MAT 449 : Problem Set 9

Due Thursday, November 29

In this problem set, we put the norm $\|.\|$ on $M_n(\mathbb{C})$ defined by $\|X\|^2 = \mathrm{Tr}(X^*X)$.

1. (2) Let $\mathcal{P}(\mathbb{Z})$ be the set of subsets of $\mathbb{Z}$. Show that there exists a finitely additive left-invariant probability measure on $\mathbb{Z}$, that is, a function $\mu : \mathcal{P}(\mathbb{Z}) \to \mathbb{R}_{\geq 0}$ such that :
   (a) If $A_1, \ldots, A_n \in \mathcal{P}(\mathbb{Z})$ are such that $A_i \cap A_j = \varnothing$ for $i \neq j$, then $\mu(A_1 \cup \ldots \cup A_n) = \mu(A_1) + \ldots + \mu(A_n)$.
   (b) $\mu(\mathbb{Z}) = 1$.
   (c) For every $A \in \mathcal{P}(Z)$ and $n \in \mathbb{Z}$, we have $\mu(n + A) = \mu(A)$.

   Solution. As $\mathbb{Z}$ is an abelian locally compact group, it is amenable by problems 4 and 6 of problem set 8. This means that there exists a left-invariant mean M on $L^\infty(\mathbb{Z})$. We define μ by $\mu(A) = M(\mathbb{1}_A)$; this function does take its values in $\mathbb{R}_{\geq 0}$ by definition of a mean. Then μ satisfies (a) because M is linear, it satisfies (b) because $M(1) = 1$ and it satisfies (c) because M is left-invariant.

   Conversely, note that the existence of a μ as in the statement implies the existence of an invariant mean.

   □

2. (2, extra credit) Is the measure of problem 1 unique ? (Hint : You need a somewhat explicit way to construct invariant means on $\mathbb{Z}$. You can for example try to exploit the sequence of (non-invariant) means $M_n : L^\infty(\mathbb{Z}) \to \mathbb{C}$, $(x_k)_{k\in\mathbb{Z}} \mapsto \frac{1}{2k+1}\sum_{k=-n}^{n} x_k$.)

   Solution. No.

   Let $V = L^\infty(\mathbb{Z})$, and consider the family of linear functionals $M_n : L^\infty(\mathbb{Z}) \to \mathbb{C}$ defined by
$$M_n((x_k)_{k\in\mathbb{Z}}) = \frac{1}{2n+1}\sum_{k=-n}^{n} x_k,$$
   for $n \in \mathbb{N}$. We have $|M_n(x)| \leq \|x\|_\infty$ for every $x \in V$, so M_n is continuous. Also, it is clear on the definition that M_n is a mean. If $a \in \mathbb{Z}$, then, for every $x \in V$ and every $n \in \mathbb{N}$, we have
$$|M_n(L_a x) - M_n(x)| \leq \frac{2|a|}{2n+1}\|x\|_\infty.$$
   So, if we could make the sequence $(M_n)_{n\geq 0}$ converge in the weak* topology of $\mathrm{Hom}(V, \mathbb{C})$, then its limit would be an invariant mean, and it would define an invariant finitely additive probability measure as in problem 1. We can always find a convergent subsequence of $(M_n)_{n\geq 0}$ converge in the weak* using the Banach-Alaoglu theorem, but we would also like to show that we can get two different limits.

Consider the element $x = (x_n)_{n\in\mathbb{Z}}$ of V defined by $x_n = 0$ for $n \leq 0$, and $x_n = (-1)^k$ if we have $2^k \leq n \leq 2^{k+1} - 1$ with $k \in \mathbb{Z}_{\geq 0}$. Then, if $n = 2^k - 1$ with $k \geq 0$, we have

$$\sum_{r=-n}^{n} x_n = \sum_{s=0}^{k-1} (-1)^s 2^s = \frac{1-(-2)^k}{3},$$

so

$$M_n(x) = \frac{1-(-2)^k}{3(2^{k+1}-1)}.$$

In particular, the sequence $(M_{2^{2l}-1}(x))_{l\geq 0}$ converges to $-\frac{1}{6}$, and the sequence $(M_{2^{2l+1}-1}(x))_{l\geq 0}$ converges to $\frac{1}{6}$.

By the Banach-Alaoglu, the sequences $(M_{2^{2l}-1})_{l\geq 0}$ $(M_{2^{2l+1}-1})_{l\geq 0}$ both have weak* limit points, say M and M'. Both M and M' are left invariant means on V, but we have $M(x) = -\frac{1}{6}$ and $M'(x) = \frac{1}{6}$ by the calculation above, so $M \neq M'$.

$\square$

3. Let d be a positive integer.

   a) (1) Let T be the intersection of $\mathbf{U}(d)$ with the set of diagonal matrices. Show that

   $$T = \left\{ \begin{pmatrix} z_1 & \dots & 0 \\ & \ddots & \\ 0 & \dots & z_d \end{pmatrix}, z_1, \dots, z_d \in S^1 \right\}.$$

   b) (1) Show that every element of $\mathbf{U}(d)$ is conjugated in $\mathbf{U}(d)$ to an element of T.

   c) (1) Show that every element of $\mathbf{SU}(d)$ is conjugated in $\mathbf{SU}(d)$ to an element of $T_0 := T \cap \mathbf{SU}(d)$.

   d) (1) Show that a finite-dimensional representation V of $\mathbf{SU}(d)$ is uniquely determined up to equivalence by $\chi_{V|T_0}$.

We now take $d = 2$. Remember the irreducible representations V_n ($n \geq 0$) of $\mathbf{SU}(2)$ defined in problem 1 of problem set 6.

   e) (1) Calculate the restriction of χ_{V_n} to T_0.

   f) (2) Let (ρ, V) be a finite-dimensional representation of $\mathbf{SU}(2)$. Show that there exists $m \geq 1$ and nonnegative integers $a_0, \dots, a_m$ such that, for every $z \in S^1$, we have

   $$\chi_V\left(\begin{pmatrix} z & 0 \\ 0 & \overline{z} \end{pmatrix}\right) = a_0 + \sum_{i=1}^{m} a_i(z^i + z^{-i}).$$

   g) (2) Show that there exist integers $c_n \in \mathbb{Z}$, $n \geq 0$, such that $c_n = 0$ for n big enough and $\chi_V = \sum_{n\geq 0} c_n \chi_{V_n}$.

   h) (1) Show that the integers c_n of (f) are all nonnegative.

   i) (1) If V is irreducible, show that there exists $n \geq 0$ such that $V \simeq V_n$.

Solution.

a) Let $x = \begin{pmatrix} z_1 & \dots & 0 \\ & \ddots & \\ 0 & \dots & z_d \end{pmatrix} \in M_d(\mathbb{C})$, with $z_1, \dots, z_d \in \mathbb{C}$. Then x is in T if and only if $xx^* = I_d$. As $x^* = \begin{pmatrix} \overline{z}_1 & \dots & 0 \\ & \ddots & \\ 0 & \dots & \overline{z}_d \end{pmatrix}$, this condition is equivalent to $|z_1| = \dots = |z_d| = 1$.

b) Let $x \in \mathbf{U}(d)$. Then x is normal, so, by the spectral theorem, it can be diagonalized in an orthonormal basis of $\mathbb{C}^n$. This means that there exists $y \in \mathbf{U}(d)$ such that yxy^{-1} is diagonal, i.e., $yxy^{-1} \in$.

c) Let $x \in \mathbf{SU}(d)$. By question (b), there exists $y \in \mathbf{U}(d)$ such that $yxy^{-1} \in T$. We have $\det(yxy^{-1}) = \det(x) = 1$, so yxy^{-1} is actually in T_0. Let $c = \det(y) \in \mathbb{C}^\times$. We choose $c' \in \mathbb{C}$ such that $(c')^d = c$; as $|c| = 1$, we also have $|c'| = 1$. Then $y' := (c')^{-1}y$ has determinant 1, hence is in $\mathbf{SU}(d)$, and $y'x(y')^{-1} = yxy^{-1}$.

d) Let V, W be two finite-dimensional representations of $\mathbf{SU}(d)$, and suppose that $\chi_V = \chi_W$ on T_0. By question (c) and the fact that χ_V and χ_W are central functions, this implies that $\chi_V = \chi_W$ on all of $\mathbf{SU}(d)$. But then V and W are equivalent by corollary IV.5.10 of the notes.

e) Let $x = \begin{pmatrix} x_1 & 0 \\ 0 & x_2 \end{pmatrix} \in T_0$. Note that $x_1x_2 = 1$, so $x_2 = x_1^{-1} = \overline{x}_1$. We calculate the action of x on the basis $(t_1^k t_2^{n-k})_{0 \le k \le n}$ of V_n. For $0 \le k \le n$, we have

$$x \cdot t_1^l t_2^{n-k} = (x_1^{-1}t_1)^k (x_2^{-1}t_2)^{n-k} = x_1^{n-2k} t_1^k t_2^{n-k}.$$

So

$$\chi_{V_n}(x) = \sum_{k=0}^{n} x_1^{n-2k}.$$

f) We embed S^1 in $\mathbf{SU}(1)$ by the continuous group morphism $z \mapsto \begin{pmatrix} z & 0 \\ 0 & \overline{z} \end{pmatrix}$. Note that this induces an isomorphism of topological groups $S^1 \xrightarrow{\sim} T_0$. Then $\rho_{|S^1}$ is a finite-dimensional representation of S^1, so it is a finite direct sum of irreducible representations. We know (from problem 5 of problem set 3) that every irreducible representation of S^1 is of the form $\rho_m : z \mapsto z^m$ with $m \in \mathbb{Z}$, so there exist nonnegative integers a_m, $m \in \mathbb{Z}$, that are 0 for all but a finite number of m, and such that $\rho_{|S^1} \simeq \bigoplus_{m\in\mathbb{Z}} \rho_m^{a_m}$. In particular, for every $z \in S^1$,

$$\chi_V\left(\begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix}\right) = \sum_{m\in\mathbb{Z}} a_m z^m.$$

Let $y = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$. Then $y \in \mathbf{SU}(2)$ and, for every $z \in S^1$, we have $y \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix} y^{-1} = \begin{pmatrix} z^{-1} & 0 \\ 0 & z \end{pmatrix}$. As V is a representation of $\mathbf{SU}(2)$, the function χ_V is central on $\mathbf{SU}(2)$, and so $\chi_V\left(\begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix}\right) = \chi_V\left(\begin{pmatrix} z^{-1} & 0 \\ 0 & z \end{pmatrix}\right)$. This implies that $a_{-m} = a_m$ for every $m \in \mathbb{Z}$, so we get the desired statement.

g) Let M be the $\mathbb{Z}$-module of functions $\chi : S^1 \to \mathbb{Z}$ that can be written $\chi(z) = a_0 + \sum_{m\ge1} a_m(z^m + z^{-m})$, with $a_0, a_1, \dots \in \mathbb{Z}$ and $a_m = 0$ for m big enough. By question (f), the restriction to S^1 of χ_V is in M.

A basis of M over $\mathbb{Z}$ is formed by the function $(\chi_0 = 1, \chi_1 = z + z^{-1}, \chi_2 = z^2 + z_2^{-2}, \ldots)$. On the other hand, we have seen in question (e) that $\chi_{V_n} = \sum_{\substack{0 \leq k \leq n \\ k = n \mod n}} \chi_k$.

So the (infinite) matrix representing $(\chi_{V_n})_{n\geq 0}$ in the basis $(\chi_n)_{n\geq 0}$ is upper triangular with ones on the diagonal, which means that it can be inverted, i.e., that $(\chi_{V_n})_{n\geq 0}$ is also a basis of M over $\mathbb{Z}$. (If you don't like that, it is also very easy from the formula expressing χ_{V_n} in the basis $(\chi_m)_{m\geq 0}$ to show by induction over n that $(\chi_{V_0}, \ldots, \chi_{V_n})$ is linearly independent and spans the same $\mathbb{Z}$-submodule as $(\chi_0, \ldots, \chi_n)$.)

The conclusion of the question follows immediately from this.

h) We know that the functions χ_{V_n} are pairwise orthogonal in $L^2(\mathbf{SU}(2))$ (by corollary IV.5.8 of the notes). So, for every $n \geq 0$,

$$c_n = \langle \chi_V, \chi_{V_n} \rangle_{L^2(\mathbf{SU}(2))}.$$

By the same corollary, the right-hand side is also equal to $\dim_{\mathbb{C}}(\mathrm{Hom}_{\mathbf{SU}(2)}(V, V_n))$, which is a nonnegative integer.

i) If V is irreducible, then, by the last formula in the proof of (h) (and Schur's lemma), we have $c_n = 0$ unless $V \simeq V_n$. So, if there were no $n \geq 0$ such that $V \simeq V_n$, we would have $\chi_V = 0$, hence $V = 0$, which is impossible.

□

4. Let G be a compact group, and let (π, V) be a faithful finite-dimensional continuous representation of G. (Remember that this means that $\pi : G \to \mathbf{GL}(V)$ is injective.) The goal of this problem is to show that, if G is finite, then every irreducible representation of G is a direct summand of a representation of the form $V^{\otimes n} \otimes (V^*)^{\otimes m}$ (for some $n, m \geq 1$), where the notation $V^{\otimes n}$ means $\underbrace{V \otimes \ldots \otimes V}_{n \text{ times}}$, and similarly for $(V^*)^{\otimes m}$.

   a) (3) Let $\mathbb{1}$ be the trivial representation of G on $\mathbb{C}$. Show that it suffices to show that every irreducible representation of G is a direct summand of a representation of the form $(V \oplus V^* \oplus \mathbb{1})^{\otimes N}$, for some $N \geq 1$.

   b) (2) Let W be an irreducible representation of G. Show that W is a direct summand of $(V \oplus V^* \oplus \mathbb{1})^{\otimes N}$ if and only if $\int_G (1 + 2\,\mathrm{Re}\,\chi_V(x))^N \overline{\chi_W(x)} dx \neq 0$.

From now on, we assume that G is finite, we fix a finite-dimensional representation W of G, and we write, for every $N \in \mathbb{Z}_{\geq 0}$,

$$S_N = \sum_{x \in G} (1 + 2\,\mathrm{Re}\,\chi_V(x))^N \overline{\chi_W(x)}.$$

Let $d = \dim V$.

   c) (2) If $x \neq 1$, show that $(1 + 2\,\mathrm{Re}\,\chi_V(x))^N \overline{\chi_W(x)} = o((1 + 2d)^N)$ as $N \to +\infty$.

   d) (2) If $W \neq 0$, show that $S_N \neq 0$ for N big enough.

Solution.

a) Let's show by induction on N that, for every $N \in \mathbb{Z}_{\geq 1}$, we have a G-equivariant isomorphism

$$(\mathbb{1} \oplus V \oplus V^*)^{\otimes N} \simeq \bigoplus_{\substack{k,l,m \geq 0 \\ k+l+m=N}} \left(V^{\otimes k} \otimes (V^*)^{\otimes l}\right)^{\frac{N!}{k!l!m!}}.$$

This clearly implies the result of (a).

For $N = 1$, it follows from the fact that $\mathbb{1} \otimes W \simeq W$ for every representation W of G. Suppose the result know for N, and let's prove it for $N + 1$. We have

$$\begin{aligned}
(\mathbb{1} \oplus V \oplus V^*)^{\otimes N+1} &\simeq (\mathbb{1} \oplus V \oplus V^*)^{\otimes N} \otimes (\mathbb{1} \oplus V \oplus V^*) \\
&\simeq (\mathbb{1} \oplus V \oplus V^*) \otimes \bigoplus_{\substack{k,l,m\geq 0 \\ k+l+m=N}} \left(V^{\otimes k} \otimes (V^*)^{\otimes l}\right)^{\frac{N!}{k!l!m!}} \\
&\simeq \bigoplus_{\substack{k,l,m\geq 0 \\ k+l+m=N}} \left(V^{\otimes k} \otimes (V^*)^{\otimes l}\right)^{\frac{N!}{k!l!m!}} \oplus \bigoplus_{\substack{k,l,m\geq 0 \\ k+l+m=N}} \left(V^{\otimes k+1} \otimes (V^*)^{\otimes l}\right)^{\frac{N!}{k!l!m!}} \\
&\quad \oplus \bigoplus_{\substack{k,l,m\geq 0 \\ k+l+m=N}} \left(V^{\otimes k} \otimes (V^*)^{\otimes l+1}\right)^{\frac{N!}{k!l!m!}} \\
&\simeq \bigoplus_{\substack{k,l,m\geq 0 \\ k+l+m=N+1}} \left(V^{\otimes k} \otimes (V^*)^{\otimes l}\right)^{\frac{N!}{k!l!(m-1)!}+\frac{N!}{(k-1)!l!m!}+\frac{N!}{k!(l-1)!m!}} \\
&\simeq \bigoplus_{\substack{k,l,m\geq 0 \\ k+l+m=N+1}} \left(V^{\otimes k} \otimes (V^*)^{\otimes l}\right)^{\frac{(N+1)!}{k!l!m!}}.
\end{aligned}$$

b) By the semisimplicity of finite-dimensional representations of G (corollary I.3.2.9 of the notes) and Schur's lemma, the representation W is a direct summand of $(\mathbb{1} \oplus V \oplus V^*)^{\otimes N}$ if and only if $\mathrm{Hom}_G((\mathbb{1} \oplus V \oplus V^*)^{\otimes N}, W) \neq 0$. By corollary IV.5.8 of the notes, this is the case if and only $\langle \chi_{(\mathbb{1}\oplus V\oplus V^*)^{\otimes N}}, \chi_W\rangle_{L^2(G)} \neq 0$. So the conclusion follows from the fact that $\chi_{(\mathbb{1}\oplus V\oplus V^*)^{\otimes N}} = (1 + \chi_V + \overline{\chi}_V)^N = (1 + 2\,\mathrm{Re}\,\chi_V)^N$, which is an immediate consequence of proposition IV.5.4 of the notes.

c) As G is compact, the representation (π, V) is unitarizable, so we can choose an isomorphism $V \simeq \mathbb{C}^d$ such that $\pi(G) \subset \mathbf{U}(d)$. Let $z_1, \dots, z_d$ be the eigenvalues of $\pi(x)$. As π is faithful, we have $\pi(x) \neq 1$, so at least one the z_i is not equal to 1 (we are using the fact that $\pi(x)$ is diagonalizable); so we may assume that $z_1 \neq 1$. As $|z_1| = 1$, this implies that $-1 \leq \mathrm{Re}\, z_1 < 1$, so $1 - 2d \leq 1 + 2\sum_{i=1}^d \mathrm{Re}(z_i) < 1 + 2d$, and $\left|1 + 2\sum_{i=1}^d \mathrm{Re}\, z_i\right| < 1 + 2d$. Finally, we get

$$|(1 + 2\,\mathrm{Re}\,\chi_V(x))^N \chi_W(x)| \leq (\dim W) \left|1 + 2\sum_{i=1}^d \mathrm{Re}\, z_i\right|^N = o((1 + 2d)^N).$$

d) As G is finite, question (c) implies that

$$\sum_{x\in G-\{1\}} (1 + 2\,\mathrm{Re}\,\chi_V(x))^N \chi_W(x) = o((1 + 2d)^N).$$

On the other hand, $(1+2\,\mathrm{Re}\,\chi_V(1))\chi_W(1) = (\dim W)(1+2d)^N$. So $S_n = (\dim W)(1+2d)^N + o((1 + 2d)^N)$, which implies that $S_n \neq 0$ for N big enough.

□

5. Consider the function $\exp : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ defined by $\exp(X) = \sum_{n\geq 0} \frac{1}{n!} X^n$; we also write e^X for $\exp(X)$. You may assume the basic properties of this function, i.e.

that the series defining it converges absolutely, that it is infinitely derivable, and that we can calculate its derivatives term by term in the sum. You may also assume that $\exp(A+B) = \exp(A)\exp(B)$ for any $A, B \in M_n(\mathbb{C})$ such that $AB = BA$; in particular, $\exp(X) \in \mathbf{GL}_n(\mathbb{C})$ for every $X \in M_n(\mathbb{C})$, and $\exp(X)^{-1} = \exp(-X)$.

We also fix a closed subgroup G of $\mathbf{GL}_n(\mathbb{C})$.

a) (2) Calculate the differential of exp at the point $0 \in M_n(\mathbb{C})$. (Remember that this is a linear operator from $M_n(\mathbb{C})$ to itself.)

b) (1) Show that exp induces a diffeomorphism from a neighborhood of 0 in $M_n(\mathbb{C})$ to a neighborhood of 1 in $\mathbf{GL}_n(\mathbb{C})$.

c) (3) Let $L = \{X \in M_n(\mathbb{C}) | \forall t \in \mathbb{R},\ \exp(tX) \in G\}$. Show that L is a $\mathbb{R}$-linear subspace of $M_n(\mathbb{C})$. (Hint : For all $X, Y \in M_n(\mathbb{C})$, show that $\exp(X+Y) = \lim_{k\to+\infty}(\exp(\frac{1}{k}X)\exp(\frac{1}{k}Y))^k$.)

d) (2) If $G = \mathbf{U}(n)$, show that $L = \{X \in M_n(\mathbb{C}) | X^* = -X\}$.

e) (2) If $G = \mathbf{SO}(n)$, show that $L = \{X \in M_n(\mathbb{C}) | X^T = -X\}$.

f) Assume again that G is any closed subgroup of $\mathbf{GL}_n(\mathbb{C})$. The goal of this question is to show the following statement : (*) There exists a neighborhood U of 0 in L such that $\exp(U)$ is a neighborhood of 1 in G and that exp induces a homeomorphism $U \xrightarrow{\sim} \exp(U)$.

   i. (2) Let L' be a $\mathbb{R}$-linear subspace of $M_n(\mathbb{C})$ such that $M_n(\mathbb{C}) = L \oplus L'$, and consider the function $\varphi : M_n(\mathbb{C}) \to \mathbf{GL}_n(\mathbb{C})$ defined by $\varphi(A+B) = e^A e^B$, for every $A \in L$ and every $B \in L'$. Show that there exist neighborhoods U_0 of 0 in L, V of 0 in L' and W of 1 in $\mathbf{GL}_n(\mathbb{C})$ such that φ induces a diffeomorphism $U_0 \times V \xrightarrow{\sim} W$.

   ii. (1) Suppose that (*) is not true. Show that there exists a decreasing sequence $U_0 \supset U_1 \supset \ldots$ of neighborhoods of 0 in L, a sequence $(A_k)_{k\geq 0}$ of elements of L and a sequence $(B_k)_{k\geq 0}$ of elements of L' such that :

   - for every $k \geq 0$, we have $A_k \in U_k$;
   - for every $k \geq 0$, we have $B_k \neq 0$;
   - for every $k \geq 0$, we have $\varphi(A_k + B_k) \in G$;
   - the limit of the sequence $(B_k)_{k\geq 0}$ is 0;
   - for every neighborhood U of 0 in L, we have $U_k \subset U$ for k big enough.

   iii. (1) Show that the sequence $(\frac{1}{\|B_k\|}B_k)_{k\geq 0}$ has a convergent subsequence, and that the limit B of this subsequence is not 0.

   iv. (1) For every $t \in \mathbb{R}$, show that $\lfloor \frac{t}{\|B_k\|} \rfloor \|B_k\| \to t$ as $k \to +\infty$. (Where, for every $c \in \mathbb{R}$, we write $\lfloor c \rfloor$ for the biggest integer that is $\leq c$.)

   v. (2) Show that $B \in L$.

g) (3, extra credit) Let (ρ, V) be a continuous finite-dimensional representation of G. For every $X \in L$, show that there exists a unique element $u(X) \in \mathrm{End}(V)$ such that $\rho(e^{tX}) = e^{tu(X)}$ for every $t \in \mathbb{R}$. Show also that the function $u : L \to \mathrm{End}(V)$ is $\mathbb{R}$-linear.

Solution.

a) Let $d\exp_0$ be the differential of exp at the point 0. By definition of the differential, for every $H \in M_n(\mathbb{C})$, we have

$$d\exp_0(H) = \lim_{t\to 0}\frac{1}{t}(e^{tH} - e^0) = \lim_{t\to 0}\frac{1}{t}(e^{tH} - I_n) = \frac{d}{dt}e^{tH}\Big|_{t=0}.$$

But we have

$$\frac{d}{dt}e^{tH} = \sum_{n\geq 0} \frac{1}{n!}\frac{d}{dt}(tH)^n = H\exp(tH) = \exp(tH)H,$$

so $d\exp_0(H) = H$. Finally, we get $d\exp_0 = \mathrm{id}_{M_n(\mathbb{C})}$.

b) As $\mathbf{GL}_n(\mathbb{C})$ is open in $M_n(\mathbb{C})$, neighborhoods of 1 in $\mathbf{GL}_n(\mathbb{C})$ are the same as small enough neighborhoods of 1 in $M_n(\mathbb{C})$. So the result follows from the fact that $d\exp_0$ is invertible and from the inversion function theorem.

c) Let's first prove the hint. Let U be a neighborhood of 0 in $M_n(\mathbb{C})$ and V be a neighborhood of 1 in $\mathbf{GL}_n(\mathbb{C})$ such that exp is a diffeomorphism from U to V. We write $\log : V \to U$ for its inverse. As $\exp(H) = 1 + H + o(H)$ as $H \to 0$, we have $\log(1 + H) = H + o(H)$ as $H \to 0$.

Let $X, Y \in M_n(\mathbb{C})$. Then $\exp(\frac{1}{k}X) = 1 + \frac{1}{k}X + O(\frac{1}{k^2})$ and $\exp(\frac{1}{k}X) = 1 + \frac{1}{k}X + O(\frac{1}{k^2})$, so $\exp(\frac{1}{k}X)\exp(\frac{1}{k}Y) = 1 + \frac{1}{k}(X + Y) + O(\frac{1}{k^2})$. If k is big enough, we have $\exp(\frac{1}{k}X)\exp(\frac{1}{k}Y) \in V$, and $\log(\exp(\frac{1}{k}X)\exp(\frac{1}{k}Y)) = \frac{1}{k}(X + Y) + o(\frac{1}{k})$. So finally

$$(\exp(\tfrac{1}{k}X)\exp(\tfrac{1}{k}Y))^k = \exp(k\log(\exp(\tfrac{1}{k}X)\exp(\tfrac{1}{k}Y))) = \exp(X+Y+o(1)) \to \exp(X+Y)$$

as $k \to +\infty$.

The set L is stable by scalar multiplication by definition. Let $X, Y \in L$. Then $c(t) := e^{tX}e^{tY} \in G$ for every $t \in \mathbb{R}$. As G is closed in $\mathbf{GL}_n(\mathbb{C})$, this implies that, for every $t \in \mathbb{R}$,

$$\exp(t(X + Y)) = \lim_{k\to+\infty}(\exp(\tfrac{t}{k}X)\exp(\tfrac{t}{k}Y))^k \in G.$$

So $X + Y \in L$.

d) Let $X \in M_n(\mathbb{C})$ such that $X = -X^*$. Then, for every $t \in \mathbb{R}$, we have $tX = -(tX)^*$ (in particular, tX and tX^* commute), hence

$$e^{tX}(e^{tX})^* = e^{tX}e^{tX^*} = e^{tX+tX^*} = e^0 = I_n,$$

i.e., $e^{tX} \in \mathbf{U}(n)$. So $X \in L$.

Conversely, let $X \in L$. Then, for every $t \in \mathbb{R}$, we have $e^{tX}e^{tX^*} = I_n$. Deriving this expression (and using the expression for the derivative from the proof of (a)) gives

$$0 = Xe^{tX}e^{tX^*} + e^{tX}e^{tX^*}X^* = X + X^*.$$

e) This is exactly the same proof as in (d), replacing "$*$" by "T".

f) i. By the inverse function theorem, it suffices to prove that the differential of φ at $0 \in M_n(\mathbb{C})$ is invertible. If $A \in L$ and $B \in L'$, we have, by definition of the differential

$$d\varphi_0(A + B) = \lim_{t\to 0}\frac{1}{t}(\varphi(tA + tB) - \varphi(0)) = \lim_{t\to 0}\frac{1}{t}(e^{tA}e^{tB} - 1) = A + B$$

(by the calculation in (a)), so $d\varphi_0 = \mathrm{id}_{M_n(\mathbb{C})}$, and this is certainly invertible.

ii. Choose a sequence of neighborhoods $U_0 \supset U_1 \supset \ldots$ (resp. $V = V_0 \supset V_1 \supset \ldots$) of 0 in L (resp. L') such that every neighborhood U (resp. V') of 0 in L contains U_k (resp. V_k) for k big enough. (For example, we could take balls with radii tending to 0 in L and in L'.) For every $k \geq 0$, the function φ is a diffeomorphism from $U_k \times V_k$ to $\varphi(U_k \times V_k)$, and in particular $\varphi(U_k \times V_k) \cap G$ is a neighborhood of 1 in G, containing $\exp(U_k)$. If (*) is not true, them $\varphi(U_k \times V) \cap G$ strictly contains $\exp(U_k)$ for every k, so we can find $A_k \in U_k$ and $B_k \in V_k$ such that $\varphi(A_k + B_k) \in G$ and $\varphi(A_k + B_k) \notin \exp(U_k)$, i.e. $B_k \neq 0$. Also, we have $B_k \to 0$ as $k \to +\infty$ because of the condition on the neighborhoods V_k.

iii. The sequence $(\frac{1}{\|B_k\|}B_k)_{k\geq 0}$ is a sequence of elements of the unit ball of L', and this unit ball is compact, so it has a convergent subsequence, whose limit is still in the unit ball (and in particular nonzero).

iv. For every $k \geq 0$, we have

$$0 \leq \frac{t}{\|B_k\|} - \lfloor \frac{t}{\|B_k\|} \rfloor < 1,$$

hence

$$0 \leq t - \lfloor \frac{t}{\|B_k\|} \rfloor \|B_k\| < \|B_k\|.$$

As $B_k \to 0$, we have $\|B_k\| \to 0$, which implies that

$$\lfloor \frac{t}{\|B_k\|} \rfloor \|B_k\| \to t.$$

v. After passing to a subsequence, we may assume that $B = \lim_{k\to+\infty} \frac{1}{\|B_k\|}B_k$. We must show that $e^{tB} \in G$ for every $t \in \mathbb{R}$. Let $t \in \mathbb{R}$. By question (iv) and the definition of B, we have

$$e^{tB} = \lim_{k\to+\infty} \exp\left(\lfloor \tfrac{t}{\|B_k\|} \rfloor \|B_k\| \tfrac{1}{\|B_k\|} B_k\right) = \exp\left(\lfloor \tfrac{t}{\|B_k\|} \rfloor B_k\right).$$

But, for every $k \geq 0$, we have $\varphi(A_k + B_k) = e^{A_k}e^{B_k} \in G$ and $e^{A_k} \in G$ because $A_k \in L$, so $e^{B_k} \in G$; as $N := \lfloor \frac{t}{\|B_k\|} \rfloor$ is an integer, this implies that $e^{NB} = (e^B)^N \in G$. Finally, as G is closed in $\mathbf{GL}_n(\mathbb{C})$, we deduce that $e^{tB} \in G$.

g) Let $X \in L$. Consider the map $\mathbb{R} \to \mathbf{GL}(V)$, $t \mapsto \rho(e^{tX})$. This a continuous morphism of groups, hence, by 5(b)(i) of problem set 3, there exists a unique $u(X) \in \mathrm{End}(V)$ such that $\rho(e^{tX}) = \exp(tu(X))$ for every $t \in \mathbb{R}$.

Let $X, Y \in L$ and $a \in \mathbb{R}$. For every $t \in \mathbb{R}$, we have

$$e^{tu(aX)} = \rho(e^{taX}) = e^{tau(X)}.$$

Taking derivatives at $t = 0$, we get $u(aX) = au(X)$. Now consider $c : \mathbb{R} \to \mathbf{GL}(V)$, $t \mapsto \rho(e^{tX})\rho(e^{tY})\rho(e^{-t(X+Y)})$. We have $c(t) = e^{tu(X)}e^{tu(Y)}e^{-tu(X+Y)}$, so c is C^∞ and $c'(0) = u(X) + u(Y) - u(X+Y)$. On the other hand, using the fact that c is C^∞, we can prove as in (c) that, for every $t \in \mathbb{R}$, we have

$$\lim_{k\to+\infty} c(\tfrac{t}{k})^k = e^{tc'(0)}.$$

So we just need to prove that this limit is equal to id_V for every $t \in \mathbb{R}$. An easy calculation with infinitesimals shows that (if t is fixed)

$$e^{\frac{t}{k}X}e^{\frac{t}{k}Y}e^{-\frac{t}{k}(X+Y)} = I_n + O(\tfrac{1}{k^2}),$$

so

$$(e^{\frac{t}{k}X}e^{\frac{t}{k}Y}e^{-\frac{t}{k}(X+Y)})^k = I_n + O(\tfrac{1}{k}),$$

and

$$c(\tfrac{t}{k}) = \rho((e^{\frac{t}{k}X}e^{\frac{t}{k}Y}e^{-\frac{t}{k}(X+Y)})^k) \xrightarrow[k\to+\infty]{} \rho(I_n) = \mathrm{id}_V.$$

$\square$

6. The goal of this problem is to generalize problem 4 to an arbitrary compact group G, assuming something about the Haar measure. Let (ρ, V) be a faithful finite-dimensional continuous representation of G. We want to show that any irreducible representation of G is a direct summand of some $V^{\otimes N} \otimes (V^*)^{\otimes M}$. We fix a normalized Haar measure μ on G.

   a) (1) Show that there exists an isomorphism $V \simeq \mathbb{C}^n$ such that ρ induces an isomorphism (of topological groups) between G and a closed subgroup of $\mathbf{U}(n)$.

   From now on, we assume that G is a closed subgroup of $\mathbf{U}(n)$, that $V = \mathbb{C}^n$ and that $\rho : G \to \mathbf{GL}_n(\mathbb{C})$ is the inclusion. Let (π, W) be a continuous nonzero finite-dimensional representation of G. Define $f : \mathbf{U}(n) \to \mathbb{C}$ and $g : G \to \mathbb{C}$ by $f(x) = 1 + \mathrm{Tr}(x) + \overline{\mathrm{Tr}(x)}$ and $g(x) = \overline{\chi_W(x)}$.

   As in problem 5, we define

$$L_0 = \{X \in M_n(\mathbb{C}) | \forall t \in \mathbb{R},\ e^{tX} \in \mathbf{U}(n)\}$$

   and

$$L = \{X \in M_n(\mathbb{C}) | \forall t \in \mathbb{R},\ e^{tX} \in G\}.$$

   Remember that we proved in problem 5 that, if Ω is a small enough neighborhood of 0 in L, then exp induces a homeomorphism between Ω and $\exp(\Omega)$, and $\exp(\Omega)$ is a neighborhood of 1 in G. Choose an isomorphism $L_0 \simeq \mathbb{R}^m$, and let dX be the Lebesgue measure on L_0 given by this isomorphism. We assume the following : (**) For Ω small enough, there exists $c \in \mathbb{R}_{>0}$ such that the inverse image by the homeomorphism $\exp : \Omega \xrightarrow{\sim} \exp(\Omega)$ of the Haar measure μ is of the form $h(X)dX$, where $h(X) = c + O(\|X\|)$ as $X \to 0$. [1]

   b) (1) Show that, for every $x \in \mathbf{U}(n)$, we have $f(x) = 1 + 2\sum_{i=1}^{n} \cos\theta_i$, where $e^{i\theta_1}, \dots, e^{i\theta_n}$ are the eigenvalues of x.

   c) (2) If Ω is a neighborhood of 0 in L_0, show that there exists $\delta > 0$ such that, for every $x \notin \exp(\Omega)$ and every $N \geq 1$, we have

$$|f(x)^N| \leq (1 + 2n - \delta)^N.$$

   d) (1) If Ω is a neighborhood of 0 in L and $U = \exp(\Omega)$, show that there exists $\delta > 0$ and $C \in \mathbb{R}_{>0}$ such that, for every $N \geq 1$, we have

$$\left| \int_{G-U} f(x)^N g(x) d\mu(x) \right| \leq C(1 + 2n - \delta)^N.$$

   e) (2) Show that

$$f(e^X) = (2n+1)e^{-K(X) + O(\|X\|^4)}$$

   as $X \to 0$ in L_0, where $K(X) = \frac{1}{1+2n}\|X\|^2 = \frac{1}{1+2n}\mathrm{Tr}(X^*X)$.

   f) (3, extra credit) Show that, if Ω is a ball (of finite radius) centered at 0 in L, there exists $D \in \mathbb{R}_{>0}$ such that

$$\int_\Omega e^{-NK(X)} g(e^X) dX \sim D \cdot N^{-\frac{1}{2}\dim L}$$

   as $N \to +\infty$. (Hint : Show that we have $g(e^X) = \dim W + O(\|X\|)$ as $X \to 0$ in L.)

[1] This is always true, but we don't have the tools to prove it.

g) (2, extra credit) Show that there exists a neighborhood U of 1 in G and $E \in \mathbb{R}_{>0}$ such that

$$\int_U f(x)^N g(x) d\mu(x) \sim E \frac{(2n+1)^N}{N^{\frac{1}{2}\dim L}}$$

as $N \to +\infty$.

h) (1, extra credit) Show that $\int_G f(x)^N g(x) d\mu(x) \neq 0$ if N is big enough.

Solution.

a) As G is compact, the representation (ρ, V) is unitarizable. This means that there exists an isomorphism $V \simeq \mathbb{C}^n$ such that $\rho(G) \subset \mathbf{U}(n)$. As the representation (ρ, V) is faithful, the morphism ρ is injective, so $\rho : G \to \mathbf{U}(n)$ is an injective and continuous map. As G is compact, this map is a homeomorphism onto its image.

b) Let D be the diagonal matrix with diagonal entries $e^{i\theta_1}, \dots, e^{i\theta_n}$. As x commutes with $x^* = x^{-1}$, the spectral theorem implies that there exists $A \in \mathbf{U}(n)$ such that $D = AxA^{-1}$. As f is clearly a central function on $\mathbf{U}(n)$ (and even on $\mathbf{GL}_n(\mathbb{C})$), we have $f(x) = f(D)$. But $f(e^D) = 1 + \sum_{j=1}^n \operatorname{Re}(e^{i\theta_j}) = 1 + 2\sum_{j=1}^n \cos(\theta_j)$.

c) By question (b), we have, for every $x \in \mathbf{U}(n)$, $f(x) \in \mathbb{R}$ and $1 - 2n \leq f(x) \leq 1 + 2n$. Moreover, the equality $f(x) = 1 + 2n$ is possible only if all te eigenvalues of x are equal to 1, which in turn implies that $x = 1$, because x is diagonalizable.

By question 5(f), we know that $\exp(\Omega)$ contains an open neighborhood V of 1 in $\mathbf{U}(n)$. As $\mathbf{U}(n)$, the continuous function f attains its supremum on $\mathbf{U}(n) - V$, and this supremum is $< 1+2n$ by the previous paragraph. So $\sup_{x \in \mathbf{U}(n) - \exp(\Omega)} |f(x)| < 1+2n$, and this implies the desired result.

d) For every $x \in G$, we have $|g(x)| \leq \dim W$. So we can take $C = \operatorname{vol}(G - U)(\dim W)$ and apply question (c).

e) Let $i\theta_1, \dots, i\theta_n$ be the eigenvalues of X. As X commutes with $X^* = -X$, there exists $A \in \mathbf{U}(n)$ such that $AXA^{-1} = D$, where D is the diagonal matrix with diagonal entries $i\theta_1, \dots, i\theta_n$. Then $Ae^X A^{-1} = e^D$ is the diagonal matrix with diagonal entries $e^{i\theta_1}, \dots, e^{i\theta_n}$, so the eigenvalues of e^X are $e^{i\theta_1}, \dots, e^{i\theta_n}$, and $f(e^X) = 1 + 2\sum_{j=1}^n \cos(\theta_j) = 1 + 2n - \sum_{j=1}^n \theta_j^2 + O(\sum_{j=1}^n \theta_j^4)$.

We have $X = A^{-1}DA$, so $X^* = -X = -A^{-1}DA$, hence $X^*X = -A^{-1}D^2A$, and finally $\operatorname{Tr}(X^*X) = -\operatorname{Tr}(D^2) = \sum_{j=1}^n \theta_j^2$. So $f(e^X) = 1 + 2n - \sum_{j=1}^n \theta_j^2 + O(\|X\|^4)$. On the other hand,

$$(2n+1)e^{-K(X)+O(\|X\|^4)} = (2n+1)(1 - \tfrac{1}{2n+1}\|X\|^2 + O(\|X\|^4)) = 2n+1 - \sum_{j=1}^n \theta_j^2 + O(\|X\|^4).$$

f) We first prove the hint. By question 5(g), there exists a $\mathbb{R}$-linear map $u : L \to \operatorname{End}(W)$ such that, for every $X \in L$, we have $\pi(e^X) = e^{tu(X)}$. As u is $\mathbb{R}$-linear, it is C^∞, and so the map $U : L \to \mathbb{C}$, $X \mapsto g(e^X) = \operatorname{Tr}(e^{u(X)})$ is also C^∞. We also have $U(0) = \operatorname{Tr}(\mathrm{id}_W) = \dim W$. So we get $U(X) = \dim W + O(\|X\|)$.

Now we evaluate the integral. Doing the change of variable $Y = N^{1/2}X$ (and observing that $NK(X) = K(Y)$), we get

$$\int_\Omega e^{-NK(X)} U(X) dX = N^{-\frac{1}{2}\dim L} \int_{N^{1/2}\Omega} e^{-K(Y)} U(N^{-1/2}Y) dY.$$

It remains to show that $\int_{N^{1/2}\Omega} e^{-K(Y)}U(N^{-1/2}Y)dY$ converges to a positive real number as $N \to +\infty$. First note that, as $\int_L e^{-K(Y)}dY$ converges and as the function g is bounded by $\dim W$, we have

$$\left|\int_{N^{1/2}\Omega - N^{1/4}\Omega} e^{-K(Y)}U(N^{-1/2}Y)dY\right| \leq \left|\int_{L-N^{1/4}\Omega} e^{-K(Y)}U(N^{-1/2}Y)dY\right| \xrightarrow[N\to+\infty]{} 0.$$

On the other hand, using the fact that $U(N^{-1/2}Y) = \dim W + O(N^{-1/4})$ for $Y \in N^{1/4}\Omega$, we get

$$\begin{aligned}\lim_{N\to+\infty}\int_{N^{1/4}\Omega} e^{-K(Y)}U(N^{-1/2}Y)dY &= \lim_{N\to+\infty}(\dim W)\int_{N^{1/4}\Omega} e^{-K(Y)}dY \\ &= (\dim W)\int_L e^{-K(Y)}dY.\end{aligned}$$

As the function $Y \mapsto e^{-K(Y)}$ takes positive real values on L, the last integral is positive and real.

g) If there exists a neighborhood Ω of 0 in L such that exp is a diffeomorphism from Ω to U (which we can always assume by making U small enough), then

$$\int_U f(x)^N g(x)dx = (2n+1)^N \int_\Omega e^{-NK(X)+NO(\|X\|^4)}g(e^X)h(X)dX,$$

with $h(X) = c + O(\|X\|)$, $c \in \mathbb{R}_{>0}$. This is equal to

$$\frac{(2n+1)^N}{(\dim L)^{N/2}}\int_{N^{1/2}\Omega} e^{-K(Y)+O(N^{-1}\|Y\|^4)}U(N^{-1/2}Y)h(N^{-1/2}Y)dY.$$

We can prove as in question (f) that, if we choose Ω to be a ball centered at 0 (which we can), then the integral converges to $c(\dim W)\int_L e^{-K(Y)}dY$ as $N \to +\infty$, which gives the conclusion.

h) By questions (d) and (f), we can decompose $\int_G f(x)^N g(x)dx$ as a sum of two terms, one of which is equivalent to a positive multiple of $\frac{(2n+1)^N}{(\dim L)^{N/2}}$ and one of which is dominated by $(1+2n-\delta)^N$, for some $\delta > 0$. As N tends to $+\infty$, the second term will become negligible with respect to the first, so the sum of the two terms cannot be 0 for N big enough.

□

7. (extra credit, 3) Show that assumption (**) in problem 6 holds for $G = \mathbf{SO}(n)$.

You can forget this problem. I'll try to write a better version in a future problem set.