \subset V$ be a closed subspace. Then the *quotient norm* on V/W is defined by

$$\|x + W\| = \inf_{w \in W} \|v + w\|.$$

If V is a Banach space, then so is V/W (for the quotient norm).

Proposition II.1.2.4. (i) *If I is a closed ideal of A , then A/I is a Banach algebra for the quotient norm.*

(ii) *If I is a proper ideal of A , then so is its closure \bar{I} .*

Proof. (i) We already know that A/I is a Banach space and an algebra, so we just need to check that its norm is submultiplicative. Let $x, y \in A$. Then

$$\begin{aligned} \|x + I\| \|y + I\| &= \inf_{a, b \in I} \|x + a\| \|y + b\| \\ &\geq \inf_{a, b \in I} \|(x + a)(y + b)\| \\ &= \inf_{a, b \in I} \|xy + (ay + xb + ab)\| \\ &\geq \inf_{c \in I} \|xy + c\| \quad (\text{because } ay + xb + ab \in I \text{ if } a, b \in I) \\ &= \|xy + I\|. \end{aligned}$$

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- (ii) Consider the open ball $B = \{x \in A \mid \|e - x\| < 1\}$. Then $B \subset A^\times$ by proposition II.1.1.2, so $B \cap I = \emptyset$. As B is open, this implies that $B \cap \bar{I} = \emptyset$, so $\bar{I} \neq A$.

□

Corollary II.1.2.5. *Let A be a commutative unital Banach algebra. If \mathfrak{m} is a maximal ideal of A , then \mathfrak{m} is closed, and $A/\mathfrak{m} = \mathbb{C}$.*

This is the Banach algebra analogue of the Nullstellensatz.

Proof. By proposition II.1.2.4, the ideal $\bar{\mathfrak{m}}$ is also proper; as \mathfrak{m} is maximal, we must have $\mathfrak{m} = \bar{\mathfrak{m}}$, i.e. \mathfrak{m} is closed. By the same proposition, A/\mathfrak{m} is a Banach algebra. Also, every nonzero element of A/\mathfrak{m} is invertible because \mathfrak{m} is maximal, so $A/\mathfrak{m} = \mathbb{C}$ by the Gelfand-Mazur theorem.

□

II.2 Spectrum of a Banach algebra

In this section, A is still a Banach algebra, but we don't assume that it has a unit.

Definition II.2.1. A *multiplicative functional* on A is a nonzero linear functional $\varphi : A \rightarrow \mathbb{C}$ such that $\varphi(xy) = \varphi(x)\varphi(y)$ for all $x, y \in A$.

The set of all multiplicative functionals on A is called the *spectrum* of A and denoted by $\sigma(A)$. We put the weak* topology on $\sigma(A)$. In other words, if $\varphi \in \sigma(A)$, then a basis of open neighborhoods of φ is given by the sets $\{\psi \in \sigma(A) \mid \forall i \in \{1, \dots, n\}, |\varphi(x_i) - \psi(x_i)| < c_i\}$, for $n \in \mathbb{Z}_{\geq 1}$, $x_1, \dots, x_n \in A$ and $c_1, \dots, c_n \in \mathbb{R}_{>0}$.

Note that we do not assume that φ is continuous; in fact, this is automatically the case, as we will see below.

Lemma II.2.2. *If A is unital, then, for every $\varphi \in \sigma(A)$, we have $\varphi(e) = 1$ and $\varphi(A^\times) \subset \mathbb{C}^\times$.*

Proof. Let $x \in A$ be such that $\varphi(x) \neq 0$. Then $\varphi(x) = \varphi(xe) = \varphi(x)\varphi(e)$, so $\varphi(e) = 1$. Also, if $y \in A^\times$, then $1 = \varphi(e) = \varphi(y)\varphi(y^{-1})$, so $\varphi(y) \in \mathbb{C}^\times$.

□

Definition II.2.3. Let A be a Banach algebra. Then we define a unital Banach algebra A_e by taking the \mathbb{C} -vector space $A \oplus \mathbb{C}e$, defining the multiplication on A_e by $(x + \lambda e)(y + \mu e) = (xy + \lambda y + \mu x) + \lambda\mu e$ (for $x, y \in A$ and $\lambda, \mu \in \mathbb{C}$) and the norm by $\|x + \lambda e\| = \|x\| + |\lambda|$ (for $x \in A$ and $\lambda \in \mathbb{C}$). If A is a Banach $*$ -algebra, we make A_e into a Banach $*$ -algebra by setting $(x + \lambda e)^* = x^* + \bar{\lambda}e$ (for $x \in A$ and $\lambda \in \mathbb{C}$).

This construction is called *adjoining an identity* to A .

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Remark II.2.4. If A already has a unit, then A_e is not equal to A . In fact, if we denote by e_A the unit of A , then the map $A_e \rightarrow A \times \mathbb{C}$ sending $x + \lambda e$ to $(x + \lambda e_A, \lambda)$ is an isomorphism of \mathbb{C} -algebras (and a homeomorphism).

Proposition II.2.5. For every $\varphi \in \sigma(A)$, we get an element $\tilde{\varphi} \in \sigma(A_e)$ by setting $\tilde{\varphi}(x + \lambda e) = \varphi(x) + \lambda$. This defines an injective map $\sigma(A) \rightarrow \sigma(A_e)$, whose image is $\sigma(A_e) - \{\varphi_\infty\}$, with φ_∞ defined by $\varphi_\infty(x + \lambda e) = \lambda$.

Later, we will identify φ and $\tilde{\varphi}$ and simply write $\sigma(A_e) = \sigma(A) \cup \{\varphi_\infty\}$.

Proof. The fact that $\tilde{\varphi}$ is a multiplicative functional follows directly from the definition of the multiplication on A_e , and $\tilde{\varphi}$ obviously determines φ . So we just need to check the statement about the image of $\sigma(A) \rightarrow \sigma(A_e)$.

Let $\psi \in \sigma(A_e)$ such that $\psi \neq \varphi_\infty$, and let $\varphi = \psi|_A$. Then we have $\psi(x + \lambda e) = \varphi(x) + \lambda$ for all $x \in A$ and $\lambda \in \mathbb{C}$; as $\psi \neq \varphi_\infty$, the linear functional $\varphi : A \rightarrow \mathbb{C}$ cannot be zero, so φ is a multiplicative functional on A , and we clearly have $\psi = \tilde{\varphi}$. □

Corollary II.2.6. Let $\varphi \in \sigma(A)$. Then φ is a bounded linear function on A , and we have $\|\varphi\|_{op} \leq 1$, with equality if A is unital.

Proof. By proposition II.2.5, the multiplicative functional extends to a multiplicative functional $\tilde{\varphi}$ on A_e . Let $x \in A$. For every $\lambda \in \mathbb{C}$ such that $|\lambda| > \|x\|$, the element $x - \lambda e$ of A_e is invertible by proposition II.1.1.2, so $\varphi(x) - \lambda = \tilde{\varphi}(x - \lambda e) \neq 0$. This implies that $|\varphi(x)| \leq \|x\|$, i.e. that φ is bounded and $\|\varphi\|_{op} \leq 1$. □

If A is unital, then $\|e\| = 1$ and $\varphi(e) = 1$, so $\|\varphi\|_{op} = 1$. □

Theorem II.2.7. Let A be a Banach algebra.

- (i) If A is unital, then the space $\sigma(A)$ is compact Hausdorff.
- (ii) In general, the space $\sigma(A)$ is locally compact Hausdorff, and $\sigma(A_e)$ is its Alexandroff compactification (a.k.a. one-point compactification).

Remember that, if X is a Hausdorff locally compact topological space, then its Alexandroff compactification is the space $X \cup \{\infty\}$ (i.e. X with one point added), and that its open subsets are the open subsets of X and the complements in $X \cup \{\infty\}$ of compact subsets of X .

Proof. By corollary II.2.6, the spectrum $\sigma(A)$ is a subset of the closed unit ball of $\text{Hom}(A, \mathbb{C})$. We know that this closed unit ball is compact Hausdorff for the weak* topology on $\text{Hom}(A, \mathbb{C})$ (this is Alaoglu's theorem), and $\sigma(A) \cup \{0\}$ is closed in this topology, because it is defined by

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the closed conditions $\varphi(xy) = \varphi(x)\varphi(y)$, for all $x, y \in A$. So $\sigma(A) \cup \{0\}$ is compact (for the weak* topology), and its open subset $\sigma(A)$ is locally compact. If A is unital, then $\sigma(A)$ is closed in $\sigma(A) \cup \{0\}$ because it is cut out by the condition $\varphi(e) = 1$, so $\sigma(A)$ is compact.

Now we show the last statement of (ii). If $\varphi \in \sigma(A)$ (resp. $\sigma(A_e)$), $x \in A$ and $c > 0$, we set

$$U(\varphi, x, c) = \{\psi \in \sigma(A) \mid |\varphi(x) - \psi(x)| < c\}$$

$$\text{(resp. } \tilde{U}(\varphi, x, c) = \{\psi \in \sigma(A_e) \mid |\varphi(x) - \psi(x)| < c\}).$$

These form a basis for the topology of $\sigma(A)$ (resp. $\sigma(A_e)$).

If $\varphi \in \sigma(A)$, $x \in A$ and $c > 0$, we have

$$\tilde{U}(\tilde{\varphi}, x, c) = \begin{cases} U(\varphi, x, c) \cup \{\varphi_\infty\} & \text{if } |\varphi(x)| < c \\ U(\varphi, x, c) & \text{otherwise.} \end{cases}$$

For the neighborhoods of φ_∞ , we get that, if $x \in A$ and $c > 0$, then

$$\begin{aligned} \tilde{U}(\varphi_\infty, x, c) &= \{\varphi_\infty\} \cup \{\varphi \in \sigma(A) \mid |\varphi(x)| < c\} \\ &= \sigma(A_e) - \{\psi \in \sigma(A_e) \mid |\psi(x)| \geq c\}. \end{aligned}$$

So the topology of $\sigma(A)$ is induced by the topology of $\sigma(A_e)$. Also, as $\{\psi \in \sigma(A_e) \mid |\psi(x)| \geq c\}$ is closed in $\sigma(A_e)$, hence compact, for all $x \in A$ and $c > 0$, the open neighborhoods of φ_∞ in $\sigma(A_e)$ are exactly the complements of the compact subsets of $\sigma(A)$. This means that $\sigma(A_e)$ is the Alexandroff compactification of $\sigma(A)$.

□

Definition II.2.8. Let A be a Banach algebra. For every $x \in A$, the map $\hat{x} : \sigma(A) \rightarrow \mathbb{C}$ defined by $\hat{x}(\varphi) = \varphi(x)$ is called the *Gelfand transform* of x .

Note that each \hat{x} is continuous on $\sigma(A)$ by definition of the topology of $\sigma(A)$. The resulting map $\Gamma : A \rightarrow \mathcal{C}(\sigma(A))$, $x \mapsto \hat{x}$ is called the *Gelfand representation* of A (or sometimes also the Gelfand transform).

Note that Γ is a morphism of \mathbb{C} -algebras by definition of the algebra operations on $\mathcal{C}(\sigma(A))$.

Theorem II.2.9. (i) The map Γ maps A into $\mathcal{C}_0(\sigma(A))$, and we have $\|\hat{x}\|_\infty \leq \|x\|$ for every $x \in A$.

(ii) The image of Γ separates the points of $\sigma(A)$.

(iii) If A is unital, then \hat{e} is the constant function 1 on $\sigma(A)$.

Proof. (i) If A is unital, then $\sigma(A)$ is compact, so $\mathcal{C}_0(\sigma(A)) = \mathcal{C}(\sigma(A))$. In general, as $\sigma(A_e)$ is the Alexandroff compactification of $\sigma(A)$, we just need to check that $\hat{x}(\varphi_\infty) = 0$ for every $x \in A$; but this follows immediately from the definitions.

Let $x \in A$. Then

$$\|\widehat{x}\|_\infty = \sup_{\varphi \in \sigma(A)} |\widehat{x}(\varphi)| = \sup_{\varphi \in \sigma(A)} |\varphi(x)| \leq \|x\|$$

by corollary II.2.6.

- (ii) Let $\varphi, \varphi' \in \sigma(A)$ such that $\varphi \neq \varphi'$. Then there exists $x \in A$ such that $\varphi(x) \neq \varphi'(x)$, i.e. $\widehat{x}(\varphi) \neq \widehat{x}(\varphi')$.
- (iii) This follows immediately from lemma II.2.2.

□

For a general Banach algebra (even a unital one), the spectrum can be empty (see problem set 5). But this cannot occur for commutative Banach algebras.

Theorem II.2.10. *Let A be a commutative unital Banach algebra. Then the map $\varphi \mapsto \text{Ker}(\varphi)$ induces a bijection from $\sigma(A)$ to the set of maximal ideals of A .*

If you have seen another definition of the spectrum (for example in algebraic geometry), this theorem shows how it is related to our definition.

Proof. If $\varphi \in \sigma(A)$, then $A/\text{Ker}(\varphi) \simeq \mathbb{C}$ (note that φ is surjective because it is nonzero), so $\text{Ker}(\varphi)$ is a maximal ideal of A . This shows that the map is well-defined.

If \mathfrak{m} is a maximal ideal, then it follows from the Gelfand-Mazur theorem that $A/\mathfrak{m} \simeq \mathbb{C}$ (see corollary II.1.2.5), so the map $\varphi : A \rightarrow A/\mathfrak{m} \simeq \mathbb{C}$ is an element of $\sigma(A)$ such that $\text{Ker}(\varphi) = \mathfrak{m}$. This shows that the map is surjective.

Now we need to check injectivity. Let $\varphi, \psi \in \sigma(A)$ such that $\mathfrak{m} := \text{Ker}(\varphi) = \text{Ker}(\psi)$. Let $x \in A$. As $A/\mathfrak{m} \simeq \mathbb{C}$, we can write $x = \lambda e + y$, with $\lambda \in \mathbb{C}$ and $y \in \mathfrak{m}$. Then we have

$$\varphi(x) = \lambda = \psi(y).$$

So $\varphi = \psi$.

□

Corollary II.2.11. *Let A be a commutative unital Banach algebra. Then, for every $x \in A$:*

- (i) $x \in A^\times$ if and only if \widehat{x} never vanishes;
- (ii) $\widehat{x}(\sigma(A)) = \sigma_A(x)$;
- (iii) $\|\widehat{x}\|_\infty = \rho(x)$.

Proof. (i) If $x \in A^\times$, then \widehat{x} cannot vanish, because we have $\widehat{xx^{-1}} = \widehat{e} = 1$. Conversely, suppose that x is not invertible. Then there exists a maximal ideal containing x , so, by theorem II.2.10, there exists $\varphi \in \sigma(A)$ such that $0 = \varphi(x) = \widehat{x}(\varphi)$.

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(ii) By (i), we have

$$\sigma_A(x) = \{\lambda \in \mathbb{C} \mid x - \lambda e \notin A^\times\} = \{\lambda \in \mathbb{C} \mid \widehat{x - \lambda} \text{ vanishes at at least one point}\} = \widehat{x}(\sigma(A)).$$

(iii) This follows from (ii) and from theorem II.1.1.3.

□

II.3 C^* -algebras and the Gelfand-Naimark theorem

Definition II.3.1. A Banach $*$ -algebra A is called a C^* -algebra if we have $\|x^*x\| = \|x\|^2$ for every $x \in A$.

Remark II.3.2. Everybody calls this a C^* -algebra, except Bourbaki who says “stellar algebra” (“algèbre stellaire”).

Lemma II.3.3. If A is a C^* -algebra, then $\|x\| = \|x^*\|$ for every $x \in A$.

Proof. Let $x \in A - \{0\}$. Then

$$\|x\|^2 = \|x^*x\| \leq \|x^*\| \|x\|,$$

so $\|x\| \leq \|x^*\|$. Applying this to x^* and using that $(x^*)^* = x$ gives $\|x^*\| \leq \|x\|$.

□

Example II.3.4. Most of the examples of example I.4.2.2 are actually C^* -algebras.

- (a) \mathbb{C} is a C^* -algebra because, for every $\lambda \in \mathbb{C}$, we have $|\lambda\bar{\lambda}| = |\lambda|^2$.
- (b) Let G be a locally compact group. Then $L^1(G)$ is not a C^* -algebra in general, though it does satisfy the conclusion of lemma II.3.3.²
- (c) Let X be a locally compact Hausdorff space. Then $\mathcal{C}_0(X)$ is a C^* -algebra, because, for every $f \in \mathcal{C}_0(X)$, we have

$$\|f^*f\|_\infty = \sup_{x \in X} |\overline{f(x)}f(x)| = \sup_{x \in X} |f(x)|^2 = \|f\|_\infty^2.$$

- (d) Let V be a Hilbert space. Then $\text{End}(V)$ is a C^* -algebra. Indeed, let $T \in \text{End}(V)$. We want to prove that $\|T^*T\|_{op} = \|T\|_{op}^2$. First note that

$$\|T^*\|_{op} = \sup_{v, w \in V, \|v\|=\|w\|=1} |\langle T^*(v), w \rangle| = \sup_{v, w \in V, \|v\|=\|w\|=1} |\langle v, T(w) \rangle| = \|T\|_{op},$$

²There is a way to modify the norm on $L^1(G)$ to make the completion for the new norm a C^* -algebra, but we won't need this here.

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so $\|T^*T\|_{op} \leq \|T^*\|_{op}\|T\|_{op} = \|T\|_{op}^2$. On the other hand,

$$\|T^*T\|_{op} = \sup_{v \in V, \|v\|=1} \|T^*T(v)\| \geq \sup_{v \in V, \|v\|=1} |\langle T^*T(v), v \rangle| = \sup_{v \in V, \|v\|=1} |\langle T(v), T(v) \rangle| = \|T\|_{op}^2.$$

Proposition II.3.5. *Let A be a C^* -algebra. Then the Gelfand representation $\Gamma : A \rightarrow \mathcal{C}_0(\sigma(A))$ is a $*$ -homomorphism.*

Remark II.3.6. The proposition says that every multiplicative functional on A is a $*$ -homomorphism. A Banach $*$ -algebra satisfying this condition is called *symmetric*. Every C^* -algebra is symmetric, but the converse is not true. (For example, if G is a locally compact commutative group, then $L^1(G)$ is symmetric, see problem set 5.)

Proof. By adjoining an identity to A , we may reduce to the case where A is unital. (See problem set 5 for the correct choice of norm on A_e . Note that changing the norm on A_e does not affect $\sigma(A_e)$, because the definition of the spectrum does not involve the norm.)

Let $x \in A$ and $\varphi \in \sigma(A)$. We want to prove that $\varphi(x^*) = \widehat{x^*}(\varphi) = \widehat{\bar{x}}(\varphi) = \overline{\varphi(x)}$. Write $\varphi(x) = a + ib$ and $\varphi(x^*) = c + id$, with $a, b, c, d \in \mathbb{R}$.

Suppose that $b + d \neq 0$. Let

$$y = \frac{1}{b + d}(x + x^* - (a + c)e) \in A.$$

Note that $y = y^*$, and that

$$\varphi(y) = \frac{1}{b + d}(a + ib + c + id - (a + c)) = i,$$

so, for every $t \in \mathbb{R}$, we have $\varphi(y + ite) = (1 + t)i$, hence

$$|1 + t| = |\varphi(y + ite)| \leq \|y + ite\|$$

(by corollary II.2.6). Using the defining property of C^* -algebras and the fact that $y = y^*$ gives, for every $t \in \mathbb{R}$,

$$(1 + t)^2 \leq \|y + ite\|^2 = \|(y + ite)(y + ite)^*\| = \|(y + ite)(y - ite)\| = \|y^2 + t^2e\| \leq \|y^2\| + t^2,$$

i.e. $1 + 2t \leq \|y\|^2$. But this implies that $\|y\|$ is infinite, which is not possible. So $b + d = 0$, i.e. $d = -b$.

Applying the same reasoning to ix and $(ix)^* = -ix^*$ (and noting that $\varphi(ix) = -b + ia$ and $\varphi((ix)^*) = d - ic$) gives $a - c = 0$, i.e. $a = c$. This finishes the proof that $\varphi(x^*) = \overline{\varphi(x)}$. □

Proposition II.3.7. *Let A be a commutative unital C^* -algebra. Then, for every $x \in A$, we have $\|x\| = \rho(x)$.*

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Proof. If $x \in A$ is such that $x = x^*$, then $\|x\|^2 = \|x^*x\| = \|x^2\|$, so $\|x^{2^n}\| = \|x\|^{2^n}$ for every $n \in \mathbb{N}$.

Now let x be any element of A . Then $(xx^*)^* = xx^*$, so the first part applies to xx^* . Also, for every $n \in \mathbb{N}$, $(xx^*)^n = x^n(x^*)^n$ (because A is commutative). So, if $n \geq 0$,

$$\|x\|^{2^{n+1}} = \|xx^*\|^{2^n} = \|(xx^*)^{2^n}\| = \|x^{2^n}(x^*)^{2^n}\| = \|x^{2^n}\|^2.$$

This implies that

$$\rho(x) = \lim_{x \rightarrow +\infty} \|x^{2^n}\|^{2^{-n}} = \|x\|.$$

□

Definition II.3.8. If A is a Banach $*$ -algebra, an element x of A is called *normal* if $xx^* = x^*x$.

Corollary II.3.9. Let A be a unital C^* -algebra, and let $x \in A$ be a normal element of A . Then $\rho(x) = \|x\|$.

In particular, if V is a Hilbert space and $T \in \text{End}(V)$ is normal, then $\|T\|_{op} = \rho(T)$.

Proof. Indeed, as x commutes with x^* , the closure of the smallest unital \mathbb{C} -algebra A' of A containing x and x^* is a commutative C^* -algebra, and $\rho(x)$ and $\|x\|$ don't change when we see x as an element of A' .

□

Theorem II.3.10 (Gelfand-Naimark theorem). Let A be a commutative unital C^* -algebra. Then the Gelfand representation $\Gamma : A \rightarrow \mathcal{C}(\sigma(A))$ is an isometric $*$ -isomorphism.

Proof. We know that Γ is a $*$ -homomorphism by proposition II.3.5, and that it is an isometry by corollary II.2.11(iii) and proposition II.3.7. In particular, Γ is injective. So it just remains to show that it is surjective. As Γ is an isometry and A is complete, the image $\Gamma(A)$ is closed in $\mathcal{C}(\sigma(A))$; but it separates points by theorem II.2.9(ii) and contains the constant functions because $\Gamma(e) = 1$, so it is equal to $\mathcal{C}(\sigma(A))$ by the Stone-Weierstrass theorem.

□

It is easy to see that the Gelfand-Naimark theorem implies the following result (but we won't need it).

Corollary II.3.11. Let A be a commutative C^* -algebra. Then the Gelfand representation $\Gamma : A \rightarrow \mathcal{C}_0(\sigma(A))$ is an isometric $*$ -isomorphism.

II.4 The spectral theorem

Theorem II.4.1. *Let V a Hilbert space, and let $T \in \text{End}(V)$ be normal. We denote by A_T the closure of the unital subalgebra of $\text{End}(V)$ generated by T and T^* ; it is commutative because T and T^* commute.*

Then there exists an isometric $$ -isomorphism $\Phi : \mathcal{C}(\sigma(T)) \xrightarrow{\sim} A_T$ such that, if ι is the injection of $\sigma(T)$ into \mathbb{C} , we have $\Phi(\iota) = T$.*

Note that we just write $\sigma(T)$ for $\sigma_{\text{End}(T)}(T)$ (this is the usual spectrum of T).

This theorem doesn't look a lot like the spectral theorem of finite-dimensional linear algebra. See problem set 5 for a way to pass between the two.

Proof. Let $A = A_T$. First we will prove the result with $\sigma_A(T)$ instead of $\sigma(T)$, then we'll show that $\sigma(T) = \sigma_A(T)$. Note that we automatically have $\sigma(T) \subset \sigma_A(T)$ (because, if $\lambda \text{id}_V - T$ is not invertible in $\text{End}(T)$, then it certainly won't be invertible in a subalgebra).

Consider the Gelfand transform of T (seen as an element of A), this is a continuous map $\widehat{T} : \sigma(A) \rightarrow \mathbb{C}$. Let's show that \widehat{T} is injective. Consider $\varphi_1, \varphi_2 \in \sigma(A)$, i.e. two multiplicative functionals on A , such that $\widehat{T}(\varphi_1) = \widehat{T}(\varphi_2)$, i.e. $\varphi_1(T) = \varphi_2(T)$. We have seen that the Gelfand representation is a $*$ -homomorphism, so we have

$$\widehat{T^*}(\varphi_1) = \overline{\widehat{T}(\varphi_1)} = \overline{\widehat{T}(\varphi_2)} = \widehat{T^*}(\varphi_2),$$

i.e. $\varphi_1(T^*) = \varphi_2(T^*)$. The multiplicative functionals φ_1 and φ_2 are equal on e , T and T^* , and they are continuous, so they are equal on all of A , which is what we wanted.

Now remember that $\sigma(A)$ is compact Hausdorff, because A is unital. So \widehat{T} induces a homeomorphism from $\sigma(A)$ to its image in \mathbb{C} , which is $\sigma_A(T)$ by corollary II.2.11. Hence composing with \widehat{T} gives an isometric $*$ -isomorphism $\Psi : \mathcal{C}(\sigma_A(T)) \xrightarrow{\sim} \mathcal{C}(\sigma(A))$.

Remember that we also have the Gelfand representation of A , which is an isometric $*$ -isomorphism $\Gamma : A \xrightarrow{\sim} \mathcal{C}(\sigma(A))$. So we get an isometric $*$ -isomorphism $\Phi : \mathcal{C}(\sigma_A(T)) \xrightarrow{\sim} A$ by setting $\Phi = \Gamma^{-1} \circ \Psi$.

Let's show that $\Phi(\iota) = T$. As $\Gamma : A \rightarrow \mathcal{C}(\sigma(A))$ is an isomorphism, it suffices to check that $\widehat{T} = \widehat{\Phi(\iota)}$, i.e. that $\widehat{T} = \Psi(\iota)$. Let $\varphi \in \sigma(A)$. We have $\Psi(\iota)(\varphi) = \iota(\widehat{T}(\varphi)) = \widehat{T}(\varphi)$, as desired.

Finally, we show that the inclusion $\sigma(T) \subset \sigma_A(T)$ is an equality. Let $\lambda \in \sigma_A(T)$, and suppose that $\lambda \notin \sigma(T)$. Let $\varepsilon > 0$, and choose $f \in \mathcal{C}(\sigma_A(T))$ such that $\|f\|_\infty = 1$, $f(\lambda) = 1$ and $f(\mu) = 0$ if $|\lambda - \mu| \geq \varepsilon > 0$. Let $U = \Phi(f) \in A$, then $\|U\|_{op} = \|f\|_\infty = 1$. Note that $\Phi(1) = \text{id}_V$ (where 1 is the constant function with value 1), because Φ is an isomorphism of algebras. So $T - \lambda \text{id}_V = \Phi(\iota - \lambda)$, and $(T - \lambda \text{id}_V) \circ U = \Phi((\iota - \lambda)f)$. As Φ is an isometry, this implies that

$$\|(T - \lambda \text{id}_V) \circ U\|_{op} = \|(\iota - \lambda)f\|_\infty \leq \varepsilon$$

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(because f is bounded by 1, and $f(\mu) = 0$ if $|\lambda - \mu| \geq \varepsilon$). On the other hand, as $\lambda \notin \sigma(T)$, the operator $T - \lambda \text{id}_V$ is invertible in $\text{End}(V)$, so we get

$$1 = \|f\|_\infty = \|U\|_{op} = \|(T - \lambda \text{id}_V)^{-1}(T - \lambda \text{id}_V)U\|_{op} \leq \varepsilon \|(T - \lambda \text{id}_V)^{-1}\|_{op}.$$

This is true for every $\varepsilon > 0$, so it implies that $1 = 0$, which is a contradiction. □

Corollary II.4.2. *Let V be a Hilbert space and $T \in \text{End}(V)$ be normal. Then the following conditions are equivalent :*

- (i) $\sigma(T)$ is a singleton;
- (ii) $T \in \mathbb{C} \text{id}_V$;
- (iii) $A_T = \mathbb{C} \text{id}_V$.

Proof.

- (i) \Rightarrow (ii) If $\sigma(T) = \{\lambda\}$, then ι is λ times the unit of $\mathcal{C}(\sigma(T))$, so $T = \Phi(\iota) = \lambda \text{id}_V$.
- (ii) \Rightarrow (iii) If $T \in \mathbb{C} \text{id}_V$, then $\mathbb{C} \text{id}_V$ is a closed unital subalgebra of A containing T and T^* , so it is equal to A_T .
- (iii) \Rightarrow (i) Suppose that $A_T = \mathbb{C} \text{id}_V$. Let $\lambda, \mu \in \sigma(T)$. If $\lambda \neq \mu$, then we can find $f_1, f_2 \in \mathcal{C}(\sigma(T))$ such that $f_1(\lambda) = 1$, $f_2(\mu) = 1$ and $f_1 f_2 = 0$. But then $\Phi(f_1)\Phi(f_2) = 0$ and $\Phi(f_1), \Phi(f_2) \neq 0$, which contradicts the fact that \mathbb{C} is a domain. □

Definition II.4.3. If A is a \mathbb{C} -algebra and $E \subset A$ is a subset, we set $Z_A(E) = \{x \in A \mid \forall y \in E, xy = yx\}$. This is called the *centralizer* of E in A .

It is easy to see that the centralizer is always a subalgebra of A .

Corollary II.4.4. *Let V be a Hilbert space, and let E be a subset of $\text{End}(V)$ such that $E^* = E$. Suppose that the only closed subspaces of V stable by all the elements of E are $\{0\}$ and V . Then $Z_{\text{End}(V)}(E) = \mathbb{C} \text{id}_V$.*

Proof. Let $A = Z_{\text{End}(V)}(E)$. It is a closed subalgebra of $\text{End}(V)$. We show that A is stable by $*$: If $T \in A$, then, for every $U \in E$, we have $U^* \in E$, hence

$$T^* \circ U = (U^* \circ T)^* = (T \circ U^*)^* = U \circ T^*,$$

so $T^* \in A$. In particular, the subalgebra A is generated by its normal elements; indeed, for every $T \in A$, we have $T = \frac{1}{2}((T + T^*) + (T - T^*))$, and both $T + T^*$ and $T - T^*$ are normal.

II.4 The spectral theorem

Remember that we want to show that each element of A is in $\mathbb{C}\text{id}_V$; by what we just showed, it suffices to prove it for the normal elements of A . So let $T \in A$ be normal. By corollary II.4.2, it suffices to show that the spectrum $\sigma(T)$ of T is a singleton. By the spectral theorem (theorem II.4.1), we have an isometric $*$ -isomorphism $\Phi : \mathcal{C}(\sigma(T)) \xrightarrow{\sim} A_T$, where A_T is the closure of the unital subalgebra of $\text{End}(V)$ generated by T and T^* , such that Φ sends the embedding $\iota : \sigma(T) \hookrightarrow \mathbb{C}$ to T . Note that $A_T \subset A$. Now suppose that $\sigma(T)$ is not a singleton. Then we can find two nonzero functions $f_1, f_2 \in \mathcal{C}(\sigma(T))$ such that $f_1 f_2 = 0$, and $\Phi(f_1), \Phi(f_2)$ are nonzero elements of $\text{End}_G(V)$ such that $\Phi(f_1)\Phi(f_2) = 0$. Let $W = \overline{\text{Im}(\Phi(f_2))}$; then $W \neq \{0\}$ because $\Phi(f_2)$ is nonzero. Also, as $\Phi(f_2)$ commutes with every element of E , the subspace W is stable by all the elements of E , so $W = V$ by hypothesis. But we also have $\Phi(f_1)(W) = 0$ because $\Phi(f_1)\Phi(f_2) = 0$, so $\Phi(f_1) = 0$, which contradicts the choice of f_1, f_2 . So $\sigma(T)$ is a singleton, and we are done.

□

III The Gelfand-Raikov theorem

The goal of this chapter is to prove the Gelfand-Raikov theorem, which says that irreducible unitary representations of locally groups separate point (i.e., if G is a locally compact group and $x \in G - \{1\}$, then there exists an irreducible unitary representation of G such that $\pi(x) \neq 1$).

In this chapter, G is a locally compact group and μ (or just “ dx ”) is a left Haar measure on G .

III.1 $L^\infty(G)$

You can safely ignore this section and assume that all groups are σ -compact.

We will be using $L^\infty(G)$ more seriously in this chapter, and we want it to be the continuous dual of $L^1(G)$, which is not true if G is not σ -compact. So we change the definition of $L^\infty(G)$ to make it true. See section 2.3 of [8].

More generally, let X be a locally compact Hausdorff topological space and let μ be a regular Borel measure. We say that $E \subset X$ is *locally Borel* if, for every Borel subset F of X such that $\mu(F) < +\infty$, we have that $E \cap F$ is a Borel subset of X . If E is locally Borel, we say that E is *locally null* if, for every Borel subset F of X such that $\mu(F) < +\infty$, we have $\mu(E \cap F) = 0$. We say that an assertion about points of X is true *locally almost everywhere* if it is true outside of a locally null subset. We saw that a function $f : X \rightarrow \mathbb{C}$ is *locally measurable* if, for every Borel subset A of \mathbb{C} , the set $f^{-1}(A)$ is locally Borel. Now we set $L^\infty(X)$ to be the space of locally measurable functions $X \rightarrow \mathbb{C}$ that are bounded locally almost everywhere, modulo the equivalence relation : $f \sim g$ if $f - g = 0$ locally almost everywhere. The norm on $L^\infty(X)$ is given by

$$\|f\|_\infty = \inf\{c \in \mathbb{R}_{\geq 0} \mid |f(x)| \leq c \text{ locally almost everywhere}\}.$$

III.2 Functions of positive type

Definition III.2.1. A *function of positive type* on G is a function $\varphi \in L^\infty(G)$ such that, for every $f \in L^1(G)$, we

$$\int_G (f^* * f)(x) \varphi(x) dx \geq 0.$$

III The Gelfand-Raikov theorem

Note that $f^* * f \in L^1(G)$ if $f \in L^1(G)$, so the integral converges.

Remark III.2.2. For every $\varphi \in L^\infty(G)$ and every $f, g \in L^1(G)$, we have

$$\begin{aligned} \int_G (f^* * g)(x) \varphi(x) dx &= \int_{G \times G} f^*(y) g(y^{-1}x) \varphi(x) dx dy \\ &= \int_{G \times G} \Delta(y)^{-1} \overline{f(y^{-1})} g(y^{-1}x) \varphi(x) dx dy \\ &= \int_{G \times G} \overline{f(y)} g(yx) \varphi(x) dx dy \\ &= \int_{G \times G} \overline{f(y)} g(x) \varphi(y^{-1}x) dx dy. \end{aligned}$$

Example III.2.3. (1) 0 is a function of positive type.

(2) If $\varphi : G \rightarrow S^1 \subset \mathbb{C}$ is a 1-dimensional representation (i.e. $\varphi(xy) = \varphi(x)\varphi(y)$ for all $x, y \in G$), then it is a function of positive type. Indeed, for every $f \in L^1(G)$, we have by remark III.2.2

$$\begin{aligned} \int_G (f^* * f)(x) \varphi(x) dx &= \int_{G \times G} \overline{f(y)} f(yx) \varphi(y^{-1}x) dx dy \\ &= \int_{G \times G} \overline{f(y)} f(x) \overline{\varphi(y)} \varphi(x) dx dy \\ &= \left| \int_G \varphi(x) f(x) dx \right|^2 \geq 0. \end{aligned}$$

We will generalize the second example in point (ii) of the following proposition.

Proposition III.2.4. (i) If $\varphi : G \rightarrow \mathbb{C}$ is a function of positive type, then so is $\overline{\varphi}$.

(ii) If (π, V) is a unitary representation of G and $v \in V$, then $\varphi : G \rightarrow \mathbb{C}, x \mapsto \langle \pi(x)(v), v \rangle$ is a continuous function of positive type.

(iii) Let $f \in L^2(G)$, and define $\tilde{f} : G \rightarrow \mathbb{C}$ by $\tilde{f}(x) = \overline{f(x^{-1})}$. Then $f * \tilde{f}$ makes sense, it is in $L^\infty(G)$, and it is a function of positive type.

Proof. (i) Let $f \in L^1(G)$. Then, by remark III.2.2,

$$\begin{aligned} \int_G (f^* * f) \overline{\varphi} d\mu &= \int_{G \times G} \overline{f(y)} f(yx) \overline{\varphi(x)} dx dy \\ &= \overline{\int_{G \times G} f(y) \overline{f(yx)} \varphi(x) dx dy} \\ &= \int_G (\overline{f}^* * \overline{f}) \varphi d\mu \geq 0. \end{aligned}$$

III.2 Functions of positive type

- (ii) The function φ is continuous because $G \rightarrow V$, $x \mapsto \pi(x)(v)$ is continuous. Let's show that it is of positive type. Note that, for all $x, y \in G$, we have

$$\varphi(y^{-1}x) = \langle \pi(y^{-1}x)(v), v \rangle = \langle \pi(x)(v), \pi(y)(v) \rangle.$$

Let $f \in L^1(G)$. Then, by remark III.2.2,

$$\begin{aligned} \int (f^* * f) \varphi d\mu &= \int_{G \times G} f(x) \overline{f(y)} \varphi(y^{-1}x) dx dy \\ &= \int_{G \times G} \langle f(x) \pi(x)(v), f(y) \pi(y)(v) \rangle dx dy \\ &= \langle \pi(f)(v), \pi(f)(v) \rangle \geq 0. \end{aligned}$$

- (iii) Let $x \in G$. Then the integral defining $f * \tilde{f}(x)$ is

$$\int_G f(y) \overline{f(x^{-1}y)} dy.$$

This integral converges, because both f and $L_x \bar{f} : y \mapsto \overline{f(x^{-1}y)}$ are in $L^2(G)$ (by left invariance of μ). Also, by the Cauchy-Schwarz inequality, we have

$$|f * \tilde{f}(x)| \leq \|f\|_2 \|L_x \bar{f}\|_2 = \|f\|_2^2.$$

So $f * \tilde{f} \in L^\infty(G)$.

Let's show that $f * \tilde{f}$ is of positive type. Let π_L be the left regular representation of G , i.e. the unitary representation of G on $L^2(G)$ given by $\pi_L(x) = L_x$. Then, for every $x \in G$, we have

$$\langle \pi_L(x)(f), f \rangle = \int_G f(x^{-1}y) \overline{f(y)} dy = \int_G \overline{\tilde{f}(y^{-1}x)} \overline{f(y)} dy = \overline{f * \tilde{f}(x)}.$$

So the result follows from (i) and (ii). □

The main result of this function is that the example in (ii) above is the only one.

Theorem III.2.5. *Let $\varphi : G \rightarrow \mathbb{C}$ be a function of positive type. Then there exists a cyclic unitary representation (π, V) of G and a cyclic vector v for V such that $\varphi(x) = \langle \pi(x)(v), v \rangle$ locally almost everywhere.*

Moreover, the representation π and the vector v are uniquely determined by φ , in the following sense : if (π', V') is another cyclic unitary representation of G and if $v' \in V'$ is a cyclic vector such that $\varphi(x) = \langle \pi'(x)(v'), v' \rangle$ locally almost everywhere, then there exists a G -equivariant isometry $T : V \rightarrow V'$ such that $T(v) = v'$.

III The Gelfand-Raikov theorem

In fact, we will give a somewhat explicit construction of (π, V) during the proof.

Before proving the theorem, let's see some easy corollaries.

Corollary III.2.6. *Let $\varphi : G \rightarrow \mathbb{C}$ be a function of positive type. Then φ agrees with a continuous function locally almost everywhere, $\|\varphi\|_\infty = \varphi(1)$ and, for every $x \in G$, we have $\varphi(x^{-1}) = \overline{\varphi(x)}$.*

Proof. The first statement follows from (ii) of proposition III.2.4. To prove the other statements, choose a cyclic unitary representation (π, V) of G and $v \in V$ such that $\varphi(x) = \langle \pi(x)(v), v \rangle$. Then, for every $x \in G$,

$$|\varphi(x)| \leq \|\pi(x)(v)\| \|v\| = \|v\|^2 = \varphi(1)$$

and

$$\varphi(x^{-1}) = \langle \pi(x^{-1})(v), v \rangle = \langle \pi(x)^*(v), v \rangle = \langle v, \pi(x)(v) \rangle = \overline{\varphi(x)}.$$

□

Now we come back to the proof of the theorem. Let $\varphi : G \rightarrow \mathbb{C}$ be a function of positive type. Define a Hermitian form $\langle \cdot, \cdot \rangle_\varphi$ on $L^1(G)$ by :

$$\langle f, g \rangle_\varphi = \int (g^* * f) \varphi = \int_{G \times G} f(x) \overline{g(y)} \varphi(y^{-1}x) dx dy$$

(see remark III.2.2). In particular, we clearly have, for all $f, g \in L^1(G)$,

$$|\langle f, g \rangle_\varphi| \leq \|f\|_1 \|g\|_1 \|\varphi\|_\infty.$$

As φ is of positive type, we have $\langle f, f \rangle_\varphi \geq 0$, that is, the Hermitian form we just defined is positive semi-definite; in particular, the Cauchy-Schwarz inequality applies to it, and it gives, for all $f, g \in L^1(G)$,

$$|\langle f, g \rangle_\varphi|^2 \leq \langle f, f \rangle_\varphi \langle g, g \rangle_\varphi.$$

Let \mathcal{N} be the kernel (or radical) of the form $\langle \cdot, \cdot \rangle_\varphi$, that is, the orthogonal of $L^1(G)$, i.e. the space of $f \in L^1(G)$ such that $\langle f, g \rangle_\varphi = 0$ for every $g \in L^1(G)$. By the Cauchy-Schwarz inequality, we have $f \in \mathcal{N}$ if and only if $\langle f, f \rangle_\varphi = 0$. Hence the form $\langle \cdot, \cdot \rangle_\varphi$ defines a positive definite Hermitian form on $L^1(G)/\mathcal{N}$, that we will still denote by $\langle \cdot, \cdot \rangle_\varphi$; we denote the associated norm by $\|\cdot\|_\varphi$. For every $f \in L^1(G)$, we have

$$\|f + \mathcal{N}\|_\varphi^2 \leq \|\varphi\|_\infty \|f\|_1^2.$$

Let V_φ be the completion of $L^1(G)/\mathcal{N}$ for the norm $\|\cdot\|_\varphi$; this is a Hilbert space.

We want to construct a unitary action of G on V_φ . We already have a continuous representation of G on $L^1(G)$, using the operators L_x . This will magically give our unitary representation. Note

III.2 Functions of positive type

first that, for every $L^1(G)$, the map $G \rightarrow L^1(G)$, $x \mapsto L_x f$ is continuous for the semi-norm $\|\cdot\|_\varphi$ because of the inequality $\|\cdot\|_\varphi \leq \|\varphi\|_\infty^{1/2} \|\cdot\|_1$ that we just proved.

Let's prove that $\langle \cdot, \cdot \rangle_\varphi$ is invariant by the action of G . Let $x \in G$ and $f, g \in L^1(G)$. Then

$$\begin{aligned} \langle L_x f, L_x g \rangle_\varphi &= \int_{G \times G} f(x^{-1}y) \overline{g(x^{-1}z)} \varphi(z^{-1}y) dy dz \\ &= \int_{G \times G} f(y) \overline{g(z)} \varphi((xz)^{-1}(xy)) dy dz \\ &= \int_{G \times G} f(y) \overline{g(z)} \varphi(z^{-1}y) dy dz = \langle f, g \rangle_\varphi. \end{aligned}$$

In particular, the radical \mathcal{N} of the form $\langle \cdot, \cdot \rangle_\varphi$ is a G -invariant subspace of $L^1(G)$, so we get an action of G on $L^1(G)/\mathcal{N}$, which preserves the Hermitian inner product and is a continuous representation by proposition I.3.1.10. We extend this action to V_φ by continuity. This gives a unitary representation of G on V_φ , which we will denote by π_φ .

Let $f, g \in L^1(G)$. Then, by example I.4.2.7, we have

$$\pi_\varphi(f)(g + \mathcal{N}) = f * g + \mathcal{N}.$$

The following lemma will imply the first statement of theorem III.2.5.

Lemma III.2.7. *There exists a cyclic vector $v = v_\varphi$ for V_φ such that :*

- (i) *for $f \in L^1(G)$, we have $\pi_\varphi(f)(v) = f + \mathcal{N}$;*
- (ii) *we have $\varphi(x) = \langle \pi_\varphi(x)(v), v \rangle_\varphi$ locally almost everywhere on G .*

Proof. By the calculation of $\pi_\varphi(f)(g + \mathcal{N})$ for $f, g \in L^1(G)$ (see above), we see that v would be the image in $L^1(G)/\mathcal{N}$ of a unit element for $*$ (i.e. a Dirac measure at $1 \in G$), if such a unit element existed. In general, it doesn't, but we can approximate it, and hope that we will get a Cauchy sequence in $L^1(G)/\mathcal{N}$.

So let $(\psi_U)_{U \in \mathcal{U}}$ be an approximate identity (see definition I.4.1.7). Note that $(\psi_U^*)_{U \in \mathcal{U}}$ is also an approximate identity, so, by proposition I.4.1.9, we have $\psi_U^* * f \xrightarrow{U \rightarrow \{1\}} f$ in $L^1(G)$ for every $f \in L^1(G)$. So, for every $f \in L^1(G)$, we have

$$\langle f, \psi_U \rangle_\varphi = \int (\psi_U^* * f) \varphi d\mu \xrightarrow{U \rightarrow \{1\}} \int f \varphi d\mu.$$

Hence $f \mapsto \int_G f \varphi d\mu$ is a bounded (for $\|\cdot\|_1$ and $\|\cdot\|_\varphi$) linear functional on $L^1(G)$ whose kernel contains \mathcal{N} . We can descend this bounded linear functional to $L^1(G)/\mathcal{N}$ and extend it to V_φ by continuity, and we get a bounded linear functional on V_φ , which must be of the form $\langle \cdot, v \rangle_\varphi$ for

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some $v \in V_\varphi$ (uniquely determined), because V_φ is a Hilbert space. By definition of v , we have, for every $f \in L^1(G)$,

$$\langle f + \mathcal{N}, v \rangle_\varphi = \int_G f \varphi d\mu,$$

and this determines v because the image of $L^1(G)$ is dense in V_φ .

Now we prove properties (i) and (ii). Let $f, g \in L^1(G)$. Then

$$\begin{aligned} \langle g, f \rangle_\varphi &= \int_G (f^* * g) \varphi d\mu \\ &= \int_{G \times G} g(x) \overline{f(y)} \varphi(y^{-1}x) dx dy \\ &= \int_{G \times G} g(yx) \overline{f(y)} \varphi(x) dx dy \\ &= \int_{G \times G} \overline{f(y)} L_{y^{-1}} g(x) \varphi(x) dx dy \\ &= \int_G \overline{f(y)} \langle \pi_\varphi(y^{-1})(g + \mathcal{N}), v \rangle_\varphi dx \\ &= \int_G \langle g + \mathcal{N}, f(y) \pi_\varphi(y)(v) \rangle_\varphi dx \\ &= \langle g + \mathcal{N}, \pi_\varphi(f)(v) \rangle_\varphi. \end{aligned}$$

As this is true for every $g \in L^1(G)$ and as the image of $L^1(G)$ is dense in V_φ , we get $\pi_\varphi(f)(v) = f + \mathcal{N}$. In particular, the span of $\{\pi_\varphi(f)(v), f \in L^1(G)\}$ is dense in V_φ , so v is a cyclic vector (by (iii) of theorem I.4.2.6).

Also, for $f \in L^1(G)$, by what we have just seen :

$$\begin{aligned} \int_G f(x) \langle \pi_\varphi(x)(v), v \rangle_\varphi dx &= \langle \int_G f(x) \pi_\varphi(x)(v) dx, v \rangle_\varphi \\ &= \langle \pi_\varphi(f)(v), v \rangle_\varphi \\ &= \langle f + \mathcal{N}, v \rangle_\varphi \\ &= \int_G f(x) \varphi(x) dx. \end{aligned}$$

As this is true for every $f \in L^1(G)$, it implies that $\varphi(x) = \langle \pi_\varphi(x)(v), v \rangle_\varphi$ locally almost everywhere. □

To finish the proof of theorem III.2.5, we just need to establish the following lemma.

Lemma III.2.8. *Let (π, V) and (π', V') be two cyclic unitary representations of G and $v \in V$, $v' \in V'$ be two cyclic vectors such that, for every $x \in G$, we have*

$$\langle \pi(x)(v), v \rangle = \langle \pi'(x)(v'), v' \rangle.$$

III.3 Functions of positive type and irreducible representations

Then there exists a G -equivariant isometry $T : V \rightarrow V'$ such that $T(v) = v'$.

Proof. Of course, we want to define $T : V \rightarrow V'$ by the formula $T(\pi(x)(v)) = \pi'(x)(v')$, for every $x \in G$. We need to make sense of this. Let $W = \text{Span}\{\pi(x)(v), x \in G\}$. By the assumption that v is cyclic, the subspace W is dense in V . Let's check that the formula above defines an isometry $T : W \rightarrow V'$. Let $x_1, \dots, x_n \in G$ and $\lambda_1, \dots, \lambda_n \in \mathbb{C}$. Then

$$\begin{aligned} \left\| \sum_{i=1}^n \lambda_i \pi(x_i)(v) \right\|^2 &= \sum_{i=1}^n \sum_{j=1}^n \lambda_i \bar{\lambda}_j \langle \pi(x_j^{-1} x_i)(v), v \rangle \\ &= \sum_{i=1}^n \sum_{j=1}^n \lambda_i \bar{\lambda}_j \langle \pi'(x_j^{-1} x_i)(v'), v' \rangle \\ &= \left\| \sum_{i=1}^n \lambda_i \pi'(x_i)(v') \right\|^2. \end{aligned}$$

In particular, if $\sum_{i=1}^n \lambda_i \pi(x_i)(v) = 0$, then we also have $\sum_{i=1}^n \lambda_i \pi'(x_i)(v') = 0$. So we can define $T : W \rightarrow V'$ by $T(\sum_{i=1}^n \lambda_i \pi(x_i)(v)) = \sum_{i=1}^n \lambda_i \pi'(x_i)(v')$, and then the calculation above shows that T is an isometry. Hence T is continuous, and so we can extend to a continuous linear operator $T : V \rightarrow V'$, which is still an isometry, hence injective and with closed image. Also, if $x \in G$ and $w \in W$, then we have $T(\pi(x)(w)) = \pi'(x)(T(w))$ by definition of T . As T is continuous and W is dense in V , this stays true for every $w \in V$; in other words, T is G -equivariant. Finally, $T(v) = v'$ by definition of T , so the image of T is dense in V' , hence equal to V' . □

III.3 Functions of positive type and irreducible representations

We have seen that cyclic unitary representations of G (together with a fixed cyclic vector) are parametrized by functions of positive type. The next natural question is “which functions of positive type correspond to the irreducible representations?”

Definition III.3.1. We denote by $\mathcal{P}(G)$ or \mathcal{P} the set of continuous functions of positive type on G . This is a convex cone in $\mathcal{C}_b(G)$.¹

Let

$$\mathcal{P}_1 = \{\varphi \in \mathcal{P} \mid \|\varphi\|_\infty = 1\} = \{\varphi \in \mathcal{P} \mid \varphi(1) = 1\}$$

and

$$\mathcal{P}_0 = \{\varphi \in \mathcal{P} \mid \|\varphi\|_\infty \leq 1\} = \{\varphi \in \mathcal{P} \mid \varphi(1) \leq 1\}.$$

¹“Cone” means that it is stable by multiplication by elements of $\mathbb{R}_{\geq 0}$.

III The Gelfand-Raikov theorem

(Remember that, by corollary III.2.6, we have $\|\varphi\|_\infty = \varphi(1)$ for every $\varphi \in \mathcal{P}$.)

Then \mathcal{P}_1 and \mathcal{P}_0 are convex subsets of $\mathcal{C}_b(G)$. We denote by $\mathcal{E}(\mathcal{P}_1)$ and $\mathcal{E}(\mathcal{P}_0)$ their sets of extremal points.

Theorem III.3.2. *Let $\varphi \in \mathcal{P}_1$. Then the unitary representation (V_φ, π_φ) constructed in the previous section is irreducible if and only if $\varphi \in \mathcal{E}(\mathcal{P}_1)$.*

Remark III.3.3. If $\varphi \in \mathcal{P}$ and $c \in \mathbb{R}_{>0}$, then we have $\langle \cdot, \cdot \rangle_{c\varphi} = c\langle \cdot, \cdot \rangle_\varphi$, so $V_{c\varphi} = V_\varphi$, $\pi_{c\varphi} = \pi_\varphi$ and $v_{c\varphi} = v_\varphi$. (But the identity of V_φ is not an isometry, because we are using two different inner products, i.e. $\langle \cdot, \cdot \rangle_\varphi$ and $\langle \cdot, \cdot \rangle_{c\varphi}$.) As each nonzero $\varphi \in \mathcal{P}$ is a of the form $c\varphi'$ for a unique $\varphi' \in \mathcal{P}_1$ (we have $c = \varphi(1)$), the theorem does answer the question at the beginning of the section.

Remark III.3.4. If G is commutative, the theorem says that $\widehat{G} = \mathcal{E}(\mathcal{P}_1)$.

Proof. In this proof, we will denote the inner form and norm of $V = V_\varphi$ by $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$, and we will write $\pi = \pi_\varphi$. (Unless this introduces confusion.)

We first suppose that π is not irreducible. Let $0 \neq W \subsetneq V$ be a closed G -invariant subspace. As π is unitary, W^\perp is also G -invariant, and we have $V = W \oplus W^\perp$. Let $v \in V$ be the cyclic vector of lemma III.2.7. As v is cyclic, it cannot be contained in W or in W^\perp (otherwise we would have $W = V$ or $W^\perp = V$). So we can write $v = v_1 + v_2$, with $v_1 \in W$, $v_2 \in W^\perp$, and $v_1, v_2 \neq 0$. Define $\varphi_1, \varphi_2 : G \rightarrow \mathbb{C}$ by $\varphi_i(x) = \langle \pi(x)(v_i), v_i \rangle$. Then $\varphi_1, \varphi_2 \in \mathcal{P}$ by (ii) of proposition III.2.4, and we have $\varphi = \varphi_1 + \varphi_2$. Let $c_1 = \|v_1\|^2$ and $c_2 = \|v_2\|^2$; we have $c_1 + c_2 = \|v\|^2 = \varphi(1) = 1$ by the Pythagorean theorem, so $c_1, c_2 \in (0, 1)$. Let $\psi_i = \frac{1}{c_i}\varphi_i$, for $i = 1, 2$. Then $\varphi = c_1\psi_1 + c_2\psi_2$, and $\psi_1, \psi_2 \in \mathcal{P}_1$ (because $\psi_1(1) = \psi_2(1) = 1$). To conclude that φ is not an extremal point of \mathcal{P}_1 , we still need to prove that $\psi_1 \neq \psi_2$, i.e. that φ_2 is not of the form $c\varphi_1$ for $c \in \mathbb{R}_{>0}$.

Let $c \in \mathbb{R}_{>0}$. Choose $\varepsilon > 0$ such that $\varepsilon < \frac{c\|v_1\|^2}{\|v_2\| + c\|v_1\|}$, i.e. such that $\varepsilon\|v_2\| < c\|v_1\|^2 - \varepsilon c\|v_1\|$. As v is a cyclic vector for V , we can find $x_1, \dots, x_n \in G$ and $a_1, \dots, a_n \in \mathbb{C}$ such that

$$\left\| \sum_{i=1}^n a_i \pi(x_i)(v) - v_1 \right\| < \varepsilon.$$

As $v = v_1 + v_2$ with $v_1 \in W$ and $v_2 \in W^\perp$, and as both W and W^\perp are stable by the action of G , we have, for $x \in G$,

$$\langle \pi(x)(v), v_1 \rangle = \langle \pi(x)(v_1) + \pi(x)(v_2), v_1 \rangle = \langle \pi(x)(v_1), v_1 \rangle.$$

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Hence

$$\begin{aligned} \left| \sum_{i=1}^n a_i \langle \pi(x_i)(v_1), v_1 \rangle - \langle v_1, v_1 \rangle \right| &= \left| \sum_{i=1}^n a_i \langle \pi(x_i)(v) - v_1, v_1 \rangle \right| \\ &= \left| \langle \sum_{i=1}^n a_i \pi(x_i)(v) - v_1, v_1 \rangle \right| \\ &< \varepsilon \|v_1\|, \end{aligned}$$

which implies that

$$\|v_1\|^2 - \varepsilon \|v_1\| < \left| \sum_{i=1}^n a_i \langle \pi(x_i)(v_1), v_1 \rangle \right| = \left| \sum_{i=1}^n a_i \varphi_1(x_i) \right|.$$

On the other hand (using the fact that $\langle \pi(x)(v), v_2 \rangle = \langle \pi(x)(v_2), v_2 \rangle$ for every $x \in G$), we have

$$\begin{aligned} \left| \sum_{i=1}^n a_i \langle \pi(x_i)(v_2), v_2 \rangle \right| &= \left| \sum_{i=1}^n a_i \langle \pi(x_i)(v) - v_1, v_2 \rangle \right| \\ &= \left| \langle \sum_{i=1}^n a_i \pi(x_i)(v) - v_1, v_2 \rangle \right| \\ &\leq \left\| \sum_{i=1}^n a_i \pi(x_i)(v) - v_1 \right\| \|v_2\| \\ &< \varepsilon \|v_2\| \\ &< c \|v_1\|^2 - \varepsilon c \|v_1\| \\ &< c \left| \sum_{i=1}^n a_i \varphi_1(x_i) \right| \end{aligned}$$

i.e.

$$\left| \sum_{i=1}^n a_i \varphi_2(x_i) \right| < c \left| \sum_{i=1}^n a_i \varphi_1(x_i) \right|.$$

So we cannot have $\varphi_2 = c\varphi_1$. As c was arbitrary, this finishes the proof that $\psi_1 \neq \psi_2$, hence that φ is not an extremal point of \mathcal{P}_1 .

Conversely, we want to show that φ is extremal in \mathcal{P}_1 if π_φ is irreducible. Suppose that $\varphi = \varphi_1 + \varphi_2$, with $\varphi_1, \varphi_2 \in \mathcal{P}$. For every $f \in L^1(G)$, we have

$$\langle f, f \rangle_{\varphi_1} = \langle f, f \rangle_\varphi - \langle f, f \rangle_{\varphi_2} \leq \langle f, f \rangle_\varphi.$$

In particular, the kernel of $\langle \cdot, \cdot \rangle_\varphi$ is contained in the kernel of $\langle \cdot, \cdot \rangle_{\varphi_1}$, so the identity of $L^1(G)$ extends to a continuous surjective map $T : V_\varphi \rightarrow V_{\varphi_1}$, and that map is G -equivariant because the action of G on both V_φ and V_{φ_1} comes from its action on $L^1(G)$ by left translations. Also, as v_φ

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(resp. v_{φ_1}) is just the limit in V_φ (resp. V_{φ_1}) of the image of an approximate identity, the operator T sends v_φ to v_{φ_1} . As $\text{Ker } T$ is a G -invariant subspace of V_φ , so is $(\text{Ker } T)^\perp$, so T defines a G -equivariant isomorphism $(\text{Ker } T)^\perp \xrightarrow{\sim} V_{\varphi_1}$, so V_{φ_1} is isomorphic to a subrepresentation of V_φ .

Now suppose that π_φ is irreducible. Then $T^*T \in \text{End}(V_\varphi)$ is G -equivariant, so it is equal to $c \text{id}_{V_\varphi}$ for some $c \in \mathbb{C}$ by Schur's lemma (theorem I.3.4.1). As $T(v_\varphi) = v_{\varphi_1}$, for every $x \in G$, we have

$$\begin{aligned}\varphi_1(x) &= \langle \pi_{\varphi_1}(x)(v_{\varphi_1}), v_{\varphi_1} \rangle_{\varphi_1} \\ &= \langle \pi_{\varphi_1}(x)(T(v_\varphi)), T(v_\varphi) \rangle_{\varphi_1} \\ &= \langle T(\pi_\varphi(x)(v_\varphi)), T(v_\varphi) \rangle_{\varphi_1} \\ &= \langle T^*T(\pi_\varphi(x)(v_\varphi)), v_\varphi \rangle_\varphi \\ &= c\varphi(x).\end{aligned}$$

As φ_1 and φ are of positive type, we must have $c \in \mathbb{R}_{\geq 0}$. We see similarly that φ_2 must be in $\mathbb{R}_{\geq 0}\varphi$. So φ is extremal. □

III.4 The convex set \mathcal{P}_1

We have seen in the previous two sections that irreducible unitary representations of G are parametrized by extremal points of \mathcal{P}_1 . Remember that we are trying to show that there enough irreducible unitary representations to separate points on G . So we want to show that \mathcal{P}_1 has a lot of extremal points. A natural ideal is to use the Krein-Milman theorem (theorem B.5.2, that says that a compact convex set is the closed convex hull of its extremal points), but \mathcal{P}_1 is not compact in general. However, the set \mathcal{P}_0 is convex and weak* compact and closely related to \mathcal{P}_1 ; this will be enough to extend the conclusion of the Krein-Milman theorem to \mathcal{P}_1 .

Remember that \mathcal{P} is a subset of $L^\infty(G)$. We identify $L^\infty(G)$ with the continuous dual of $L^1(G)$ and consider the weak* topology on it and on its subsets \mathcal{P} , \mathcal{P}_0 and \mathcal{P}_1 . For $f \in L^\infty(G)$, a basis of neighborhoods of f is given by the sets $U_{g_1, \dots, g_n, c} = \{f' \in L^\infty(G) \mid |\int_G (f - f')g_i d\mu| < c, 1 \leq i \leq n\}$, for $n \in \mathbb{Z}_{\geq 1}$, $g_1, \dots, g_n \in L^1(G)$ and $c > 0$. The second main result of this section is that the weak* topology coincides with the topology of compact convergence on \mathcal{P}_1 .

Theorem III.4.1. *The convex hull of $\mathcal{E}(\mathcal{P}_1)$ is dense in \mathcal{P}_1 for the weak* topology.*

Lemma III.4.2. *We have $\mathcal{E}(\mathcal{P}_0) = \mathcal{E}(\mathcal{P}_1) \cup \{0\}$.*

Proof. First we show that every point of $\mathcal{E}(\mathcal{P}_1) \cup \{0\}$ is extremal in \mathcal{P}_0 . Let $\varphi_1, \varphi_2 \in \mathcal{P}_0$ and $c_1, c_2 \in (0, 1)$ such that $c_1 + c_2 = 1$. If $c_1\varphi_1 + c_2\varphi_2 = 0$, then $0 = c_1\varphi_1(1) + c_2\varphi_2(1)$, so $\varphi_1(1) = \varphi_2(1) = 0$, so $\|\varphi_1\|_\infty = \|\varphi_2\|_\infty = 0$, i.e. $\varphi_1 = \varphi_2 = 0$. This shows that 0 is

extremal. Suppose that $\varphi := c_1\varphi_1 + c_2\varphi_2 \in \mathcal{E}(\mathcal{P}_1)$. Then $1 = \varphi(1) = c_1\varphi_1(1) + c_2\varphi_2(1)$, so $\varphi_1(1) = \varphi_2(1) = 1$, so $\varphi_1, \varphi_2 \in \mathcal{P}_1$; as φ is extremal in \mathcal{P}_1 , this implies that $\varphi_1 = \varphi_2$. So φ is also extremal in \mathcal{P}_0 .

Now we show that every extremal point of \mathcal{P}_0 is in $\mathcal{E}(\mathcal{P}_1) \cup \{0\}$. Let $\varphi \in \mathcal{P}_0 - (\mathcal{E}(\mathcal{P}_1) \cup \{0\})$. If $\varphi \in \mathcal{P}_1$, it is not extremal. If $\varphi \notin \mathcal{P}_1$, then $0 < \varphi(1) < 1$, so $\varphi = (1 - c)0 + c\frac{1}{\varphi(1)}\varphi$, with $c = \varphi(1) \in (0, 1)$ and $\frac{1}{\varphi(1)}\varphi \in \mathcal{P}_0$; this shows that φ is not extremal.

□

Proof of the theorem. Note that the conditions defining \mathcal{P} in $L^\infty(G)$ are weak* closed conditions, so \mathcal{P} is a weak* closed subset of $L^\infty(G)$. As \mathcal{P}_0 is the intersection of \mathcal{P} with the closed unit ball of $L^\infty(G)$, it is weak* closed in this closed unit ball, hence weak* compact by the Banach-Alaoglu theorem (theorem B.4.1). As \mathcal{P}_0 is also convex, the Krein-Milman theorem (theorem B.5.2) says that the convex hull of $\mathcal{E}(\mathcal{P}_0)$ is weak* dense in \mathcal{P}_0 . Also, the lemma above says that $\mathcal{E}(\mathcal{P}_0) = \mathcal{E}(\mathcal{P}_1) \cup \{0\}$.

Let $\varphi \in \mathcal{P}_1$, and let U be a weak* neighborhood of φ of the form $\{\psi \in \mathcal{P}_1 \mid |\int_G(\varphi - \psi)g_id\mu| < c, 1 \leq i \leq n\}$, with $n \in \mathbb{Z}_{\geq 1}$, $g_1, \dots, g_n \in L^1(G)$ and $c > 0$. We want to find a point of U that is in the convex hull of $\mathcal{E}(\mathcal{P}_1)$. Choose $\varepsilon > 0$ (we will see later how small it needs to be). By the first paragraph and the fact that closed balls in $L^\infty(G)$ are weak* closed (a consequence of the Hahn-Banach theorem), we can find ψ in the convex hull of $\mathcal{E}(\mathcal{P}_1) \cup \{0\}$ such that, for every $i \in \{1, \dots, n\}$, we have $|\int_G(\varphi - \psi)g_id\mu| < c/2$ and such that $\|\psi\|_\infty \geq 1 - \varepsilon$. Write $\psi = c_1\psi_1 + \dots + c_r\psi_r$, with $c_1, \dots, c_r \in [0, 1]$, $\psi_1, \dots, \psi_r \in \mathcal{E}(\mathcal{P}_1)$ and $c_1 + \dots + c_r \leq 1$. Let $a = \frac{1}{\|\psi\|_\infty}$. Then $a\psi = (ac_1)\psi_1 + \dots + (ac_r)\psi_r$ and $ac_1 + \dots + ac_r = 1$, so $a\psi$ is in the convex hull of $\mathcal{E}(\mathcal{P}_1)$. Let's show that $a\psi \in U$. If $i \in \{1, \dots, n\}$, we have

$$\begin{aligned} \left| \int_G(\varphi - a\psi)g_id\mu \right| &\leq \left| \int_G(\varphi - \psi)g_id\mu \right| + \left| \int_G(\psi - a\psi)g_id\mu \right| \\ &< c/2 + |1 - a| \left| \int_G \psi g_id\mu \right| \\ &< c/2 + \varepsilon \left(c/2 + \left| \int_G \varphi g_id\mu \right| \right). \end{aligned}$$

So, if we choose ε small enough so that $\varepsilon (c/2 + |\int_G \varphi g_id\mu|) < c/2$ for every $i \in \{1, \dots, n\}$, the function $a\psi$ will be in U .

□

As \mathcal{P} is a subspace of the space $\mathcal{C}(G)$, we can also consider the topology of compact convergence on \mathcal{P} , that is, of convergence on compact subsets of G . If $\varphi \in \mathcal{P}$, a basis of neighborhoods of φ for this topology is given by $\{\psi \in \mathcal{P} \mid \sup_{x \in K} |\varphi(x) - \psi(x)| < c\}$, for all compact subsets K of G and all $c > 0$.

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Theorem III.4.3. (Raikov) *On the subset \mathcal{P}_1 of \mathcal{P} , the topology of compact convergence and the weak* topology coincide.*

Remark III.4.4. This theorem generalizes problem 6 of problem set 3. (See remark III.3.4.)

Note that the theorem is *not* true for \mathcal{P}_0 . For example, if $G = \mathbb{R}$, then the topology of compact convergence and the weak* topology do not coincide on $\widehat{G} \cup \{0\}$ (see the remark at the end of the solution of problem set 3).

Corollary III.4.5. *The convex hull of $\mathcal{E}(\mathcal{P}_1)$ is dense in \mathcal{P}_1 for the topology of compact convergence.*

Proof of the theorem. We first show that the topology of compact convergence on \mathcal{P}_1 is finer than the weak* topology (this is the easier part). Let $\varphi \in \mathcal{P}_1$. Let $f \in L^1(G)$ and $c > 0$, and let $U = \{\psi \in \mathcal{P}_1 \mid \left| \int_G f(\varphi - \psi) d\mu \right| < c\}$. We want to find a neighborhood of φ in the topology of compact convergence that is contained in U . Let $K \subset G$ be a compact subset such that $\int_{G \setminus K} |f| d\mu < c/3$, and let $V = \{\psi \in \mathcal{P}_1 \mid \sup_{x \in K} |\varphi(x) - \psi(x)| \leq \frac{c}{3\|f\|_1 + 1}\}$. Then, if $\psi \in V$, we have

$$\begin{aligned} \left| \int_G f(\varphi - \psi) d\mu \right| &\leq \left| \int_K f(\varphi - \psi) d\mu \right| + \left| \int_{G \setminus K} f(\varphi - \psi) d\mu \right| \\ &\leq \|f\|_1 \sup_{x \in K} |\varphi(x) - \psi(x)| + 2 \int_{G \setminus K} |f| d\mu \\ &< c \end{aligned}$$

so $\psi \in U$ (on the second line, we use the fact that $\|\varphi\|_\infty = \|\psi\|_\infty = 1$).

Now let's prove the hard direction, i.e. the fact that the weak* topology on \mathcal{P}_1 is finer than the topology of compact convergence. Let $\varphi \in \mathcal{P}_1$, and let $V = \{\psi \in \mathcal{P}_1 \mid \sup_{x \in K} |\varphi(x) - \psi(x)| < c\}$, with $K \subset G$ compact and $c > 0$. Let $\delta > 0$ be such that $\delta + 4\sqrt{\delta} < c$. Let Q be a compact neighborhood of 1 in G such that

$$\sup_{x \in Q} |\varphi(x) - 1| \leq \delta.$$

(Such a Q exists because φ is continuous and $\varphi(1) = 1$.) As Q contains an open set, we have $\mu(Q) \neq 0$. Let $f = \frac{1}{\mu(Q)} \mathbf{1}_Q$. By the first lemma below (applied to $V = L^1(G)$ and $B = \mathcal{P}_1$) and the fact that $G \rightarrow L^1(G)$, $x \mapsto L_{x^{-1}}f$ is continuous (hence $\{L_{x^{-1}}f, x \in K\} \subset L^1(G)$ is compact), we can find a weak* neighborhood U_1 of φ in \mathcal{P}_1 such that, for every $x \in K$ and every $\psi \in U_1$, we have

$$\left| \int_G (\overline{\varphi - \psi}) L_{x^{-1}}f \right| \leq \delta.$$

Then, for every $x \in K$ and every $\psi \in U_1$, we have

$$\begin{aligned} |f * \varphi(x) - f * \psi(x)| &= \left| \int_G f(xy)(\varphi(y^{-1}) - \psi(y^{-1}))dy \right| \\ &= \left| \int_G L_{x^{-1}}f(y)(\overline{\varphi(y) - \psi(y)})dy \right| \quad (\text{see corollary III.2.6}) \\ &\leq \delta. \end{aligned}$$

Let $U_2 = \{\psi \in \mathcal{P}_1 \mid \left| \int_G (\varphi - \psi)f d\mu \right| < \delta\}$. (This is a weak* neighborhood of φ .) Let $\psi \in U_1 \cap U_2$. Then

$$\begin{aligned} \left| \int_G (1 - \psi)f d\mu \right| &\leq \left| \int_G (1 - \varphi)f d\mu \right| + \left| \int_G (\varphi - \psi)f d\mu \right| \\ &\leq \frac{1}{\mu(Q)} \left| \int_Q (1 - \varphi(x))dx \right| + \delta \\ &\leq \sup_{x \in Q} |1 - \varphi(x)| + \delta \\ &\leq 2\delta. \end{aligned}$$

On the other hand, for every $x \in G$, we have

$$\begin{aligned} |f * \psi(x) - \psi(x)| &= \left| \frac{1}{\mu(Q)} \int_G \mathbf{1}_Q(y)\psi(y^{-1}x)dy - \frac{1}{\mu(Q)} \int_Q \psi(x)dy \right| \\ &= \frac{1}{\mu(Q)} \left| \int_Q (\psi(y^{-1}x) - \psi(x))dy \right| \\ &\leq \frac{1}{\mu(Q)} \int_Q |\psi(y^{-1}x) - \psi(x)|dy \\ &\leq \frac{1}{\mu(Q)} \int_Q \sqrt{2(1 - \operatorname{Re}(\psi(y)))}dy \quad (\text{see the second lemma below}) \\ &\leq \frac{\sqrt{2}}{\mu(Q)} \left(\int_Q (1 - \operatorname{Re}(\psi(y)))dy \right)^{1/2} \left(\int_Q dy \right)^{1/2} \quad (\text{Cauchy-Schwarz}) \\ &\leq \frac{\sqrt{2}}{\sqrt{\mu(Q)}} \left| \int_Q (1 - \psi(y))dy \right|^{1/2} \\ &= \sqrt{2} \left| \int_G (1 - \psi)f d\mu \right|^{1/2} \end{aligned}$$

As $\psi \in U_2$, the previous calculation shows that this is $\leq 2\sqrt{\delta}$. Note that this also applies to $\psi = \varphi$, because of course φ is in $U_1 \cap U_2$. Putting all these bounds together, we get, is $\psi \in U_1 \cap U_2$ and $x \in K$,

$$\begin{aligned} |\psi(x) - \varphi(x)| &\leq |\psi(x) - f * \psi(x)| + |f * \psi(x) - f * \varphi(x)| + |f * \varphi(x) - \varphi(x)| \\ &\leq \delta + 4\sqrt{\delta} \\ &< c. \end{aligned}$$

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So $U_1 \cap U_2 \subset V$, and we are done. □

Lemma III.4.6. *Let V be a Banach space, and let B be a norm-bounded subset of $\text{Hom}(V, \mathbb{C})$. Then the topology of compact convergence (i.e. of uniform convergence on compact subsets of V) and the weak* topology coincide on B .*

Proof. We want to compare the topology of pointwise convergence on V (i.e. the weak* topology) and the topology of compact convergence on V . The second one is finer than the first one on all of $\text{Hom}(V, \mathbb{C})$, so we just need to show that the first one is finer than the second on B .

Let $T_0 \in B$, let $K \subset V$ be compact and let $c > 0$. We want to find a weak* neighborhood of T_0 in B contained in $\{T \in B \mid \sup_{x \in K} |T(x) - T_0(x)| < c\}$. Let $M = \sup_{T \in B} \|T\|_{op}$ (this is finite because B is bounded). Let $x_1, \dots, x_n \in K$ such that K is contained in the union of the open balls centered at the x_i with radius $\frac{c}{3M}$. Let $T \in B$ be such that $|(T - T_0)(x_i)| < c/3$ for $i = 1, \dots, n$ (this defines a weak* neighborhood of T). For every $x \in K$, there exists $i \in \{1, \dots, n\}$ such that $\|x - x_i\| < \frac{c}{3M}$, and then we have

$$\begin{aligned} |T(x) - T_0(x)| &\leq |T(x - x_i)| + |(T - T_0)(x_i)| + |T_0(x - x_i)| \\ &\leq \|T\|_{op} \|x - x_i\| + c/3 + \|T_0\|_{op} \|x - x_i\| \\ &< c/3 + 2M \frac{c}{3M} = c. \end{aligned}$$

So $T \in U$. □

Lemma III.4.7. *Let $\varphi \in \mathcal{P}_1$. Then, for all $x, y \in G$, we have*

$$|\varphi(x) - \varphi(y)|^2 \leq 2 - 2 \operatorname{Re}(\varphi(yx^{-1})).$$

Proof. By theorem III.2.5, we can find a unitary representation (π, V) of G and $v \in V$ such that $\varphi(x) = \langle \pi(x)(v), v \rangle$ for every $x \in G$. Also, as $\varphi(1) = 1$, we have $\|v\| = 1$. So, for all $x, y \in G$, we have

$$\begin{aligned} |\varphi(x) - \varphi(y)|^2 &= |\langle (\pi(x) - \pi(y))(v), v \rangle|^2 \\ &= |\langle v, (\pi(x^{-1}) - \pi(y^{-1}))(v) \rangle|^2 \\ &\leq \|(\pi(x^{-1}) - \pi(y^{-1}))(v)\|^2 \\ &= \|\pi(x^{-1})(v)\|^2 + \|\pi(y^{-1})(v)\|^2 - 2 \operatorname{Re}(\langle \pi(x^{-1})(v), \pi(y^{-1})(v) \rangle) \\ &= 2 - 2 \operatorname{Re}(\langle \pi(x^{-1})(v), \pi(y^{-1})(v) \rangle) \\ &= 2 - 2 \operatorname{Re}(\langle \pi(yx^{-1})(v), v \rangle) \\ &= 2 - 2 \operatorname{Re}(\varphi(yx^{-1})). \end{aligned}$$

□

III.5 The Gelfand-Raikov theorem

Theorem III.5.1. (Gelfand-Raikov) *Let G be a locally compact group. Then, for all $x, y \in G$ such that $x \neq y$, there exists an irreducible unitary representation π of G such that $\pi(x) \neq \pi(y)$.*

More precisely, there exists an irreducible unitary representation (π, V) of G and a vector $v \in V$ such that $\langle \pi(x)(v), v \rangle \neq \langle \pi(y)(v), v \rangle$.

Proof. Let $x, y \in G$. Suppose that $\langle \pi(x)(v), v \rangle = \langle \pi(y)(v), v \rangle$ for every irreducible unitary representation (π, V) of G and every $v \in V$. By theorem III.3.2, this implies that $\varphi(x) = \varphi(y)$ for every $\varphi \in \mathcal{E}(\mathcal{P}_1)$, hence for every $\varphi \in \mathcal{P}_1$ by corollary III.4.5 (and the fact that $\{x, y\}$ is a compact subset of G), hence for every $\varphi \in \mathcal{P}$ because $\mathcal{P} = \mathbb{R}_{\geq 0} \cdot \mathcal{P}_1$.

Let π_L be the left regular representation of G , i.e. the representation of G on $L^2(G)$ defined by $\pi_L(z)(f) = L_z f$ for $z \in G$ and $f \in L^2(G)$. This is a unitary representation of G , so, by the first paragraph and by proposition III.2.4, we have $\langle \pi_L(x)(f), f \rangle = \langle \pi_L(y)(f), f \rangle$ for every $f \in L^2(G)$. Let $f_1, f_2 \in L^2(G)$. Then

$$\langle \pi_L(x)(f_1 + f_2), f_1 + f_2 \rangle = \langle \pi_L(x)(f_1), f_1 \rangle + \langle \pi_L(x)(f_2), f_2 \rangle + \langle \pi_L(x)(f_1), f_2 \rangle + \langle \pi_L(x)(f_2), f_1 \rangle$$

and

$$\langle \pi_L(x)(f_1 + if_2), f_1 + if_2 \rangle = \langle \pi_L(x)(f_1), f_1 \rangle + \langle \pi_L(x)(f_2), f_2 \rangle - i\langle \pi_L(x)(f_1), f_2 \rangle + i\langle \pi_L(x)(f_2), f_1 \rangle,$$

so

$$\begin{aligned} 2\langle \pi_L(x)(f_1), f_2 \rangle &= \langle \pi_L(x)(f_1 + f_2), f_1 + f_2 \rangle + i\langle \pi_L(x)(f_1 + if_2), f_1 + if_2 \rangle \\ &\quad - (1 + i)(\langle \pi_L(x)(f_1), f_1 \rangle + \langle \pi_L(x)(f_2), f_2 \rangle). \end{aligned}$$

We have a similar identity for $\pi_L(y)$, and this shows that

$$\langle \pi_L(x)(f_1), f_2 \rangle = \langle \pi_L(y)(f_1), f_2 \rangle.$$

Now note that

$$\begin{aligned} \langle \pi_L(x)(f_1), f_2 \rangle &= \int_G L_x f_1(z) \overline{f_2(z)} dz \\ &= \int_G f_1(x^{-1}z) \overline{f_2(z)} dz \\ &= \int_G \overline{f_2(z)} \widetilde{f_1}(z^{-1}x) dz \\ &= \overline{f_2 * \widetilde{f_1}(x)} \end{aligned}$$

(remember that $\widetilde{f_1} \in L^2(G)$ is defined by $\widetilde{f_1}(z) = \overline{f_1(z^{-1})}$), so $f_2 * \widetilde{f_1}(x) = \overline{\langle \pi_L(x)(f_1), f_2 \rangle}$. This calculation also shows that $f_2 * \widetilde{f_1}$ makes sense and is continuous.

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As $f \mapsto \tilde{f}$ is an involution on $L^2(G)$, we deduce that $f_1 * f_2(x) = f_1 * f_2(y)$ for all $f_1, f_2 \in L^2(G)$, and in particular for all $f_1, f_2 \in \mathcal{C}_c(G)$. Let $f \in \mathcal{C}_c(G)$, and let $(\psi_U)_{U \in \mathcal{U}}$ be an approximate identity. We have $\psi_U \in \mathcal{C}_c(G)$ for every $U \in \mathcal{U}$, and $\psi_U * f \xrightarrow{U \rightarrow \{1\}} f$ for $\|\cdot\|_\infty$ by proposition I.4.1.9 (and the fact that f is uniformly continuous, which is proposition I.1.12). As $\psi_U * f(x) = \psi_U * f(y)$ for every $U \in \mathcal{U}$, this implies that $f(x) = f(y)$. But then we must have $x = y$ (by Urysohn's lemma).

□

IV The Peter-Weyl theorem

IV.1 Compact operators

Definition IV.1.1. Let V and W be Banach spaces, and let B be the closed unit ball in V . A continuous linear operator $T : V \rightarrow W$ is called *compact* if $\overline{T(B)}$ is compact.

Example IV.1.2. (1) If $\text{Im}(T)$ is finite-dimensional (i.e. if T has finite rank), then T is compact.

(2) If T is a limit of operators of finite rank, then T is compact (see problem 6 of problem set 5). Conversely, if W is a Hilbert space, then every compact operators $T : V \rightarrow W$ is a limit of operators of finite rank.¹

(3) The identity of V is compact if and only if V is finite-dimensional. (This is a consequence of Riesz's lemma, see theorem B.4.2.)

In this class, we will only need to use self-adjoint compact endomorphisms of Hilbert space. A much simpler version of the spectral theorem holds for them.

Theorem IV.1.3. Let V be a Hilbert space over \mathbb{C} , and let $T : V \rightarrow V$ be a continuous endomorphism of V . Assume that T is compact and self-adjoint, and write $V_\lambda = \text{Ker}(T - \lambda \text{id}_V)$ for every $\lambda \in \mathbb{C}$.

Then :

- (i) If $V_\lambda \neq 0$, then $\lambda \in \mathbb{R}$.
- (ii) If $\lambda, \mu \in \mathbb{C}$ and $\lambda \neq \mu$, then $V_\mu \subset V_\lambda^\perp$.
- (iii) If $\lambda \in \mathbb{C} - \{0\}$, then $\dim_{\mathbb{C}} V_\lambda < +\infty$.
- (iv) $\{\lambda \in \mathbb{C} | V_\lambda \neq 0\}$ is finite or countable, and its only possible limit point is 0.
- (v) $\bigoplus_{\lambda \in \mathbb{C}} V_\lambda$ is dense in V .

Proof. We prove (i). Let $\lambda \in \mathbb{C}$ such that $V_\lambda \neq 0$, and choose $v \in V_\lambda$ nonzero. Then

$$\lambda \|v\|^2 = \langle \lambda v, v \rangle = \langle T(v), v \rangle = \langle v, T^*(v) \rangle = \langle v, T(v) \rangle = \bar{\lambda} \|v\|^2.$$

¹This is not true in general, see <https://mathscinet.ams.org/mathscinet-getitem?mr=402468>.

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As $\|v\| \neq 0$, this implies that $\lambda \in \mathbb{R}$.

We prove (ii). Let $\lambda, \mu \in \mathbb{C}$ such that $\lambda \neq \mu$, and let $v \in V_\lambda$ and $w \in V_\mu$. We want to prove that $\langle v, w \rangle = 0$. By (i), it suffices to treat the case where $\lambda, \mu \in \mathbb{R}$ (otherwise $v = w = 0$). In that case, we have

$$\lambda \langle v, w \rangle = \langle T(v), w \rangle = \langle v, T(w) \rangle = \bar{\mu} \langle v, w \rangle = \mu \langle v, w \rangle,$$

so $\langle v, w \rangle = 0$.

Let $r > 0$. Let $W = \overline{\bigoplus_{|\lambda| \geq r} V_\lambda}$. We want to show that $\dim W < +\infty$, which will imply (iii) and (iv). Choose a Hilbert basis $(e_i)_{i \in I}$ of W made up of eigenvectors of T , i.e. such that, for every $i \in I$, we have $T(e_i) = \lambda_i e_i$ with $|\lambda_i| \geq r$. If I is infinite, then the family $(T(e_i))_{i \in I}$ cannot have a convergent (non-stationary) subsequence. Indeed, if we had an injective map $\mathbb{N} \rightarrow I$, $n \mapsto i_n$, such that $(T(e_{i_n}))_{n \geq 0}$ converges to some vector v of V , then $\lambda_{i_n} e_{i_n} \rightarrow v$, so v is in the closure of $\text{Span}(e_{i_n}, n \geq 0)$. But on the other hand, for every $n \geq 0$, $\langle v, e_{i_n} \rangle = \lim_{m \rightarrow +\infty} \langle \lambda_{i_m} e_{i_m}, e_{i_n} \rangle = 0$, so $v \in \text{Span}(e_{i_n}, n \geq 0)^\perp$. This forces $v = 0$. But $\|v\| = \lim_{n \rightarrow +\infty} \|\lambda_{i_n} e_{i_n}\| \geq r > 0$, contradiction. As T is compact, this shows that I cannot be infinite, i.e. that $\dim(W) < +\infty$.

Let's prove (v). Let $W' = \bigoplus_{\lambda \in \mathbb{C}} V_\lambda$, and $W = W'^\perp$. We want to show that $W = 0$. So suppose that $W \neq 0$. As T is self-adjoint and W' is clearly stable by T , we have $T(W) \subset W$. (If $v \in W$, then for every $w \in W'$, $\langle T(v), w \rangle = \langle v, T(w) \rangle = 0$.) By definition of W , we have $\text{Ker}(T|_W) = \{0\}$, hence $\|T|_W\|_{op} > 0$. Let $B = \{x \in W \mid \|x\| = 1\}$. As $\|T|_W\|_{op} = \sup_{x \in B} |\langle T(x), x \rangle|$ by the lemma below, there exists a sequence $(x_n)_{n \geq 0}$ of elements of B such that $\langle T(x_n), x_n \rangle \rightarrow \lambda$ as $n \rightarrow +\infty$, where $\lambda = \pm \|T|_W\|_{op}$. Then

$$0 \leq \|T(x_n) - \lambda x_n\|^2 = \|T(x_n)\|^2 + \lambda^2 \|x_n\|^2 - 2\lambda \langle T(x_n), x_n \rangle \leq 2\lambda^2 - 2\lambda \langle T(x_n), x_n \rangle$$

converges to 0 as $n \rightarrow +\infty$, so $T(x_n) - \lambda x_n$ itself converges to 0. As T is compact, we may assume that the sequence $(T(x_n))_{n \geq 0}$ has a limit in W , say w . Then $T(w) - \lambda w = 0$. By definition of W , we must have $w = 0$. But then $T(x_n) \rightarrow 0$, so $\langle T(x_n), x_n \rangle \rightarrow 0$, so $\lambda = 0 = \|T|_W\|_{op}$, a contradiction.

□

Lemma IV.1.4. *Let V be a Hilbert space, and let $T \in \text{End}(V)$ be self-adjoint. Then*

$$\|T\|_{op} = \sup_{x \in V, \|x\|=1} |\langle T(x), x \rangle|.$$

Proof. Let $c = \sup_{x \in V, \|x\|=1} |\langle T(x), x \rangle|$. We have $c \leq \|T\|_{op}$ by definition of $\|T\|_{op}$. As $\|T\|_{op} = \sup_{x, y \in V, \|x\|=\|y\|=1} |\langle T(x), y \rangle|$, to prove the other inequality, it suffices to show that $|\langle T(x), y \rangle| \leq c$ for all $x, y \in V$ such that $\|x\| = \|y\| = 1$. Let $x, y \in V$. After multiplying y by a norm 1 element of \mathbb{C} (which doesn't change $\|y\|$), we may assume that $\langle T(x), y \rangle \in \mathbb{R}$. Then

$$\langle T(x), y \rangle = \frac{1}{4} (\langle T(x+y), x+y \rangle - \langle T(x-y), x-y \rangle),$$

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so

$$|\langle T(x), y \rangle| \leq \frac{c}{4}(\|x + y\|^2 + \|x - y\|^2) = \frac{c}{2}(\|x\|^2 + \|y\|^2)$$

(the last equality is the parallelogram identity). This shows the desired result. \square

Here are some results that are true for compact operators in greater generality (see [16] 4.16-4.25).

Theorem IV.1.5. *Let V be a Banach space, and let $T \in \text{End}(V)$ be a compact endomorphism. We write $\sigma(T)$ for the spectrum of T in $\text{End}(V)$, i.e.*

$$\sigma(T) = \{\lambda \in \mathbb{C} \mid \lambda \text{id}_V - T \notin \text{End}(V)^\times\}.$$

Then :

- (i) *For every $\lambda \neq 0$, the image of $T - \lambda \text{id}_V$ is closed.*
- (ii) *For every $\lambda \in \sigma(T) - \{0\}$, we have $\text{Im}(T - \lambda \text{id}_V) \neq V$ and $\dim(\text{Ker}(T - \lambda \text{id}_V)) = \dim(V / \text{Im}(T - \lambda \text{id}_V))$.² In particular, $\text{Ker}(T - \lambda \text{id}_V) \neq \{0\}$.*
- (iii) *For every $\lambda \neq 0$, the increasing sequence $(\text{Ker}((T - \lambda \text{id}_V)^n))_{n \geq 1}$ stabilizes, and its limit is finite-dimensional.*
- (iv) *If $\dim_{\mathbb{C}} V = +\infty$, then $0 \in \sigma(T)$.*
- (v) *The subset $\sigma(T) - \{0\}$ of $\mathbb{C} - \{0\}$ is discrete. In particular, for every $r > 0$, there are only finitely many $\lambda \in \sigma(T)$ such that $|\lambda| \geq r$.*

In particular, if V is a Hilbert space and T is self-adjoint, then (v) of theorem IV.1.3 become

$$V = \overline{\bigoplus_{\lambda \in \sigma(T)} \text{Ker}(T - \lambda \text{id}_V)}.$$

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The goal of this section is to prove the following theorem. (Compare with proposition I.3.3.3.)

²Note that this generalizes the rank-nullity theorem.

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Theorem IV.2.1. *Let G be a compact group, and let V be a unitary representation of G . Then there exists a family $(W_i)_{i \in I}$ of pairwise orthogonal subrepresentations of V such that each W_i is irreducible and that*

$$V = \overline{\bigoplus_{i \in I} W_i}.$$

We already saw the crucial construction in problem set 5. Let's summarize it in a proposition.

Proposition IV.2.2. *(See problem 6 of problem set 5.) Let G be a compact group, let dx be the normalized Haar measure on G , and let (π, V) be a unitary representation of G . If $u \in V$, then the formula*

$$T(v) = \int_G \langle v, \pi(x)(u) \rangle \pi(x)(u) dx$$

defines a continuous G -equivariant self-adjoint compact endomorphism of V , and we have $T = 0$ if and only if $u = 0$.

In fact, we even know that T is positive, i.e. that $\langle T(v), v \rangle \geq 0$ for every $v \in V$.

Corollary IV.2.3. *Let V be a nonzero unitary representation of a compact group G . Then V contains an irreducible representation of G .*

Proof. If V is finite-dimensional, then any nonzero G -invariant subspace of V of minimal dimension has to be irreducible.

In the general case, choose $u \in V - \{0\}$, and let $T \in \text{End}(V)$ be the endomorphism of V constructed in the proposition. By the spectral theorem for self-adjoint compact operators (theorem IV.1.3), we have

$$V = \overline{\bigoplus_{\lambda \in \mathbb{C}} \text{Ker}(T - \lambda \text{id}_V)}.$$

As $T \neq 0$, the closed subspace $\text{Ker}(T)$ of V is not equal to V . By the equality above, there exists $\lambda \in \mathbb{C} - \{0\}$ such that $W := \text{Ker}(T - \lambda \text{id}_V) \neq 0$. Then W is a nonzero closed subspace of V , and it is G -invariant because T is G -equivariant, and stable by T by definition. Also, the space W is finite-dimensional by (iii) of theorem IV.1.3. So W has an irreducible subrepresentation by the beginning of the proof, and we are done. □

Proof of the theorem. By Zorn's lemma, we can find a maximal collection $(W_i)_{i \in I}$ of pairwise orthogonal irreducible subrepresentations of V . Suppose that the direct sum of the W_i is not dense in V , then $W := (\bigoplus_{i \in I} W_i)^\perp$ is a nonzero closed invariant subspace of V (see lemma I.3.2.6). By the corollary above, the representation W has an irreducible subrepresentation, which contradicts the maximality of the family $(W_i)_{i \in I}$. Hence $V = \overline{\bigoplus_{i \in I} W_i}$. □

IV.2 Semisimplicity of unitary representations of compact groups

We finish this section with a remark on two different notions of equivalence for unitary representations. Remember that two continuous representations V_1 and V_2 of a topological group G are called *equivalent* (or *isomorphic*) if there exists a continuous G -equivariant isomorphism $V_1 \rightarrow V_2$ with a continuous inverse.

Definition IV.2.4. Two unitary representations V_1 and V_2 of a topological group G are called *unitarily equivalent* if there exists a G -equivariant isomorphism $V_1 \rightarrow V_2$ that is an isometry.

Two unitarily equivalent representations are clearly equivalent.

Example IV.2.5. Let G be a locally compact group, let μ be a left Haar measure on G , and let ν be the right Haar measure defined by $\nu(E) = \mu(E^{-1})$.

Then the left and right regular representations of G are unitarily equivalent, by sending $f \in L^2(G, \mu)$ to the element $x \mapsto \Delta(x)^{-1/2} f(x^{-1})$ of $L^2(G, \nu)$. (See proposition I.2.12.)

Proposition IV.2.6. Suppose that V_1 and V_2 are irreducible unitary representations of G . Then they are equivalent if and only if they are unitarily equivalent.

Proof. Suppose that V_1 and V_2 are equivalent, and let $U : V_1 \rightarrow V_2$ be a G -equivariant isomorphism. We denote by $\langle \cdot, \cdot \rangle_1$ and $\langle \cdot, \cdot \rangle_2$ the inner products of V_1 and V_2 . Let $B : V_1 \times V_1 \rightarrow \mathbb{C}$, $(v, w) \mapsto \langle U(v), U(w) \rangle_2$. This is a Hermitian sesquilinear form on V_1 , and it is bounded because U is bounded. By the lemma below, there exists a self-adjoint endomorphism $T \in \text{End}(V_1)$ such that, for all $v, w \in V$, we have $B(v, w) = \langle T(v), w \rangle_1$. Let's prove that T is G -equivariant. Let $v \in V$ and $x \in G$. For every $w \in V$, we have

$$\begin{aligned} \langle T(\pi_1(x)(v)), w \rangle_1 &= B(\pi_1(x)(v), w) \\ &= \langle U(\pi_1(x)(v)), U(w) \rangle_2 \\ &= \langle \pi_2(x)(U(v)), U(w) \rangle_2 \\ &= \langle U(v), \pi_2(x^{-1})U(w) \rangle_2 \\ &= \langle U(v), U(\pi_1(x)^{-1}(w)) \rangle_2 \\ &= B(v, \pi_1(x^{-1})(w)) \\ &= \langle T(v), \pi_1(x)^{-1}(w) \rangle_1 \\ &= \langle \pi_1(x)(T(v)), w \rangle_1, \end{aligned}$$

so $T(\pi_1(x)(v)) = \pi_1(x)(T(v))$. As V_1 is irreducible, Schur's lemma (theorem I.3.4.1) implies that $T = \lambda \text{id}_{V_1}$ for some $\lambda \in \mathbb{C}$. As $\langle T(v), v \rangle_1 = \langle U(v), U(v) \rangle_2 > 0$ for every nonzero $v \in V_1$, we must have $\lambda \in \mathbb{R}_{>0}$. Then $\lambda^{-1/2}U$ is an isometry, so V_1 and V_2 are unitarily equivalent. □

Lemma IV.2.7. Let V be a Hilbert space, and let $B : V \times V \rightarrow \mathbb{C}$ be a bounded sesquilinear form (i.e. B is \mathbb{C} -linear in the first variable and \mathbb{C} -antilinear in the second variable; the boundedness conditions means that $\sup_{v, w \in V, \|v\|=\|w\|=1} |B(v, w)| < +\infty$).

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Then there exists a unique $T \in \text{End}(V)$ such that, for all $v, w \in V$,

$$B(v, w) = \langle T(v), w \rangle.$$

Moreover, T is self-adjoint if and only if B is Hermitian (which means that $B(w, v) = \overline{B(v, w)}$ for all $v, w \in V$).

Proof. The uniqueness of T is clear (it follows from the fact that $V^\perp = \{0\}$.)

If $v \in V$, then the map $V \rightarrow \mathbb{C}$, $w \mapsto \overline{B(v, w)}$ is a continuous linear functional on V , so there exists a unique $T(v) \in V$ such that $B(v, w) = \langle T(v), w \rangle$ for every $w \in V$. The linearity of T follows from the fact that B is linear in the first variable. Moreover, for every $v \in V$, we have

$$\|T(v)\| = \sup_{w \in V, \|w\|=1} |\langle T(v), w \rangle| = \sup_{w \in V, \|w\|=1} |B(v, w)| \leq C\|v\|,$$

where

$$C = \sup_{x, y \in V, \|x\|=\|y\|=1} |B(x, y)|.$$

So T is bounded.

Finally, T is self-adjoint if and only, for all $v, w \in V$, we have

$$B(v, w) = \langle T(v), w \rangle = \langle v, T(w) \rangle = \overline{B(w, v)}.$$

This proves the last statement. □

Definition IV.2.8. We denote by \widehat{G} the set of equivalence (or unitary equivalence) classes of irreducible unitary representations of G , and call it the *unitary dual* of G .

If $(\pi, V) \in \widehat{G}$, we write $\dim(\pi)$ and $\text{End}(\pi)$ for $\dim(V)$ and $\text{End}(V)$.

Note that this notation agrees with the one used in problem set 3 for a commutative group.

IV.3 Matrix coefficients

Definition IV.3.1. Let (π, V) be a unitary representation of a topological group G . A *matrix coefficient* of (π, V) is a function $G \rightarrow \mathbb{C}$ of the form $x \mapsto \langle \pi(x)(u), v \rangle$, where $u, v \in V$.

Note that matrix coefficients are continuous functions. We denote by \mathcal{E}_π or \mathcal{E}_V the subspace of $\mathcal{C}(G)$ spanned by the matrix coefficients of π .

We start by proving some general results that are true for any group G .

Proposition IV.3.2. *Let (π, V) be a unitary representation of G .*

- (i) *The subspace \mathcal{E}_π of $\mathcal{C}(G)$ only depends on the unitary equivalence class of π , and it is invariant by the operators L_x and R_x , for every $x \in G$.*
- (ii) *If V is finite-dimensional, then \mathcal{E}_π is finite-dimensional and $\dim(\mathcal{E}_\pi) \leq (\dim V)^2$.*
- (iii) *If $V = V_1 \oplus \dots \oplus V_n$ with the V_i G -invariant and pairwise orthogonal, then $\mathcal{E}_\pi = \sum_{i=1}^n \mathcal{E}_{V_i}$.*

In particular, we get an action of $G \times G$ on \mathcal{E}_π by making $(x, y) \in G \times G$ act by $L_x \circ R_y = R_y \circ L_x$.

Proof. (i) The first statement is obvious. To prove the second statement, let $v, w \in V$ and $x \in G$. Then, for every $y \in G$,

$$\langle \pi(x^{-1}y)(v), w \rangle = \langle \pi(y)(v), \pi(x)(w) \rangle$$

and

$$\langle \pi(yx)(v), w \rangle = \langle \pi(y)(\pi(x)(v)), w \rangle,$$

so the functions $y \mapsto \langle \pi(x^{-1}y)(v), w \rangle$ and $y \mapsto \langle \pi(yx)(v), w \rangle$ are also matrix coefficients of π .

- (ii) Let (e_1, \dots, e_n) be a basis of V . For $i, j \in \{1, \dots, n\}$, write φ_{ij} for the function $G \rightarrow \mathbb{C}$, $x \mapsto \langle \pi(x)(e_i), e_j \rangle$. If $v, w \in V$, we can write $v = \sum_{i=1}^n a_i e_i$ and $w = \sum_{j=1}^n b_j e_j$, and then we have, for every $x \in G$,

$$\langle \pi(x)(v), w \rangle = \sum_{i,j=1}^n a_i \bar{b}_j \varphi_{ij}(x).$$

So the family $(\varphi_{ij})_{1 \leq i,j \leq n}$ spans \mathcal{E}_π .

- (iii) For every $i \in \{1, \dots, n\}$, we clearly have $\mathcal{E}_{V_i} \subset \mathcal{E}_\pi$. So $\sum_{i=1}^n \mathcal{E}_{V_i} \subset \mathcal{E}_\pi$. Conversely, let $v, w \in V$, and write $v = \sum_{i=1}^n v_i$ and $w = \sum_{i=1}^n w_i$, with $v_i, w_i \in V_i$. Then, for every $x \in G$,

$$\langle \pi(x)(v), w \rangle = \sum_{i,j=1}^n \langle \pi(x)(v_i), w_j \rangle = \sum_{i=1}^n \langle \pi(x)(v_i), w_i \rangle.$$

So the function $x \mapsto \langle \pi(x)(v), w \rangle$ is in $\sum_{i=1}^n \mathcal{E}_{V_i}$.

□

Definition IV.3.3. Let (π, V) and (π', V') be continuous representation of V . We define an action ρ of $G \times G$ on $\text{Hom}(V, V')$ by

$$\rho(x, y)(T) = \pi'(y) \circ T \circ \pi(x)^{-1},$$

for $T \in \text{Hom}(V, V')$ and $x, y \in G$.

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Proposition IV.3.4. *We have*

$$\mathrm{Hom}_G(V, V') = \{T \in \mathrm{Hom}(V, V') \mid \forall x \in G, \rho(x, x)(T) = T\}.$$

Moreover, if the maps $G \rightarrow \mathrm{End}(V)$, $x \mapsto \pi(x)$ and $G \rightarrow \mathrm{End}(V')$, $x \mapsto \pi'(x)$ are continuous (for example if V and V' are finite-dimensional, see proposition I.3.5.1), then the action defined above is a continuous representation of $G \times G$ on $\mathrm{Hom}(V, V')$.

Proof. The first statement is obvious. The second statement follows from the continuity of the composition on Hom spaces, and of inversion on G . □

In particular, we get actions of $G \times G$ on $\mathrm{End}(V)$ and $V^* := \mathrm{Hom}(V, \mathbb{C})$ (using the trivial action of G on \mathbb{C}); the second one gives an action of G on V^* by restriction to the first factor (if $x \in G$ and $\Lambda \in V^*$, then (x, Λ) is sent to $\Lambda \circ \pi(x)^{-1}$). This will be the default action on these spaces.

Definition IV.3.5. Let (π, V) and (π', V') be continuous representations of V . We define an action ρ of $G \times G$ on the algebraic tensor product $V \otimes_{\mathbb{C}} V'$ by

$$\rho(x, y)(v \otimes w) = \pi(x)(v) \otimes \pi'(y)(w),$$

for $x, y \in G$, $v \in V$ and $w \in V'$.

This action is well-defined because, for all $x, y \in G$, the map $V \times V' \rightarrow V \otimes_{\mathbb{C}} V'$, $(v, w) \mapsto \pi(x)(v) \otimes \pi'(y)(w)$ is bilinear, hence induces a map $\rho(x, y) : V \otimes_{\mathbb{C}} V' \rightarrow V \otimes_{\mathbb{C}} V'$. If V and V' are finite-dimensional, the resulting action of $G \times G$ on $V \otimes_{\mathbb{C}} V'$ is continuous by proposition I.3.5.1.

Note that, if we restrict the action of $G \times G$ on $V \otimes_{\mathbb{C}} W$ to the first (resp. the second) factor, we get a representation equivalent to $V^{\oplus \dim(W)}$ (resp. $W^{\oplus \dim(V)}$).

Proposition IV.3.6. *Let V, W be continuous representations of G .*

- (i) *The map $V^* \otimes_{\mathbb{C}} W \rightarrow \mathrm{Hom}(V, W)$ sending $\Lambda \otimes w$ (with $\Lambda \in V^*$, $w \in W$) to the linear operator $V \rightarrow W$, $v \mapsto \Lambda(v)w$ is well-defined and $G \times G$ -equivariant. If V and W are finite-dimensional, it is an equivalence of continuous representations.*
- (ii) *The map $V^* \otimes_{\mathbb{C}} V \rightarrow \mathcal{C}(G)$ sending $\Lambda \otimes v$ (with $\Lambda \in V^*$, $v \in V$) to the function $G \rightarrow \mathbb{C}$, $x \mapsto \Lambda(\pi(x)(v))$ is well-defined and $G \times G$ -equivariant, and its image is \mathcal{E}_V if V is unitary.*

In particular, if V is finite-dimensional and unitary, we get a surjective $G \times G$ -equivariant map $\mathrm{End}(V) \rightarrow \mathcal{E}_V$.

Remark IV.3.7. Point (ii) suggests a way to generalize the definition of a matrix coefficients to the non-unitary case : just define a matrix coefficient as the image of a pure tensor by the map $V^* \otimes_{\mathbb{C}} V \rightarrow \mathcal{C}(G)$.

Proof. In this proof, we will denote all the actions of G and $G \times G$ by \cdot (this should not cause confusion, as each space has at most one action).

- (i) The map is well-defined, because the map $V^* \otimes_{\mathbb{C}} W \rightarrow \text{Hom}(V, W)$ sending (Λ, w) to $(v \mapsto \Lambda(v)w)$ is bilinear. Let's denote it by φ . To check that it is $G \times G$ -equivariant, it suffices to check it on pure tensors (because they generate $V^* \otimes_{\mathbb{C}} W$). So let $\Lambda \in V^*$, $w \in W$, $x, y \in G$. For every $v \in V$, we have

$$\varphi((x, y) \cdot (\Lambda \otimes w))(v) = \varphi((y \cdot \Lambda) \otimes (x \cdot w))(v) = \Lambda(y^{-1} \cdot v)(x \cdot w)$$

and

$$((x, y) \cdot \varphi(\Lambda \otimes w))(v) = x \cdot (\varphi(\Lambda \otimes w)(y^{-1} \cdot v)) = x \cdot (\Lambda(y^{-1} \cdot v)w) = \Lambda(y^{-1} \cdot v)(x \cdot w).$$

So

$$\varphi(x \cdot (\Lambda \otimes w)) = x \cdot \varphi(\Lambda \otimes w).$$

Suppose that V is finite-dimensional, let (e_1, \dots, e_n) be a basis of V , and let (e_1^*, \dots, e_n^*) be the dual basis. Define $\psi : \text{Hom}(V, W) \rightarrow V^* \otimes_{\mathbb{C}} W$ by sending T to $\sum_{i=1}^n e_i^* \otimes T(e_i)$. Let's show that ψ is the inverse of φ .

If $j \in \{1, \dots, n\}$ and $w \in W$, then

$$\psi(\varphi(e_j^* \otimes w)) = \sum_{i=1}^n e_i^* \otimes (\varphi(e_j^* \otimes w)(e_i)) = e_j^* \otimes w.$$

As the elements $e_j^* \otimes w$, for $j \in \{1, \dots, n\}$ and $w \in W$, generate $V^* \otimes_{\mathbb{C}} W$, this shows that $\psi \circ \varphi = \text{id}$.

Conversely, if $T \in \text{Hom}(V, W)$, then, for every $v \in V$,

$$\varphi(\psi(T)) = \sum_{i=1}^n \varphi(e_i^* \otimes T(e_i))(v) = \sum_{i=1}^n e_i^*(v)T(e_i) = T(v),$$

because $v = \sum_{i=1}^n e_i^*(v)e_i$. So $\varphi(\psi(T)) = T$.

This shows that, if V is finite-dimensional, the map $V^* \otimes_{\mathbb{C}} W \rightarrow \text{Hom}(V, W)$ is an isomorphism. The last statement follows immediately.

- (ii) The map is well-defined because the map $V^* \times V \rightarrow \mathcal{C}(G)$ sending (Λ, v) to the function $x \mapsto \Lambda(\pi(x)(v))$ is bilinear. Let's denote it by α . We show that α is $G \times G$ -equivariant.

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As before, it suffices to check it on pure tensors. So let $\Lambda \in V^*$, $v \in V$ and $x, y \in G$. For every $z \in G$, we have

$$\alpha((x, y) \cdot (\Lambda \otimes v))(z) = \Lambda(x^{-1} \cdot (z \cdot (y \cdot v))) = \Lambda((x^{-1}zy) \cdot v) = ((L_x \circ R_y)(\alpha(\Lambda \otimes v)))(z),$$

$$\text{hence } \alpha((x, y) \cdot (\Lambda \otimes v)) = (L_x \circ R_y)(\alpha(\Lambda \otimes v)).$$

Finally, we show that the image of α is \mathcal{E}_V if V is unitary. Let $\Lambda \in V^*$. As V is a Hilbert space, there exists a unique $v \in V$ such that $\Lambda = \langle \cdot, v \rangle$. So, for every $w \in V$ and every $x \in G$, we have

$$\alpha(\Lambda \otimes w)(v) = \langle \pi(x)(w), v \rangle.$$

This shows that $\alpha(\Lambda \otimes w)$ is a matrix coefficient of π , and also that we get all the matrix coefficients of π in this way. □

Now we prove stronger results that are only true for compact groups. If G is a compact group, we fix a normalized Haar measure on G , and we denote by $L^p(G)$ the L^p space for this measure. Note that we have $\mathcal{C}(G) \subset L^p(G)$ for every p .

Theorem IV.3.8. *Let G be a compact group, and let (π, V) be an irreducible unitary representation of G . Remember that V is finite-dimensional (by problem 6 of problem set 5).*

- (i) (Schur orthogonality) *If (π', V') is another irreducible unitary representation of G that is not equivalent to (π, V) , then \mathcal{E}_π and $\mathcal{E}_{\pi'}$ are orthogonal as subspaces of $L^2(G)$.*
- (ii) *We have $\dim(\mathcal{E}_\pi) = (\dim V)^2$. More precisely, if (e_1, \dots, e_d) is an orthonormal basis of V and if we denote by φ_{ij} the function $G \rightarrow \mathbb{C}$, $x \mapsto \langle \pi(x)(e_j), e_i \rangle$, then $\{\sqrt{d}\varphi_{ij}, 1 \leq i, j \leq d\}$ is an orthonormal basis of \mathcal{E}_π for the L^2 inner product.*
- (iii) *The $G \times G$ -equivariant map $\text{End}(V) \rightarrow \mathcal{E}_\pi$ defined above is an isomorphism.*

Proof. Note that (iii) follows immediately from (ii), because $\text{End}(V) \rightarrow \mathcal{E}_\pi$ is surjective and (ii) says that $\dim(\mathcal{E}_\pi) = (\dim V)^2 = \dim(\text{End}(V))$.

We prove (i) and (ii). Let (π', V') be an irreducible unitary representation of G , that could be equal to (π, V) . If $A \in \text{Hom}(V, V')$, we define $\tilde{A} \in \text{Hom}(V, V')$ by

$$\tilde{A} = \int_G \pi'(x)^{-1} \circ A \circ \pi(x) dx$$

(note that there is no problem with the integral, because the representations are finite-

dimensional). Then, for every $y \in G$, we have

$$\begin{aligned}\tilde{A} \circ \pi(y) &= \int_G \pi'(x)^{-1} \circ A \circ \pi(xy) dx \\ &= \int_G \pi'(xy^{-1})^{-1} \circ A \circ \pi(x) dx \quad (\text{right invariance of } dx) \\ &= \pi'(y) \circ \tilde{A}.\end{aligned}$$

In other words, \tilde{A} is G -equivariant.

Let $v \in V$ and $v' \in V'$, and define $A \in \text{Hom}(V, V')$ by $A(u) = \langle u, v \rangle v'$. Then, for all $u \in V$ and $u' \in V$, we have

$$\begin{aligned}\langle \tilde{A}(u), u' \rangle &= \int_G \langle (\pi'(x)^{-1} \circ A \circ \pi(x))(u), u' \rangle dx \\ &= \int_G \langle \langle \pi(x)(u), v \rangle \pi'(x)^{-1}(v'), u' \rangle dx \\ &= \int_G \langle \pi(x)(u), v \rangle \overline{\langle \pi'(x)(u'), v' \rangle} dx.\end{aligned}$$

Suppose that π and π' are not equivalent. Then, by Schur's lemma, we have $\tilde{A} = 0$ for every $A \in \text{Hom}(V, V')$, and so, by the calculation above, for all $u, v \in V$ and $u', v' \in V'$,

$$\int_G \langle \pi(x)(u), v \rangle \overline{\langle \pi'(x)(u'), v' \rangle} dx = 0.$$

This proves (i).

Suppose that $\pi = \pi'$, and use the notation of (ii). Take $v = e_i$ and $v' = e_{i'}$ with $i, i' \in \{1, \dots, d\}$, and define A as above. By Schur's lemma again, there exists $c \in \mathbb{C}$ such that $\tilde{A} = c \text{id}_V$. So, taking $u = e_j$ and $u' = e_{j'}$, we get from the calculation above that

$$\langle \varphi_{i,j}, \varphi_{i',j'} \rangle_{L^2(G)} = \langle c e_j, e_{j'} \rangle = \begin{cases} c & \text{if } j = j' \\ 0 & \text{otherwise.} \end{cases}$$

On the other hand, we have

$$cd = \text{Tr}(\tilde{A}) = \int_G \text{Tr}(\pi(x)^{-1} \circ A \circ \pi(x)) dx = \int_G \text{Tr}(A) dx = \text{Tr}(A).$$

As A is defined by $A(w) = \langle w, e_i \rangle e_{i'}$, we have $\text{Tr}(A) = 0$ if $i \neq i'$, and $\text{Tr}(A) = 1$ if $i = i'$. This finishes the proof that $\{\sqrt{d} \varphi_{ij}, 1 \leq i, j \leq d\}$ is an orthonormal basis of \mathcal{E}_π .

□

IV.4 The Peter-Weyl theorem

Let G be a compact group. We see $L^2(G)$ as a representation of $G \times G$ by making $(x, y) \in G \times G$ act by $L_x \circ R_y = R_y \circ L_x$. The restriction of this to the first (resp. second) factor is the left (resp. right) regular representation of G .

Theorem IV.4.1. *If G is compact, then $\mathcal{E} := \bigoplus_{\pi \in \widehat{G}} \mathcal{E}_\pi$ is a dense subalgebra of $\mathcal{C}(G)$. (For the usual pointwise multiplication and the norm $\|\cdot\|_\infty$.)*

Proof. Let's prove that \mathcal{E} is stable by multiplication. Note that, by (iii) of proposition IV.3.2 and theorem IV.2.1, for every finite-dimensional unitary representation π of G , we have $\mathcal{E}_\pi \subset \mathcal{E}$. Let (π, V) and (π', V') be irreducible unitary representations of G , and let $v, w \in V$ and $v', w' \in V'$. Remember that we have defined an action $\pi \otimes \pi'$ of G on $V \otimes_{\mathbb{C}} V'^3$ and an inner product on $V \otimes_{\mathbb{C}} V'$ in problem 1 of problem set 7.⁴ By definition of these, for every $x \in G$, we have

$$\langle (\pi \otimes \pi')(x)(v \otimes w), v' \otimes w' \rangle = \langle \pi(x)(v), w \rangle \langle \pi'(x)(v'), w' \rangle.$$

This proves that the product of a matrix coefficient of π and a matrix coefficient of π' is a matrix coefficient of $\pi \otimes \pi'$. By the observation above, every matrix coefficient of $\pi \otimes \pi'$ is in \mathcal{E} , and we are done.

Now we prove that \mathcal{E} is dense in $\mathcal{C}(G)$. We have shown that \mathcal{E} is a subalgebra, it contains the constants (they are the matrix coefficients of the trivial representation of G on \mathbb{C}) and it separates points on G by the Gelfand-Raikov theorem. So it is dense in $\mathcal{C}(G)$ by the Stone-Weierstrass theorem. □

Corollary IV.4.2. *For every $p \in [1, +\infty)$, the subspace \mathcal{E} of $L^p(G)$ is dense for the L^p norm.*

In particular, we have a canonical $G \times G$ -equivariant isomorphism

$$L^2(G) = \overline{\bigoplus_{(\pi, V) \in \widehat{G}} \text{End}(V)}.$$

The last statement is what is usually called the Peter-Weyl theorem. It implies that the left and right regular representations of G are both isomorphic to the completion of $\bigoplus_{\pi \in \widehat{G}} \pi^{\oplus \dim(\pi)}$.

Remark IV.4.3. The Peter-Weyl theorem actually predates the Gelfand-Raikov theorem, and the original proof uses the fact that the operators $f * \cdot$ are compact on $L^2(G)$, for $f \in L^2(G)$.

³This is just the restriction to the diagonal of $G \times G$ of the action defined above.

⁴We don't need to complete the tensor product here, because V and V' are finite-dimensional.

IV.5 Characters

Definition IV.5.1. Let (π, V) be a continuous finite-dimensional representation of a topological group G . The *character* of π is the continuous map $\chi_V = \chi_\pi : G \rightarrow \mathbb{C}, x \mapsto \text{Tr}(\pi(x))$.

Remark IV.5.2. If (π, V) is a finite-dimensional representation of G and (e_1, \dots, e_n) is an orthonormal basis of V , then, for every $x \in G$, we have

$$\chi_\pi(x) = \sum_{i=1}^n \langle \pi(x)(e_i), e_i \rangle.$$

So $\chi_\pi \in \mathcal{C}_\pi$.

Definition IV.5.3. We say that a function $f : G \rightarrow \mathbb{C}$ is a *central function* or a *class function* if $f(xy x^{-1}) = f(y)$ for all $x, y \in G$.

These functions are called central because they are central for the convolution product, as we will see in section IV.7.

Proposition IV.5.4. Let G be a topological group, and let (π, V) and (π', V') be continuous finite-dimensional representations of G . Then :

- (i) χ_π is a central function, and it only depends on the equivalence class of π .
- (ii) $\chi_{V \oplus V'} = \chi_V + \chi_{V'}$.
- (iii) For every $x \in G$, $\chi_{V^*}(x) = \chi(x^{-1})$.
- (iv) For all $x, y \in G$, we have

$$\chi_{V \otimes_{\mathbb{C}} V'}(x, y) = \chi_V(x) \chi_{V'}(y) \quad \text{and} \quad \chi_{\text{Hom}(V, V')}(x, y) = \chi_V(x^{-1}) \chi_{V'}(y).$$

- (v) If (π, V) is unitarizable (for example if G is compact), then $\chi_V(x^{-1}) = \overline{\chi_V(x)}$ for every $x \in G$.

Proof. Point (i) just follows from the properties of the trace, i.e. the fact that $\text{Tr}(AB) = \text{Tr}(BA)$ for all $A, B \in M_n(\mathbb{C})$.

Put arbitrary Hermitian inner products on V and V' . Let (e_1, \dots, e_n) (resp. (e'_1, \dots, e'_m)) be an orthonormal basis of V (resp. V'). Then $(e_1, \dots, e_n, e'_1, \dots, e'_m)$ is an orthonormal basis of $V \oplus V'$, so, for every $x \in G$,

$$\chi_{V \oplus V'}(x) = \sum_{i=1}^n \langle \pi(x)(e_i), e_i \rangle + \sum_{j=1}^m \langle \pi'(x)(e'_j), e'_j \rangle = \chi_V(x) + \chi_{V'}(x).$$

This proves (ii).

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Let (e_1^*, \dots, e_n^*) be the dual basis of (e_1, \dots, e_n) . Let $x \in G$. Then, if A is the matrix of $\pi(x^{-1})$ in the basis (e_1, \dots, e_n) , the matrix of the endomorphism $\Lambda \mapsto \Lambda \circ \pi(x^{-1})$ in the basis (e_1^*, \dots, e_n^*) is A^T , and we have

$$\chi_{V^*}(x) = \text{Tr}(A^T) = \text{Tr}(A) = \chi_V(x^{-1}).$$

This proves (iii).

We prove the formula for $\chi_{V \otimes_{\mathbb{C}} V'}$. We have seen in problem 1 of problem set 7 how to put an inner product on $V \otimes_{\mathbb{C}} V'$ for which $(e_i \otimes e'_j)_{1 \leq i \leq n, 1 \leq j \leq m}$ is an orthonormal basis. So, for all $x, y \in G$, we have

$$\begin{aligned} \chi_{V \otimes_{\mathbb{C}} V'}(x, y) &= \sum_{i=1}^n \sum_{j=1}^m \langle \pi(x)(e_i) \otimes \pi'(y)(e'_j), e_i \otimes e'_j \rangle \\ &= \sum_{i=1}^n \sum_{j=1}^m \langle \pi(x)(e_i), e_i \rangle \langle \pi'(y)(e'_j), e'_j \rangle \\ &= \chi_V(x) \chi_{V'}(y). \end{aligned}$$

Now the formula for $\chi_{\text{Hom}(V, V')}$ follows from this, from (iii) and from proposition IV.3.6(i).

Finally, we prove (v). If V is unitarizable, we can choose the Hermitian inner form on V to be invariant by G . Then, for every $x \in G$, we have

$$\chi_V(x^{-1}) = \sum_{i=1}^n \langle \pi(x)^{-1}(e_i), e_i \rangle = \sum_{i=1}^n \langle e_i, \pi(x)(e_i) \rangle = \sum_{i=1}^n \overline{\langle \pi(x)(e_i), e_i \rangle} = \overline{\chi_V(x)}.$$

□

Notation IV.5.5. If (π, V) is a representation of a topological group G (continuous or not), we write

$$V^G = \{v \in V \mid \forall x \in G, \pi(x)(v) = v\}.$$

This is a closed G -invariant subspace of V .

Example IV.5.6. If V and W are two representations of G , then

$$\text{Hom}(V, W)^G = \text{Hom}_G(V, W).$$

Theorem IV.5.7. Let G be a compact group and (π, V) be a finite-dimensional continuous representation of G . Then

$$\int_G \chi_V(x) dx = \dim(V^G).$$

Proof. As V is finite-dimensional, we can find a finite family $(V_i)_{i \in I}$ of irreducible subrepresentations of V such that $V = \bigoplus_{i \in I} V_i$. (Cf. corollary I.3.2.9.) We have $\chi_V = \sum_{i \in I} \chi_{V_i}$ by proposition IV.5.4, and $V^G = \bigoplus_{i \in I} V_i^G$. So it suffices to prove the theorem for V irreducible.

Suppose that V is an irreducible representation of V . As V^G is a G -invariant subspace of V , we have $V^G = V$ or $V^G = \{0\}$. If $V^G = V$, then G acts trivially on V , so every linear subspace of V is invariant by G , so we must have $\dim V = 1$. On the other hand, we have $\chi_V(x) = \text{Tr}(1) = 1$ for every $x \in G$, so $\int_G \chi_V(x) dx = 1$. Suppose that V is irreducible and that $V^G = \{0\}$. Let π_0 be the trivial representation of G on \mathbb{C} . Then, by theorem IV.3.8(i), the subspaces \mathcal{E}_π and \mathcal{E}_{π_0} of $L^2(G)$ are orthogonal. But \mathcal{E}_{π_0} is the subspace of constant functions, and we saw above (remark IV.5.2) that $\chi_V \in \mathcal{E}_\pi$. So χ_V is orthogonal to the constant function 1, which means exactly that $\int_G \chi_V(x) dx = 0$.

□

Corollary IV.5.8. *Let G be a compact group, and let (π, V) and (σ, W) be two continuous finite-dimensional representations of G .*

- (i) *We have $\langle \chi_W, \chi_V \rangle_{L^2(G)} = \dim_{\mathbb{C}}(\text{Hom}_G(V, W))$.*
- (ii) *If V and W are irreducible and not equivalent, then $\langle \chi_V, \chi_W \rangle_{L^2(G)} = 0$.*
- (iii) *The representation V is irreducible if and only $\|\chi\|_{L^2(G)} = 1$.*

Proof. (i) Make G act on $\text{Hom}(V, W)$ by $x \cdot T = \rho(x) \circ T \circ \pi(x)^{-1}$. We know (cf. proposition IV.3.4) that $\text{Hom}_G(V, W) = \text{Hom}(V, W)^G$. Applying the theorem to the representation $\text{Hom}(V, W)$ and using points (iv) and (v) of proposition IV.5.4 to calculate the character of this representation, we get :

$$\begin{aligned} \dim_{\mathbb{C}}(\text{Hom}_G(V, W)) &= \dim_{\mathbb{C}}(\text{Hom}(V, W)^G) \\ &= \int_G \chi_{\text{Hom}(V, W)}(x) dx \\ &= \int_G \overline{\chi_V(x)} \chi_W(x) dx \\ &= \langle \chi_W, \chi_V \rangle_{L^2(G)}. \end{aligned}$$

- (ii) This follows from (i) and from Schur's lemma (theorem I.3.4.1), or from the fact that $\chi_V \in \mathcal{E}_V$, $\chi_W \in \mathcal{E}_W$ and \mathcal{E}_V and \mathcal{E}_W are orthogonal in $L^2(G)$ (see theorem IV.3.8).
- (iii) If V is irreducible, then Schur's lemma implies that $\text{End}_G(V)$ is 1-dimensional, so we have $\|\chi_V\|_{L^2(G)} = 1$ by (i). Conversely, suppose that $\|\chi_V\|_{L^2(G)} = 1$. We write $V = \bigoplus_{i \in I} V_i$, where I is finite and the V_i are irreducible subrepresentations of V . By (ii), the characters of non-isomorphic irreducible representations of G are orthogonal in $L^2(G)$, so we have

$$\|\chi_V\|_{L^2(G)}^2 = \sum_{W \in \hat{G}} n_W \|\chi_W\|_{L^2(G)}^2 = \sum_{W \in \hat{G}} n_W,$$

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where, for every $W \in \widehat{G}$,

$$n_W = \text{card}(\{i \in I \mid V_i \simeq W\}).$$

As $\|\chi_V\|_{L^2(G)} = 1$, there is a unique $W \in \widehat{G}$ such that $n_W \neq 0$, and we must have $n_W = 1$. By the definition of the integers n_W , this means that $V \simeq W$, so V is irreducible.

□

Corollary IV.5.9. *Let G be a compact group. Then the family $(\chi_V)_{V \in \widehat{G}}$ of elements of $L^2(G)$ (or $\mathcal{C}(G)$) is linearly independent.*

Proof. This follows from (ii) of the previous corollary.

□

Corollary IV.5.10. *Let π and π' be two continuous finite-dimensional representations of a compact group G . Then π and π' are equivalent if and only if $\chi_\pi = \chi_{\pi'}$.*

Proof. If π and π' are equivalent, we already know that $\chi_\pi = \chi_{\pi'}$. Conversely, suppose that $\chi_\pi = \chi_{\pi'}$. We decompose π and π' as direct sums of irreducible representations :

$$\pi \simeq \bigoplus_{\rho \in \widehat{G}} \rho^{n_\rho}$$

and

$$\pi' \simeq \bigoplus_{\rho \in \widehat{G}} \rho^{m_\rho},$$

with $n_\rho, m_\rho \in \mathbb{Z}_{\geq 0}$ and $n_\rho = m_\rho = 0$ for all but a finite number of $\rho \in \widehat{G}$. By corollary IV.5.8, we have $\chi_\pi = \sum_{\rho \in \widehat{G}} n_\rho \chi_\rho$ and $\chi_{\pi'} = \sum_{\rho \in \widehat{G}} m_\rho \chi_\rho$ (and these are finite sums). By the linear independence of the χ_ρ , the equality $\chi_\pi = \chi_{\pi'}$ implies that $n_\rho = m_\rho$ for every $\rho \in \widehat{G}$, which in turn implies that π and π' are equivalent.

□

IV.6 The Fourier transform

We still assume that G is a compact group.

By propositions I.4.3.4 and I.4.1.3, the space $L^2(G)$ is actually a Banach algebra for the convolution product. This section answers the question “how can we see the algebra structure in the decomposition given by the Peter-Weyl theorem ?”.

Definition IV.6.1. Let $f \in L^2(G)$. For every $(\pi, V) \in \widehat{G}$, the *Fourier transform of f at π* is the endomorphism

$$\widehat{f}(\pi) = \int_G f(x) \pi(x^{-1}) dx = \int_G f(x) \pi(x)^* dx$$

of V .

This is clearly a \mathbb{C} -linear endomorphism of V .

Example IV.6.2. Suppose that $G = S^1$. Then we have seen in problem 5 of problem set 3 that $\widehat{G} \simeq \mathbb{Z}$, where $n \in \mathbb{Z}$ corresponds to the representation $G \rightarrow \mathbb{C}^\times$, $e^{2i\pi t} \mapsto e^{2i\pi n t}$ (with $t \in \mathbb{R}$). So, if $f \in L^1(G)$, its Fourier transform is the function $\widehat{f} : \mathbb{Z} \rightarrow \mathbb{C}$ sending n to

$$\widehat{f}(n) = \int_0^1 f(e^{2i\pi t}) e^{-2i\pi n t} dt.$$

Theorem IV.6.3. (i) For every $\pi \in \widehat{G}$, the map $L^2(G) \rightarrow \text{End}(\pi)$, $f \mapsto \widehat{f}(\pi)$ is a $G \times G$ -equivariant $*$ -homomorphism from $L^2(G)$ to the opposite algebra of $\text{End}(\pi)$. (Note that $L^2(G) \subset L^1(G)$, because G is compact. The involution of $L^1(G)$ defined in example I.4.2.2 restricts to an involution of $L^2(G)$.)

In other words, we have, for $f, g \in L^2(G)$ and $x \in G$:

$$\begin{aligned} \widehat{f * g}(\pi) &= \widehat{g}(\pi) \circ \widehat{f}(\pi), \\ \widehat{f^*}(\pi) &= (\widehat{f}(\pi))^*, \\ \widehat{L_x f}(\pi) &= \widehat{f}(\pi) \circ \pi(x)^{-1} \text{ and } \widehat{R_x f}(\pi) = \pi(x) \circ \widehat{f}(\pi). \end{aligned}$$

(Compare with (i) of theorem I.4.2.6.)

(ii) Let $f \in L^2(G)$. Then, for every $\pi \in \widehat{G}$, the function $\dim(\pi) \text{Tr}(\widehat{f}(\pi) \circ \pi(\cdot)) \in L^2(G)$ is the orthogonal projection of f on \mathcal{E}_π , and the series

$$\sum_{\pi \in \widehat{G}} \dim(\pi) \text{Tr}(\widehat{f}(\pi) \circ \pi(\cdot))$$

converges to f in $L^2(G)$ (Fourier inversion formula).

(iii) For every $f \in L^2(G)$, we have

$$\|f\|_2^2 = \sum_{\pi \in \widehat{G}} \dim(\pi) \text{Tr}(\widehat{f}(\pi)^* \circ \widehat{f}(\pi))$$

(Parseval formula).

Example IV.6.4. Take $G = S^1$. Then (ii) and (iii) say that, for every $f \in L^2(S^1)$, the series $\sum_{n \in \mathbb{Z}} \widehat{f}(n) e^{2i\pi n t}$ converges to f in $L^1(S^1)$ and that

$$\int_0^1 |f(e^{2i\pi t})|^2 dy = \sum_{n \in \mathbb{Z}} |\widehat{f}(n)|^2.$$

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Proof. (i) We have

$$\begin{aligned}
 \widehat{f * g}(\pi) &= \int_G (f * g)(x) \pi(x^{-1}) dx \\
 &= \int_{G \times G} f(y) g(y^{-1}x) \pi(x^{-1}) dx dy \\
 &= \int_{G \times G} f(y) g(x) \pi(x^{-1}y^{-1}) dx dy \quad (\text{change of variable } x' = y^{-1}x) \\
 &= \widehat{g}(\pi) \circ \widehat{f}(\pi).
 \end{aligned}$$

Remember that $f^*(x) = \overline{f(x^{-1})}$, because $\Delta = 1$. So

$$\begin{aligned}
 \widehat{f^*}(\pi) &= \int_G \overline{f(x^{-1})} \pi(x)^* dx \\
 &= \int_G \overline{f(x)} \pi(x^{-1})^* dx \\
 &= (\widehat{f}(\pi))^*.
 \end{aligned}$$

Finally,

$$\begin{aligned}
 \widehat{L_x f}(\pi) &= \int_G f(x^{-1}y) \pi(y^{-1}) dy \\
 &= \int_G f(y) \pi(y^{-1}x^{-1}) dy \\
 &= \widehat{f}(\pi) \circ \pi(x^{-1})
 \end{aligned}$$

and

$$\begin{aligned}
 \widehat{R_x f}(\pi) &= \int_G f(yx) \pi(y^{-1}) dy \\
 &= \int_G f(y) \pi(xy^{-1}) dy \\
 &= \pi(x) \circ \widehat{f}(\pi).
 \end{aligned}$$

- (ii) It is enough to prove the first statement (the second will follow by the Peter-Weyl theorem). Let $(\pi, V) \in \widehat{G}$. As in theorem IV.3.8, fix an orthonormal basis (e_1, \dots, e_d) of V and denote by φ_{ij} the function $G \rightarrow \mathbb{C}$, $x \mapsto \langle \pi(x)(e_j), e_i \rangle$. Then we have seen (in (ii) of theorem IV.3.8) that $\{\sqrt{d}\varphi_{ij}, 1 \leq i, j \leq d\}$ is an orthonormal basis of \mathcal{E}_π for the L^2 inner product. So the orthogonal projection of f on \mathcal{E}_π is

$$d \sum_{i,j=1}^d \langle f, \varphi_{ij} \rangle_{L^2(G)} \varphi_{ij}.$$

For all $i, j \in \{1, \dots, d\}$, we have

$$\begin{aligned}\langle f, \varphi_{i,j} \rangle_{L^2(G)} &= \int_G f(x) \langle e_i, \pi(x)(e_j) \rangle dx \\ &= \int_G f(x) \langle \pi(x)^*(e_i), e_j \rangle dx \\ &= \langle \widehat{f}(\pi)(e_i), e_j \rangle.\end{aligned}$$

Let $y \in G$, and let $(\widehat{\pi}(f)_{i,j})$ and $(\pi(y)_{i,j})$ be the matrices of $\widehat{f}(\pi)$ and $\pi(y)$ in the basis (e_1, \dots, e_d) . Then

$$\widehat{f}(\pi)_{i,j} = \langle \widehat{f}(\pi)(e_j), e_i \rangle = \langle f, \varphi_{j,i} \rangle_{L^2(G)}$$

and

$$\pi(y)_{i,j} = \langle \pi(y)(e_j), e_i \rangle = \varphi_{ij}(y),$$

so

$$\mathrm{Tr}(\widehat{f}(\pi) \circ \pi(y)) = \sum_{i,j=1}^d \widehat{f}(\pi)_{j,i} \pi(y)_{i,j} = \sum_{i,j=1}^d \langle \widehat{f}(\pi), \varphi_{i,j} \rangle_{L^2(G)} \varphi_{i,j}(y).$$

This gives the desired formula for the orthogonal projection of f on \mathcal{E}_π .

- (iii) Let $\pi \in \widehat{G}$, and use the notation of the proof of (ii). Let $g = d\mathrm{Tr}(\widehat{f}(\pi)^* \circ \widehat{f}(\pi))$. It suffices to show that $\|g\|_2^2 = d\mathrm{Tr}(\widehat{f}(\pi)^* \circ \widehat{f}(\pi))$ (because the \mathcal{E}_π for non-isomorphic π are orthogonal, by theorem IV.3.8). We have

$$\mathrm{Tr}(\widehat{f}(\pi)^* \circ \widehat{f}(\pi)) = \sum_{i,j=1}^d |\widehat{f}(\pi)_{i,j}|^2 = \sum_{i,j=1}^d |\langle f, \varphi_{i,j} \rangle_{L^2(G)}|^2.$$

On the other hand, as $g = d \sum_{i,j=1}^d \langle f, \varphi_{ij} \rangle_{L^2(G)} \varphi_{ij}$, we get

$$\|g\|_{L^2(G)}^2 = d^2 \sum_{i,j=1}^d |\langle f, \varphi_{i,j} \rangle_{L^2(G)}|^2 = d \cdot d\mathrm{Tr}(\widehat{f}(\pi)^* \circ \widehat{f}(\pi)).$$

□

IV.7 Characters and Fourier transforms

To finish this chapter, we relate characters and the Fourier transform, and give an explanation of the name “central function”.

Proposition IV.7.1. *Let $f \in L^2(G)$. Then, for every $x \in G$, we have*

$$\mathrm{Tr}(\widehat{f}(\pi) \circ \pi(x)) = f * \chi_\pi(x).$$

IV The Peter-Weyl theorem

By theorem IV.6.3, this says that the orthogonal projection of f in \mathcal{E}_π is $\dim(\pi)f * \chi_\pi$, so we have

$$f = \sum_{\pi \in \widehat{G}} \dim(\pi) f * \chi_\pi$$

in $L^2(G)$.

Proof. We have

$$\begin{aligned} \text{Tr}(\widehat{f}(\pi) \circ \pi(x)) &= \int_G f(y) \text{Tr}(\pi(y)^{-1} \pi(x)) dy \\ &= \int_G f(y) \chi_\pi(y^{-1}x) \\ &= f * \chi_\pi(x). \end{aligned}$$

□

Corollary IV.7.2. For all $\pi, \pi' \in \widehat{G}$, we have

$$\chi_\pi * \chi_{\pi'} = \begin{cases} \dim(\pi)^{-1} \chi_\pi & \text{if } \pi \simeq \pi' \\ 0 & \text{otherwise.} \end{cases}$$

Proof. We know that $\chi_\pi \in \mathcal{E}_\pi$ for every $\pi \in \widehat{G}$, that \mathcal{E}_π and $\mathcal{E}_{\pi'}$ are orthogonal for $\pi \not\simeq \pi'$, and the proposition says that $\dim(\pi)\chi_\pi * \chi_{\pi'}$ is the orthogonal projection of χ_π on $\mathcal{E}_{\pi'}$. This immediately implies the formula of the corollary.

□

Definition IV.7.3. For $1 \leq p < +\infty$, we denote by $ZL^p(G)$ the subspace of central functions in $L^p(G)$. We also denote by $Z\mathcal{C}(G)$ the subspace of central functions in $\mathcal{C}(G)$.

Proposition IV.7.4. The space $L^p(G)$, $1 \leq p < +\infty$ (resp. $\mathcal{C}(G)$) is a Banach algebra for the convolution product, and $ZL^p(G)$ (resp. $Z\mathcal{C}(G)$) is its center.

Proof. Let $p \in [1, +\infty)$, and let $q \in [1, +\infty)$ be such that $p^{-1} + q^{-1} = 1$. As G is compact, the constant function 1 is in $L^q(G)$ and has L^q norm equal to 1, so, by Hölder's inequality, $f = f \cdot 1$ is in $L^1(G)$, and $\|f\|_1 \leq \|f\|_p$. Now corollary I.4.3.2 says that, for every $g \in L^p(G)$, the function $f * g$ exists and is in $L^p(G)$, and that we have $\|f * g\|_p \leq \|f\|_1 \|g\|_p \leq \|f\|_p \|g\|_p$. This shows that $L^p(G)$ is a Banach algebra for $*$.

We show that $\mathcal{C}(G)$ is also a Banach algebra for $*$. If $f, g \in \mathcal{C}(G)$, then $f * g$ clearly exists, and, for every $x \in G$,

$$|f * g(x)| \leq \int_G |f(y)| |g(y^{-1}x)| dy \leq \|f\|_\infty \|g\|_\infty \int_G 1 dy = \|f\|_\infty \|g\|_\infty.$$

So $\|f * g\|_\infty \leq \|f\|_\infty \|g\|_\infty$.

Finally, we show the statement about the centers. Let $f \in L^p(G)$, and suppose that $f * g = g * f$ for every $g \in L^p(G)$. Then, for every $x \in G$ and every $g \in L^p(G)$, we have

$$\int_G f(xy)g(y^{-1})dy = \int_G g(y)f(y^{-1}x)dy = \int_G f(yx)g(y^{-1})dy.$$

This holds if and only if $f(xy) = f(yx)$ almost everywhere on $G \times G$. The proof for $f \in \mathcal{C}(G)$ is the same. □

Corollary IV.7.5. *The family $(\chi_\pi)_{\pi \in \widehat{G}}$ is an orthonormal basis of $ZL^2(G)$.*

Proof. We already know that the χ_π are in $ZL^2(G)$ and that they are pairwise orthogonal, so it just remains to show that a central function orthogonal to all the χ_π has to be 0. Let $f \in ZL^2(G)$. By the lemma below, we have $(\dim \pi)f * \chi_\pi = \langle f, \chi_\pi \rangle_{L^2(G)} \chi_\pi$ for every $\pi \in \widehat{G}$, so, if f is orthogonal to every χ_π , then its projection on all the spaces \mathcal{E}_π is 0 by proposition IV.7.1, hence $f = 0$ by theorem IV.4.1. □

Lemma IV.7.6. *If $f \in ZL^1(G)$ and $\pi \in \widehat{G}$, then $(\dim \pi)f * \chi_\pi = \langle f, \chi_\pi \rangle_{L^2(G)} \chi_\pi$.*

Proof. We know that $f * \chi_\pi = \text{Tr}(\widehat{f}(\pi) \circ \pi(\cdot))$ by proposition IV.7.1. For every $x \in G$, we have

$$\begin{aligned} \widehat{f}(\pi) \circ \pi(x) &= \int_G f(y)\pi(y^{-1}x)dy \\ &= \int_G f(xy^{-1})\pi(y)dy \\ &= \int_G f(y^{-1}x)\pi(y)dy \quad (\text{because } f \text{ is central}) \\ &= \int_G f(y)\pi(xy^{-1})dy \\ &= \pi(x) \circ \widehat{f}(\pi). \end{aligned}$$

So $\widehat{f}(\pi) \in \text{End}(\pi)$ is G -equivariant. By Schur's lemma, this implies that $\widehat{f}(\pi) = c \text{id}$, with $c \in \mathbb{C}$. Taking the trace gives

$$c(\dim \pi) = \text{Tr}(\widehat{f}(\pi)) = \int_G f(y)\text{Tr}(\pi(y^{-1}))dy = \langle f, \chi_\pi \rangle_{L^2(G)}.$$

So

$$\langle f, \chi_\pi \rangle_{L^2(G)} \chi_\pi = (\dim \pi) \text{Tr}(\widehat{f}(\pi) \circ \pi(\cdot)) = (\dim \pi) f * \chi_\pi.$$

□

IV The Peter-Weyl theorem

Remark IV.7.7. In fact, we can even show that the family $(\chi_\pi)_{\pi \in \hat{G}}$ spans a dense subspace in $ZL^p(G)$ for every $p \in [1, +\infty)$ and in $Z\mathcal{C}(G)$. (See proposition 5.25 of [8].)

Remark IV.7.8. If G is finite, then $L^2(G)$ is the space of all functions from G to \mathbb{C} , and $ZL^2(G)$ is the space of functions that are constant on the conjugacy classes of G . So the proposition above says that $|\hat{G}|$ is equal to the number of conjugacy classes in G , and the Peter-Weyl theorem says that $|G| = \sum_{\pi \in \hat{G}} (\dim \pi)^2$.

Remark IV.7.9. We have shown in particular that the Banach algebras $(ZL^p(G), *)$ (for $1 \leq p < +\infty$) and $(Z\mathcal{C}(G), *)$ are commutative. We could ask what their spectrum is. In fact, the answer is very simple (see theorem 5.26 of [8]) : For every $\pi \in \hat{G}$, the formula $f \mapsto (\dim \pi) \int_G f \overline{\chi_\pi} d\mu$ defines a multiplicative functional on $ZL^p(G)$ (resp. $Z\mathcal{C}(G)$), and this induces a homeomorphism from the *discrete* set \hat{G} to the spectrum of $ZL^p(G)$ (resp. $Z\mathcal{C}(G)$).

V Gelfand pairs

In this chapter, G will always be a locally compact group, and K a compact subgroup of G . We fix a left Haar measure $\mu = \mu_G$ on G and a normalized Haar measure μ_K on K .

V.1 Invariant and bi-invariant functions

Definition V.1.1. A function f on G is called *left invariant* (resp. *right invariant*, resp. *bi-invariant*) by K if, for every $x \in K$, we have $L_x f = f$ (resp. $R_x f = f$, resp. $L_x f = R_x f = f$).

If $\mathcal{F}(G)$ is a space of functions on G (for example $\mathcal{C}_c(G)$), we denote by $\mathcal{F}(K \backslash G)$ (resp. $\mathcal{F}(G/K)$, resp. $\mathcal{F}(K \backslash G/K)$) its subspace of left invariant (resp. right invariant, resp. bi-invariant) functions.

Let Δ_G be the modular function of G . As K is compact, we have $\Delta_{G|K} = 1$, so we can use the results of problem 1 of problem set 2. In particular :

Proposition V.1.2. Let $f \in \mathcal{C}(G)$, and define two functions $f^K : G \rightarrow \mathbb{C}$ and ${}^K f : G \rightarrow \mathbb{C}$ by setting

$$f^K(x) = \int_K f(xk) dk$$

and

$${}^K f(x) = \int_K f(kx) dk.$$

Then f^K is right invariant and ${}^K f$ is left invariant.

Proposition V.1.3. There exists a unique regular Borel measure $\mu_{G/K}$ (resp. $\mu_{K \backslash G}$) on G/K (resp. $K \backslash G$) such that, for every $f \in \mathcal{C}_c(G)$, we have

$$\int_G f(x) dx = \int_{G/K} f^K(x) d\mu_{G/K}(x)$$

$$(\text{resp. } \int_G f(x) dx = \int_{K \backslash G} {}^K f(x) d\mu_{K \backslash G}(x)).$$

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Definition V.1.4. If f is a continuous function on G , we write

$${}^K f^K = {}^K(f^K) = ({}^K f)^K.$$

In other words, this is the continuous function on G defined by :

$${}^K f^K(x) = \int_{K \times K} f(kxk') dk dk'.$$

Note that ${}^K f^K$ is obviously a bi-invariant function.

Proposition V.1.5. *Let $f \in \mathcal{C}(G)$. Then f is left invariant (resp. right invariant, resp. bi-invariant) if and only if $f = {}^K f$ (resp. $f = f^K$, resp. $f = {}^K f^K$).*

Proof. This follows immediately from proposition V.1.2 and from the fact that the measure on K is normalized. □

Lemma V.1.6. *For every $f \in \mathcal{C}_c(G)$, we have*

$$\int_G f(x) dx = \int_G {}^K f^K(x) dx.$$

Proof. We have

$$\int_G {}^K f^K(x) dx = \int_{G \times K^2} f(kxk') dx dk dk' = \int_G f(x) dx,$$

because, for all $k, k' \in K$,

$$\int_G f(kxk') dx = \Delta(k')^{-1} \int_G f(x) dx = \int_G f(x) dx$$

(by proposition I.2.8). □

Proposition V.1.7. *Let (π, V) be a unitary representation of G , and let $P_K : V \rightarrow V$ be the orthogonal projection on V^K . Then we have, for every $v \in V$,*

$$P_K(v) = \int_K \pi(k)(v) dk.$$

Moreover, if $f \in \mathcal{C}_c(G)$ and $v \in V$, then $\pi(f)(P_K(v)) = \pi(f^K)(v)$ and $P_K(\pi(f)(v)) = \pi({}^K f)(v)$. In particular :

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(i) If $f \in \mathcal{C}_c(G)$ and $v \in V^K$, we have $\pi(f)(v) = \pi(f^K)(v)$.

(ii) If $f \in \mathcal{C}_c(K \setminus G)$ and $v \in V$, then $\pi(f)(v) \in V^K$.

(Remember that $\pi : L^1(G) \rightarrow \text{End}(V)$ is defined in theorem I.4.2.6.)

Proof. Let $v \in V$. The existence of the integral $w := \int_K \pi(k)(v)dk$ follows from problem 2 of problem set 4. If $x \in K$, then we have

$$\pi(x)(w) = \int_K \pi(xk)(v)dk = \int_K \pi(k)(v)dk = w,$$

so $w \in V^K$. Also, if $w' \in V^K$, then

$$\langle w, w' \rangle = \int_K \langle \pi(k)(v), w' \rangle dk = \int_K \langle v, \pi(k^{-1})(w') \rangle dk = \int_K \langle v, w' \rangle dk = \langle v, w' \rangle.$$

So w is the orthogonal projection of v on V^K .

Now we prove the last statement. Let $f \in \mathcal{C}_c(G)$ and $v \in V$. Then :

$$\begin{aligned} \pi(f^K)(v) &= \int_G f^K(x) \pi(x)(v) dx = \int_G \int_K f(xk) \pi(x)(v) dx dk \\ &= \int_G \int_K f(x) \pi(x) \pi(k)^{-1}(v) dx dk \\ &= \int_G \int_K f(x) \pi(x) \pi(k)(v) dx dk \quad (K \text{ is unimodular}) \\ &= \pi(f)(P_K(v)). \end{aligned}$$

On the other hand :

$$\begin{aligned} P_K(\pi(f)(v)) &= \int_K \int_G f(x) \pi(kx)(v) dk dx \\ &= \int_K \int_G f(k^{-1}x) \pi(x)(v) dk dx \\ &= \int_K \int_G f(kx) \pi(x)(v) dk dx \\ &= \pi(Kf)(v). \end{aligned}$$

□

The same proof gives :

Proposition V.1.8. Let $f, g \in \mathcal{C}_c(G)$. Then

$${}^K(f * g) = ({}^K f) * g \quad \text{and} \quad (f * g)^K = f * (g^K).$$

In particular, if f and g are bi-invariant, then $f * g$ is also bi-invariant, so $\mathcal{C}_c(K \setminus G / K)$ is a subalgebra of $\mathcal{C}_c(G)$ for the convolution product.

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Remark V.1.9. Let $L^p(K \backslash G/K)$ be the subspace of bi-invariant functions in $L^p(G)$. Then, if $1 \leq p < +\infty$, if $f \in L^1(K \backslash G/K)$ and $g \in L^p(K \backslash G/K)$, then their convolution product $f * g$ is in $L^p(K \backslash G/K)$. This is clear on the formulas defining $f * g$ (see proposition I.4.1.3); indeed, we have

$$f * g(x) = \int_G f(y)g(y^{-1}x)dx = \int_G f(xy^{-1})g(y)dy$$

(the first formula shows that $f * g$ is right invariant, and the second that $f * g$ is left invariant).

In particular, the subspace $L^1(K \backslash G/K)$ of $L^1(G)$ is a subalgebra, and we have a similar result for the L^2 spaces if G is compact.

Remark V.1.10. All this is easier to remember if we extend the convolution product and the representation π to the space $\mathcal{M}(G)$ of Radon measures on G . (See remark I.4.1.6.) We can see μ_K as an element of $\mathcal{M}(G)$ by identifying it to the Radon measure $\mathcal{C}_c(G) \rightarrow \mathbb{C}, f \mapsto \int_K f(x)d\mu_K(x)$. Then we have $\mu_K * \mu_K = \mu_K$, $f^K = f * \mu_K$, ${}^K f = \mu_K * f$ and $P_K = \pi(\mu_K)$, so, for example, the last part of proposition V.1.7 just follows from the fact that π is a $*$ -homomorphism.

V.2 Definition of a Gelfand pair

Definition V.2.1. We say that (G, K) is a *Gelfand pair* if the algebra $\mathcal{C}_c(K \backslash G/K)$ is commutative for the convolution product.

Remark V.2.2. If $p \in [1, +\infty)$, $f \in L^p(K \backslash G/K)$ and $g \in \mathcal{C}_c(G)$, then

$$\begin{aligned} \|f - {}^K g^K\|_p^p &= \int_G \left| f(x) - \int_{K \times K} g(kxk')dkdk' \right|^p dx \\ &= \int_G \left| \int_{K \times K} (f(kxk') - g(kxk'))dkdk' \right|^p dx. \end{aligned}$$

So, by Minkowski's formula (see problem 7 of PS 4), we have

$$\|f - {}^K g^K\|_p \leq \int_{K \times K} \|L_k R_{k'} f - L_k R_{k'} g\|_p dkdk' = \|f - g\|_p.$$

As $\mathcal{C}_c(G)$ is dense in $L^p(G)$, every function of $L^p(K \backslash G/K)$ can be approximated by elements of $\mathcal{C}_c(G)$, hence, by the calculation above, by elements of $\mathcal{C}_c(K \backslash G/K)$. In other words, the space $\mathcal{C}_c(K \backslash G/K)$ is dense in $L^p(K \backslash G/K)$. So, in the definition of a Gelfand pair, we could have replaced the condition “ $\mathcal{C}_c(K \backslash G/K)$ is commutative for the convolution product” by the condition “ $L^1(K \backslash G/K)$ is commutative for the convolution product” (or, for G , we could have used “ $L^2(K \backslash G/K)$ is commutative for the convolution product”).

Example V.2.3. If G is abelian, then $(G, \{1\})$ is a Gelfand pair.

Here are other examples (but we will not prove yet that they are Gelfand pairs) :

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- $(\mathrm{SO}(n+1), \mathrm{SO}(n))$, where $\mathrm{SO}(n)$ is identified to a subgroup of $\mathrm{SO}(n+1)$ by sending $x \in \mathrm{SO}(n)$ to the $(n+1) \times (n+1)$ matrix $\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}$;
- $(\mathfrak{S}_{n+m}, \mathfrak{S}_n \times \mathfrak{S}_m)$;
- $(\mathrm{GL}_n(\mathbb{Q}_p), \mathrm{GL}_n(\mathbb{Z}_p))$.

Proposition V.2.4. *Let (G, K) be a Gelfand pair. Then G is unimodular.*

Proof. By proposition I.2.12, we have, for every $f \in \mathcal{C}_c(G)$,

$$\int_G f(x)dx = \int_G \Delta(x)^{-1} f(x^{-1})dx.$$

So it suffices to prove that $\int_G f(x)dx = \int_G f(x^{-1})dx$ for every $f \in \mathcal{C}_c(G)$. First note that

$$\int_G {}^K f^K(x)dx = \int_G f(x)dx$$

and

$$\int_G {}^K f^K(x^{-1})dx = \int_G f(x^{-1})dx,$$

by lemma V.1.6. So it suffices to show that $\int_G f(x)dx = \int_G f(x^{-1})dx$ for every $f \in \mathcal{C}_c(K \backslash G/K)$. Fix $f \in \mathcal{C}_c(K \backslash G/K)$. We can find $g \in \mathcal{C}_c(K \backslash G/K)$ such that g is equal to 1 on $(\mathrm{supp} f) \cup (\mathrm{supp} f)^{-1}$ (because $\mathrm{supp} f = K(\mathrm{supp} f)K$). Then

$$f * g(1) = \int_G f(y)g(y^{-1})dy = \int_{\mathrm{supp} f} f(y)dy = \int_G f(y)dy$$

and

$$g * f(1) = \int_G g(y)f(y^{-1})dy = \int_{(\mathrm{supp} f)^{-1}} f(y^{-1})dy = \int_G f(y^{-1})dy.$$

But $f * g = g * f$ because (G, K) is a Gelfand pair, so this implies the desired result. □

The following criterion will allow us to find more Gelfand pairs.

Proposition V.2.5. *Suppose that there exists a continuous automorphism $\theta : G \rightarrow G$ such that :*

- (a) $\theta^2 = \mathrm{id}_G$ (i.e. θ is an involution);
- (b) for every $x \in G$, we have $\theta(x) \in Kx^{-1}K$.

Then (G, K) is Gelfand pair.

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Proof. Consider the linear functional $\mathcal{C}_c(G) \rightarrow \mathbb{C}$, $f \mapsto \int_G f(\theta(x))dx$. This is a left-invariant positive linear functional on $\mathcal{C}_c(G)$, so, by the uniqueness statement in theorem I.2.7, there exists $c \in \mathbb{R}_{>0}$ such that, for every $f \in \mathcal{C}_c(G)$, we have

$$\int_G f(\theta(x))dx = c \int_G f(x)dx.$$

As $\theta^2 = \text{id}_G$, we must have $c^2 = 1$, so $c = 1$.

Let $f, g \in \mathcal{C}_c(G)$. On the one hand, we have, for every $x \in G$,

$$\begin{aligned} (f \circ \theta) * (g \circ \theta)(x) &= \int_G f(\theta(y))g(\theta(y)^{-1}\theta(x))dy \\ &= \int_G f(y)g(y^{-1}\theta(x))dy \\ &= (f * g) \circ \theta(x) \end{aligned}$$

(the second equality follows from the first paragraph of this proof). On the other hand, for every $x \in G$, we have

$$(g * f)(x^{-1}) = \int_G g(x^{-1}y)f(y)dy = \int_G f'(y^{-1})g'(yx)dy = (f' * g')(x),$$

where $f'(z) = f(z^{-1})$ and $g'(z) = g(z^{-1})$. (We used the fact that G is unimodular to do the change of variables $y \mapsto y^{-1}$.)

Suppose that f and g are bi-invariant. Then we have $f(\theta(x)) = f(x^{-1})$ and $g(\theta(x)) = g(x^{-1})$ by condition (b), and a similar equality for $g * f$ because $g * f$ is also bi-invariant, so, for every $x \in G$,

$$(f * g)(\theta(x)) = ((f \circ \theta) * (g \circ \theta))(x) = (f' * g')(x) = (g * f)(x^{-1}) = (g * f)(\theta(x)).$$

As θ is an automorphism, this implies that $f * g = g * f$.

□

Example V.2.6. (1) If G is abelian, then we can take $\theta : x \mapsto x^{-1}$, so (G, K) is a Gelfand pair for any compact subgroup K , and in particular for $K = \{1\}$.

(2) If G is compact, then $(G \times G, \{(x, x), x \in G\})$ is a Gelfand pair. Indeed, it suffices to apply the proposition above with $\theta(x, y) = (y, x)$. Indeed, for every $(x, y) \in G \times G$, we have $\theta(x, y) = (x, x)(x^{-1}, y^{-1})(y, y)$.

V.3 Gelfand pairs and representations

In this section, we will give two representation-theoretic criteria for (G, K) to be a Gelfand pair, one valid in general and one for G compact.

V.3.1 Gelfand pairs and vectors fixed by K

Theorem V.3.1.1. *The couple (G, K) is a Gelfand pair if and only if, for every irreducible unitary representation (π, V) of G , we have $\dim(V^K) \leq 1$.*

We will need the following variant of the Gelfand-Raikov theorem.

Lemma V.3.1.2. *Let $f \in \mathcal{C}_c(G)$. If $f \neq 0$, then there exists $\varphi \in \mathcal{E}(\mathcal{P}_1)$ (see section III.3) such that $\int_G f(x)\varphi(x)dx \neq 0$.*

Proof. Suppose that $\int_G f\varphi d\mu = 0$ for every $\varphi \in \mathcal{E}(\mathcal{P}_1)$. By theorem III.4.1, we have $\int_G f\varphi d\mu = 0$ for every function of positive type φ . By theorem III.2.5, for every unitary representation (π, V) of G and any $v \in V$, we have $\langle \pi(f)(v), v \rangle = 0$. Applying this to the left regular representation of G , we get that, for every $g \in L^2(G)$, we have $\langle f * g, g \rangle_{L^2(G)} = 0$. As in the proof of theorem III.5.1, we see that this implies that $\langle f * g_1, g_2 \rangle_{L^2(G)} = 0$ for all $g_1, g_2 \in L^2(G)$. Again as in the proof of that theorem, we see that, for all $g_1, g_2 \in L^2(G)$, we have $\langle f * g_1, g_2 \rangle_{L^2(G)} = \langle f, g_2 * \tilde{g}_1 \rangle_{L^2(G)}$, where $\tilde{g}_1(x) = \overline{g_1(x^{-1})}$. So we get $\langle f, g_1 * g_2 \rangle_{L^2(G)} = 0$ for all $g_1, g_2 \in L^2(G)$. Applying this to $g_1 = f$ and to $g_2 = \psi_U$, where $(\psi_U)_{U \in \mathcal{U}}$ is an approximate identity, we finally get $\langle f, f \rangle_{L^2(G)} = 0$, hence $f = 0$. □

We also need the following variant of Schur's lemma.

Lemma V.3.1.3. *Let A be a commutative Banach $*$ -algebra, and let $u : A \rightarrow \text{End}(V)$ be a representation of A on a nonzero Hilbert space V . Suppose that the only closed subspaces of V that are fixed by all the $u(x)$, $x \in A$ are $\{0\}$ and V . Then $\dim V = 1$.*

Proof. By assumption, the subset $u(A)$ satisfies the hypothesis of corollary II.4.4, so its centralizer in $\text{End}(V)$ is equal to $\mathbb{C}\text{id}_V$. But as A is commutative, every element of $u(A)$ is in the centralizer in $u(A)$, so this implies that $\text{Im}(u) \subset \mathbb{C}\text{id}_V$. In particular, every subspace of V is invariant by all the elements of $u(A)$, so V has no nontrivial closed subspaces, which is only possible if $\dim V \leq 1$. □

Lemma V.3.1.4. *Let (π, V) be a unitary representation of G . Then $\pi(f)$ sends V^K to itself for every $f \in L^1(K \backslash G/K)$. If moreover π is irreducible, then the only closed subspaces of V^K stable by all the $\pi(f)$, $f \in L^1(K \backslash G/K)$, are $\{0\}$ and V^K .*

Proof. By proposition V.1.7, for every $f \in \mathcal{C}_c(K \backslash G/K)$ and every $v \in V^K$, we have $\pi(f)(v) \in V^K$. The first statement follows from the fact that $\mathcal{C}_c(K \backslash G/K)$ is dense in $L^1(K \backslash G/K)$.

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To prove the second statement, it suffices to show that, for every $v \in V^K - \{0\}$, the space $\{\pi(f)(v), f \in \mathcal{C}_c(K \backslash G/K)\}$ is dense in V^K . Let $w \in V^K$, and let $\varepsilon > 0$. As V is irreducible, the space $\{\pi(f)(v), f \in L^1(G)\}$ is dense in V . As $\mathcal{C}_c(G)$ is dense in $L^1(G)$, there exists $f \in \mathcal{C}_c(G)$ such that $\|\pi(f)(v) - w\| \leq \varepsilon$. By proposition V.1.7 again, we have $\pi(f)(v) = \pi(f^K)(v)$, and so $\pi(K f^K)(v) = P_K(\pi(f)(v))$, where P_K is the orthogonal projection of V on V^K . As $w \in V^K$, we get $\|\pi(K f^K)(v) - w\| = \|P_K(\pi(f)(v)) - w\| \leq \|\pi(f)(v) - w\| \leq \varepsilon$.

□

Proof of theorem V.3.1.1. Suppose that (G, K) is a Gelfand pair. Let (π, V) be an irreducible unitary representation of G . By lemma V.3.1.4, π defines a $*$ -homomorphism from $L^1(K \backslash G/K)$ to $\text{End}(V^K)$, and the only closed subspaces of V^K stable by all the elements of $L^1(K \backslash G/K)$ are $\{0\}$ and V^K . As $L^1(K \backslash G/K)$ is commutative, lemma V.3.1.3 implies that $\dim(V^K) \leq 1$.

We prove the converse. Suppose that $\dim(V^K) \leq 1$ for every irreducible unitary representation (π, V) of G . Let $f \in \mathcal{C}_c(K \backslash G/K)$ be nonzero. By lemma V.3.1.2, there exists $\varphi \in \mathcal{E}(\mathcal{P}_1)$ such that $\int_G f \varphi d\mu \neq 0$. Let (π, V) be a cyclic unitary representation of G and $v \in V$ be a cyclic vector such that $\varphi(x) = \langle \pi(x)(v), v \rangle$ for every $x \in G$ (see theorem III.2.5). Then we have

$$\int_G f(x) \varphi(x) dx = \int_G f(x) \langle \pi(x)(v), v \rangle dx = \langle \pi(f)(v), v \rangle,$$

so $\pi(f)(v) \neq 0$. By theorem III.3.2, the representation (π, V) is irreducible. By lemma V.3.1.4, the endomorphism $\pi(f)$ of V preserves V^K and, by proposition V.1.7, if w is the orthogonal projection of v on V^K , then $\pi(f)(w) = \pi(f)(v) \neq 0$. In particular, the subspace V^K of V is nonzero, so $\dim(V^K) = 1$ by assumption. Hence $\text{End}(V^K) = \mathbb{C}$, which means that we have found a $*$ -homomorphism $u : \mathcal{C}_c(K \backslash G/K) \rightarrow \mathbb{C}$ (sending g to $\pi(g)|_{V^K}$) such that $u(f) \neq 0$.

Now let $f_1, f_2 \in \mathcal{C}_c(K \backslash G/K)$. As \mathbb{C} is commutative, we have $u(f_1 * f_2 - f_2 * f_1) = u(f_1)u(f_2) - u(f_2)u(f_1) = 0$ for every morphism of algebras $u : \mathcal{C}_c(K \backslash G/K) \rightarrow \mathbb{C}$. By the preceding paragraph, this implies that $f_1 * f_2 - f_2 * f_1 = 0$, and we are done.

□

V.3.2 Gelfand pairs and multiplicity-free representations

Definition V.3.2.1. Let (π, V) be a unitary representation of G , and suppose that we can write $V = \bigoplus_{i \in I} V_i$, with the V_i closed G -invariant subspaces of V that are irreducible as representations of V .¹ Then we say that (π, V) is *multiplicity-free* if, for every irreducible unitary representation W of G , the set of $i \in I$ such that V_i and W are equivalent has cardinality ≤ 1 .

Note that the group G acts by left translations on the homogenous space G/K , so, if $x \in G$ and f is a function on G/K , we can define $L_x f$ by $L_x f(y) = f(x^{-1}y)$.

¹This is always the case if G is compact, see theorem IV.2.1.

Definition V.3.2.2. The *quasiregular representation* of G on $L^2(G/K)$ is the representation defined by $x \cdot f = L_x f$, for every $x \in G$ and every $f \in L^2(G/K)$.

Proposition V.3.2.3. *The definition above makes sense, and gives a unitary representation of G .*

Proof. By definition of the measure on G/K , we have $\int_{G/K} f d\mu_{G/K} = \int_{G/K} L_x f d\mu_{G/K}$ for every $f \in \mathcal{C}_c(G/K)$ and every $x \in G$. As $\mathcal{C}_c(G/K)$ is dense in $L^2(G/K)$, this implies that the operators L_x preserve $L^2(G/K)$ and are isometries. By proposition I.3.1.10, it suffices to prove that, for every $f \in L^2(G/K)$, the map $G \rightarrow L^2(G/K)$, $x \mapsto L_x f$ is continuous. As in the proof of proposition I.3.1.13, it suffices to prove this for $f \in \mathcal{C}_c(G/K)$, in which case it follows from proposition I.1.12. □

Remark V.3.1. If we make G act on $L^2(G)$ by the right regular representation, then $L^2(G/K)$ is the space of K -invariant vectors in $L^2(G)$. The quasi-regular regular representation is then the restriction of the left regular representation to $L^2(G/K)$.

We could also define a quasiregular representation on $L^2(K \backslash G)$ (this is the space of K -invariant vectors in $L^2(G)$ if K acts via the left regular representation, and it gets an action of G via the right regular representation). The representation we get is unitarily equivalent to the quasiregular representation on $L^2(G/K)$.

Theorem V.3.2.4. *Assume that G is compact. Then (G, K) is a Gelfand pair if and only if the quasiregular representation of G on $L^2(G/K)$ is multiplicity-free.*

Also, if (G, K) is a Gelfand pair, then we have a G -equivariant isomorphism

$$L^2(G/K) \simeq \widehat{\bigoplus_{\substack{(\pi, V) \in \widehat{G} \\ V^K \neq 0}} V}.$$

Proof. First observe that $L^2(G/K)$ is the space of vectors of $L^2(G)$ that are K -invariant if K acts by the right regular representation. The Peter-Weyl theorem (corollary IV.4.2) says that, as a representation of $G \times G$, the space $L^2(G)$ is isomorphic to the completion of $\bigoplus_{(\pi, V) \in \widehat{G}} \text{End}(V) = \bigoplus_{(\pi, V) \in \widehat{G}} V^* \otimes_{\mathbb{C}} V$. So $L^2(G/K)$ is isomorphic as a representation of G to the completion of

$$\bigoplus_{\substack{(\pi, V) \in \widehat{G} \\ V^K \neq 0}} (V^*)^{\dim(V^K)}.$$

Note that, for every $(\pi, V) \in \widehat{G}$, the representation V^* is also irreducible; this follows for example from (iii) of corollary IV.5.8, because $\chi_{V^*} = \overline{\chi_V}$, so $\|\chi_{V^*}\|_2 = \|\chi_V\|_2$. So the representation $L^2(G/K)$ is multiplicity-free if and only if, for every irreducible unitary representation (π, V) of G , we have either $V^K = 0$ or $\dim(V^K) = 1$. Hence the first statement of the theorem follows from theorem V.3.1.1.

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We now prove the second statement. We have already seen that

$$L^2(G/K) \simeq \widehat{\bigoplus_{\substack{(\pi, V) \in \widehat{G} \\ V^K \neq 0}} V^*},$$

so we just need to show that, if (π, V) is a finite-dimensional representation of G , then $V^K \neq 0$ if and only if $(V^*)^K \neq 0$. Applying theorem IV.5.7 to the restrictions of the representations V and V^* to K , we get

$$\dim(V^K) = \int_K \chi_V(k) dk$$

and

$$\dim((V^*)^K) = \int_K \chi_{V^*}(k) dk = \int_K \overline{\chi_V(k)} dk = \overline{\dim(V^K)} = \dim(V^K).$$

□

V.4 Spherical functions

In this section, we assume that (G, K) is a Gelfand pair.

Definition V.4.1. Let $\varphi \in \mathcal{C}(K \backslash G/K)$. We say that φ is a *spherical function* if the linear functional $\chi_\varphi : \mathcal{C}_c(K \backslash G/K) \rightarrow \mathbb{C}$, $f \mapsto f * \varphi(1) = \int_G f(x) \varphi(x^{-1}) dx$ is a multiplicative functional, where the multiplication on $\mathcal{C}_c(K \backslash G/K)$ is the convolution product.

In other words, the function φ is spherical if $\varphi \neq 0$ and if, for all $f, g \in \mathcal{C}_c(K \backslash G/K)$, we have $\chi_\varphi(f * g) = \chi_\varphi(f) \chi_\varphi(g)$.

Example V.4.2. If G is commutative and $K = \{1\}$, then every continuous morphism of groups $\varphi : G \rightarrow \mathbb{C}^\times$ is a spherical function. Indeed, let $f, g \in \mathcal{C}_c(G)$. Then :

$$\begin{aligned} \int_G (f * g)(x) \varphi(x^{-1}) dx &= \int_{G \times G} f(y) g(y^{-1}x) \varphi(x^{-1}) dx \\ &= \int_{G \times G} f(y) g(z) \varphi(z^{-1}y^{-1}) dy dz \\ &= \left(\int_G f(y) \varphi(y^{-1}) dy \right) \left(\int_G g(z) \varphi(z^{-1}) dz \right). \end{aligned}$$

These are actually the only spherical functions in this case. (This follows immediately from the next proposition.)

Proposition V.4.3. Let $\varphi \in \mathcal{C}(K \backslash G/K)$. The following conditions are equivalent :

- (i) The function φ is spherical.

(ii) The function φ is nonzero and, for all $x, y \in G$, we have

$$\int_K \varphi(xky)dk = \varphi(x)\varphi(y).$$

(iii) We have :

$$(a) \quad \varphi(1) = 1;$$

$$(b) \quad \text{for every } f \in \mathcal{C}_c(K \backslash G / K), \text{ there exists } \chi(f) \in \mathbb{C} \text{ such that } f * \varphi = \chi(f)\varphi.$$

Proof. We can extend χ_φ to $\mathcal{C}_c(G)$ by using the same formula, i.e., $\chi_\varphi(f) = \int_G f(x)\varphi(x^{-1})dx$. Note that, for all $f, g \in \mathcal{C}_c(G)$, we have

$$\begin{aligned} \chi_\varphi(f * g) &= \int_{G \times G} f(y)g(y^{-1}x)\varphi(x^{-1})dx \\ &= \int_{G \times G} f(y)g(z)\varphi(z^{-1}y^{-1})dydz, \end{aligned}$$

hence

$$\chi_\varphi(f * g) - \chi_\varphi(f)\chi_\varphi(g) = \int_{G \times G} f(y)g(z)(\varphi(z^{-1}y^{-1}) - \varphi(z^{-1})\varphi(y^{-1}))dydz.$$

Let $f, g \in \mathcal{C}_c(G)$. Applying the calculation above to $f' := {}^K f^K$ and $g' := {}^K g^K$ and using the bi-invariance of φ (and the fact that the measure on K is normalized), we get that $\chi_\varphi(f' * g') - \chi_\varphi(f')\chi_\varphi(g')$ is equal to

$$\begin{aligned} &\int_{G^2 \times K^4} f(k_1 x k_2)g(k_3 y k_4)(\varphi(y^{-1}x^{-1}) - \varphi(y^{-1})\varphi(x^{-1}))dx dy dk_1 dk_2 dk_3 dk_4 \\ &= \int_{G^2 \times K^2} f(x)g(y)(\varphi(y^{-1}k_3 k_2 x^{-1}) - \varphi(x^{-1})\varphi(y^{-1}))dx dy dk_2 dk_3 \\ &= \int_{G^2} f(x)g(y) \left(\int_K \varphi(y^{-1}kx^{-1})dk - \varphi(x^{-1})\varphi(y^{-1}) \right) dx dy. \end{aligned}$$

This shows that χ_φ is multiplicative if and only if $\int_G \varphi(y^{-1}kx^{-1})dk = \varphi(y^{-1})\varphi(x^{-1})$ for all $x, y \in G$. As $\chi_\varphi \neq 0$ if and only if $\varphi \neq 0$, this proves that (i) and (ii) are equivalent.

Suppose that φ satisfies conditions (a) and (b) of (iii). Then, for every $f \in \mathcal{C}_c(K \backslash G / K)$, we have

$$\chi_\varphi(f) = f * \varphi(1) = \chi(f).$$

As $f \mapsto \chi(f)$ is multiplicative (by the associativity of the convolution product), this implies that φ is spherical.

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Finally, suppose that φ is spherical. We want to prove that conditions (a) and (b) of (iii) are satisfied. Let $f \in \mathcal{C}_c(K \backslash G / K)$. Then we have, for every $x \in G$,

$$\begin{aligned} f * \varphi(x) &= \int_G f(y) \varphi(y^{-1}x) dy \\ &= \int_{G \times K} f(y) \varphi(y^{-1}kx) dy dk \quad (\text{by left invariance of } f) \\ &= \int_G f(y) \varphi(y)^{-1} \varphi(x) dy \quad (\text{by (ii)}) \\ &= \chi_\varphi(f) \varphi(x). \end{aligned}$$

This shows condition (b). Choosing $f \in \mathcal{C}_c(K \backslash G / K)$ such that $\chi_\varphi(f) \neq 0$, and applying the equality above to $x = 1$, we get $\chi_\varphi(f) = \chi_\varphi(f) \varphi(1)$, hence $\varphi(1) = 1$. □

Remember that $L^1(G)$ is a Banach $*$ -algebra, for the convolution product and the involution given by $f^*(x) = \overline{f(x^{-1})}$.² We have seen that $L^1(K \backslash G / K)$ is a commutative Banach subalgebra of $L^1(G)$, and it is clear that it is also preserved by the involution. So it is natural to ask what the spectrum of $L^1(K \backslash G / K)$ is.

If $\varphi \in \mathcal{C}_b(K \backslash G / K)$ (note the boundedness condition), then the integral $\int_G f(x) \varphi(x^{-1}) dx$ converges for every $f \in L^1(G)$, so we can extend the linear functional χ_φ on $\mathcal{C}_c(K \backslash G / K)$ to a bounded linear functional on $L^1(K \backslash G / K)$, that we will still denote by χ_φ .

Theorem V.4.4. *The map $\varphi \mapsto \chi_\varphi$ identifies the set of bounded spherical functions to $\sigma(L^1(K \backslash G / K))$.*

Example V.4.5. If G is commutative and $K = \{1\}$, a bounded spherical function is a bounded continuous morphism of groups $G \rightarrow \mathbb{C}^\times$, that is, a continuous morphism of groups $G \rightarrow S^1$, i.e. an irreducible unitary representation of G . So we get a canonical bijection $\widehat{G} \xrightarrow{\sim} \sigma(L^1(G))$. In particular, every multiplicative functional on $L^1(G)$ is a $*$ -homomorphism in this case, that is, the Banach $*$ -algebra $L^1(G)$ is symmetric. This recovers the result of question 4(c) of problem set 5.

If G is compact, we will see (in theorem V.7.1) that it is still true that every spherical function defines a $*$ -homomorphism of $L^1(K \backslash G / K)$, i.e. that $L^1(K \backslash G / K)$ is symmetric. But in general, this is not true.

Proof of theorem V.4.4. If φ is a bounded spherical function, then χ_φ is multiplicative on $\mathcal{C}_c(K \backslash G / K)$, hence also on $L^1(K \backslash G / K)$ because $\mathcal{C}_c(K \backslash G / K)$ is dense in $L^1(K \backslash G / K)$.

Conversely, let $\chi : L^1(K \backslash G / K) \rightarrow \mathbb{C}$ be a multiplicative functional. By corollary II.2.6, the linear functional χ is continuous and has norm ≤ 1 .

²As (G, K) is a Gelfand pair, the group G is automatically unimodular by proposition V.2.4, so we don't need the factor $\Delta(x)^{-1}$.

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By remark V.2.2, the linear operator $\mathcal{C}_c(G) \rightarrow \mathcal{C}_c(K \backslash G/K)$, $f \mapsto {}^K f^K$ decreases the L^1 norm, so it extends to a continuous linear operator $L^1(G) \rightarrow L^1(K \backslash G/K)$, that we will still denote by $f \mapsto {}^K f^K$. Then $f \mapsto \chi({}^K f^K)$ is a continuous linear functional on $L^1(G)$, and its norm is equal to that of χ , so there exists a unique $\varphi \in L^\infty(G)$ such that $\|\varphi\|_\infty = \|\chi\|_{op}$ and that, for every $f \in L^1(G)$, we have

$$\int_G f(x) \varphi(x^{-1}) dx = \chi({}^K f^K).$$

In particular, for all $k, k' \in K$ and every $f \in L^1(G)$, we have

$$\begin{aligned} \int_G f(x) \varphi(kx^{-1}k') dx &= \int_G f((k')^{-1}xk^{-1}) \varphi(x^{-1}) dx \\ &= \chi({}^K (L_{k'} R_{k^{-1}} f)^K) \\ &= \chi({}^K f^K) \\ &= \int_G f(x) \varphi(x^{-1}) dx. \end{aligned}$$

So φ is bi-invariant.

Let $f, g \in L^1(K \backslash G/K)$. We have

$$\begin{aligned} \chi(f * g) &= \int_G (f * g)(x) \varphi(x^{-1}) dx \\ &= \int_{G \times G} f(y) g(y^{-1}x) \varphi(x^{-1}) dx dy \\ &= \int_{G \times G} f(y) \varphi(y^{-1}z) g(z^{-1}) dy dz \\ &= \int_G (f * \varphi)(z) g(z^{-1}) dz. \end{aligned}$$

As $\chi(f * g) = \chi(f)\chi(g) = \chi(f) \int_G \varphi(z) g(z^{-1}) dz$, this implies that

$$\int_G ((f * \varphi) - \chi(f)\varphi)(z) g(z^{-1}) dz = 0.$$

Hence, for every $f \in L^1(K \backslash G/K)$, we have $f * \varphi = \chi(f)\varphi$. Choose $f \in \mathcal{C}_c(K \backslash G/K)$ such that $\chi(f) \neq 0$. Then $\chi(f) = f * \varphi(1) = \chi(f)\varphi(1)$, so $\varphi(1) = 1$. Also, the function $f * \varphi$ is continuous, because it is left uniformly continuous (note that, for every $x \in G$, we have $\|L_x(f * \varphi) - f * \varphi\|_\infty = \|(L_x f - f) * \varphi\|_\infty \leq \|L_x f - f\|_1 \|\varphi\|_\infty$ and apply proposition I.3.1.13). So φ is locally almost everywhere equal to a bi-invariant continuous bounded function, and this continuous bounded function is spherical by proposition V.4.3.

Finally, let φ' be another bounded spherical function such that, for every $f \in L^1(K \backslash G/K)$, we have

$$\int_G f(x) \varphi'(x^{-1}) dx = \int_G f(x) \varphi(x^{-1}) dx.$$

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We have seen above that, for every $f \in L^1(G)$, we have

$$\int_G f(x)\varphi(x^{-1})dx = \chi(K f^K) = \int_G K f^K(x)\varphi(x^{-1})dx,$$

and we have a similar equality for φ' . So

$$\int_G f(x)\varphi'(x^{-1})dx = \int_G f(x)\varphi(x^{-1})dx$$

for every $f \in L^1(G)$, and this implies that $\varphi' = \varphi$.

□

V.5 Spherical functions of positive type

For the first result, we don't need to assume that (G, K) is a Gelfand pair.

Proposition V.5.1. *Let φ be a function of positive type on G , and let (π_φ, V_φ) and $v_\varphi \in V_\varphi$ be the unitary representation of G and the cyclic vector associated to φ . (See section III.2.)*

Then $v_\varphi \in V_\varphi^K$ if and only if φ is bi-invariant.

Proof. For all $k, k' \in K$ and $x \in G$, we have

$$\varphi(kxk') = \langle \pi_\varphi(kxk')(v_\varphi), v_\varphi \rangle = \langle \pi_\varphi(x)(\pi_\varphi(k')(v_\varphi)), \pi_\varphi(k^{-1})(v_\varphi) \rangle.$$

So, if $v_\varphi \in V_\varphi^K$, we get $\varphi(kxk') = \varphi(x)$. Conversely, suppose that φ is bi-invariant. Taking $k' = 1$ in the equation above, we see that, for every $k \in K$ and every $x \in G$,

$$\varphi(x) = \langle \pi_\varphi(x)(v_\varphi), v_\varphi \rangle = \varphi(k^{-1}x) = \langle \pi_\varphi(x)(v_\varphi), \pi_\varphi(k)(v_\varphi) \rangle.$$

As v_φ is a cyclic vector, the span of $\{\pi_\varphi(x)(v_\varphi), x \in G\}$ is dense in V_φ , and so this implies that $\pi_\varphi(k)(v_\varphi) = v_\varphi$, for every $k \in K$.

□

Theorem V.5.2. *Assume again that (G, K) is a Gelfand pair. Let φ be a continuous bi-invariant function on G .*

If φ is a normalized function of positive type (i.e. $\varphi \in \mathcal{P}_1$), then φ is spherical if and only if $\varphi \in \mathcal{E}(\mathcal{P}_1)$, that is, if and only if the representation (π_φ, V_φ) is irreducible.

Proof. We write (π, V) and v for (π_φ, V_φ) and v_φ . As φ is bi-invariant, we know that $v \in V^K$ by proposition V.5.1. Suppose first that $\varphi \in \mathcal{E}(\mathcal{P}_1)$, i.e., that π is irreducible. By theorem V.3.1.1,

we have $\dim(V^K) = 1$. Let $f \in \mathcal{C}_c(K \backslash G/K)$. Then we have, for every $x \in G$,

$$\begin{aligned} f * \varphi(x) &= \int_G f(y) \langle \pi(y^{-1}x)(v), v \rangle dy \\ &= \int_G f(y) \langle \pi(x)(v), \pi(y)v \rangle dy \\ &= \langle \pi(x)(v), \pi(f)(v) \rangle. \end{aligned}$$

As $\pi(f)(v) \in V^K$, we can write $\pi(f)(v) = \overline{\chi(f)}v$, with $\chi(f) \in \mathbb{C}$, and we get, for every $x \in G$,

$$f * \varphi(x) = \chi(f) \langle \pi(x)(v), v \rangle = \chi(f) \varphi(x).$$

By proposition V.4.3, this implies that φ is spherical.

Conversely, assume that φ is spherical. Then, by proposition V.4.3 again, there exists a map $\chi : \mathcal{C}_c(K \backslash G/K) \rightarrow \mathbb{C}$ such that, for every $f \in \mathcal{C}_c(K \backslash G/K)$, we have $f * \varphi = \chi(f)\varphi$. In other words, for every $f \in \mathcal{C}_c(K \backslash G/K)$ and every $x \in G$, we have

$$\begin{aligned} f * \varphi(x) &= \int_G f(y) \langle \pi(y^{-1}x)(v), v \rangle dy = \langle \pi(x)(v), \pi(f)(v) \rangle \\ &= \chi(f) \varphi(x) \\ &= \chi(f) \langle \pi(x)(v), v \rangle. \end{aligned}$$

As v is a cyclic vector, this implies that $\pi(f)(v) = \overline{\chi(f)}v \in V^K$. But we have seen in the proof of lemma V.3.1.4 that the space $\{\pi(f)(v), f \in \mathcal{C}_c(K \backslash G/K)\}$ is dense in V^K (if v is cyclic), so $\dim(V^K) = 1$. By lemma V.5.3, this implies that (π, V) is irreducible. □

Lemma V.5.3. *We don't assume that (G, K) is a Gelfand pair. Let (π, V) be a unitary representation of G , and suppose that there is a cyclic vector in V^K . If $\dim(V^K) \leq 1$, then (π, V) is irreducible.*

Proof. It suffices to prove that $\text{End}_G(V) = \mathbb{C}\text{id}_V$. Indeed, if V has a closed G -invariant subspace W such that $W \neq \{0\}, V$, then the orthogonal projection on W is a G -equivariant endomorphism of V (by lemma I.3.4.3) that is not a multiple of id_V .

So let $T \in \text{End}_G(V)$. Then, by proposition V.1.7, the operator T commutes with the orthogonal projection on V^K , so it preserves V^K . Choose a cyclic vector $v \in V^K$. As $\dim(V^K) = 1$, we have $T(v) = \lambda v$, with $\lambda \in \mathbb{C}$. As T is G -equivariant, we get that $T(\pi(x)(v)) = \lambda \pi(x)(v)$ for every $x \in G$. As v is cyclic, this implies that $T = \lambda \text{id}_V$. □

Corollary V.5.4. *Assume that (G, K) is a Gelfand pair. Then $\varphi \mapsto (\pi_\varphi, V_\varphi)$ induces a bijection from the set of spherical functions in $\mathcal{E}(\mathcal{P}_1)$ to the set of unitary equivalence classes of irreducible unitary representations (π, V) of G such that $V^K \neq \{0\}$.*

Proof. The only statement that doesn't follow immediately from proposition V.5.1 and theorem V.5.2 is the fact that, if two spherical functions in $\mathcal{E}(\mathcal{P}_1)$ give rise to unitarily equivalent representations, then they must be equal. Let $\varphi_1, \varphi_2 \in \mathcal{E}(\mathcal{P}_1)$ be spherical, and suppose that there is an isometric G -equivariant isomorphism $T : V_{\varphi_1} \rightarrow V_{\varphi_2}$. By proposition V.5.1, the vectors v_{φ_1} and v_{φ_2} are K -invariant. Also, as (G, K) is a Gelfand pair, the spaces $V_{\varphi_1}^K$ and $V_{\varphi_2}^K$ are both of dimension ≤ 1 , hence of dimension 1 because they contain nonzero vectors. But T is G -equivariant, so we have $T(V_{\varphi_1}^K) \subset V_{\varphi_2}^K$, which implies that $T(v_{\varphi_1}) = \lambda v_{\varphi_2}$ for some $\lambda \in \mathbb{C}$. As $\|v_{\varphi_1}\| = \|v_{\varphi_2}\| = 1$, we must have $|\lambda| = 1$. So, for every $x \in G$, we get

$$\begin{aligned} \varphi_2(x) &= \langle \pi_{\varphi_2}(x)(v_{\varphi_2}), v_{\varphi_2} \rangle \\ &= \langle \pi_{\varphi_2}(x)(\lambda^{-1}T(v_{\varphi_1})), \lambda^{-1}T(v_{\varphi_1}) \rangle \\ &= \langle T(\pi_{\varphi_1}(x)(v_{\varphi_1})), T(v_{\varphi_1}) \rangle \\ &= \langle \pi_{\varphi_1}(x)(v_{\varphi_1}), v_{\varphi_1} \rangle. \end{aligned}$$

□

V.6 The dual space and the spherical Fourier transform

In this section, we suppose that (G, K) is a Gelfand pair. We will state a few results on the (spherical) Fourier transform without proof. In the next section, we will give proofs of some version of these results if G is compact.

Definition V.6.1. The *dual space* of (G, K) is the set Z of spherical functions in $\mathcal{E}(\mathcal{P}_1)$, with the weak* topology coming from the embedding $\mathcal{E}(\mathcal{P}_1) \subset L^\infty(G) \simeq \text{Hom}(L^1(G), \mathbb{C})$.

Example V.6.2. If G is commutative and $K = \{1\}$, then $Z = \widehat{G}$, the dual group of G . (See problem set 3.)

Proposition V.6.3. *The space Z is locally compact, and its topology coincides with the topology of convergence on compact subsets of G .*

Proof. For the first statement, note first that $\mathcal{P}_0 = \{\psi \text{ of positive type} \mid \psi(1) \leq 1\}$ is weak* compact, because it is weak* closed in the closed unit ball of $L^\infty(G)$. By the proof of theorem V.4.4, the subset $\mathcal{P}_0 \cap \mathcal{C}(K \setminus G/K)$ is the set of $\varphi \in \mathcal{P}_0$ such that, for every $f \in L^1(G)$, we have $\int_G f(x)\varphi(x^{-1})dx = \int_G^K f^K(x)\varphi(x^{-1})dx$. These are weak* closed conditions, so $\mathcal{P}_0 \cap \mathcal{C}(K \setminus G/K)$ is weak* closed in \mathcal{P}_0 , hence weak* compact. Finally, by theorem V.5.2, the set $Z \cup \{0\}$ is the set of $\varphi \in \mathcal{P}_0 \cap \mathcal{C}(K \setminus G/K)$ such that $\int_G (f * g)(x)\varphi(x^{-1})dx = (\int_G f(x)\varphi(x^{-1})dx)(\int_G g(x)\varphi(x^{-1})dx)$ for all $f, g \in L^1(K \setminus G/K)$. This is a weak* closed condition, so $Z \cup \{0\}$ is weak* compact, and Z is locally compact. Note that this also proves that $Z \cup \{0\}$ is the Alexandroff compactification of Z .

The second statement follows immediately from Raikov's theorem (theorem III.4.3).

□

Definition V.6.4. Let $f \in L^1(K \backslash G/K)$. The (spherical) Fourier transform of f is the function $\widehat{f} : Z \rightarrow \mathbb{C}$ defined by

$$\widehat{f}(\varphi) = \int_G f(x) \varphi(x^{-1}) dx = \chi_\varphi(f).$$

Proposition V.6.5. The Fourier transform has the following properties :

- (i) For every $f \in L^1(K \backslash G/K)$, the function \widehat{f} is in $\mathcal{C}_0(Z)$, and we have $\|\widehat{f}\|_\infty \leq \|f\|_1$.
- (ii) The map $L^1(K \backslash G/K) \rightarrow \mathcal{C}_0(Z)$, $f \mapsto \widehat{f}$ is \mathbb{C} -linear and it has dense image.
- (iii) For all $f, g \in L^1(K \backslash G/K)$, we have $\widehat{f * g} = \widehat{f} \widehat{g}$.
- (iv) For every $f \in L^1(K \backslash G/K)$, we have $\widehat{f^*} = \overline{\widehat{f}}$.

Proof. (i) The continuity \widehat{f} follows immediately from the definition of the weak* topology. In fact, we can extend \widehat{f} (by the same formula) to a continuous linear functional on the whole space $L^\infty(G)$. But have seen in the proof of proposition V.6.3 that $Z \cup \{0\}$ is the Alexandroff compactification of Z , so this implies that $\widehat{f} \in \mathcal{C}(Z)$ vanishes at ∞ . The inequality $\|\widehat{f}\|_\infty \leq \|f\|_1$ just follows from the fact that $\|\varphi\|_\infty = 1$ for every $\varphi \in Z$.

(iii) and (iv) This is just expressing the fact that χ_φ is a *-homomorphism from $L^1(K \backslash G/K)$ to \mathbb{C} , for every $\varphi \in Z$.

(ii) The linearity is clear. The second statement follows from the Stone-Weierstrass theorem : indeed, the image of the spherical Fourier transform is a \mathbb{C} -subalgebra of $\mathcal{C}_0(Z)$ by (iii), it is stable by complex conjugation by (iv), it separates points (because, by theorem V.4.4, the map $Z \rightarrow \sigma(L^1(K \backslash G/K))$, $\varphi \mapsto (f \mapsto f(\varphi))$ is injective), and it vanishes nowhere (for every $\varphi \in Z$, the map $f \mapsto \widehat{f}(\varphi)$ is a multiplicative functional on $L^1(K \backslash G/K)$, so it is nonzero).

□

Theorem V.6.6. (Fourier inversion) ³ Let $\mathcal{V}^1(K \backslash G/K)$ be the space of L^1 functions that are complex linear combinations of bi-invariant functions of positive type on G .

Then there exists a unique measure ν on Z , called the Plancherel measure, such that, for every $f \in \mathcal{V}^1(K \backslash G/K)$, we have $\widehat{f} \in L^1(Z, \nu)$ and, for every $x \in G$,

$$f(x) = \int_Z \varphi(x) \widehat{f}(\varphi) d\nu.$$

Theorem V.6.7. (Plancherel formula) ⁴ For every $f \in \mathcal{C}_c(K \backslash G/K)$, we have $\widehat{f} \in L^2(Z, \nu)$, and

$$\int_G |f(x)|^2 dx = \int_Z |\widehat{f}(\varphi)|^2 d\nu(\varphi).$$

³See [18] Theorem 6.4.5.

⁴See [18] Theorem 6.4.6.

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In particular, the map $f \mapsto \hat{f}$ extends to an isometry $L^2(K \backslash G / K) \rightarrow L^2(Z, \nu)$, and this is an isomorphism.

Remark V.6.8. If G is commutative and $K = \{1\}$, then $Z = \hat{G}$ is a locally compact group, the measure ν is a Haar measure on \hat{G} , and the Pontrjagin duality theorem says that the canonical map $G \rightarrow \hat{\hat{G}}$, $x \mapsto (\varphi \mapsto \varphi(x))$ is an isomorphism of topological groups. (See for example [8] Theorems 4.22 and 4.32, or [18] Theorems 5.5.1 and 5.7.1.)

But in general, the Plancherel measure ν could be supported on a strict subset of Z .

V.7 The case of compact groups

In this section, we assume that (G, K) is a Gelfand pair, and that G is compact. We also assume that the Haar measure on G is normalized.

Theorem V.7.1. (i) *The dual space Z of (G, K) is discrete, and it is an orthogonal subset of $L^2(G)$.*

(ii) *Every spherical function on G is of positive type (hence in $\mathcal{E}(\mathcal{P}_1)$ by theorem V.5.2). In other words, the set Z is in canonical bijection (via $\varphi \mapsto (\pi_\varphi, V_\varphi)$) with the set of equivalence classes of irreducible unitary representations of G such that $\dim(V_\varphi^K) = 1$.*

(iii) *For every $\varphi \in Z$, we have*

$$\varphi(x) = \int_K \chi_{\pi_\varphi}(xk) dk$$

for $x \in G$, and

$$\int_G |\varphi(x)|^2 = \frac{1}{\dim V_\varphi}.$$

(iv) *If $f \in L^2(K \backslash G / K)$ and $(\pi, V) \in \hat{G}$, then $f * \chi_\pi = 0$ if $V^K = \{0\}$, and otherwise $f * \chi_\pi$ is a multiple of the element φ_π of Z corresponding to π by corollary V.5.4.*

Proof. Let $\varphi, \varphi' \in Z$ such that $\varphi \neq \varphi'$. We know by corollary V.5.4 (and proposition IV.2.6) that the representations V_φ and $V_{\varphi'}$ are unitary and not equivalent. We also know (by construction of the representation) that φ and φ' are matrix coefficients of V_φ and $V_{\varphi'}$, respectively. By Schur orthogonality (theorem IV.3.8), this implies that $\langle \varphi, \varphi' \rangle_{L^2(G)} = 0$.

We prove that Z is discrete. Let $\varphi \in Z$, and consider $U = \{\varphi' \in Z \mid \|\varphi - \varphi'\|_\infty < \|\varphi\|_2\}$. This is open in the topology of convergence on compact subsets of G (because G is compact), hence

is an open subset of Z by Raikov's theorem (theorem III.4.3). Also, if $\varphi' \in U$, then we have

$$\begin{aligned} |\langle \varphi, \varphi' \rangle_{L^2(G)}| &= |\langle \varphi, \varphi \rangle_{L^2(G)} - \langle \varphi, \varphi' - \varphi \rangle_{L^2(G)}| \\ &\geq \|\varphi\|_2^2 - \|\varphi\|_2 \|\varphi - \varphi'\|_2 \\ &\geq \|\varphi\|_2^2 - \|\varphi\|_2 \|\varphi - \varphi'\|_\infty \\ &> 0, \end{aligned}$$

hence, by the first paragraph, $\varphi' = \varphi$. This means that $U = \{\varphi\}$, i.e., that φ is an isolated point of Z .

Let (π, V) be an irreducible unitary representation of G and let $f \in L^2(K \backslash G/K)$. We want to calculate $f * \chi_\pi$. Let (v_1, \dots, v_d) be an orthonormal basis; then, for every $x \in G$, we have $\chi_\pi(x) = \sum_{i=1}^d \langle \pi(x)(e_i), e_i \rangle$. Hence, for every $x \in G$,

$$f * \chi_\pi(x) = \int_G f(y) \sum_{i=1}^d \langle \pi(y^{-1}x)(e_i), e_i \rangle = \sum_{i=1}^d \langle \pi(x)(e_i), \pi(\bar{f})(e_i) \rangle.$$

Let $P_K \in \text{End}(V)$ be the orthogonal projection on V^K . As \bar{f} is bi-invariant, we have $\pi(\bar{f}) = P_K \circ \pi(\bar{f}) \circ P_K$ by proposition V.1.7. Suppose first that $V^K = \{0\}$. Then the formula above gives $f * \chi_\pi = 0$. Now suppose that $V^K \neq \{0\}$. Then, by corollary V.5.4, there is a unique spherical function of positive type φ_π whose associated representation is (π, V) , and a unitary cyclic vector $v \in V^K$ such that $\varphi_\pi(x) = \langle \pi(x)(v), v \rangle$. We may choose the orthonormal basis such that $v_1 = v$. Then $P_K(v_i) = 0$ for $i \geq 2$ and $P_K(v_1) = v_1$, for every $x \in G$, we have

$$f * \chi_\pi(x) = \sum_{i=1}^d \langle \pi(x)(v_i), P_K(\pi(\bar{f})(P_K(v_i))) \rangle = \langle \pi(x)(v_1), P_K(\pi(\bar{f})(v_1)) \rangle.$$

As V^K is 1-dimensional, the vector $P_K(\pi(\bar{f})(v_1))$ is a multiple of v_1 , and so $f * \chi_\pi$ is a multiple of φ_π . This proves (iv). Note also that, for every $x \in G$, we have

$$\begin{aligned} \int_K \chi_\pi(kx) dk &= \int_K \sum_{i=1}^d \langle \pi(kx)(v_i), v_i \rangle dk \\ &= \sum_{i=1}^d \left\langle \pi(x)(v_i), \int_K \pi(k^{-1})(v_i) dk \right\rangle \\ &= \sum_{i=1}^d \langle \pi(x)(v_i), P_K(v_i) \rangle \quad (\text{by proposition V.1.7}) \\ &= \langle \pi(x)(v_1), v_1 \rangle \\ &= \varphi_\pi(x), \end{aligned}$$

which gives the first part of (iii). The second part of (iii) is contained in point (ii) of proposition IV.3.8.

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Now consider a spherical function φ on G . By proposition IV.7.1 (i.e. the Fourier inversion formula), we have an equality (in $L^2(G)$)

$$\varphi = \sum_{\pi \in \widehat{G}} \dim(\pi) \varphi * \chi_\pi.$$

By the calculations above, only the $\pi \in \widehat{G}$ with nonzero K -invariant vectors appear in the sum above, and then $\varphi * \chi_\pi$ is a multiple of the function that was denoted by φ_π in the previous paragraph. In other words, using corollary V.5.4 again, we get

$$\varphi = \sum_{\psi \in Z} c_\psi \psi,$$

for some $c_\psi \in \mathbb{C}$. If we denote by χ_φ (resp. χ_ψ) the linear functional $f \mapsto f * \varphi(1)$ (resp. $f \mapsto f * \psi(1)$) on $L^1(K \backslash G/K)$, we know that it is multiplicative (for χ_ψ , this uses theorem V.5.2). Also, as $\varphi = \sum_{\psi \in Z} c_\psi \psi$, we have $\chi_\varphi = \sum_{\psi \in Z} c_\psi \chi_\psi$. Let $\psi, \psi' \in Z$ such that $\psi \neq \psi'$. Then $\psi * \psi' = \psi' * \psi$ is a multiple of both ψ and ψ' (by proposition V.4.3), so $\psi * \psi' = 0$. In particular, we have $\chi_\psi(\psi') = \chi_{\psi'}(\psi) = 0$. This implies that $\chi_\varphi(\psi) = c_\psi \chi_\psi(\psi)$ for every $\psi \in Z$; note also that

$$\chi_\psi(\psi) = \int_G \psi(x) \psi(x^{-1}) dx = \int_G \psi(x) \overline{\psi(x)} dx > 0.$$

Hence, if $\psi, \psi' \in Z$ and $\psi \neq \psi'$, then

$$0 = \chi_\varphi(\psi * \psi') = \chi_\varphi(\psi) \chi_\varphi(\psi') = c_\psi c_{\psi'} \chi_\psi(\psi) \chi_{\psi'}(\psi'),$$

so $c_\psi c_{\psi'} = 0$. So at most of one the c_ψ can be nonzero, i.e., there exists $\psi \in Z$ such that $\varphi = c_\psi \psi$. As $\varphi(1) = 1 = \psi(1)$, we must also have $c_\psi = 1$, so finally we see that $\varphi = \psi$ is of positive type. This finishes the proof of (ii). □

Corollary V.7.2. (i) *We have a G -equivariant isomorphism*

$$L^2(G/K) \simeq \widehat{\bigoplus_{\varphi \in Z} V_\varphi}.$$

(ii) *The family $((\dim V_\varphi)^{1/2} \varphi)_{\varphi \in Z}$ is a Hilbert basis of $L^2(K \backslash G/K)$.*

(iii) *For every $f \in L^2(K \backslash G/K)$, we have*

$$f = \sum_{\varphi \in Z} \dim(V_\varphi) \widehat{f}(\varphi) \varphi$$

(Fourier inversion formula) and

$$\|f\|_{L^2(G)}^2 = \sum_{\varphi \in Z} \dim(V_\varphi) |\widehat{f}(\varphi)|^2$$

(Parseval formula).

Proof. Point (i) is just a reformulation of the last statement of theorem V.3.2.4.

For (ii), we already know that the family $(\sqrt{\dim(V_\varphi)}\varphi)_{\varphi \in Z}$ is orthonormal in $L^2(G)$. Also, if $f \in L^2(K \backslash G/K)$, we have

$$f = \sum_{(\pi, V) \in \widehat{G}} \dim(V) f * \chi_\pi$$

by proposition IV.7.1, so f is in the closure of $\text{Span}(Z)$ by point (iv) of the theorem, which means that $\text{Span}(Z)$ is dense in $L^2(K \backslash G/K)$.

The second formula of (iii) follows from the first formula and from (ii). To prove the first formula, it only remains to show that, for every $f \in L^2(K \backslash G/K)$ and every $\varphi \in Z$, we have $f * \chi_{\pi_\varphi} = \widehat{f}(\varphi)\varphi$. As we already know that $f * \chi_{\pi_\varphi}$ is a multiple of φ , we just need to check that $f * \chi_{\pi_\varphi}(1) = \widehat{f}(\varphi)$. By point (iii) of the theorem, we have $\varphi(x) = \int_K \chi_{\pi_\varphi}(kx) dk$ for every $x \in G$. So :

$$\begin{aligned} f * \chi_{\pi_\varphi}(1) &= \int_G f(x) \chi_{\pi_\varphi}(x^{-1}) dx \\ &= \int_{G \times K} f(k^{-1}x) \overline{\chi_{\pi_\varphi}(x)} dx dk \quad (f \text{ is left invariant and } \text{vol}(K) = 1) \\ &= \int_{G \times K} f(x) \overline{\chi_{\pi_\varphi}(kx)} dx dk \\ &= \int_G f(x) \overline{\varphi(x)} dx \\ &= \int_G f(x) \varphi(x^{-1}) dx \\ &= \widehat{f}(\varphi). \end{aligned}$$

□

Remark V.7.3. The corollary says in particular that the Plancherel measure ν on Z is given by $|\nu(\{\varphi\})| = \dim(V_\varphi)$.

VI Application of Fourier analysis to random walks on groups

We will mostly be interested in the case of finite groups in this chapter, but we will give some results for more general groups in the last section.

VI.1 Finite Markov chains

We fix once and for all a probability space Ω (i.e. a measure space with total volume one).

Definition VI.1.1. Let X be a measurable space (i.e. a space with a σ -algebra). A *random variable* with values in X is a measurable function $X : \Omega \rightarrow X$.

For every measurable subset A of X , we write $\mathbb{P}(X \in A)$ for the measure of $X^{-1}(A)$. (We think of this as the probability that X is in A .) The *distribution* of X is the probability distribution μ on X defined by $\mu(A) = \mathbb{P}(X \in A)$.

We think of random variables as representing the outcome of some experiment or observation. The probability space Ω is usually not specified (you can think of it as something like “all the possible universes”). For example, we could think of the outcome of flipping a coin as a random variable with values in the finite set $\{\text{heads}, \text{tails}\}$. If the coin is unbiased, the distribution of that random variable is given by $\mu(\{\text{heads}\}) = \mu(\{\text{tails}\}) = \frac{1}{2}$.

In this notes, we will only be concerned with the case where X is finite and its σ -algebra is the set of all subsets of X . We can (and will) think of measures on X as functions $\mu : X \rightarrow \mathbb{R}_{\geq 0}$.

From now on, we assume that X is finite.

Definition VI.1.2. A matrix $P = (P_{i,j}) \in M_n(\mathbb{R})$ is called *stochastic* if $P_{i,j} \geq 0$ for all $i, j \in \{1, \dots, n\}$ and $\sum_{j=1}^n P_{i,j} = 1$ for every $i \in \{1, \dots, n\}$.

If $P : X \times X \rightarrow \mathbb{R}$ is a function, we think of it as a matrix of size $|X| \times |X|$ and we call it *stochastic* if $P(x, y) \geq 0$ for all $x, y \in X$ and $\sum_{y \in X} P(x, y) = 1$ for every $x \in X$.

Definition VI.1.3. Let $P : X^2 \rightarrow \mathbb{R}$ be a stochastic function and ν be a probability distribution on X . A (discrete-time homogeneous) *Markov chain* with *state space* X , *initial distribution* ν and *transition matrix* P is a sequence $(X_n)_{n \geq 0}$ of random variables with values in X such that :

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- (a) The distribution of X_0 is ν .
- (b) For every $n \geq 0$ and all $x_0, \dots, x_{n+1} \in X$, if $\mathbb{P}(X_n = x_n, \dots, X_0 = x_0) > 0$, then

$$\mathbb{P}(X_{n+1} = x_{n+1} | X_n = x_n, \dots, X_0 = x_0) = P(x_n, x_{n+1}).$$

Let $P, Q : X^2 \rightarrow \mathbb{R}$ be two functions. We write PQ for the function $X^2 \rightarrow \mathbb{R}$ defined by

$$PQ(x, y) = \sum_{z \in X} P(x, z)Q(z, y).$$

(If we see functions on X^2 as matrices, this is the usual matrix product.)

In particular, we write P^n for the product $PP \dots P$ (n times); by convention, P^0 is the characteristic function of the diagonal.

Lemma VI.1.4. *Let $(X_n)_{n \geq 0}$ be a Markov chain on X with initial distribution ν and transition matrix P . Then, for every $x \in X$, we have*

$$\mathbb{P}(X_n = x) = \sum_{y \in X} \nu(y)P^n(y, x).$$

In other words, if we see P as a matrix and ν as a row vector, then the distribution of X_n is νP^n .

Proof. We prove the result by induction on n . It is obvious for $n = 0$. Suppose that we know it for some n , and let's prove it for $n + 1$. Let $x \in X$. Then

$$\begin{aligned} \mathbb{P}(X_{n+1} = x) &= \sum_{y \in X, \mathbb{P}(X_n = y) > 0} \mathbb{P}(X_{n+1} = x | X_n = y) \\ &= \sum_{y \in X} \mathbb{P}(X_n = y) \mathbb{P}(X_{n+1} = x | X_n = y) \\ &= \sum_{y \in X} \mathbb{P}(X_n = y) P(y, x). \end{aligned}$$

Using the induction hypothesis, we get

$$\mathbb{P}(X_{n+1} = x) = \sum_{y \in X} P(y, x) \sum_{z \in X} \nu(z) P^n(z, y) = \sum_{z \in X} \nu(z) P^{n+1}(z, x).$$

□

Example VI.1.5. (1) Random walk on the discrete circle : We take $X = \mathbb{Z}/r\mathbb{Z}$, $\nu = \delta_0$ and P defined by

$$P(x, y) = \begin{cases} \frac{1}{2} & \text{if } x - y \in \{\pm 1\} \\ 0 & \text{otherwise.} \end{cases}$$

The Markov chain is modeling a random walk on the “discrete circle” $\mathbb{Z}/n\mathbb{Z}$ where we start at 0 with probability 1, and then, at each time n , we have a 50% chance to go to the preceding point on the discrete circle and a 50% chance to go to the next point of the circle.

- (2) Mixing a deck of cards using random transpositions : We are trying to understand the following situation : We have a deck of N cards. At each time n , we randomly (uniformly and independently) choose two cards and switch their positions in the deck. How long will it take to mix the deck ?

This problem is modeled by a Markov chain with state space \mathfrak{S}_N (representing all the possible orderings of the deck), initial distribution the Dirac measure supported at our starting position, and transition matrix P given by

$$P(\tau\sigma, \sigma) = \begin{cases} \frac{1}{N} & \text{if } \tau = 1 \\ \frac{2}{N^2} & \text{if } \tau \text{ is a transposition} \\ 0 & \text{otherwise.} \end{cases}$$

- (3) The Bernoulli-Laplace diffusion model : We have two urns labeled by 0 and 1. At the start, urn 0 contains r red balls and urn 1 contains b blue balls. At each time n , we choose a ball in each urn (uniformly and independently) and switch them. How long will it take to mix the balls ?

We model this problem using a Markov chain with state space $\mathfrak{S}_N / \mathfrak{S}_r \times \mathfrak{S}_b$, where $N = r + b$, and $\mathfrak{S}_r \times \mathfrak{S}_b$ is embedded in \mathfrak{S}_N via the obvious bijection $\{1, \dots, r\} \times \{1, \dots, b\} \simeq \{1, \dots, N\}$. Indeed, we can think of the N balls as the set $\{1, \dots, N\}$, where the first r balls are red and the last b balls are blue. A state of the process described above is a subset A of $\{1, \dots, N\}$ such that $|A| = r$ (the content of urn 0); note that switching two balls between the urns does not change the number of balls in each urn. The group \mathfrak{S}_N acts transitively on the set Ω_r of cardinality r subsets of $\{1, \dots, N\}$, and its subgroup $\mathfrak{S}_r \times \mathfrak{S}_b$ is the stabilizer of $\{1, \dots, r\}$, so the state set is indeed in bijection with $\mathfrak{S}_N / \mathfrak{S}_r \times \mathfrak{S}_b$. The initial distribution is the Dirac measure concentrated at $\{1, \dots, r\}$. The transition matrix P is given by

$$P(A', A) = \begin{cases} \frac{(r-1)!(b-1)!}{(r+b)!} & \text{if } r - |A \cap A'| = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Indeed, we need to calculate the number of pairs (A, A') of subsets of cardinality r of $\{1, \dots, N\}$ such that $r - |A \cap A'| = 1$; note that the condition means that $A' - A$ and $A - A'$ both have exactly one element. There are $\frac{(r+b)!}{r!b!}$ choices for A , b choices for the element of $A' - A$ and r choices for the element of $A - A'$.

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We have been asking if the chains described in the examples converge, but the first question should be : to what distribution(s) can they converge ?

Definition VI.1.6. Consider a stochastic function $P : X^2 \rightarrow \mathbb{R}$. A *stationary distribution* for P is a probability distribution μ on X such that, for every $y \in X$, we have

$$\mu(y) = \sum_{x \in X} \mu(x)P(x, y).$$

If we think of P as a $|X| \times |X|$ matrix and of μ as a row vector of size $|X|$, then the condition becomes $\mu P = \mu$.

If a Markov chain with transition matrix P converges in any reasonable sense, then the distribution of its limit should be a stationary distribution of P .

Finally, we define the distance that we will use on random variables. Note that this definition makes just as much sense if X is a general measure space, and the lemma following it stays true with essentially the same proof.

Definition VI.1.7. Let μ and ν be two probability distributions on X . Their *total variation distance* is

$$\|\mu - \nu\|_{TV} = \max_{A \subset X} |\mu(A) - \nu(A)|.$$

This is clearly a metric on the set of probability distributions, and in fact it is closely related to the L^1 metric.

Lemma VI.1.8. Let μ and ν be two probability distributions on X . Then we have

$$\|\mu - \nu\|_{TV} = \frac{1}{2} \sum_{x \in X} |\mu(x) - \nu(x)|.$$

Proof. Let $B = \{x \in X | \mu(x) \geq \nu(x)\}$. For every $A \subset X$, we have

$$\begin{aligned} \mu(A) - \nu(A) &= \mu(A \cap B) - \nu(A \cap B) + \sum_{x \in A - A \cap B} (\mu(x) - \nu(x)) \\ &\leq \mu(A \cap B) - \nu(A \cap B) \\ &= \mu(B) - \nu(B) - \sum_{x \in B - A \cap B} (\mu(x) - \nu(x)) \\ &\leq \mu(B) - \nu(B). \end{aligned}$$

Similarly, we have

$$\nu(A) - \mu(A) \leq \nu(X - B) - \mu(X - B) = \mu(B) - \nu(B).$$

Hence $|\mu(A) - \nu(A)| \leq \mu(B) - \nu(B)$, with equality if $A = B$ or $A = X - B$, and we get

$$\begin{aligned} \|\mu - \nu\|_{TV} &= \mu(B) - \nu(B) = \frac{1}{2}(\mu(B) - \nu(B) + \nu(X - B) - \mu(X - B)) \\ &= \frac{1}{2} \sum_{x \in B} |\mu(x) - \nu(x)| + \frac{1}{2} \sum_{x \in X-B} |\mu(x) - \nu(x)| \\ &= \frac{1}{2} \|\mu - \nu\|_1. \end{aligned}$$

□

VI.2 The Perron-Frobenius theorem and convergence of Markov chains

Notation VI.2.1. Let $A, B \in M_{nm}(\mathbb{R})$. We say that $A \geq B$ (resp. $A > B$) if $A_{ij} \geq B_{ij}$ (resp. $A_{ij} > B_{ij}$) for every $(i, j) \in \{1, \dots, n\} \times \{1, \dots, m\}$. We also denote by $|A|$ the $n \times m$ matrix $(|A_{ij}|)$.

Definition VI.2.2. We say that a matrix $P = (P_{ij}) \in M_n(\mathbb{R})$ is *positive* if $P > 0$.

Definition VI.2.3. We say that a stochastic matrix $P \in M_n(\mathbb{R})$ is *ergodic* if there exists a positive integer r such that P^r is positive.

Remember the following classical theorem from linear algebra :

Theorem VI.2.4 (Perron-Frobenius theorem). *Let $P = (P_{ij}) \in M_n(\mathbb{R})$ be an ergodic stochastic matrix. Then :*

- (i) *The matrix P has 1 as a simple eigenvalue, and every complex eigenvalue λ of P satisfies $|\lambda| < 1$.*
- (ii) *The space of row vectors $w \in M_{1n}(\mathbb{R})$ such that $wP = w$ is 1-dimensional, and it has a generator $v = (v_1, \dots, v_n)$ such that $v_i > 0$ for every i and $v_1 + \dots + v_n = 1$.*
- (iii) *Let P_∞ be the $n \times n$ matrix all of whose rows are equal to the vector v of (ii). Then $P^r \rightarrow P_\infty$ as $r \rightarrow +\infty$. More precisely, let $\rho = \max\{|\lambda|, \lambda \neq 1 \text{ eigenvalue of } P\}$; by (i), we know that $\rho < 1$. Fix any norm $\|\cdot\|$ on $M_n(\mathbb{R})$. Then there exists a polynomial $f \in \mathbb{Z}[t]$ such that*

$$\|P^k - P_\infty\| \leq f(k)\rho^k.$$

Lemma VI.2.5. *Let $A = (A_{ij}) \in M_n(\mathbb{R})$ be a positive matrix, let*

$$Z = \{x = (x_1, \dots, x_n) \in \mathbb{R}^n \mid x \geq 0 \text{ and } x_1 + \dots + x_n = 1\},$$

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and let

$$\Lambda = \{\lambda \in \mathbb{R} \mid \exists x \in Z, Ax \geq \lambda x\}.$$

Then the real number $\lambda_0 = \sup \Lambda$ is positive and a simple root of the characteristic polynomial of A , and it has an eigenvector all of whose entries are positive. Moreover, for any complex eigenvalue $\lambda \neq \lambda_0$ of A , we have $|\lambda| < \lambda_0$.

Proof. Note that $\Lambda \neq \emptyset$ because $0 \in \Lambda$, and Λ is bounded above by the sum of all the entries of A . So λ_0 is well-defined and nonnegative. Let $(\mu_n)_{n \geq 0}$ be a sequence of elements of Λ converging to λ_0 ; for every $n \geq 0$, choose $x^{(n)} \in Z$ such that $Ax^{(n)} \geq \mu_n x^{(n)}$. As Z is compact, we may assume that the sequence $(x^{(n)})_{n \geq 0}$ converges to some $x \in Z$, and then we have $Ax \geq \lambda_0 x$. Suppose that $Ax \neq \lambda_0 x$, then, as $A > 0$, we get $A(Ax) > \lambda_0 Ax$. As $Ax \geq 0$ and $Ax \neq 0$, we can multiply Ax by a positive scalar to get a vector $y \in Z$ such that $Ay > \lambda_0 y$, which contradicts the definition of λ_0 . So $Ax = \lambda_0 x$. Also, as x has at least one positive entry and $A > 0$, the vector $\lambda_0 x = Ax$ has all its entries positive, which implies that $\lambda_0 > 0$ and $x > 0$.

Next we show that every complex eigenvalue $\lambda \neq \lambda_0$ of A satisfies $|\lambda| < \lambda_0$. Let λ be a complex eigenvalue of A . Then there exists a nonzero vector $y = (y_1, \dots, y_n) \in \mathbb{C}^n$ such that $Ay = \lambda y$. For every $i \in \{1, \dots, n\}$, we have

$$|\lambda| |y_i| = \left| \sum_{j=1}^n A_{i,j} y_j \right| \leq \sum_{j=1}^n A_{i,j} |y_j|.$$

In other words, we have $A|y| \geq |\lambda| |y|$. As we can normalize $|y|$ to get an element of Z , this shows that $|\lambda| \leq \lambda_0$. Suppose that $|\lambda| = \lambda_0$. As $A > 0$, there exists a positive real number δ such that $A' := A - \delta I_n > 0$. Then $\mu \mapsto \mu - \delta$ induces a bijection between the eigenvalues and those of A' , and in particular $\lambda_0 - \delta$ is the biggest real eigenvalue of A' (and it is positive because $A' > 0$). By applying the beginning of the paragraph to A' , we see that $|\lambda - \delta| \leq \lambda_0 - \delta$. But then

$$\lambda_0 = |\lambda| = |\lambda - \delta + \delta| \leq |\lambda - \delta| + \delta \leq \lambda_0,$$

so $|\lambda - \delta| + \delta = |\lambda|$, so $\lambda \in \mathbb{R}_{\geq \delta}$, and we must have $\lambda = \lambda_0$.

Let's show that the eigenspace $E_{\lambda_0} := \text{Ker}(A - \lambda_0 I_n)$ has dimension 1. Suppose that there exists $y = (y_1, \dots, y_n) \in E_{\lambda_0}$ (with real entries) such that the family $\{x, y\}$ is linearly independent. We may assume that y has at least one positive entry. Write $x = (x_1, \dots, x_n)$, and let $\mu = \sup\{\nu \in \mathbb{R} \mid \forall i \in \{1, \dots, n\}, x_i \geq \nu y_i\}$. Then $x - \mu y \geq 0$ and $x - \mu y \not\geq 0$. The vector $x - \mu y$ is nonzero because x and y are linearly independent, and we have $A(x - \mu y) = \lambda_0(x - \mu y)$. As $A > 0$, $x - \mu y \geq 0$ and λ_0 , this implies $x - \mu y > 0$, contradicting the choice of μ .

Now we show that λ_0 is a simple root of the characteristic polynomial $\chi_A(t)$ of A . We can find $g \in \text{GL}_n(\mathbb{R})$ such that $g^{-1}Ag$ is of the form $\begin{pmatrix} \lambda_0 & * \\ 0 & B \end{pmatrix}$, with $B \in M_{n-1}(\mathbb{R})$. We have $\chi_A(t) = (t - \lambda_0)\chi_B(t)$. Suppose that the multiplicity of λ_0 as a root of $\chi_A(t)$ is ≥ 2 . Then λ_0

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is a root of $\chi_B(t)$, so there exists $z \in \mathbb{R}^{n-1}$ such that $Bz = \lambda_0 z$. Let $y = g \begin{pmatrix} 0 \\ z \end{pmatrix} \in \mathbb{R}^n$, then $Ay = \lambda_0 y + \alpha x$ for some $\alpha \in \mathbb{R}$. As $\dim(E_{\lambda_0}) \neq 0$, the vector y cannot be an eigenvector of A , so $\alpha \neq 0$. An easy induction (using the fact that $Ax = \lambda_0 x$) shows that, for every positive integer r , we have $A^r y = \lambda_0^r y + r\alpha \lambda_0^{r-1} x$. As $A^r > 0$, this implies that

$$A^r |y| \geq |A^r y| = |\lambda_0^r y + r\alpha \lambda_0^{r-1} x| \geq |r\alpha \lambda_0^{r-1} x| - \lambda_0^r |y| = \lambda_0^{r-1} (r|\alpha x| - \lambda_0 |y|).$$

As $\alpha \neq 0$ and $x > 0$, there exists a positive integer r such that $r|\alpha x| - \lambda_0 |y| > \lambda_0 |y|$, and then we have $A^r |y| > \lambda_0^r |y|$. As $A^r > 0$, applying the beginning of the proof to A^r , we see that this implies that A^r has a real eigenvalue $> \lambda_0^r$. But this is impossible, because the eigenvalues of A^r are the r th powers of the eigenvalues of A , so they all have absolute value $\leq \lambda_0^r$.

□

Proof of the theorem. We prove (i). Let $v_0 = (1, \dots, 1) \in \mathbb{R}^n$. Then the fact that P is stochastic is equivalent to the fact $P \geq 0$ and $Pv_0 = v_0$. As all the matrices P^r for $r \geq 1$ have nonnegative entries and satisfy $P^r v_0 = v_0$, they are all stochastic. Also, if $x = (x_1, \dots, x_n) \in (\mathbb{R}_{\geq 0})^n$ and $Q = (Q_{ij}) \in M_n(\mathbb{R})$ is stochastic, then, for every $i \in \{1, \dots, n\}$, we have

$$(Qx)_i = \sum_{j=1}^n Q_{ij} x_j \leq \sup_{1 \leq j \leq n} x_j.$$

Fix an integer $r \geq 1$ such that $P^r > 0$. By the lemma, the matrix P^r has a simple real positive eigenvalue λ_0 such that every complex eigenvalue $\lambda \neq \lambda_0$ of P^r satisfies $|\lambda| < \lambda_0$. By the definition of λ_0 in the lemma and the observation above about stochastic matrices, we have $\lambda_0 \leq 1$. On the other hand, we have $Pv_0 = v_0$, so 1 is an eigenvalue of P , hence also of P^r , and so $\lambda_0 = 1$. Let $\lambda \in \mathbb{C}$ be an eigenvalue of P , and $y \in \mathbb{C}^n$ be an eigenvector for this eigenvalue. Then $P^r y = \lambda^r y$, so λ^r is an eigenvalue of P^r . If $\lambda^r \neq 1$, then $|\lambda^r| < 1$ by the lemma, hence $|\lambda| < 1$. If $\lambda^r = 1$, then the eigenvector y must be in $\text{Ker}(P^r - I_n)$, and we know (again by the lemma) that this space is 1-dimensional. As $v_0 \in \text{Ker}(P^r - I_n)$, the vector y must be a multiple of v_0 , and then $\lambda = 1$.

Finally, if the characteristic polynomial of P is $\chi_P(t) = (t - \lambda_1) \dots (t - \lambda_n)$, then that of P^r is $\chi_{P^r}(t) = (t - \lambda_1^r) \dots (t - \lambda_n^r)$. So the multiplicity of 1 in $\chi_P(t)$ is at most its multiplicity in $\chi_{P^r}(t)$, which we know is 1 by the lemma. This finishes the proof of (i).

Let's prove (ii). As P and P^T have the same characteristic polynomial, we know that 1 is a simple eigenvalue of P^T by (i), so the space of row vectors w such that $wP = w$ has dimension 1. Let $w = (w_1, \dots, w_n)$ be a nonzero vector in this space. Then we also have $|w|P = |w|$. Indeed, for every $j \in \{1, \dots, n\}$, we have

$$|w_j| = \left| \sum_{i=1}^n w_i P_{ij} \right| \leq \sum_{i=1}^n |w_i| P_{i,j}$$

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(because all the P_{ij} are nonnegative). Suppose that $|w| \neq |w|P$. Then there exists $j_0 \in \{1, \dots, n\}$ such that $|w_{j_0}| < \sum_{i=1}^n |w_i| P_{ij_0}$, and this implies that

$$\sum_{i=1}^n |w_i| = \sum_{i,j=1}^n |w_i| P_{ij} > \sum_{j=1}^n |w_j|,$$

a contradiction. As $w \neq 0$, at least one of the $|w_i|$ is positive. If we choose as before $r \geq 1$ such that $P^r > 0$, then $|w| = P^r |w|$, so, for every $j \in \{1, \dots, n\}$, we have $|w_j| = \sum_{i=1}^n (P^r)_{ij} |w_i| > 0$. This finishes the proof of (ii).

We finally prove (iii). As all the norms on $M_n(\mathbb{R})$ are equivalent, it suffices to prove the statement for a particular norm. By the existence of the Jordan normal form (actually by the Jordan-Chevalley decomposition), there exists a matrix $g \in \text{GL}_n(\mathbb{R})$ with $g^{-1}Pg = A = \begin{pmatrix} 1 & 0 \\ 0 & B \end{pmatrix}$, with $B \in M_{n-1}(\mathbb{R})$ such that $B = D + N$, with D a diagonal matrix, N a nilpotent matrix and $DN = ND$. Choose the operator norm $\|\cdot\|$ on $M_n(\mathbb{R})$ coming from the usual Euclidian norm on \mathbb{R}^n . The entries of D are the eigenvalues of P different from 1, so $\|D\| = \rho$. As D and N commute, we have, for every $k \in \mathbb{Z}_{\geq 0}$,

$$B^k = (D + N)^k = \sum_{j=0}^k \binom{k}{j} D^{k-j} N^j.$$

If $k \geq n$ (in fact, $k \geq n-1$ suffices), then this simplifies to $\sum_{j=0}^n \binom{k}{j} D^{k-j} N^j$, because $N^j = 0$ for $j \geq n$. Hence, if $k \geq n$,

$$\|B^k\| \leq \sum_{j=0}^n \binom{k}{j} \|D\|^{k-j} \|N\|^j \leq \rho^{k-n} \sum_{j=0}^n k^j \|N\|^j.$$

Let $A_\infty = \begin{pmatrix} 1 & 0 \\ 0 & B_\infty \end{pmatrix}$, with $B_\infty = 0 \in M_{n-1}(\mathbb{R})$. Then $\|A^k - A_\infty\| = \|B^k\|$ for every $k \geq 0$, so $A^k \rightarrow A_\infty$ as $k \rightarrow +\infty$ (because $\rho < 1$). This implies that $P^k \rightarrow P' := gA_\infty g^{-1}$ as $k \rightarrow +\infty$. Also,

$$\|P^k - P'\| = \|g^{-1}(A^k - A_\infty)g\| \leq \|g\| \|g^{-1}\| \|B^k\|,$$

which is bounded by the product of ρ^k and of a polynomial in k . So it only remains to show that $P' = P_\infty$. As $P' = \lim_{k \rightarrow +\infty} P^k$, we have $P'P = PP' = P'$. Remember that 1 is a simple eigenvalue of P and of P^T . So all the rows of P' are multiples of the corresponding eigenvector of P^T , i.e. of v . Also, as P^k is stochastic for every $k \geq 0$, its limit P' is stochastic. So all the rows of P' have nonnegative entries whose sum is 1, which means that they are all equal to v , and that $P' = P_\infty$.

□

Definition VI.2.6. A Markov chain with transition matrix P is called *ergodic* if P is ergodic.

In example VI.1.5, all the chains are ergodic, except the Markov chain of (1) when r is even.

Corollary VI.2.7. *Let $(X_n)_{n \geq 0}$ be an ergodic Markov chain with transition matrix P . Then P has a unique stationary distribution μ , and, if μ_n is the distribution of X_n , we have*

$$\|\mu_n - \mu\|_{TV} \leq f(n)\rho^n,$$

where f is a polynomial and $\rho = \max\{|\lambda|, \lambda \neq 1 \text{ eigenvalue of } P\} < 1$.

Proof. Let ν be the initial distribution of the Markov chain. By lemma VI.1.4, we have $\mu_n = \nu P^n$. Let P_∞ be the limit of the sequence $(P^n)_{n \geq 0}$. All the rows of P_∞ are equal to μ , so $\nu P_\infty = \mu$. If we use the L^1 norm on the space of functions from X to \mathbb{R} (for the counting measure on X) to define the operator norm $\|\cdot\|$ on the space of matrices, then we have

$$\|\mu_n - \mu\|_{TV} = \frac{1}{2} \|\mu_n - \mu\|_1 = \frac{1}{2} \|\nu P^n - \nu P_\infty\|_1 \leq \frac{1}{2} \|\nu\|_1 \|P^n - P_\infty\|,$$

so the bound on $\|\mu_n - \mu\|_{TV}$ follows immediately from (iii) of the theorem. □

Remark VI.2.8. Although the bound on $\|\mu_n - \mu\|_{TV}$ looks quite good (it is exponential), it is useless if we want to know when exactly μ_n becomes close to the stationary distribution. We need to analyse the problem more closely to answer that kind of question. This is what we will now try to do in some particular cases.

Example VI.2.9. The chain of example VI.1.5(2) is ergodic. Indeed, let $T \subset \mathfrak{S}_n$ be the union of $\{1\}$ and of the set of transpositions. Then, for $r \geq 1$ and $\sigma, \sigma' \in \mathfrak{S}_n$, we have $P^r(\sigma', \sigma) > 0$ if and only if $\sigma'\sigma^{-1}$ can be written as a product of exactly r elements of T ; as $1 \in T$, this is equivalent to the condition that $\sigma'\sigma^{-1}$ can be written as a product of s transpositions, for some $s \leq r$. So if $r \geq \frac{n(n-1)}{2}$ (the length of the longest element of \mathfrak{S}_n), then $P^r(\sigma', \sigma) > 0$ for all $\sigma, \sigma' \in \mathfrak{S}_n$.

VI.3 A criterion for ergodicity

The definitions and results of this section are not used in the next sections.

Remember the following definitions :

Definition VI.3.1. A (finite unoriented) graph is a pair $G = (X, E)$, where X is a finite set and E is a set of unordered pairs $\{x, y\}$ of distinct elements of X . We say that X is the set of *vertices* of G and that E is the set of *edges*.

Let $x, y \in X$. A path connecting x and y in the graph G is a sequence $p = (e_0, \dots, e_n)$ of edges of G such that we can write $e_i = \{x_i, y_i\}$ with $x_0 = x$, $y_n = y$ and $y_i = x_{i+1}$ for every

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$i \in \{0, \dots, n-1\}$. We call the integer $n+1$ the *length* of the path p and denoted by $|p|$. If $x = y$, we say that the path is a *closed path* or a *loop* based at x .

We say that the graph G is *connected* if for every $x, y \in X$, there exists a path connected x and y . We say that G is *bipartite* if there exists a surjective function $\phi : X \rightarrow \{-1, 1\}$ such that, for every edge $e = \{x, y\}$ of G , we have $\phi(x) \neq \phi(y)$. (In other parts, we can partition X into two nonempty subsets X_0 and X_1 such that every edge connects an element of X_0 to an element of X_1 .)

For every $x, y \in X$, the *distance* $d(x, y)$ between x and y is the length of the shortest path connecting x and y ; if there is no such path, then we set $d(x, y) = +\infty$. Note that this defines a metric on X if G is connected.

The following result is classical.

Proposition VI.3.2. *Let $G = (X, E)$ be a connected graph such that $|X| \geq 2$. Then the following conditions are equivalent :*

- (i) G is bipartite;
- (ii) every loop in G has even length;
- (iii) there exists x_0 such that every loop based at x_0 has even length.

Proof. We show that (i) implies (ii). Suppose that G is bipartite, and let $\phi : X \rightarrow \{-1, 1\}$ be as in the definition above. Let (e_0, \dots, e_n) be a loop in G . We write $e_i = \{x_i, y_i\}$ with $x_i = y_{i+1}$ for $0 \leq i \leq n-1$ and $y_n = x_0$. Then an easy induction on i shows that, if i is even, we have $\phi(x_i) = \phi(x_0)$ and $\phi(y_i) \neq \phi(x_0)$, and, if i is odd, we have $\phi(x_i) \neq \phi(x_0)$ and $\phi(y_i) = \phi(x_0)$. But $y_n = x_0$, so $\phi(y_n) = \phi(x_0)$, so n is odd, and the loop has even length.

It is obvious that (ii) implies (iii). Now assume (iii) and let's show (i). Pick $x_0 \in X$ such that every loop based at x_0 has even length. We want to define a function $\phi : X \rightarrow \{-1, 1\}$. Let $y \in X$. As G is connected, there exists a path $p = (e_0, \dots, e_n)$ connecting x_0 and x , and we set $\phi(x) = (-1)^{|p|}$. We need to show that this does not depend on the path. Let $q = (f_0, \dots, f_m)$ be another path connecting x_0 and x . Then $(e_0, \dots, e_n, f_m, \dots, f_0)$ is a loop based at x_0 , so it has even length by assumption, so $|p| + |q|$ is even, and $(-1)^{|p|} = (-1)^{|q|}$. Note that $\phi(x_0) = 1$ and that $\phi(x) = -1$ if $\{x_0, x\}$ is an edge (such an edge must exist because G is connected and $|X| \geq 2$). So ϕ is surjective. Let $e = \{x, y\}$ be an edge of G . Let $p = (e_0, \dots, e_n)$ be a path connecting x_0 and x . Then $p' := (e_0, \dots, e_n, e)$ is a path connecting x_0 and y , and $|p'| = |p| + 1$, so $\phi(x) \neq \phi(y)$. This shows that G is bipartite. □

We now come to the connection with Markov chains.

Proposition VI.3.3. *Let X be a finite set and $P : X \times X \rightarrow \mathbb{R}$ be a stochastic function. We define a graph $G = (X, E)$ in the following way : a pair $\{x, y\}$ of distinct elements of X is an edge of G if and only if $P(x, y) > 0$.*

Suppose that G is connected and that it is not bipartite. Then the function P is ergodic.

Proof. Note that, for every $x, y \in X$ and every $n \geq 1$, we have $P^n(x, y) > 0$ if and only if there exists a path of length n connecting x and y .

By proposition VI.3.2, for every $x \in X$, there exists a loop p_x of odd length based at x . Write $2m + 1 = \max_{x \in X} |p_x|$, with $m \in \mathbb{Z}_{\geq 0}$. Let $x \in X$. Let's show that, for every $n \geq 2m$, there is a loop of length n based at x . Let $\{x, z\}$ be an edge. For every $r \geq 0$, write q_{2r} for the loop of length $2r$ given by $q_{2r} = (\{x, z\}, \{z, x\}, \dots, \{x, z\}, \{z, x\})$. Let $n \geq 2m$. If n is even, then q_n is a loop of length n based at x . If n is odd, then $r := \frac{n - |p_x|}{2}$ is a nonnegative integer, and the loop obtained by concatenating p_x and q_{2r} has length n and contains x .

Let $\delta = \max_{x, y \in X} d(x, y)$. (This is called the *diameter* of the graph G .) Let $x, y \in X$ and $n \geq 2m + \delta$, and let's show that there is a path of length n connecting x and y (this will finish the proof). Let p be any path connecting x and y . Then $|p| \leq \delta$, so, by the previous paragraph, there exists a loop q of length $n - |p|$ based at x . The concatenation of p and q is the desired path. □

Corollary VI.3.4. (i) The chain of example VI.1.5(1) is ergodic if and only if r is odd.

(ii) The chain of example VI.1.5(3) is ergodic if $r \leq n - 1$.

We will reprove (ii) by a different method in section VI.5.

Proof. (i) The graph corresponding to the chain has $\mathbb{Z}/r\mathbb{Z}$ as set of vertices, and there is an edge between $a, b \in \mathbb{Z}/r\mathbb{Z}$ if and only if $a - b \in \{\pm 1\}$. This graph is obviously connected, and it is easy to see that it is bipartite if and only if r is even. In particular, if r is odd, then the proposition implies that the chain is ergodic.

Now assume that r is even. An easy induction on n shows that, for every $n \geq 1$ and all $a, b \in \mathbb{Z}/r\mathbb{Z}$, we have $P^n(a, b) = 0$ if the image of $n + a + b$ in $\mathbb{Z}/2\mathbb{Z}$ is nonzero. Indeed, this follows from the definition of P if $n = 1$. Suppose the result known up to some $n \geq 1$, and let's prove it for $n + 1$. Let $a, b \in \mathbb{Z}/r\mathbb{Z}$ be such that $P^{n+1}(a, b) \neq 0$. As $P^{n+1}(a, b) = \sum_{c \in \mathbb{Z}/r\mathbb{Z}} P(a, c)P^n(c, b)$, there exists $c \in \mathbb{Z}/r\mathbb{Z}$ such that $P(a, c) \neq 0$ and $P^n(c, b) \neq 0$. By the induction hypothesis and the case $n = 1$, this implies that $a + c \not\equiv 0 \pmod{2}$ and $n + c + b \not\equiv 0 \pmod{2}$, and then $n + a + b + 2c \equiv n + a + b \equiv 0 \pmod{2}$.

(ii) The graph corresponding to the Markov chain has the set Ω_r of cardinality r subsets of $\{1, \dots, n\}$ as its set of vertices, and there is an edge linking $A, A' \in \Omega_r$ if and only if $|A \cap A'| = r - 1$. Let $A_0 = \{1, \dots, r\}$. We first show that the graph is connected. Let $A \in \Omega_r$. We write $A = \{n_1, \dots, n_r\}$, and we choose the ordering of the elements such that $A \cap A_0 = \{1, \dots, n_s\}$, with $s = |A \cap A_0|$. Let m_1, \dots, m_{r-s} be the elements of $A_0 - A$. For $0 \leq i \leq r - s$, let $B_i = \{n_1, \dots, n_{s+i}, m_{i+1}, \dots, m_{r-s}\}$. Then $B_0 = A_0$, $B_{r-s} = A$, and there is an edge between B_i and B_{i+1} for every $i \in \{0, \dots, r - s - 1\}$. So the graph is connected.

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Now we show that the graph is not bipartite, by finding a loop of odd length. Let $A = \{1, \dots, r-1, r+1\}$ and $B = \{2, \dots, r, r+1\}$. Then $\{A_0, A\}$, $\{A, B\}$ and $\{B, A_0\}$ are edges, so we have found a loop of length 3.

□

VI.4 Random walks on homogeneous spaces

Now suppose that we have a finite group G acting transitively (on the left) on the finite set X . Fix $x_0 \in X$, and let K be the stabilizer of x_0 in G , so that we have a bijection $G/K \simeq X$, $g \mapsto g \cdot x_0$.

Warning : We will be using the counting measure on G to define convolution products and L^p norms in this section. Beware constants ! (The reason for this choice is that we want the convolution of two probability distributions to be a probability distribution.)

Definition VI.4.1. If π is a probability distribution on G , we denote by $P_\pi : X \times X \rightarrow \mathbb{R}$ the function defined by

$$P_\pi(xK, yK) = \pi(yKx^{-1}),$$

for all $x, y \in G$.

Definition VI.4.2. A *left-invariant random walk* on X driven by π and with initial distribution ν is a Markov chain with state space X , initial distribution ν and transition matrix P_π .

Here is the description of this Markov chain $(X_n)_{n \geq 0}$ in words : We choosing a starting point on X according to the probability distribution ν . At time n , we choose an element of G using the probability distribution π and act on our position by this element to get to the position at time $n+1$.

Remark VI.4.3. The matrix P_π is actually *bistochastic*, i.e. both P_π and its transpose are stochastic. Indeed, for every $y \in G$, we have

$$\begin{aligned} \sum_{x \in G/K} P_\pi(xK, yK) &= \sum_{x \in G/K} \pi(yKx^{-1}) \\ &= \sum_{x \in G} \pi(yx^{-1}) \\ &= 1. \end{aligned}$$

In particular, the uniform probability distribution on X is an invariant distribution for P_π . If P_π is ergodic, it is the only invariant distribution.

If the homogeneous space is G itself, we can give a simple criterion for ergodicity. (See lemma 16.20 and proposition 16.21 of [1].)

Proposition VI.4.4. *Suppose that $X = G$, and let $S = \text{supp}(\pi)$. Write G_S for the set of elements of G that can be written as $g_1 \dots g_{2r}$ for some $r \geq 0$, with exactly r of the g_i in S and r of the g_i in S^{-1} .*

Then G_S is a subgroup of G , and the function P_π is ergodic if and only if $G = G_S$.

In particular, if $\pi(1) \neq 0$, then P_π is ergodic if and only if S generates G . More generally, if S generates G and is not contained in a coset of a strict subgroup of G , then P_π is ergodic. (Note that we have $S \subset gG_S$ for every $g \in S$.)

Proposition VI.4.5. *For every $n \geq 1$, we have $P_\pi^n = P_{\pi^{*n}}$, where π^{*n} is the n -fold convolution product of π .*

Proof. We prove the result by induction on n . It is just the definition of P_π if $n = 1$. Suppose the equality known for some $n \geq 1$, and let's prove it for $n + 1$. Let $x, y \in X$. Then

$$\begin{aligned} P_\pi^{n+1}(xK, yK) &= \sum_{z \in G/K} P_\pi(xK, zK) P_\pi^n(zK, yK) \\ &= \sum_{z \in G/K} \pi(zKx^{-1}) \pi^{*n}(yKz^{-1}) \\ &= \sum_{z \in G, h \in K} \pi(zx^{-1}) \pi^{*n}(yhz^{-1}) \\ &= \sum_{h \in K} \pi^{*(n+1)}(yhx^{-1}) \\ &= \pi^{*(n+1)}(x, y). \end{aligned}$$

□

Corollary VI.4.6. *Let π a probability measure on G , and suppose that π is right invariant by K . Consider a left-invariant random walk $(X_n)_{n \geq 0}$ driven by π and with initial distribution the Dirac measure concentrated at $x_0 \in X$. Let μ_n be the distribution of X_n , and let μ be the uniform probability distribution on X .*

Then, for every $n \geq 0$, we have

$$\|\mu_n - \mu\|_{TV}^2 \leq \frac{1}{4} \sum_{(\rho, V) \in \widehat{G} | V^K \neq 0 \text{ and } \rho \neq \mathbf{1}} \dim(V) \text{Tr}((\widehat{\pi}(\rho)^*)^n \circ \widehat{\pi}(\rho)^n),$$

where we denote by $\mathbf{1}$ the trivial representation of G .

Remember that, if $(\rho, V) \in \widehat{G}$ is an irreducible unitary representation of G and $f : G \rightarrow \mathbb{C}$ is a function, then $\widehat{f}(\rho)$ is then endomorphism of V defined by

$$\widehat{f}(\rho) = \sum_{x \in G} f(x) \rho(x^{-1}).$$

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Proof. Fix $n \geq 0$. For every $x \in G$, we have

$$\mu_n(x) = P_\pi^n(x_0, x) = \pi^{*n}(xK)$$

by lemma VI.1.4 and proposition VI.4.5. Let π_0 be the uniform probability distribution on G . By lemma VI.1.8, we have

$$\begin{aligned} \|\mu_n - \mu\|_{TV}^2 &= \frac{1}{4} \left(\sum_{x \in G/K} |\mu_n(x) - \mu(x)| \right)^2 \\ &= \frac{1}{4} \left(\sum_{x \in G} |\pi^{*n}(x) - \pi_0(x)| \right)^2 \\ &= \frac{1}{4} \|\pi^{*n} - \pi_0\|_1^2 \\ &\leq \frac{|G|}{4} \|\pi^{*n} - \pi_0\|_2^2, \end{aligned}$$

where the last inequality is the Cauchy-Schwarz inequality. (Note that we are using the counting measure on G to define the L^p norms.) Let $f = \pi^{*n} - \pi_0 \in L^2(G)$. By the Parseval formula (theorem IV.6.3(iii), note the factor $\frac{1}{|G|}$ coming from the unnormalized Haar measure), we have

$$\|f\|^2 = \frac{1}{|G|} \sum_{(\rho, V) \in \widehat{G}} \dim(V) \text{Tr}(\widehat{f}(\rho)^* \circ \widehat{f}(\rho)).$$

So we need to calculate the $\widehat{f}(\rho)$. Note that we have

$$\widehat{f}(\rho) = \widehat{\pi}(\rho)^n - \widehat{\mu}(\rho)$$

for every $\rho \in \widehat{G}$.

Suppose first that $\rho = \mathbf{1}$. Then $\widehat{\pi}(\rho) = \widehat{\mu}(\rho) = 1$, so $\widehat{f}(\rho) = 0$.

Let $(\rho, V) \in \widehat{G}$, and suppose that $\rho \neq \mathbf{1}$. Then $\widehat{\mu}(\rho) = \sum_{x \in G} \rho(x^{-1})$ is an element of $\text{End}(V)$ that is G -equivariant, hence a multiple of id_V by Schur's lemma, and has trace equal to $\frac{1}{|G|} \sum_{x \in G} \chi(x) = 0$ (by corollary IV.5.8). So $\widehat{\mu}(\rho) = 0$, and $\widehat{f}(\rho) = \widehat{\pi}(\rho)^n$. To finish the proof, we just need to show that $\widehat{\pi}(\rho) = 0$ if $V^K = 0$. Let $T = \widehat{\pi}(\rho) = \sum_{x \in G} \pi(x) \rho(x^{-1})$ and $P_K = \sum_{x \in K} \rho(x)$. As π is right invariant by K , we have $\rho(x) \circ T = T$ for every $x \in K$, so $P_K \circ T = |K|T$. But P_K is the orthogonal projection on V^K by proposition V.1.7, so $\text{Im}(T) \subset V^K$, and so $T = 0$ if $V^K = 0$.

□

Corollary VI.4.7. *With the notation of the previous corollary, suppose that (G, K) is a Gelfand pair and that π is bi- K -invariant. As in section V.6, let Z be the dual space of (G, K) (i.e. the set of spherical functions by theorem V.7.1).*

VI.5 Application to the Bernoulli-Laplace diffusion model

Then, for every $n \geq 0$, we have

$$\|\mu_n - \mu\|_{TV}^2 \leq \frac{1}{4} \sum_{\varphi \in Z, \varphi \neq 1} \dim(V_\varphi) |\hat{\pi}(\varphi)|^{2n},$$

where now, if $f \in \mathcal{C}(K \backslash G/K)$ and $\varphi \in Z$, the scalar $\hat{f}(\varphi) \in \mathbb{C}$ is the spherical Fourier transform, defined by

$$\hat{f}(\varphi) = \sum_{x \in G} f(x) \varphi(x^{-1}).$$

Proof. The proof is almost the same as for the previous corollary, except that we use the Parseval formula of corollary V.7.2 to calculate $\|\pi^{*n} - \pi_0\|_2^2$. By this formula, we have

$$\|\pi^{*n} - \pi_0\|_2^2 = \frac{1}{|G|} \sum_{\varphi \in Z} \dim(V_\varphi) |\hat{f}(\varphi)|^2,$$

where $f = \pi^{*n} - \pi_0$. If $\varphi = 1$ is the spherical function corresponding to the trivial representation, then $\hat{\pi}(\varphi) = \hat{\pi}_0(\varphi) = 1$, so $\hat{f}(\varphi) = 0$. If $\varphi \neq 1$, then

$$\hat{\pi}_0(\varphi) = \sum_{x \in G} \varphi(x^{-1}) = \langle 1, \varphi \rangle_{L^2(G)} = 0$$

(by (i) of theorem V.7.1 for example). So $\hat{f}(\varphi) = \hat{\pi}(\varphi)^n$, which finishes the proof. □

VI.5 Application to the Bernoulli-Laplace diffusion model

Remember that the Bernoulli-Laplace diffusion model was described in example VI.1.5(3). We have two positive integers r and b . This model is a Markov chain $(X_n)_{n \geq 0}$ on the set Ω_r of subsets of cardinality r of $\{1, \dots, r+b\}$ with initial distribution the Dirac distribution concentrated at $\{1, \dots, r\}$. The group $G := \mathfrak{S}_{r+b}$ acts transitively on Ω_r , and the stabilizer of $A_0 := \{1, \dots, r\}$ is $K := \mathfrak{S}_r \times \mathfrak{S}_b$. The transition matrix P of the chain is given by

$$P(A', A) = \begin{cases} \frac{(r-1)!(b-1)!}{(r+b)!} & \text{if } r - |A \cap A'| = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Remember that we have defined in 1(e) of problem set 11 a metric d on Ω_r by $d(A, A') = r - |A \cap A'|$, and that we have proved in 1(d) (and 1(f)) of the same problem set that the orbits of K on $G/K \simeq \Omega_r$ are the spheres with center A_0 for this metric. Bi-invariant

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probability distributions π on G correspond bijectively to probability distributions on the set $K \backslash G / K$ of K -orbits on G / K , and the description of P implies easily that $P = P_\pi$, where π is the bi-invariant probability distribution that corresponding to the uniform distribution on the sphere with center A_0 and radius 1.

If μ_n is the distribution of X_n and μ is the uniform distribution of Ω_r , then, by corollary VI.4.7, we have

$$\|\mu_n - \mu\|_{TV}^2 \leq \frac{1}{4} \sum_{\varphi \in Z - \{1\}} \dim(V_\varphi) |\widehat{\pi}(\varphi)|^{2n}.$$

We calculated all these terms in problem set 11. Suppose for example that $r \leq b$ (if not, we can just switch r and b and we get an equivalent problem). Then we saw how to decompose the quasi-regular representation of G on $L^2(\Omega_r)$ into irreducible subrepresentations in problem 3 of problem set 11 (see 3(j) and 3(k)), and we have exactly $r + 1$ of them. We denote the corresponding spherical functions by $\varphi_0, \dots, \varphi_r$, as in problem 4. In particular, the function φ_0 is just the constant function 1. We calculated these functions in 3(f), but actually we only need 3(g). Indeed, we only care about $\widehat{\pi}(\varphi_s)$, for $1 \leq s \leq r$. As π corresponds to the uniform distribution on the sphere of radius 1 centered at A_0 , the number $\widehat{\pi}(\varphi_s)$ is just the coefficient of $\sigma_{1,r-1}(A_0)$ in φ_s (with the notation of problem 3), that is,

$$\widehat{\pi}(\varphi_s) = 1 - \frac{s(r+b-s+1)}{rb}.$$

Also, 3(f) says that

$$\dim(V_{\varphi_s}) = \binom{r+b}{s} - \binom{r+b}{s-1}$$

if $1 \leq s \leq r$.

So corollary VI.4.7 gives

$$\|\mu_n - \mu\|_{TV} \leq \frac{1}{4} \sum_{s=1}^r \left(\binom{r+b}{s} - \binom{r+b}{s-1} \right) \left(1 - \frac{s(r+b-s+1)}{rb} \right)^{2n}.$$

With some more effort, we can get the following result.

Theorem VI.5.1. (See theorem 10 of chapter 3F of [7].) *There exists a universal constant $a \in \mathbb{R}_{>0}$ such that, if $n = \frac{r+b}{4}(\log(2(r+b)) + c)$ with $c \geq 0$, then we have*

$$\|\mu_n - \mu\|_{TV} \leq ae^{-c/2}.$$

A different calculation (still using spherical functions) gives the following theorem :

Theorem VI.5.2. (See theorem 6.3.2 of [6].) *If $r = b$ is large enough, then, for $n = \frac{r+b}{4}(\log(2(r+b)) - c)$ with $0 < c < \log(2(r+b))$, we have*

$$\|\mu_n - \mu\|_{TV} \geq 1 - 32e^{-c}.$$

VI.6 Random walks on locally compact groups

In this section, we will see a few results (mostly without proofs) about random walks on more general groups. A good reference for many questions that we did not touch on here is Breuillard's survey [5].

We fix a locally compact group G and a left Haar measure on G .

VI.6.1 Setup

Definition VI.6.1.1. (See remark I.4.1.6.) A (complex) Radon measure on G is a bounded linear functional on $\mathcal{C}_0(G)$ (with the norm $\|\cdot\|_\infty$). We denote by $\mathcal{M}(G)$ the space of Radon measures and by $\|\cdot\|$ its norm (which is the operator norm); this is a Banach space. If μ is a Radon measure, we write $f \mapsto \int_G f(x) d\mu(x)$ for the corresponding linear functional on $\mathcal{C}_0(G)$.

Example VI.6.1.2. (1) Any regular Borel measure is a Radon measure on G (such measure are called “positive” when we want to distinguish them from general Radon measures).

(2) If $\varphi \in L^1(G)$, then the linear functional $f \mapsto \int_G f(x)\varphi(x)dx$ is a Radon measure on G , often denoted by $\varphi(x)dx$ or φdx .

(3) For every $x \in G$, the linear functional $\mathcal{C}_0(G) \rightarrow \mathbb{C}$, $f \mapsto f(x)$ is a Radon measure on G , called the Dirac measure at x .

We define the convolution product $\mu * \nu$ of two Radon measures μ and ν to be the linear functional

$$f \mapsto \int_{G \times G} f(xy) d\mu(x) d\nu(y).$$

Then it is not very hard to check that $\|\mu * \nu\| \leq \|\mu\| \|\nu\|$ and that the convolution product is associative on $\mathcal{M}(G)$. This makes $\mathcal{M}(G)$ into a Banach algebra, and the Dirac measure at 1 is a unit element of $\mathcal{M}(G)$.

If $\mu = \varphi dx$ and $\mu' = \varphi' dx$, then it is easy to check that $\mu * \mu' = (\varphi * \varphi') dx$, where $\varphi * \varphi'$ is the usual convolution in $L^1(G)$.

We denote by \widehat{G} the set of unitary equivalence classes of irreducible unitary representations of G . We can extend the Fourier transform (both the ordinary and the spherical versions) to $\mathcal{M}(G)$:

(1) If $\mu \in \mathcal{M}(G)$ and $(\pi, V) \in \widehat{G}$, define $\widehat{\mu}(\pi) \in \text{End}(V)$ by

$$\widehat{\mu}(\pi)(v) = \int_G \pi(x^{-1})(v) d\mu(x).$$

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- (2) Suppose that G is the first entry of a Gelfand pair (G, K) , and that φ is a spherical function of positive type on G . Then, for every $\mu \in \mathcal{M}(G)$, we define $\widehat{\mu}(f) \in \mathbb{C}$ by :

$$\widehat{\mu}(f) = \int_G \varphi(x^{-1}) d\mu(x).$$

For both versions of the Fourier transform, the equality

$$\widehat{\mu * \mu'} = \widehat{\mu} \widehat{\mu'}$$

for all $\mu, \mu' \in \mathcal{M}(G)$ (where the product on the right is composition of endomorphisms in the first case and multiplication in the second case).

The following theorem is a generalization of Lévy's convergence criterion. We say that a sequence $(\mu_n)_{n \geq 0}$ of Radon measures converges weakly if it converges in the weak* topology of $\mathcal{M}(G)$.

Theorem VI.6.1.3. (See [9], section 5.2, theorem 5.2.)

- (i) If $\mu, \mu' \in \mathcal{M}(G)$ are such that $\widehat{\mu}(\pi) \widehat{\mu'}(\pi)$ for every $\pi \in \widehat{G}$, then $\mu = \mu'$.
- (ii) Let $(\mu_n)_{n \geq 0}$ be a sequence of (positive) probability measures on G and μ be another probability measure on G . If $(\mu_n)_{n \geq 0}$ converges weakly to μ , then, for every $(\pi, V) \in \widehat{G}$ and every $v \in V$, we have $\lim_{n \rightarrow +\infty} \widehat{\mu}_n(\pi)(v) = \widehat{\mu}(\pi)(v)$. Conversely, if, for every $(\pi, V) \in \widehat{G}$ and all $v, w \in V$, we have $\lim_{n \rightarrow +\infty} \langle \widehat{\mu}_n(\pi)(v), w \rangle = \langle \widehat{\mu}(\pi)(v), w \rangle$, then $(\mu_n)_{n \geq 0}$ converges weakly to μ .

VI.6.2 Random walks

We fix a regular Borel probability measure μ on G , and we want to understand the behavior of μ^{*n} as $n \rightarrow +\infty$.

The connection with random walks is that μ^{*n} is the distribution of the n th step of a Markov chain with state space G , initial distribution δ_1 and “transition matrix” $\mu(yx^{-1})$. (We are choosing δ_1 as initial distribution to simplify the notation, but this is not really necessary for most results.) In other words, we consider a sequence $(g_n)_{n \geq 1}$ of independent random variables with values in G and distribution μ . The Markov chain $(X_n)_{n \geq 0}$ is defined by $X_n = g_1 n \dots g_1$ (so X_0 is the constant function 1). We could also consider random walks on a space G/K , where K is a subgroup of G : take $(g_n)_{n \geq 1}$ as before, fix some initial random variable X_0 with values in G/K (for example a constant function) and set $X_n = g_n \dots g_1 X_0$.

VI.6.3 Compact groups

In this section, we suppose that G is compact. We start with a general convergence result, due to Ito and Kawada ([11], see also theorem 2.3 of [5]).

Remember that the support of μ is by definition the set of $x \in G$ such that, for every neighborhood U of x , we have $\mu(U) > 0$.

Theorem VI.6.3.1. *Suppose that the support of μ generates a dense subgroup of G and is not contained in any (left or right) coset of a proper closed subgroup of G . Then the sequence $(\mu^{*n})_{n \geq 0}$ converges weakly to the normalized Haar measure on G .*

The proof is based on the convergence criterion of theorem VI.6.1.3(ii). We must show that, for every $(\pi, V) \in \widehat{G}$ nontrivial, the sequence $\widehat{\mu^{*n}}(\pi) = \widehat{\mu}(\pi)^n$ converges to 0 in $\text{End}(V)$. Note that V is finite-dimensional (because G is compact), so all the notions of convergence in $\text{End}(V)$ are equivalent, and we just need to prove that all the eigenvalues of $\widehat{\mu}(\pi)$ are < 1 in absolute value. Suppose that this not the case, then there exists a unit vector $v \in V$ such that

$$\int_G \pi(x^{-1})(v) d\mu(x) = \lambda v,$$

with $|\lambda| = 1$. It is not hard to see that this forces $\pi(x^{-1})(v)$ to be equal to λv μ -almost everywhere and contradicts the hypothesis of the theorem.

Note that this result is much weaker than proposition VI.4.4 (and the Perron-Frobenius theorem), because it only guarantees the weak convergence of $(\mu^{*n})_{n \geq 0}$ and says nothing about convergence for other topologies (such as the one induced by the total variation distance) or about the speed of convergence. If G is finite, all the notions of convergence on the set of probability measures on G coincide (it's just a convex subset of the space of functions on G , which is finite-dimensional); also, it follows from the upper bound lemma (corollary VI.4.6) that the speed of convergence is exponential and controlled by the biggest eigenvalue of a $\widehat{\mu}(\pi)$ that is $\neq 1$. But if G is infinite, then \widehat{G} is also infinite, so, also $\widehat{\mu}(\pi)$ has all its eigenvalues < 1 (in absolute value), we can get eigenvalues that are arbitrarily close to 1. In fact, there is a special name for when this doesn't happen :

Definition VI.6.3.2. We say that the probability measure μ on G has a *spectral gap* if there exists $\varepsilon > 0$ such that, for every $\pi \in \widehat{G}$ nontrivial and for every eigenvalue λ of $\widehat{\mu}(\pi)$, we have $|\lambda| < 1 - \varepsilon$.

Let's first look at some examples.

Example VI.6.3.3. If $\mu = \varphi dx$ with $\varphi \in L^2(G)$, then μ has a spectral gap. In fact, the upper bound lemma (corollary VI.4.6) holds with essentially the same proof : for every $n \geq 0$, we have

$$\|\mu^{*n} - \mu_G\|_{TV}^2 \leq \frac{1}{4} \sum_{(\rho, V) \in \widehat{G} | \rho \neq \mathbb{1}} \dim(V) \text{Tr}((\widehat{\pi}(\rho)^*)^n \circ \widehat{\pi}(\rho)^n),$$

where we denote by $\mathbb{1}$ the trivial representation of G and by μ_G the normalized Haar measure on G . (We could also prove a version for random walks on spaces G/K .) So we have convergence in total variation distance and with exponential speed in this case.

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At the other extreme, we have measures with finite support.

Example VI.6.3.4. Take $G = S^1$. Let $\lambda_1, \dots, \lambda_r \in \mathbb{R}$, and consider the measure

$$\mu = \frac{1}{2r} \sum_{s=1}^r (\delta_{e^{2i\pi\lambda_s}} + \delta_{e^{-2i\pi\lambda_s}})$$

on G . Remember that $\widehat{G} = \mathbb{Z}$ (where $n \in \mathbb{Z}$ corresponds to the representation $z \mapsto z^n$ of G). For every $n \in \mathbb{Z}$, we have

$$\widehat{\mu}(n) = \frac{1}{2r} \sum_{s=1}^r (e^{2i\pi n\lambda_s} + e^{-2i\pi n\lambda_s}).$$

Suppose that the family $(1, \lambda_1, \dots, \lambda_r)$ is \mathbb{Q} -linearly independent. Then Kronecker's theorem (see for example chapter XXIII of [10]) says that the set $\{(e^{2i\pi n\lambda_1}, \dots, e^{2i\pi n\lambda_r}), n \in \mathbb{Z}\}$ is dense in $(S^1)^r$. So we can find $n \neq 0$ such that $\widehat{\mu}(n)$ is arbitrarily close to 1. In other words, the measure μ has no spectral gap.

The question of which measures on nice groups like $SU(d)$ have a spectral gap is a very difficult and an active area of research. We'll give some (difficult) recent results, due to Bourgain and Gamburd (cf. [4] and [3]) for $G = SU(d)$ and to Benoist and de Saxcé (cf. [2]) for a general simple compact Lie group.

Theorem VI.6.3.5. *Let G be a simple compact Lie group (for example $G = SU(d)$ for $d \geq 2$ or $G = SO(d)$ for $d = 3$ or $d \geq 5$), and let μ be a probability measure on G . We say that μ is almost Diophantine if there exists $c_1, c_2 > 0$ such that for every proper closed subgroup H of G and for every $n \in \mathbb{Z}_{\geq 0}$ big enough, we have $\mu^{*n}(\{x \in G | d(x, H) \leq e^{-c_1 n}\}) \leq e^{-c_2 n}$ (where d is any metric on G).*

Then μ has a spectral gap if and only if it is almost Diophantine.

Although the next version has a generalization to any simple compact Lie group, we'll just state it for $SU(d)$ for simplicity.

Theorem VI.6.3.6. *Let $G = SU(d)$, and let μ be a probability measure on G such that the support of μ generates a dense subgroup of G (such a measure is sometimes called “adapted”).*

If every element of the support of μ has algebraic entries, then μ has a spectral gap.

In fact, Benoist and de Saxcé conjecture that the algebraicity condition is not necessary (so every adapted measure should have a spectral gap), see the introduction of [2].

Remark VI.6.3.7. The spectral gap question is also connected to the Banach-Ruziewicz problem (see chapter 2 of Sarnak's book [17] for the connection; another good reference on the Banach-Ruziewicz problem is Lubotzky's book [12]). This problem asks whether Lebesgue measure is

the only (up to a constant) finitely additive $\mathrm{SO}(n+1)$ -invariant measure on Lebesgue measurable subsets of the sphere $S^n \subset \mathbb{R}^n$. The answer is known to be “no” for $n = 1$ and “yes” for $n \geq 2$. For $n \geq 4$, it is due to Margulis and Sullivan and uses the fact that $\mathrm{SO}(n+1)$ has a finitely generated dense subgroup that satisfies Kazhdan’s property (T) for $n \geq 4$ (in fact, the same methods will show that Haar measure is the only left-invariant mean on any simple compact Lie group that is not $\mathrm{SO}(n)$ with $n \geq 4$). For $n = 2, 3$, the solution is originally due to Drinfeld and uses the Jacquet-Langlands correspondence and the Ramanujan-Petersson conjecture. (All this and more is explained in [12].)

VI.6.4 Convergence of random walks with Fourier analysis

We now present some example of random walks on compact groups (or homogeneous spaces) that can be analyzed using Fourier analysis, in the spirit of section VI.5.

As we noted before (in example VI.6.3.3), the upper bound lemma (corollary VI.4.6) still holds for general compact groups.

As for finite groups, Fourier analysis works best if the measure μ is conjugation or if μ is bi- K -invariant and (G, K) is a Gelfand pair.

Random reflections in $\mathrm{SO}(n)$

The reference for this result is Rosenthal’s paper [13]. Fix $n \geq 2$ and $\theta \in (0, 2\pi)$. Let

$$R_\theta = \begin{pmatrix} \cos \theta & \sin \theta & & 0 \\ -\sin \theta & \cos \theta & & \\ & & 1 & \\ & & & \ddots \\ 0 & & & & 1 \end{pmatrix} \in \mathrm{SO}(n),$$

and let μ_θ be the unique conjugation-invariant probability measure concentrated on the conjugacy class of R_θ (in other words, the measure μ_θ is the image of the normalized Haar measure of $\mathrm{SO}(n)$ by the map $\mathrm{SO}(n) \rightarrow \mathrm{SO}(n)$, $x \mapsto xR_\theta x^{-1}$).

Theorem VI.6.4.1. (i) *There exist $\Gamma, \Delta > 0$ (with Δ independent of θ) such that, for every $n \geq 1$ and every $c > 0$, if $k = \frac{1}{2(1-\cos \theta)}(n \log n - cn)$, then*

$$\|\mu_\theta^{*k} - dx\|_{TV} \geq 1 - \Gamma e^{-2c} - \Delta \frac{\log n}{n}.$$

(ii) *Suppose that $\theta = \pi$. Then there exist $\Lambda, \gamma > 0$ such that, for every $n \geq 3$ and every $c > 0$, if $k = \frac{1}{4}n \log n + cn$, then*

$$\|\mu_\theta^{*k} - dx\|_{TV} \leq \Lambda e^{-\gamma c}.$$

The Gelfand pair case

The reference for this result is Su's paper [?].

Fix $\theta \in (0, \pi)$ and consider the following random process on $S^2 \simeq \text{SO}(3)/\text{SO}(2)$:

- X_0 is constant with value the North pole;
- to go from X_n to X_{n+1} , choose a direction (independently and uniformly) and move a distance of θ following the geodesic (= big circle) in that direction.

This random walk is not driven by a measure on $\text{SO}(3)$, but it is equivalent to one that is (see section 3 of [?]). Let μ_n be the distribution of X_n and λ be the unique $\text{SO}(3)$ -invariant probability measure on S^2 . Then we have the following result :

Theorem VI.6.4.2. *If $n = \frac{c}{\sin^2 \theta}$ with $c \geq 0$, then*

$$0.433e^{-c/2} \leq \|\mu_l - \lambda\|_{DD} \leq 4.442e^{-c/8}.$$

In this theorem, $\|\cdot\|_{DD}$ is the *discrepancy distance* : If X is a metric space and μ, μ' are two (Borel) probability measures on X , then

$$\|\mu - \mu'\|_{DD} = \sup_{B \subset X \text{ ball}} |\mu(B) - \mu'(B)|.$$

It is bounded above by the total variation distance, but it can see some phenomena that the total variation distance misses (see the next subsection).

Remark about the different types of convergence

The reference for this subsection is Su's paper [?].

Consider a random walk $(X_n)_{n \geq 0}$ on the circle S^1 driven by the measure $\mu = \frac{1}{2}(\delta_{e^{2i\pi\alpha}} + \delta_{e^{-2i\pi\alpha}})$, for some $\alpha \in \mathbb{R}$ irrational, and let μ_n be the distribution of X_n . Then :

- The general convergence result of Ito-Kawada (theorem VI.6.3.1) says that $(\mu_n)_{n \geq 0}$ converges weakly to the normalized Haar measure dx on S^1 .
- On the other hand, we have seen in example VI.6.3.4 that μ has no spectral gap, so the convergence cannot be too good. In fact, $(\mu_n)_{n \geq 0}$ does not converge to dx in total variation distance.
- On the third hand, $(\mu_n)_{n \geq 0}$ does converge to dx (but not exponentially fast) in discrepancy distance in many cases. More precisely, we have :

Theorem VI.6.4.3. *Let η be the type of α , i.e.*

$$\eta = \sup\{\gamma > 0 \mid \lim_{m \rightarrow +\infty} \inf m^\gamma \{m\alpha\} = 0\}$$

(where $\{\cdot\}$ is the fractional part). Then we have, for every $\varepsilon > 0$,

$$\|\mu_n - dx\|_{DD} = O(n^{-1/2\eta+\varepsilon})$$

and

$$\|\mu_n - dx\|_{DD} = \Omega(n^{-1/2\eta-\varepsilon}).$$

If α is irrational quadratic, we can do better : there exist constants $c_1, c_2 > 0$ such that, for every $n \geq 1$, we have

$$\frac{c_1}{\sqrt{n}} \leq \|\mu_n - dx\|_{DD} \leq \frac{c_2}{\sqrt{n}}.$$

It is known that $\eta = 1$ if α is algebraic, and also that the subset of type 1 elements of $[0, 1]$ has Lebesgue measure 1.

VI.6.5 Random walks on noncompact groups

We don't assume that G is compact anymore. We fix a probability measure μ on G . One of the many questions we can ask is whether a random walk on G driven by μ goes to infinity, and if so, how fast.

The results of this section are proved in the third problem of the final problem set, so a reference will be added after this problem set is due.

First, we define a continuous linear operator $P_\mu : L^2(G) \rightarrow L^2(G)$ by setting

$$P_\mu(f)(x) = \int_G f(yx) d\mu(y)$$

if $f \in \mathcal{C}_c(G)$ and $x \in G$; this extends to $L^2(G)$ by continuity. (If $\mu = \varphi dx$ with $\varphi \in L^1(G)$, this is just the construction of theorem I.4.2.6(i) applied to the right regular representation of G .)

We denote by $\rho(P_\mu)$ the spectral radius of P_μ , seen as an element of the Banach algebra $\text{End}(L^2(G))$. We always have $\rho(P_\mu) \leq 1$ (because μ is a probability measure).

Theorem VI.6.5.1. (i) If G is amenable, then $\rho(P_\mu) = 1$.

(ii) Let H be the closure of the subgroup of G generated by the support of μ . If $\rho(P_\mu) = 1$, then H is amenable.

Definition VI.6.5.2. We say that G is *compactly generated* if there exists a compact subset K of G that generates G .

If G is discrete, this just means that G is finitely generated.

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Definition VI.6.5.3. Suppose that G is compactly generated, and let K be a symmetric compact subset generating G . We define $j_K : G \rightarrow \mathbb{Z}$ by

$$j_K(x) = \min\{n \in \mathbb{Z}_{\geq 0} \mid x \in K^n\}$$

(with the convention that $K^0 = \{1\}$).

Lemma VI.6.5.4. If K and L are two symmetric compact subsets generating G , then there exists $a > 0$ such that $j_L \leq aj_K$.

Corollary VI.6.5.5. Suppose that G is compactly generated, and let K be a symmetric compact subset generating G . Let μ be a probability measure on G , and let $(g_n)_{n \geq 1}$ be a sequence of independent random variables valued on G with distribution μ .

Let H be the closure of the subgroup of G generated by the support of μ . If H is not amenable, then there exist $\alpha, \varepsilon > 0$ such that, for every $n \geq 1$, we have

$$\mathbb{P}(j_K(g_n \dots g_1) \leq \varepsilon n) = o(e^{-\alpha n}).$$

In particular, by the Borel-Cantelli lemma (see section 17.1 of [14]), if n is large enough, we have $j_K(g_n \dots g_1) \geq \varepsilon n$ almost surely.

We finish with an example. We say that a subgroup H of $\mathrm{SL}_2(\mathbb{R})$ is *non-elementary* if no conjugate of H is contained in $\mathrm{SO}(2)$, in

$$\left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}, a \in \mathbb{R}^\times, b \in \mathbb{R} \right\}$$

or in

$$\left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}, a \in \mathbb{R}^\times \right\} \cup \left\{ \begin{pmatrix} 0 & a \\ a^{-1} & 0 \end{pmatrix}, a \in \mathbb{R}^\times \right\}.$$

(An equivalent condition is that H is not compact and does not fix a line in \mathbb{R}^2 or the union of two lines in \mathbb{R}^2 . Here the action of $\mathrm{SL}_2(\mathbb{R})$ in \mathbb{R}^2 is the standard one, given by the inclusion $\mathrm{SL}_2(\mathbb{R}) \subset \mathrm{GL}_2(\mathbb{R})$.)

Proposition VI.6.5.6. A closed subgroup of $\mathrm{SL}_2(\mathbb{R})$ is non-amenable if and only if it is non-elementary.

Example VI.6.5.7. If $t \in \mathbb{R}^\times$, we set

$$a_t = \begin{pmatrix} t & 0 \\ t^{-1} & 0 \end{pmatrix}.$$

If $\theta \in \mathbb{R}$, we set

$$r_\theta = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}.$$

Then, if $s, t > 1$ and $0 < \theta < \pi/2$, the subgroup of $\mathrm{SL}_2(\mathbb{R})$ spanned by a_s and $r_\theta a_t r_\theta^{-1}$ is non-elementary, and so corollary VI.6.5.5 applies to a random walk driven by the measure $\mu = \frac{1}{2}(\delta_{a_t} + \delta_{r_\theta a_t r_\theta^{-1}})$.

A Urysohn's lemma and some consequences

A.1 Urysohn's lemma

Definition A.1.1. A topological space X is called *normal* if whenever we have two disjoint closed subsets A and B of X , there exist open subsets U and V of X such that $A \subset U$ and $B \subset V$.

Proposition A.1.2. Any topological space that is compact Hausdorff or metrizable is normal.

Theorem A.1.3 (Urysohn's lemma). Let X be a normal topological space, and let A, B be two disjoint closed subsets of X . Then there exists a continuous function $f : X \rightarrow [0, 1]$ such that $f(x) = 0$ for every $x \in A$ and $f(x) = 1$ for every $x \in B$.

A.2 The Tietze extension theorem

Corollary A.2.1 (Tietze extension theorem). Let X be a normal topological space, A be a closed subset of X and $f : A \rightarrow \mathbb{C}$ be a continuous function. Then there exists a continuous function $F : X \rightarrow \mathbb{R}$ such that $F|_A = f$ and that $\sup_{x \in X} |F(x)| = \sup_{x \in A} |f(x)|$.

A.3 Applications

Corollary A.3.1. Let X be a locally compact Hausdorff topological space, and let $K \subset U$ be two subsets of X such that K is compact and U is open. Then there exists a continuous function with compact support $f : X \rightarrow [0, 1]$ such that $f|_K = 1$ and $\text{supp } f \subset U$.

Proof. As X is locally, for every $x \in K$, we can find an open neighborhood V_x of x such that $\overline{V_x}$ is compact and contained in U . We have $K \subset \bigcup_{x \in K} V_x$; as K is compact, we can find $x_1, \dots, x_n \in K$ such that $K \subset \bigcup_{i=1}^n V_{x_i}$. Set $K' = \bigcup_{i=1}^n \overline{V_{x_i}}$. Then K' is a compact subset of X , it is contained in U and its interior contains K . Applying the same procedure to $K' \subset U$, we can find a compact subset $K'' \subset U$ of X whose interior contains K' .

A Urysohn's lemma and some consequences

The space K'' is compact, hence normal, and its subsets K and $K'' - \overset{\circ}{K}'$ are closed and disjoint, so, by Urysohn's lemma, we have a continuous function $f : K'' \rightarrow [0, 1]$ such that $f|_K = 1$ and $f|_{K'' - \overset{\circ}{K}'} = 0$. We extend f to X by setting $f(x) = 0$ if $x \in X - K''$. Then f is equal to 0 (hence continuous) on $X - K'$, and it is also continuous on $\overset{\circ}{K}''$. As $X - K'$ and $\overset{\circ}{K}''$ are open subset whose union is X , the function f is continuous on X . It is clear from the construction of f that it satisfies all the desired properties.

□

Corollary A.3.2. *Let X be a locally compact Hausdorff topological space, and let $K \subset U$ be two subsets of X such that K is compact and U is open. Then, for every continuous function $f : K \rightarrow \mathbb{C}$, there exists a continuous function with compact support $F : X \rightarrow \mathbb{C}$ such that :*

- (a) $\text{supp}(F) \subset U$;
- (b) $F|_K = f$;
- (c) $\sup_{x \in X} |F(x)| = \sup_{x \in K} |f(x)|$.

Proof. By corollary A.3.1, we can find a continuous function with compact support $\psi : X \rightarrow [0, 1]$ such that $\psi|_K = 1$ and $\text{supp}(\psi) \subset U$. On the other hand, we can find, as in the proof of corollary A.3.1, a compact set $K' \subset U$ whose interior contains $\text{supp} \psi$. Applying the Tietze extension to the normal space K' , we get a continuous function $f' : K' \rightarrow \mathbb{C}$ such that $f'|_K = f$ and $\sup_{x \in K'} |f'(x)| = \sup_{x \in K} |f(x)|$. We define a function $F : X \rightarrow \mathbb{C}$ by $F(x) = f'(x)\psi(x)$ if $x \in K'$, and $F(x) = 0$ if $x \in X - K'$. This function F clearly satisfies conditions (a)-(c), so we just need to check that it is continuous. But this follows from the fact that F is continuous on the open sets $X - \text{supp}(\psi)$ (because it is zero on that set) and $\overset{\circ}{K}'$, and that the union of these open sets is X .

□

B Useful things about normed vector spaces

B.1 The quotient norm

Cf. [15] 18.15 or [16] 1.40, 1.41.

Definition B.1.1. Let V be a normed vector space and $W \subset V$ be a subspace. Then the *quotient seminorm* on V/W is defined by

$$\|x + W\| = \inf_{w \in W} \|v + w\|.$$

If W is closed, this is called the *quotient norm*.

Proposition B.1.2. (i) The formula of the preceding definition gives a seminorm on V/W , which is a norm if and only if W is closed in V .

(ii) If V is a Banach space and W is closed in V , then V/W is a Banach space for the quotient norm.

Proof. (i) Let $v, v' \in V$ and $\lambda \in \mathbb{C}$. Then we have

$$\|v + v' + W\| = \inf_{x \in W} \|v + v' + w\| \leq \inf_{w \in W} \|v + w\| + \inf_{w \in W} \|v' + w\| = \|v + W\| + \|v' + W\|.$$

If $\lambda = 0$, then $\lambda v \in W$, so $\|\lambda v + W\| = 0$; otherwise,

$$\|\lambda v + W\| = \inf_{w \in W} \|\lambda v + w\| = \inf_{w \in W} \|\lambda(v + w)\| = |\lambda| \inf_{w \in W} \|v + w\| = |\lambda| \|v + W\|.$$

This shows that the quotient seminorm is indeed a seminorm on V/W . Now let's prove that $\|v + W\| = 0$ if and only if $v \in \overline{W}$, which will imply the last statement. By definition of $\|v + W\|$ (and the fact that W is a subspace), we have $\|v + W\| = 0$ if and only if, for every $\varepsilon > 0$, there exists $w \in W$ such that $\|v - w\| < \varepsilon$. This is equivalent to $v \in \overline{W}$.

(ii) Let $(v_n)_{n \geq 0}$ be a sequence in V such that $(v_n + W)_{n \geq 0}$ is a Cauchy sequence in V/W . Up to replacing $(v_n)_{n \geq 0}$ by a subsequence, we may assume that $\|v_{n+1} - v_n + W\| < 2^{-n}$ for every $n \geq 0$. We define another sequence $(v'_n)_{n \geq 0}$ such that $v'_n \in v_n + W$ for $n \geq 0$ and $\|v'_n - v'_{n-1}\| < 2^{-n+1}$ for $n \geq 1$, in the following way :

B Useful things about normed vector spaces

- Take $v'_0 = v_0$.
- Suppose that we have v'_0, \dots, v'_n satisfying the two required conditions, with $n \geq 0$. Then we have $\|v_{n+1} - v'_n + W\| = \|v_{n+1} - v_n + W\| < 2^{-n}$, so, by definition of the quotient norm, we can find $w \in W$ such that $\|v_{n+1} - v'_n + w\| < 2^{-n}$. Take $v'_{n+1} = v_{n+1} + w$.

By the second condition, $(v'_n)_{n \geq 0}$ is a Cauchy sequence, so it has a limit v in V . By the first condition, $v'_n + W = v_n + W$ for every $n \geq 0$, so $v + W$ is the limit of the sequence $(v_n + W)_{n \geq 0}$ in V/W .

□

B.2 The open mapping theorem

This is also known as the Banach-Schauder theorem. See for example theorem 5.10 of [15].

Theorem B.2.1. *Let V and W be Banach spaces, and let $T : V \rightarrow W$ be a bounded linear transformation that is bijective. Then $T^{-1} : W \rightarrow V$ is also bounded.*

B.3 The Hahn-Banach theorem

See [15] Theorem 5.16 or [16] Theorems 3.2-3.7.

Theorem B.3.1 (Hahn-Banach theorem, analytic version, real case). *Let V be a vector space over \mathbb{R} , let $p : V \rightarrow \mathbb{R}$ such that :*

- (a) $p(v + v') \leq p(v) + p(v')$ for all $v, v' \in V$ (i.e. p is subadditive);
- (b) $p(\lambda v) = \lambda p(v)$ for every $v \in V$ and every $\lambda \in \mathbb{R}_{>0}$.

Let $E \subset V$ be a K -subspace and let $f : E \rightarrow K$ be a linear functional such that, for every $x \in E$, we have $f(x) \leq p(x)$.

Then there exists a linear function $F : V \rightarrow K$ such that $F|_E = f$ and $F(x) \leq p(x)$ for every $x \in V$.

Note that, in this version, there is no norm or topology on V and no continuity condition on the linear functionals.

Proof. Consider the set X of pairs (W, g) , where $W \supset E$ is a subspace of V and $g : W \rightarrow \mathbb{R}$ is a linear functional such that $g|_E = f$ and $g(x) \leq p(x)$ for every $x \in W$. We order this set by saying that $(W, g) \leq (W', g')$ if $W \subset W'$ and $g = g'|_W$. Suppose that $(W_i, g_i)_{i \in I}$ is a nonempty

totally ordered family in X , and let's show that it has an upper bound. We set $W = \bigcup_{i \in I} W_i$; this is a subspace of V because $(W_i)_{i \in I}$ is totally ordered (so, for all $i, j \in I$, we have $W_i \subset W_j$ or $W_j \subset W_i$). We define $g : W \rightarrow \mathbb{R}$ in the following way : If $v \in W$, then there exists $i \in I$ such that $v \in W_i$, and we set $g(v) = g_i(v)$. We obviously have $g(v) \leq p(v)$. Also, if $j \in I$ is another element such that $v \in W_j$, then we have $(W_i, f_i) \leq (W_j, f_j)$ or $(W_j, f_j) \leq (W_i, f_i)$, and in both cases this forces $g_i(v) = g_j(v)$, so the definition makes sense. It is also easy to see that g is \mathbb{R} -linear, so that $(W, g) \in X$. This is an upper bound for the family.

So we can apply Zorn's lemma to the set X . Let (W, g) be a maximal element of X , and let's show that $W = V$. Suppose that $W \neq V$, and choose $v \in V - W$. We want to extend g to a linear functional h on $W \oplus \mathbb{R}v$ such that $h \leq p$, which will contradict the maximality of (W, g) . This just means that we have to choose the value of $h(v)$, say $h(v) = \alpha \in \mathbb{R}$. The condition $h \leq p$ means that we want, for every $w \in W$ and every $t \in \mathbb{R}$:

$$h(w + tv) = g(w) + t\alpha \leq p(w + tv).$$

If the inequality above is true for a $t \in \mathbb{R}$ and all $w \in W$, it is also true for all ct , $c \in \mathbb{R}_{>0}$, and for all $w \in W$ (because W is a subspace and the values of both g and p are multiplied by c when their argument is multiplied by c). So it suffices to check it for $t = \pm 1$, which means that we want, for every $w \in W$:

$$g(w) + \alpha \leq p(w + v) \text{ and } g(w) - \alpha \leq p(w - v).$$

In other words, we want to have :

$$\sup_{w \in W} (g(w) - p(w - v)) \leq \alpha \leq \inf_{w \in W} (p(w + v) - g(w)).$$

We can find such a α because we have, for all $w, w' \in W$,

$$g(w) + g(w') = g(w + w') \leq p(w + w') \leq p(w + v) + p(w' - v),$$

i.e.

$$g(w') - p(w' - v) \leq p(w + v) - g(w).$$

So we get our contradiction, we can conclude that W was equal to V after all, and we are done. \square

Theorem B.3.2 (Hahn-Banach theorem, analytic version, complex case). *Let V be a vector space over \mathbb{C} , let $p : V \rightarrow \mathbb{R}_{\geq 0}$ be a semi-norm,¹ let $E \subset V$ be a \mathbb{C} -subspace and let $f : E \rightarrow \mathbb{C}$ be a linear functional such that, for every $x \in E$, we have $|f(x)| \leq p(x)$.*

Then there exists a linear function $F : V \rightarrow \mathbb{C}$ such that $F|_E = f$ and $|F(x)| \leq p(x)$ for every $x \in V$.

¹This means that, for all $x, y \in V$ and all $\lambda \in \mathbb{C}$, we have $p(x + y) \leq p(x) + p(y)$ and $p(\lambda x) = |\lambda|p(x)$.

B Useful things about normed vector spaces

Proof. We see V and E as \mathbb{R} -vector spaces, and define a \mathbb{R} -linear functional $h : E \rightarrow \mathbb{R}$ by

$$h(v) = \frac{1}{2}(f(v) + \overline{f(v)}).$$

Then we have, for every $v \in E$,

$$h(v) \leq \frac{1}{2}(|f(v)| + |\overline{f(v)}|) \leq p(v).$$

Note that satisfies conditions (a) and (b) of theorem B.3.1. By that theorem, we can find a \mathbb{R} -linear functional $H : V \rightarrow \mathbb{R}$ such that $H|_E = h$ and that $H(v) \leq p(v)$ for every $v \in V$. Define $F : V \rightarrow \mathbb{C}$ by

$$F(v) = H(v) + \frac{1}{i}H(iv),$$

and let's show that it has all the desired properties.

(i) F is \mathbb{R} -linear by construction, and, for every $v \in V$, we have

$$F(iv) = H(iv) + \frac{1}{i}H(i(iv)) = iF(v).$$

So F is \mathbb{C} -linear.

(ii) If $v \in E$, then

$$F(v) = h(v) + \frac{1}{i}h(iv) = \frac{1}{2}(f(v) + \overline{f(v)} - if(iv) - \overline{if(iv)}) = f(v)$$

(because f is \mathbb{C} -linear), so $F|_E = f$.

(iii) Let $v \in V$ and choose $\theta \in \mathbb{R}$ such that $e^{i\theta}F(v) \in \mathbb{R}_{\geq 0}$. Then we have

$$|F(v)| = e^{i\theta}F(v) = F(e^{i\theta}v) = H(e^{i\theta}v) - iH(ie^{i\theta}v) \in \mathbb{R}.$$

As $H(e^{i\theta}v) \in \mathbb{R}$ and $iH(ie^{i\theta}v) \in i\mathbb{R}$, we must have $iH(ie^{i\theta}v) = 0$. So

$$|F(v)| = H(e^{i\theta}v) \leq p(e^{i\theta}v) = p(v).$$

□

Corollary B.3.3. *Let V be a normed vector space (over \mathbb{R} or \mathbb{C}), let W be a subspace of V , and let T_W be a bounded linear functional on W . Then there exists a bounded linear functional T on V such that $T|_W = T_W$ and $\|T\|_{op} = \|T_W\|_{op}$.*

Proof. Let $C = \|T_W\|_{op}$. Apply the Hahn-Banach with $p(v) = C\|v\|$. We get a linear functional $T : V \rightarrow \mathbb{C}$ extending T_W and such that $|T(v)| \leq C\|v\|$ for every $v \in V$, which means that T is bounded and $\|T\|_{op} \leq \|T_W\|_{op}$. As the inequality $\|T_W\|_{op} \leq \|T\|$ is obvious, we are done.

□

Corollary B.3.4. (See Theorem 5.20 and Remark 5.21 of [15].) Let V be a normed vector space over $K = \mathbb{R}$ or \mathbb{C} . We write $V^* = \text{Hom}(V, K)$. Then the map $V \rightarrow V^{**}$ sending $v \in V$ to the linear functional $\widehat{v} : V^* \rightarrow \mathbb{C}, T \mapsto T(v)$ is an isometry.

In particular, this map is injective, which means that bounded linear functionals on V separate points.

We can now deduce the geometric versions of the Hahn-Banach theorem. (In finite dimension, these are sometimes called “the hyperplane separation theorem”).

Definition B.3.5. Let V be a vector space over the field K , with $K = \mathbb{R}$ or \mathbb{C} . We say that V is a *topological vector space* over K if it has a topology such that :

- $(V, +)$ is a topological group;
- the map $K \times V \rightarrow V, (a, v) \mapsto av$ is continuous.

We say that a topological vector space is *locally convex* if every point has a basis of convex neighborhoods.

Example B.3.6. (a) Any normed vector is a locally convex topological vector, as is its dual for the weak* topology.

- (b) Let (X, μ) be a measure space, let $p \in (0, 1)$, and consider the space $L^p(X, \mu)$, with the metric given by

$$d(f, g) = \int_X |f(x) - g(x)|^p d\mu(x).$$

This makes $L^p(X, \mu)$ into a topological vector space, which is not locally convex if μ is atomless and finite (for example if μ is Lebesgue measure on a bounded subset of \mathbb{R}^n , or the Haar measure on a compact group).

Theorem B.3.7 (Hahn-Banach theorem, first geometric version). Let V be a topological \mathbb{R} -vector space, and let A and B be two nonempty convex subsets of V . Suppose that A is open and that $A \cap B = \emptyset$.

Then there exists a continuous linear functional $f : V \rightarrow \mathbb{R}$ and $c \in \mathbb{R}$ such that, for every $x \in A$ and every $y \in B$, we have

$$f(x) \leq c \leq f(y).$$

We are going to use as our function p what is called the *gauge* of an open convex set $C \ni 0$.

Lemma B.3.8. Let C be a nonempty open convex subset of V , and suppose that $0 \in C$. We define the gauge $p : V \rightarrow \mathbb{R}_{\geq 0}$ of C by

$$p(v) = \inf\{\alpha > 0 \mid v \in \alpha C\}.$$

Then p satisfies conditions (a) and (b) of theorem B.3.1, and moreover :

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(c) If V is a normed vector space, then there exists $M \in \mathbb{R}_{\geq 0}$ such that, for every $v \in V$,

$$0 \leq p(v) \leq M\|v\|.$$

(d) $C = \{v \in V \mid p(v) < 1\}$.

Proof. The fact that $p(\lambda v) = \lambda p(v)$ for every $\lambda \in \mathbb{R}_{>0}$ and every $v \in V$ follows immediately from the definition (and doesn't use the convexity or openness of C).

Let's prove (c). As C is open and $0 \in C$, there exists $r > 0$ such that $C \supset \{v \in V \mid \|v\| < r\}$. Then, for every $v \in V - \{0\}$, we have $\frac{r}{\|v\|}v \in C$, so $p(v) \leq \frac{1}{r}\|v\|$.

Let's prove (d). Let $v \in C$. As C is open, there exists $\varepsilon > 0$ such that $(1 + \varepsilon)v \in C$. So $p(v) \leq \frac{1}{1+\varepsilon} < 1$. Conversely, let $v \in V$ such that $p(v) < 1$. Then there exists $\alpha \in (0, 1)$ such that $x \in \alpha C$, i.e. $\frac{1}{\alpha}v \in C$, and then we have $v = \alpha(\frac{1}{\alpha}v) + (1 - \alpha)0 \in C$, because C is convex.

Finally, we prove that p is subadditive, i.e. condition (b). Let $v, w \in V$. Let $\varepsilon > 0$. By (b) (and the first property we proved), we have $\frac{1}{p(v)+\varepsilon}v \in C$ and $\frac{1}{p(w)+\varepsilon}w \in C$. As C is convex, this implies that, for every $t \in [0, 1]$, we have

$$\frac{t}{p(v)+\varepsilon}v + \frac{1-t}{p(w)+\varepsilon}w \in C.$$

Taking $t = \frac{p(v)+\varepsilon}{p(v)+p(w)+2\varepsilon}$, we get that

$$\frac{1}{p(v) + p(w) + 2\varepsilon}(v + w) \in C,$$

i.e. that $p(v + w) \leq p(v) + p(w) + 2\varepsilon$. As $\varepsilon > 0$ was arbitrary, this implies that $p(v + w) \leq p(v) + p(w)$.

□

Lemma B.3.9. Let $C \subset V$ be a nonempty open convex subset, and let $v_0 \in V - C$.

Then there exists a continuous linear functional F on V such that, for every $v \in C$, we have $F(v) < F(v_0)$.

Proof. We may assume $0 \in C$ (by translating the situation). Let $p : V \rightarrow \mathbb{R}_{\geq 0}$ be the gauge of C , i.e. the function defined in the preceding lemma.

Let $E = \mathbb{R}v_0$, and let $f : E \rightarrow \mathbb{R}$ be the linear functional defined by $f(\lambda v_0) = \lambda$, for every $\lambda \in \mathbb{R}$. Let's show that $f \leq p$. If $\lambda \leq 0$, then $f(\lambda v_0) \leq 0 \leq p(\lambda v_0)$. If $\lambda > 0$, then $\lambda = g(\lambda v_0) \leq p(\lambda v_0)$ because $\frac{1}{\lambda}(\lambda v_0) = v_0 \notin C$.

So we can apply the analytic form of the Hahn-Banach theorem to get a linear function $F : V \rightarrow \mathbb{R}$ such that $F(v) \leq p(v)$ for every $v \in V$. In particular, $F(v_0) = 1$, and, if $v \in C$, then $F(v) \leq p(v) < 1$ (by (d) in the first lemma).

Finally, we show that F is continuous. Note that, if $v \in -C$, we have $-F(v) = F(-v) < 1$. So, for every v in the open neighborhood $U := C \cap (-C)$ of 0, we have $|F(v)| < 1$. If $\varepsilon > 0$, then εU is an open neighborhood of 0 in V , and we have $|F(v)| < \varepsilon$ for every $v \in \varepsilon U$. So F is continuous at 0. As F is linear and translations are continuous on V , this implies that F is continuous at every point of V .

□

Proof of the theorem. Let $C = A - B = \{x - y, x \in A, y \in B\}$. Then C is clearly convex, C is open because it is equal to $\bigcup_{y \in B} (A - y)$, and $0 \notin C$ because $A \cap B = \emptyset$. So we can apply the second lemma above to get a continuous linear functional $f : V \rightarrow \mathbb{R}$ such that $f(x) < 0$ for every $x \in C$. Then, for every $x \in A$ and every $y \in B$, we have $f(x) < f(y)$. So the conclusion is true for f and for $c = \sup_{x \in A} f(x)$.

□

Theorem B.3.10 (Hahn-Banach theorem, second geometric version). *Let V be a locally convex topological \mathbb{R} -vector space, and let A and B be two nonempty convex subsets of V . Suppose that A is closed, that B is compact, and that $A \cap B = \emptyset$.*

Then there exists a continuous linear functional $f : V \rightarrow \mathbb{R}$ and $c \in \mathbb{R}$ such that, for every $x \in A$ and every $y \in B$, we have

$$f(x) < c < f(y).$$

Proof. We first find a convex open neighborhood U of 0 in V such that $(A + U) \cap (B + U) = \emptyset$. (Note : this only uses that V is locally and that A is closed and B compact, but not the fact that A and B are convex.)

For every $x \in B$, choose a symmetric convex open neighborhood U_x of 0 such that $(x + U_x + U_x + U_x) \cap A = \emptyset$; as U_x is symmetric, this is equivalent to saying that $(x + U_x + U_x) \cap (A + U_x) = \emptyset$. As B is compact, we can find $x_1, \dots, x_n \in B$ such that $B \subset \bigcup_{i=1}^n (x_i + U_{x_i})$. Let $U = \bigcap_{i=1}^n U_{x_i}$. Then U is a convex open neighborhood of 0, and we have $B + U \subset \bigcup_{i=1}^n (x_i + U_{x_i} + U)$ and $A + U \subset \bigcap_{i=1}^n (A + U_{x_i})$, so $(B + U) \cap (A + U) = \emptyset$.

The sets $A + U$ and $B + U$ are convex and open, so, by theorem B.3.7, there exists a continuous linear functional $f : V \rightarrow \mathbb{R}$ and $c' \in \mathbb{R}$ such that $f(x) \leq c' \leq f(y)$ for every $x \in A + U$ and every $y \in B + U$. As B is compact and f continuous, there exists $y_0 \in B$ such that $f(y_0) = \min_{y \in B} f(y)$. In particular, $c' < \min_{y \in B} f(y)$. Choose $c \in \mathbb{R}$ such that $c' < c < \min_{y \in B} f(y)$. Then we have $f(x) < c < f(y)$ for every $x \in A$ and every $y \in B$.

□

B.4 The Banach-Alaoglu theorem

See section 15.1 of [14] or Theorem 3.15 of [16]. This theorem is also called “Alaoglu’s theorem”.

Theorem B.4.1. *Let V be a normed vector space. Then the closed unit ball in $\text{Hom}(V, \mathbb{C})$ is compact Hausdorff for the weak* topology.*

Compare with the following results, usually called “Riesz’s lemma” or “Riesz’s theorem” (see section 13.3 of [14] or Theorem 1.22 of [16]) :

Theorem B.4.2. *Let V be normed vector space. Then the closed unit ball of V is compact if and only if V is finite-dimensional.*

B.5 The Krein-Milman theorem

See section 14.6 of [14] (or theorem 3.23 of [16]).

Definition B.5.1. Let V be a \mathbb{R} -vector space and C be a convex subset of V . We say that $x \in C$ is *extremal* if, whenever $x = ty + (1 - t)z$ with $t \in (0, 1)$ and $y, z \in C$, we must have $y = z = x$.

Theorem B.5.2. *Let V be a locally convex topological \mathbb{R} -vector space, and let K be a nonempty compact convex subset of V . Then K is the closure of the convex hull of its set of extremal points.*

Lemma B.5.3. *Let V be a locally convex topological \mathbb{R} -vector space, and let K be a nonempty compact convex subset of V . Then K has an extremal point.*

Proof. We say that a subset S of K is extremal if for every $x \in S$, if we have $x = ty + (1 - t)z$ with $y, z \in K$ and $t \in (0, 1)$, then we must have $y, z \in S$. (Note that a point $x \in K$ is extremal if and only if $\{x\}$ is extremal.)

Let X be the set of nonempty closed extremal subsets of K , ordered by reverse inclusion. Let Y a nonempty totally ordered subset of X , and let’s show that it has a maximal element. As Y is totally ordered, for all $T_1, \dots, T_n \in Y$, there exists $i \in \{1, \dots, n\}$ such that $T_i \subset T_j$ for every $j \in \{1, \dots, n\}$, and then $T_1 \cap \dots \cap T_n \supset T_i \neq \emptyset$. As K is compact, this implies that $S := \bigcap_{T \in Y} T$ is not empty. The set S is clearly closed, so if we can show that it is extremal, we will be done. Let $x \in S$, and write $x = ty + (1 - t)z$, with $y, z \in K$ and $t \in (0, 1)$. For every $T \in Y$, as T is extremal, we must have $y, z \in T$. So $y, z \in S$, and S is indeed extremal.

By Zorn’s lemma, the set X has a maximal element, let’s call it S . To finish the proof, we just need to show that S is a singleton. If $|S| \geq 2$, let $x, y \in S$ such that $x \neq y$. By the geometric version of the Hahn-Banach theorem (theorem B.3.10), there exists a continuous linear functional $f : V \rightarrow \mathbb{R}$ such that $f(x) < f(y)$. As S is compact, the continuous function f reaches its

minimum on S . Let $m = \min_{z \in S} f(z)$, and let $S' = \{z \in S \mid f(z) \leq m\}$. Then S' is closed, it is nonempty by the observation we just made, and $S' \neq S$ because $y \notin S'$. Let's show that S' is extremal, which will give a contradiction (and imply that S had to be a singleton). Let $z \in S'$, and write $z = tz' + (1 - t)z''$, with $z', z'' \in K$ and $t \in (0, 1)$. As S , we have $z', z'' \in S$. By definition of m , we have $m = f(z) = tf(z') + (1 - t)f(z'') \leq tm + (1 - t)m$, which forces $m = f(z') = f(z'')$, i.e. $z', z'' \in S'$.

□

Proof of the theorem. Let L be the closure of the convex hull of the set of extremal points of K . Then L is convex, closed and contained in K ; in particular, L is also compact. Suppose that $L \neq K$, and let $x \in K \setminus L$. By the geometric version of the Hahn-Banach theorem (theorem B.3.10), there exists a continuous linear functional $f : V \rightarrow \mathbb{R}$ such that $\max_{y \in L} f(y) < f(x)$. Let $M = \max_{z \in K} f(z)$, and let $K' = \{z \in K \mid f(z) = M\}$. Then K' is a closed convex subset of K (hence it is compact), and $K' \cap L = \emptyset$. By the lemma, K' must have an extremal point z , and it is easy to see (as in the proof of the lemma) that z is also an extremal point of K . But then z should be in L , contradiction.

□

B.6 The Stone-Weierstrass theorem

See section 12.3 of [14] or Theorem 5.7 of [16] for the case of a compact space.

Theorem B.6.1. *Let X be a locally compact Hausdorff topological space, and let A be a \mathbb{C} -subalgebra of $\mathcal{C}_0(X)$ such that :*

- (a) *for every $f \in A$, the function $x \mapsto \overline{f(x)}$ is also in A ;*
- (b) *for all $x, y \in X$ such that $x \neq y$, there exists $f \in A$ such that $f(x) \neq f(y)$ (“ A separates the points of X ”);*
- (c) *for every $x \in X$, there exists $f \in A$ such that $f(x) \neq 0$ (“ A vanishes nowhere on X ”).*

Then A is dense in $\mathcal{C}_0(X)$.

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Bibliography

- [1] Ehrhard Behrends. *Introduction to Markov chains*. Advanced Lectures in Mathematics. Friedr. Vieweg & Sohn, Braunschweig, 2000. With special emphasis on rapid mixing.
- [2] Yves Benoist and Nicolas de Saxcé. A spectral gap theorem in simple Lie groups. *Invent. Math.*, 205(2):337–361, 2016.
- [3] J. Bourgain and A. Gamburd. A spectral gap theorem in $SU(d)$. *J. Eur. Math. Soc. (JEMS)*, 14(5):1455–1511, 2012.
- [4] Jean Bourgain and Alex Gamburd. On the spectral gap for finitely-generated subgroups of $SU(2)$. *Invent. Math.*, 171(1):83–121, 2008.
- [5] Emmanuel Breuillard. Random walks on lie groups. <https://www.math.u-psud.fr/~breuilla/part0gb.pdf>, 2004.
- [6] Tullio Ceccherini-Silberstein, Fabio Scarabotti, and Filippo Tolli. *Harmonic analysis on finite groups*, volume 108 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2008. Representation theory, Gelfand pairs and Markov chains.
- [7] Persi Diaconis. *Group representations in probability and statistics*, volume 11 of *Institute of Mathematical Statistics Lecture Notes—Monograph Series*. Institute of Mathematical Statistics, Hayward, CA, 1988.
- [8] Gerald B. Folland. *A course in abstract harmonic analysis*. Textbooks in Mathematics. CRC Press, Boca Raton, FL, second edition, 2016.
- [9] Ulf Grenander. *Probabilities on algebraic structures*. Second edition. Almqvist & Wiksell, Stockholm; John Wiley & Sons Inc., New York-London, 1968.
- [10] G. H. Hardy and E. M. Wright. *An introduction to the theory of numbers*. Oxford University Press, Oxford, sixth edition, 2008. Revised by D. R. Heath-Brown and J. H. Silverman, With a foreword by Andrew Wiles.
- [11] Yukiyosi Kawada and Kiyosi Itô. On the probability distribution on a compact group. I. *Proc. Phys.-Math. Soc. Japan (3)*, 22:977–998, 1940.
- [12] Alexander Lubotzky. *Discrete groups, expanding graphs and invariant measures*. Modern Birkhäuser Classics. Birkhäuser Verlag, Basel, 2010. With an appendix by Jonathan D. Rogawski, Reprint of the 1994 edition.
- [13] Jeffrey S. Rosenthal. Random rotations: characters and random walks on $SO(N)$. *Ann.*

Bibliography

- Probab.*, 22(1):398–423, 1994.
- [14] H. L. Royden. *Real analysis*. Macmillan Publishing Company, New York, third edition, 1988.
 - [15] Walter Rudin. *Real and complex analysis*. McGraw-Hill Book Co., New York, third edition, 1987.
 - [16] Walter Rudin. *Functional analysis*. International Series in Pure and Applied Mathematics. McGraw-Hill, Inc., New York, second edition, 1991.
 - [17] Peter Sarnak. *Some applications of modular forms*, volume 99 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 1990.
 - [18] Gerrit van Dijk. *Introduction to harmonic analysis and generalized Gelfand pairs*, volume 36 of *De Gruyter Studies in Mathematics*. Walter de Gruyter & Co., Berlin, 2009.