MAT 449 : Problem Set 9

Due Thursday, November 29

In this problem set, we put the norm $\|.\|$ on $M_n(\mathbb{C})$ defined by $\|X\|^2 = \mathrm{Tr}(X^*X)$.

1. (2) Let $\mathcal{P}(\mathbb{Z})$ be the set of subsets of $\mathbb{Z}$. Show that there exists a finitely additive left-invariant probability measure on $\mathbb{Z}$, that is, a function $\mu : \mathcal{P}(\mathbb{Z}) \to \mathbb{R}_{\geq 0}$ such that :
   (a) If $A_1, \ldots, A_n \in \mathcal{P}(\mathbb{Z})$ are such that $A_i \cap A_j = \varnothing$ for $i \neq j$, then $\mu(A_1 \cup \ldots \cup A_n) = \mu(A_1) + \ldots + \mu(A_n)$.
   (b) $\mu(\mathbb{Z}) = 1$.
   (c) For every $A \in \mathcal{P}(Z)$ and $n \in \mathbb{Z}$, we have $\mu(n + A) = \mu(A)$.
2. (2, extra credit) Is the measure of problem 1 unique ? (Hint : You need a somewhat explicit way to construct invariant means on $\mathbb{Z}$. You can for example try to exploit the sequence of (non-invariant) means $M_n : L^\infty(\mathbb{Z}) \to \mathbb{C}$, $(x_k)_{k\in\mathbb{Z}} \mapsto \frac{1}{2k+1}\sum_{k=-n}^{n} x_k$.)
3. Let d be a positive integer.
   a) (1) Let T be the intersection of $\mathbf{U}(d)$ with the set of diagonal matrices. Show that
   $$T = \left\{ \begin{pmatrix} z_1 & \ldots & 0 \\ & \ddots & \\ 0 & \ldots & z_d \end{pmatrix}, z_1, \ldots, z_d \in S^1 \right\}.$$
   b) (1) Show that every element of $\mathbf{U}(d)$ is conjugated in $\mathbf{U}(d)$ to an element of T.
   c) (1) Show that every element of $\mathbf{SU}(d)$ is conjugated in $\mathbf{SU}(d)$ to an element of $T_0 := T \cap \mathbf{SU}(d)$.
   d) (1) Show that a finite-dimensional representation V of $\mathbf{SU}(d)$ is uniquely determined up to equivalence by $\chi_{V|T_0}$.

   We now take $d = 2$. Remember the irreducible representations V_n ($n \geq 0$) of $\mathbf{SU}(2)$ defined in problem 1 of problem set 6.
   e) (1) Calculate the restriction of χ_{V_n} to T_0.
   f) (2) Let (ρ, V) be a finite-dimensional representation of $\mathbf{SU}(2)$. Show that there exists $m \geq 1$ and nonnegative integers $a_0, \ldots, a_m$ such that, for every $z \in S^1$, we have
   $$\chi_V\left(\begin{pmatrix} z & 0 \\ 0 & \overline{z} \end{pmatrix}\right) = a_0 + \sum_{i=1}^{m} a_i(z^i + z^{-i}).$$
   g) (2) Show that there exist integers $c_n \in \mathbb{Z}$, $n \geq 0$, such that $c_n = 0$ for n big enough and $\chi_V = \sum_{n\geq 0} c_n \chi_{V_n}$.
   h) (1) Show that the integers c_n of (f) are all nonnegative.

i) (1) If V is irreducible, show that there exists $n \geq 0$ such that $V \simeq V_n$.

4. Let G be a compact group, and let (π, V) be a faithful finite-dimensional continuous representation of G. (Remember that this means that $\pi : G \to \mathbf{GL}(V)$ is injective.) The goal of this problem is to show that, if G is finite, then every irreducible representation of G is a direct summand of a representation of the form $V^{\otimes n} \otimes (V^*)^{\otimes m}$ (for some $n, m \geq 1$), where the notation $V^{\otimes n}$ means $\underbrace{V \otimes \ldots \otimes V}_{n \text{ times}}$, and similarly for $(V^*)^{\otimes m}$.

   a) (3) Let $\mathbb{1}$ be the trivial representation of G on $\mathbb{C}$. Show that it suffices to show that every irreducible representation of G is a direct summand of a representation of the form $(V \oplus V^* \oplus \mathbb{1})^{\otimes N}$, for some $N \geq 1$.

   b) (2) Let W be an irreducible representation of G. Show that W is a direct summand of $(V \oplus V^* \oplus \mathbb{1})^{\otimes N}$ if and only if $\int_G (1 + 2 \operatorname{Re} \chi_V(x))^N \overline{\chi_W(x)} dx \neq 0$.

   From now on, we assume that G is finite, we fix a finite-dimensional representation W of G, and we write, for every $N \in \mathbb{Z}_{\geq 0}$,

$$S_N = \sum_{x \in G} (1 + 2 \operatorname{Re} \chi_V(x))^N \overline{\chi_W(x)}.$$

   Let $d = \dim V$.

   c) (2) If $x \neq 1$, show that $(1 + 2 \operatorname{Re} \chi_V(x))^N \overline{\chi_W(x)} = o((1 + 2d)^N)$ as $N \to +\infty$.

   d) (2) If $W \neq 0$, show that $S_N \neq 0$ for N big enough.

5. Consider the function $\exp : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ defined by $\exp(X) = \sum_{n \geq 0} \frac{1}{n!} X^n$; we also write e^X for $\exp(X)$. You may assume the basic properties of this function, i.e. that the series defining it converges absolutely, that it is infinitely derivable, and that we can calculate its derivatives term by term in the sum. You may also assume that $\exp(A + B) = \exp(A) \exp(B)$ for any $A, B \in M_n(\mathbb{C})$ such that $AB = BA$; in particular, $\exp(X) \in \mathbf{GL}_n(\mathbb{C})$ for every $X \in M_n(\mathbb{C})$, and $\exp(X)^{-1} = \exp(-X)$.

   We also fix a closed subgroup G of $\mathbf{GL}_n(\mathbb{C})$.

   a) (2) Calculate the differential of exp at the point $0 \in M_n(\mathbb{C})$. (Remember that this is a linear operator from $M_n(\mathbb{C})$ to itself.)

   b) (1) Show that exp induces a diffeomorphism from a neighborhood of 0 in $M_n(\mathbb{C})$ to a neighborhood of 1 in $\mathbf{GL}_n(\mathbb{C})$.

   c) (3) Let $L = \{X \in M_n(\mathbb{C}) | \forall t \in \mathbb{R},\ \exp(tX) \in G\}$. Show that L is a $\mathbb{R}$-linear subspace of $M_n(\mathbb{C})$. (Hint : For all $X, Y \in M_n(\mathbb{C})$, show that $\exp(X + Y) = \lim_{k \to +\infty} (\exp(\frac{1}{k} X) \exp(\frac{1}{k} Y))^k$.)

   d) (2) If $G = \mathbf{U}(n)$, show that $L = \{X \in M_n(\mathbb{C}) | X^* = -X\}$.

   e) (2) If $G = \mathbf{SO}(n)$, show that $L = \{X \in M_n(\mathbb{C}) | X^T = -X\}$.

   f) Assume again that G is any closed subgroup of $\mathbf{GL}_n(\mathbb{C})$. The goal of this question is to show the following statement : (*) There exists a neighborhood U of 0 in L such that $\exp(U)$ is a neighborhood of 1 in G and that exp induces a homeomorphism $U \xrightarrow{\sim} \exp(U)$.

      i. (2) Let L' be a $\mathbb{R}$-linear subspace of $M_n(\mathbb{C})$ such that $M_n(\mathbb{C}) = L \oplus L'$, and consider the function $\varphi : M_n(\mathbb{C}) \to \mathbf{GL}_n(\mathbb{C})$ defined by $\varphi(A + B) = e^A e^B$, for every $A \in L$ and every $B \in L'$. Show that there exist neighborhoods U_0 of 0 in L, V of 0 in L' and W of 1 in $\mathbf{GL}_n(\mathbb{C})$ such that φ induces a diffeomorphism $U_0 \times V \xrightarrow{\sim} W$.

ii. (1) Suppose that (*) is not true. Show that there exists a decreasing sequence $U_0 \supset U_1 \supset \dots$ of neighborhoods of 0 in L, a sequence $(A_k)_{k\geq 0}$ of elements of L and a sequence $(B_k)_{k\geq 0}$ of elements of L' such that :
   - for every $k \geq 0$, we have $A_k \in U_k$;
   - for every $k \geq 0$, we have $B_k \neq 0$;
   - for every $k \geq 0$, we have $\varphi(A_k + B_k) \in G$;
   - the limit of the sequence $(B_k)_{k\geq 0}$ is 0;
   - for every neighborhood U of 0 in L, we have $U_k \subset U$ for k big enough.

iii. (1) Show that the sequence $(\frac{1}{\|B_k\|}B_k)_{k\geq 0}$ has a convergent subsequence, and that the limit B of this subsequence is not 0.

iv. (1) For every $t \in \mathbb{R}$, show that $\lfloor \frac{t}{\|B_k\|} \rfloor \|B_k\| \to t$ as $k \to +\infty$. (Where, for every $c \in \mathbb{R}$, we write $\lfloor c \rfloor$ for the biggest integer that is $\leq c$.)

v. (2) Show that $B \in L$.

g) (3, extra credit) Let (ρ, V) be a continuous finite-dimensional representation of G. For every $X \in L$, show that there exists a unique element $u(X) \in \mathrm{End}(V)$ such that $\rho(e^{tX}) = e^{tu(X)}$ for every $t \in \mathbb{R}$. Show also that the function $u : L \to \mathrm{End}(V)$ is $\mathbb{R}$-linear.

6. The goal of this problem is to generalize problem 4 to an arbitrary compact group G, assuming something about the Haar measure. Let (ρ, V) be a faithful finite-dimensional continuous representation of G. We want to show that any irreducible representation of G is a direct summand of some $V^{\otimes N} \otimes (V^*)^{\otimes M}$. We fix a normalized Haar measure μ on G.

   a) (1) Show that there exists an isomorphism $V \simeq \mathbb{C}^n$ such that ρ induces an isomorphism (of topological groups) between G and a closed subgroup of $\mathbf{U}(n)$.

   From now on, we assume that G is a closed subgroup of $\mathbf{U}(n)$, that $V = \mathbb{C}^n$ and that $\rho : G \to \mathbf{GL}_n(\mathbb{C})$ is the inclusion. Let (π, W) be a continuous nonzero finite-dimensional representation of G. Define $f : \mathbf{U}(n) \to \mathbb{C}$ and $g : G \to \mathbb{C}$ by $f(x) = 1 + \mathrm{Tr}(x) + \overline{\mathrm{Tr}(x)}$ and $g(x) = \overline{\chi_W(x)}$.

   As in problem 5, we define

$$L_0 = \{X \in M_n(\mathbb{C}) | \forall t \in \mathbb{R},\ e^{tX} \in \mathbf{U}(n)\}$$

   and

$$L = \{X \in M_n(\mathbb{C}) | \forall t \in \mathbb{R},\ e^{tX} \in G\}.$$

   Remember that we proved in problem 5 that, if Ω is a small enough neighborhood of 0 in L, then exp induces a homeomorphism between Ω and $\exp(\Omega)$, and $\exp(\Omega)$ is a neighborhood of 1 in G. Choose an isomorphism $L_0 \simeq \mathbb{R}^m$, and let dX be the Lebesgue measure on L_0 given by this isomorphism. We assume the following : (**) For Ω small enough, there exists $c \in \mathbb{R}_{>0}$ such that the inverse image by the homeomorphism $\exp : \Omega \xrightarrow{\sim} \exp(\Omega)$ of the Haar measure μ is of the form $h(X)dX$, where $h(X) = c + O(\|X\|)$ as $X \to 0$. [1]

   b) (1) Show that, for every $x \in \mathbf{U}(n)$, we have $f(x) = 1 + 2\sum_{i=1}^{n} \cos\theta_i$, where $e^{i\theta_1}, \dots, e^{i\theta_n}$ are the eigenvalues of x.

   c) (2) If Ω is a neighborhood of 0 in L_0, show that there exists $\delta > 0$ such that, for every $x \notin \exp(\Omega)$ and every $N \geq 1$, we have

$$|f(x)^N| \leq (1 + 2n - \delta)^N.$$

[1] This is always true, but we don't have the tools to prove it.

d) (1) If Ω is a neighborhood of 0 in L and $U = \exp(\Omega)$, show that there exists $\delta > 0$ and $C \in \mathbb{R}_{>0}$ such that, for every $N \geq 1$, we have

$$\left| \int_{G-U} f(x)^N g(x) d\mu(x) \right| \leq C(1 + 2n - \delta)^N.$$

e) (2) Show that

$$f(e^X) = (2n+1)e^{-K(X)+O(\|X\|^4)}$$

as $X \to 0$ in L_0, where $K(X) = \frac{1}{1+2n}\|X\|^2 = \frac{1}{1+2n}\operatorname{Tr}(X^*X)$.

f) (3, extra credit) Show that, if Ω is a ball (of finite radius) centered at 0 in L, there exists $D \in \mathbb{R}_{>0}$ such that

$$\int_\Omega e^{-NK(X)} g(e^X) dX \sim D \cdot N^{-\frac{1}{2}\dim L}$$

as $N \to +\infty$. (Hint : Show that we have $g(e^X) = \dim W + O(\|X\|)$ as $X \to 0$ in L.)

g) (2, extra credit) Show that there exists a neighborhood U of 1 in G and $E \in \mathbb{R}_{>0}$ such that

$$\int_U f(x)^N g(x) d\mu(x) \sim E \frac{(2n+1)^N}{N^{\frac{1}{2}\dim L}}$$

as $N \to +\infty$.

h) (1, extra credit) Show that $\int_G f(x)^N g(x) d\mu(x) \neq 0$ if N is big enough.

7. (extra credit, 3) Show that assumption (**) in problem 6 holds for $G = \mathbf{SO}(n)$.

You can forget this problem. I'll try to write a better version in a future problem set.