MAT 449 : Problem Set 8

Due Sunday, November 18

Let (X, μ) be a measure space. and let E be a closed linear subspace of $L^\infty(X)$ containing the constant functions and closed under the map $\varphi \mapsto \overline{\varphi}$. A *mean* on E is a linear functional $M : E \to \mathbb{C}$ such that :

(i) $M(\mathbb{1}_X) = 1$;

(ii) if $f \geq 0$ (locally) almost everywhere, then $M(f) \geq 0$.

If $X = G$ is a locally compact group, we say that a mean M on E is G-*invariant* if for every $f \in E$ and every $x \in G$, we have $L_x f \in E$ and $M(L_x f) = M(f)$.

The group G is called *amenable* is there exists a G-invariant mean on $L^\infty(G)$.

Let V be a locally convex topological vector space (see definition B.3.5 of the notes), let K be a convex subset of V. We say that a map $f : K \to K$ is *affine* if, for all $v, w \in K$ and every $t \in [0, 1]$, we have $f(tv + (1-t)w) = tf(v) + (1-t)f(w)$. Let $G \times K \to K$, $(x, v) \mapsto x \cdot v$ be a continuous left action of G on X. We say that this action is an *affine action* if, for every $x \in G$, the map $K \to K$, $v \mapsto x \cdot v$ is affine.

We say that the group G has the *fixed point property* if every affine action of G on a nonempty compact convex subset of a locally convex topological vector space has a fixed point.

Note : The Hahn-Banach theorem is your friend in this problem set. Also the fact that, if V is a topological vector space, then any weak* continuous linear functional on $\mathrm{Hom}(V, \mathbb{C})$ is of the form $\Lambda \mapsto \Lambda(v)$, for some $v \in V$. (See theorem 3.10 of Rudin's *Functional analysis*.)

1. (Some basic properties.)

   a) (2) If (X, μ) is a measure space and E is a subspace of $L^\infty(X)$ containing the constant functions, show that any mean M on E is automatically continuous (for the topology given by the norm $\|.\|_\infty$) and that $\|M\|_{op} = 1$.

   We now suppose that G is a locally compact group.

   b) (1) If G is compact, show that left invariant means on $\mathcal{C}(G)$ are in natural bijection with normalized Haar measures on G.

   c) (3) Let $L^1(G)_{1,+}$ be the convex subset of $f \in L^1(G)$ such that $f \geq 0$ almost everywhere and $\|f\|_1 = 1$. We identify $L^1(G)$ to a subspace of the continuous dual of $L^\infty(G)$ in the usual way (i.e. a function $f \in L^1(G)$ corresponds to the continuous linear functional $\varphi \mapsto \int_G f\varphi d\mu$ on $L^\infty(G)$). Show that $L^1_{1,+}(G)$ is weak* dense in the set of means on $L^\infty(G)$.

   d) (2) Let $UCB(G)$ be the subspace of $L^\infty(G)$ composed of the left uniformly continuous bounded functions on G. For every $x \in G$, we write δ_x for the linear functional $\mathcal{C}(G) \to \mathbb{C}$, $f \mapsto f(x)$. Show that the set of convex combinations of functionals δ_x (that is, the set of sums $\sum_{i=1}^n a_i \delta_{x_i}$, with $x_1, \ldots, x_n \in G$ and $a_1, \ldots, a_n \in [0, 1]$ such that $a_1 + \ldots + a_n = 1$) is weak* dense in the set of means on $UCB(G)$.

Solution.

a) Let M be a mean on E. Let $\varphi \in E$, and suppose that $\varphi(x) \in \mathbb{R}$ for almost every x. We have $\|\varphi\|_\infty \mathbb{1}_X \in E$, because it is a multiple of the constant function $\mathbb{1}_X$, and the functions $\|\varphi\|_\infty \mathbb{1}_X - \varphi$ and $\|\varphi\|_\infty \mathbb{1}_X + \varphi$ are ≥ 0 almost everywhere, so their image by M is ≥ 0, that is, $M(\varphi) \in \mathbb{R}$ and

$$-\|\varphi\|_\infty \leq M(\varphi) \leq \|\varphi\|_\infty,$$

i.e. $|M(\varphi)| \leq \|\varphi\|_\infty$.

Now let φ be any element of E. Choose $c \in \mathbb{C}$ such that $|c| = 1$ and $M(c\varphi) \in \mathbb{R}$. Let $\varphi_1 = \frac{1}{2}(c\varphi + \overline{c\varphi})$ and $\varphi_2 = \frac{1}{2i}(c\varphi - \overline{c\varphi})$. Then φ_1, φ_2 have real values and $c\varphi = \varphi_1 + i\varphi_2$. We have

$$|\varphi(x)| = \sqrt{\varphi_1(x)^2 + \varphi_2(x)^2} \geq \max(|\varphi_1(x)|, |\varphi_2(x)|)$$

for every $x \in X$, so $\|\varphi\|_\infty \geq \max(\|\varphi_1\|_\infty, \|\varphi_2\|_\infty)$. On the other hand, $M(c\varphi) = M(\varphi_1) + iM(\varphi_2)$ and $M(\varphi_1), M(\varphi_2) \in \mathbb{R}$, so $M(\varphi_2) = 0$, and

$$|M(\varphi)| = |M(c\varphi)| = |M(\varphi_1)| \leq \|\varphi_1\|_\infty \leq \|\varphi\|_\infty.$$

This shows that M is continuous and that $\|M\|_{op} \leq 1$. As $M(\mathbb{1}_X) = 1 = \|\mathbb{1}_X\|_\infty$, we have $\|M\|_{op} = 1$.

b) This is just the Riesz representation theorem and proposition I.2.6 of the notes.

c) Let $\mathcal{M}$ be the set of means on $L^\infty(G)$. It is clearly a convex subset of $\mathrm{Hom}(L^\infty(G), \mathbb{C})$. By question (a), the set $\mathcal{M}$ is contained in the closed unit ball of $\mathrm{Hom}(L^\infty(G), \mathbb{C})$. Also, as the conditions characterizing a mean are all closed for the weak* topology, the set $\mathcal{M}$ is weak* closed in $\mathrm{Hom}(L^\infty(G), \mathbb{C})$. So $\mathcal{M}$ is weak* compact.

By definition of $L^1(G)_{1,+}$, for every $f \in L^1(G)_{1,+}$, the corresponding linear form on $L^\infty(G)$ is an element of $\mathcal{M}$. Note also that $L^1(G)_{1,+}$ is a convex subset of $L^1(G)$, so its image in $\mathrm{Hom}(L^\infty(G), \mathbb{C})$ is also convex. Let $\mathcal{M}'$ be the weak* closure of this image. We have $\mathcal{M}' \subset \mathcal{M}$, so $\mathcal{M}'$ is convex and weak* compact. Suppose that $\mathcal{M}' \neq \mathcal{M}$. Then, by the Hahn-Banach theorem (second geometric form), there exists $M \in \mathcal{M}$ and a weak* continuous $\mathbb{R}$-linear operator $\Lambda : \mathrm{Hom}(L^\infty(G), \mathbb{C}) \to \mathbb{R}$ such that $\Lambda(M) > \sup_{M' \in \mathcal{M}'} \Lambda(M')$. Note that the linear operator $\Lambda' : M' \mapsto \Lambda(M) + \frac{1}{i}\Lambda(iM')$ is weak* continuous and $\mathbb{C}$-linear, so there exists $\varphi \in L^\infty(G)$ such that $\Lambda'(M') = M'(\varphi)$ for every $M' \in \mathrm{Hom}(L^\infty(G), \mathbb{C})$, which gives $\Lambda(M') = \mathrm{Re}(M'(\varphi))$. Then we have

$$\mathrm{Re}\, M(\varphi) > \sup_{f \in L^1(G)_{1,+}} \left(\mathrm{Re} \int_G f \varphi d\mu \right).$$

Write $\varphi = \varphi_1 + i\varphi_2$, with $\varphi_1 = \mathrm{Re}\, \varphi$ and $\varphi_2 = \mathrm{Im}\, \varphi$. Then

$$M(\varphi_1) > \sup_{f \in L^1(G)_{1,+}} \int_G f \varphi_1 d\mu$$

(because $M(\varphi_1), M(\varphi_2) \in \mathbb{R}$ by the solution of question (a)). Let

$$c = \inf\{d \in \mathbb{R} | \varphi_1 \leq d\mathbb{1}_G \text{ locally almost everywhere}\}.$$

If $\varphi_1 \leq d\mathbb{1}_G$ locally almost everywhere, then $M(\varphi_1) \leq M(d\mathbb{1}_G) = d$. So $M(\varphi_1) \leq c$. Let $\delta > 0$ such that $M(\varphi_1) - \delta > \sup_{f \in L^1(G)_{1,+}} \int_G f \varphi_1 d\mu$. By definition of c, there exists a measurable subset A of G such that $\mu(A) > 0$ and $\varphi_{1|A} \geq (c + \delta)\mathbb{1}_A$. Let $f = \mu(A)^{-1}\mathbb{1}_A$. Then $f \in L^1(G)_{1,+}$ and $\int_G \varphi_1 f d\mu \geq c + \delta \geq M(\varphi_1) + \delta$, a contradiction.

d) Let $\mathcal{M}$ be the set of means on $UCB(G)$. We see as in the solution of (c) that $\mathcal{M}$ is a convex and weak* compact subset of $\mathrm{Hom}(UCB(G), \mathbb{C})$. Let $\mathcal{M}'$ be the weak* closure of the convex hull of the δ_x, $x \in G$; then $\mathcal{M}' \subset \mathcal{M}$ because each δ_x is in $\mathcal{M}$. If $\mathcal{M}' \neq \mathcal{M}$, then, by the Hahn-Banach theorem (second geometric version), there exists an element M of $\mathcal{M}$ and a continuous $\mathbb{R}$-linear functional $\Lambda : \mathrm{Hom}(UCB(G), \mathbb{C}) \to \mathbb{R}$ such that

$$\Lambda(M) > \sup_{M' \in \mathcal{M}'} \Lambda(M').$$

As in the solution of (c), we see that we can find a function $\varphi \in UCB(G)$ having real values and such that $\Lambda(M') = M'(\varphi)$ for every $M' \in \mathcal{M}$. So we have

$$M(\varphi) > \sup_{M' \in \mathcal{M}'} M'(\varphi) \geq \sup_{x \in G} \delta_x(\varphi) = \sup_{x \in G} \varphi(x).$$

Let $\delta > 0$ be such that $M(\varphi) - \delta \geq \sup_{x \in G} \varphi(x)$. Then $\varphi \leq (M(\varphi) - \delta)\mathbb{1}_G$, and so $M(\varphi) \leq M(\varphi) - \delta$, a contradiction.

□

2. Let G be an amenable locally compact group. The goal of this problem is to prove that G has the fixed point property.

So let V be a locally convex topological vector space, let K be a nonempty compact convex subset of V, and let $G \times K \to K$, $(x, v) \mapsto x \cdot v$ be a continuous affine action.

a) (1) Show that there exists a left invariant mean on $UCB(G)$.

b) (3) Fix a point $v_0 \in K$ and define $t : G \to K$ by $t(x) = x \cdot v_0$. If M is a mean on $UCB(G)$, show that there exists a unique regular Borel measure μ_M on K such that, for every $f \in \mathcal{C}(K)$, we have

$$\int_K f d\mu_M = M(f \circ t).$$

c) (2) Show that the integral $b_M = \int_K v d\mu_M(v)$ exists and that $b_M \in K$.

d) (1) Let $\mathcal{M}$ be the set of all means on $UCB(G)$, equipped with the weak* topology (where the topology on $UCB(G)$ is given by $\|.\|_\infty$). Show that, for every continuous linear functional $\Lambda : V \to \mathbb{C}$, the map $\mathcal{M} \to K$, $M \mapsto \Lambda(b_M)$ is continuous.

e) (1) If $M = \delta_x$ for some $x \in G$, calculate b_M.

f) (3) Show that, for every $M \in \mathcal{M}$ and every $x \in G$, we have $b_{M \circ L_{x^{-1}}} = x \cdot b_M$.

g) (1) Show that the action of G on K has a fixed point.

Solution.

a) Just take the restriction of a left invariant mean on $L^\infty(G)$.

b) We first show that $f \circ t \in UCB(G)$ for every $f \in \mathcal{C}(K)$, and that the linear operator $\mathcal{C}(K) \to UCB(G)$, $f \mapsto f \circ t$ is continuous. So let $f \in \mathcal{C}(K)$. Note that the function $t : G \to K$ is continuous by assumption, so $f \circ t$ is continuous. Also, we clearly have $\|f \circ t\|_\infty \leq \|f\|_\infty$. It remains to show that $f \circ t$ is left uniformly continuous. We denote by $a : G \times K \to K$ the action map. Let $\varepsilon > 0$. For every $v \in K$, there exists an open neighborhood Ω of $a^{-1}(v)$ such that $|f(x \cdot w) - f(v)| < \varepsilon$ for every $(x, w) \in \Omega$; as $(1, v) \in a^{-1}(v)$, we may assume that $\Omega = U_v \times V_v$, with U_v an open neighborhood of 1 in G and V_v an open neighborhood of v in K. As K is compact, we can find $v_1, \ldots, v_n \in K$ such that $K = \bigcup_{i=1}^n V_{v_i}$. Let $U = \bigcap_{i=1}^n U_{v_i}$. Let $x \in U$ and $v \in K$. Then there exists $i \in \{1, \ldots, n\}$ such that $v \in V_{v_i}$, and we have

$$|f(x \cdot v) - f(v)| \leq |f(x \cdot v) - f(v_i)| + |f(v_i) - f(1 \cdot v)| < 2\varepsilon.$$

So, for every $x \in U$ and every $y \in G$, we have

$$|f \circ t(xy) - f \circ t(y)| = |f(x \cdot t(y)) - f(t(y))| < 2\varepsilon.$$

This shows that $f \circ t$ is uniformly continuous.

Let M be a mean on $UCB(G)$. Composing M with the continuous linear operator $\mathcal{C}(K) \to UCB(G)$, $f \mapsto f \circ t$, we get a mean on $\mathcal{C}(K)$. By the Riesz representation theorem, there is a unique regular Borel measure μ_M on K such that $M(f \circ t) = \int_K f d\mu_M$ for every $f \in \mathcal{C}(K)$.

c) Note that $\mu(K) = \int_K 1 d\mu_K = M(\mathbb{1}_G) = 1$. The function id_K is a continuous function with compact on K, so, by problem 2 of problem set 4, its integral $b_M = \int_K v d\mu_M$ with respect to μ_M exists, and $\mu(K)^{-1} b_M = b_M$ is in closure of the convex hull of K, i.e. in K.

d) By definition of the integral, for every $\Lambda \in \mathrm{Hom}(V, \mathbb{C})$ and every mean M on $UCB(G)$, we have

$$\Lambda(b_M) = \int_G \Lambda(v) d\mu_M = M(\Lambda \circ t).$$

This is continuous in M for the weak* topology by the very definition of the weak* topology.

e) Let $x \in G$, and let $M = \delta_x$. Then, for every $f \in Cf(K)$, we have

$$\int_K f d\mu_M = M(f \circ t) = f(x \cdot v_0).$$

Taking $f = \mathrm{id}_K$, we get

$$b_M = \int_K v d\mu_M = x \cdot v_0.$$

f) Let $\mathcal{M}$ be the set of means on $UCB(G)$. Fix $M \in \mathcal{M}$. Let $x \in G$, and let $\Lambda \in \mathrm{Hom}(V, \mathbb{C})$. The map $L_{x^{-1}}$ sends $UCB(G)$ to itself, so $M \circ L_{x^{-1}}$ makes sense. For every $M \in \mathcal{M}$, using the fact that the map $K \to K$, $x \mapsto x \cdot v$ is continuous and affine, we get

$$\begin{aligned}\Lambda(x \cdot b_M) = \Lambda\left(\int_K x \cdot v d\mu_M\right) &= \int_K \Lambda(x \cdot v) d\mu_M = M(\Lambda(x \cdot t)) \\ &= M(L_{x^{-1}}(\Lambda \circ t)) \\ &= \Lambda(b_{M \circ L_{x^{-1}}}).\end{aligned}$$

As continuous linear functionals separate points (by the Hahn-Banach theorem), this implies that $x \cdot b_M = b_{M \circ L_{x^{-1}}}$.

g) Let M be an invariant mean on $UCB(G)$ (this exists by question (a)). Then, by question (f), the point $b_M \in K$ is a fixed point for the action of G.

$\square$

3. (extra credit) Let G be a locally compact group, and let dx be a left Haar measure on G. Let $f \in L^1(G)$ and $\varphi \in L^\infty(G)$.

a) (2) Show that $f * \varphi$ exists and is left uniformly continuous and bounded.

b) (2) If $\varphi \in UCB(G)$, show that the integral $\int_G f(y) L_y \varphi dy$ exists and is equal to $f * \varphi$.

Solution.

a) Let $x \in G$. Then the integral defining $f * \varphi(x)$ is

$$\int_G f(y)\varphi(y^{-1}x)d\mu(y),$$

which converges because $|f(y)\varphi(y^{-1}x)| \leq \|\varphi\|_\infty |f(y)|$ for every $y \in G$. This also shows that

$$|f * \varphi(x)| \leq \|\varphi\|_\infty \|f\|_1$$

for every $x \in G$, so $f * \varphi$ is bounded and

$$\|f * \varphi\|_\infty \leq \|\varphi\|_\infty \|f\|_1,$$

Now we show that $f * \varphi$ is left uniformly continuous. Let $x \in G$. By proposition I.4.1.3 of the notes, we have $L_x(f * \varphi) = (L_x f) * \varphi$, so

$$\|L_x(f * \varphi) - f * \varphi\|_\infty = \|(L_x f - f) * \varphi\|_\infty \leq \|L_x f - f\|_1 \|\varphi\|_\infty.$$

By proposition I.3.1.13 of the notes, this tends to 0 as x tends to 1 in G, which exactly means that $f * \varphi$ is left uniformly continuous.

b) Suppose that $\varphi \in UCB(G)$. Then the map $G \to UCB(G)$, $y \mapsto L_y \varphi$ is continuous (see remark I.1.13 of the notes), so the integral $\int_G f(y) L_y \varphi d\mu(y)$ exists in $UCB(G)$ by problem 3 of problem set 4. Let $h = \int_G f(y) L_y \varphi d\mu(y)$.

For every $g \in L^1(G)$, the map $\psi \mapsto \int_G g\psi d\mu$ is a continuous linear functional on $UCB(G)$. So, by definition of the integral, we have

$$\int_G g h d\mu = \int_{G \times G} h(x) f(y) \varphi(y^{-1}x) d\mu(x) d\mu(y) = \int_G g(f * \varphi) d\mu.$$

As the linear functionals defined by the elements of $L^1(G)$ separate points on $L^\infty(G)$, this implies that $h = f * \varphi$.

□

4. Let G be a locally compact group, and suppose that G has the fixed point property. The goal of this problem is to show that G is amenable. (You will need problem 3, so at least read it.)

   a) (3) Let $\mathcal{M}$ be the set of all means on $UCB(G)$. Show that this is a nonempty weak* compact convex subset of the continuous dual of $UCB(G)$, and that the action of G on $\mathcal{M}$ given by $x \cdot M(f) = M(L_{x^{-1}} f)$ for $x \in G$, $M \in \mathcal{M}$ and $f \in UCB(G)$, is continuous and affine. (For the weak* topology on $\mathcal{M}$.)

   b) (1) Show that there exists a left invariant mean m on $UCB(G)$. [1]

   c) (1) Show that, if $f \in L^1(G)_{1,+}$ and $\varphi \in UCB(G)$, then $m(f * \varphi) = m(\varphi)$.

   d) (2) Show that, if $f, f' \in L^1(G)_{1,+}$ and $\varphi \in L^\infty(G)$, then $m(f * \varphi) = m(f' * \varphi)$.

[1] If G is a general topological group, it is called amenable if such a mean exists. One of the things we prove in this problem is that, for G locally compact, this is equivalent to the other definition.

e) (2) Let $f_0 \in L^1(G)_{1,+}$. Show that the formula $\varphi \mapsto m(f_0 * \varphi)$ defines a mean $\widetilde{m}$ on $L^\infty(G)$, and that we have $\widetilde{m}(f * \varphi) = \widetilde{m}(\varphi)$ for every $f \in L^1(G)_{1,+}$ and every $\varphi \in L^\infty(G)$.

Let $E = \prod_{f \in L^1(G)_{1,+}} L^1(G)$. We consider two topologies on E :

- The product of the weak* topology on $L^1(G)$ (that we get by seeing $L^1(G)$ as a subspace of the continuous dual of $L^\infty(G)$). We will call this the *weak topology* on E.
- The product of the topology on $L^1(G)$ defined by the norm $\|.\|_1$. We will call this the *strong topology* on E.

f) (2) Let
$$\Sigma = \{(f * g - g)_{f \in L^1(G)_{1,+}},\ g \in L^1(G)_{1,+}\} \subset E.$$
Show that the closure of Σ in the weak topology contains 0.

g) (3) Show that the closure of Σ in the strong topology contains 0. (Hint : Any strongly continuous linear functional Λ on E can be written as $\Lambda((g_f)_{f \in L^1(G)_{1,+}}) = \sum_{f \in L^1(G)_{1,+}} \int_G g_f \varphi_f d\mu$, with the φ_f in $L^\infty(G)$ and $\varphi_f = 0$ for all but a finite number of f.)

h) (2) Let $Q \ni 1$ be a compact subset of G, $\varepsilon > 0$ and $f \in L^1(G)_{1,+}$. Show that there exists $g \in L^1(G)_{1,+}$ such that
$$\sup_{x \in Q} \|(L_x f) * g - g\|_1 \leq \varepsilon.$$

i) (1) Find a function $h \in L^1(G)_{1,+}$ such that
$$\sup_{x \in Q} \|L_x h - h\|_1 \leq 2\varepsilon.$$

j) (2) Show that there exists a left invariant mean on $L^\infty(G)$. (If you are uncomfortable with nets, you may assume that G is σ-compact, i.e. a countable union of compact subsets.)

Solution.

a) We already saw that $\mathcal{M}$ is a weak* compact convex subset of $\mathrm{Hom}(UCB(G), \mathbb{C})$ in the solution of 1(d), and $\mathcal{M}$ is not empty because it contains all the linear functionals δ_x, $x \in G$.

If $x \in G$, the morphism $\Lambda \mapsto \Lambda \circ L_{x^{-1}}$ from $\mathrm{Hom}(UCB(G), \mathbb{C})$ to itself is linear, and it clearly preserves $\mathcal{M}$, so its restriction to $\mathcal{M}$ is affine.

It remains to show that the map $G \times \mathcal{M} \to \mathcal{M}$, $(x, M) \mapsto M \circ L_{x^{-1}}$ is continuous. As we are using the weak* topology on $\mathcal{M}$, this means that, for every $\varphi \in UCB(G)$, the map $G \times \mathcal{M} \to \mathbb{C}$, $(x, M) \mapsto M(L_{x^{-1}}\varphi)$ is continuous. Fix $\varphi \in UCB(G)$, and let $\varepsilon > 0$. As φ is left uniformly continuous, there exists an open neighborhood U of 1 in G such that, for every $y \in U$, we have $\|L_{y^{-1}}\varphi - \varphi\|_\infty < \varepsilon$. Note that, for every $y \in U$ and every $x \in G$, we have
$$\|L_{x^{-1}y^{-1}}\varphi - L_{x^{-1}}\varphi\|_\infty = \|L_{y^{-1}}\varphi - \varphi\|_\infty < \varepsilon.$$

Let $(x, M) \in G \times \mathcal{M}$. Let $V = \{M' \in \mathcal{M} | |M(L_{x^{-1}}\varphi) - M'(L_{x^{-1}}\varphi)| < \varepsilon\}$. This is weak* neighborhood of M, so $Ux \times V$ is a neighborhood of (x, M) in $G \times \mathcal{M}$. If

$y \in U$ and $M' \in V$, we have

$$\begin{aligned}|M(L_{x^{-1}}\varphi) - M'(L_{(yx)^{-1}}\varphi)| &\leq |M(L_{x^{-1}}\varphi) - M'(L_{x^{-1}}\varphi)| + |M'(L_{x^{-1}}\varphi) - M'(L_{x^{-1}y^{-1}}\varphi)| \\ &< \varepsilon + \|M'\|_{op}\|L_{x^{-1}}\varphi - L_{x^{-1}y^{-1}}\varphi\|_\infty \\ &< 2\varepsilon\end{aligned}$$

(using 1(a) to see that $\|M'\|_{op} = 1$). This shows the desired result.

b) A left invariant mean on $UCB(G)$ is exactly a fixed point of the action of G on $\mathcal{M}$ defined by $x \cdot M = M \circ L_{x^{-1}}$. So the existence of such a mean follows from (a) and from the fact that G has the fixed point property.

c) By problem 3, we have $f * \varphi = \int_G f(y)L_y\varphi dy$. By problem 1, the linear functional m on $UCB(G)$ is continuous. Applying the definition of the integral and the left invariance of m, we get

$$m(f * \varphi) = \int_G f(y)m(L_y\varphi)dy = \int_G f(y)m(\varphi)dy = m(\varphi)\int_G f d\mu = m(\varphi).$$

d) Let $(\psi_U)_{U\in\mathcal{U}}$ be an approximate identity on G. Note that $\psi_U \in L^1(G)_{1,+}$ for every $U \in \mathcal{U}$. Let $\varphi \in L^\infty(G)$ and $f, f' \in L^1(G)_{1,+}$. By question 3(a), we have $\psi_U * \varphi \in UCB(G)$ for every $U \in \mathcal{U}$, so, by question (c), we get

$$m(f * \psi_U * \varphi) = m(\psi_U * \varphi) = m(f' * \psi_U * \varphi).$$

Also, by proposition I.4.1.9 of the notes, we have $\lim_{U\to\{1\}} f*\psi_U = f$ and $\lim_{U\to\{1\}} f'*\psi_U = f'$. Taking the limit as $U \to \{1\}$ in the equality above (forgetting the middle term) and using the fact that the convolution product from $L^1(G) \times L^\infty(G)$ to $UCB(G)$ is continuous in both variables (by the solution of 3(a)) and that m is continuous, we get that $m(f * \varphi) = m(f' * \varphi)$.

e) The map $\widetilde{m}$ is well-defined by 3(a), and it is clearly $\mathbb{C}$-linear. If $\varphi = \mathbb{1}_G$, then, for every $x \in G$,

$$f_0 * \varphi(x) = \int_G f_0(y)d\mu(y) = 1,$$

so $\widetilde{m}(\varphi) = m(\mathbb{1}_G) = 1$. If $\varphi \geq 0$ locally almost everywhere, then $f_0 * \varphi \geq 0$ almost everywhere, so $\widetilde{m}(\varphi) \geq 0$. This shows that $\widetilde{m}$ is a mean on $L^\infty(G)$.

Let $f \in L^1(G)_{1,+}$ and $\varphi \in L^\infty(G)$. Then

$$\widetilde{m}(f * \varphi) = m(f * f_0 * \varphi) \quad \text{and} \quad \widetilde{m}(\varphi) = m(f_0 * \varphi).$$

By (d), to show that these are equal, it suffices to show that $f * f_0 \in L^1(G)_{1,+}$. We already know that $f * f_0 \in L^1(G)$ by proposition I.4.1.2 of the notes, and the fact that $f * f_0 \geq 0$ almost everywhere is clear from the formula defining $f * f_0$. Finally, we have

$$\begin{aligned}\int_G f * f_0(x)dx = \int_{G\times G} f(y)f_0(y^{-1}x)dxdy &= \int_G f(y)\left(\int_G f_0(y^{-1}x)dx\right)dy \\ &= \int_G f(y)dy = 1.\end{aligned}$$

f) A piece of useful notation : for every $f \in L^1(G)$, we will denote by M_f the linear functional $\varphi \mapsto \int_G f\varphi d\mu$ on $L^\infty(G)$.

We want to show the following statement : For every $n \geq 1$, for all $f_1, \ldots, f_n \in L^1(G)_{1,+}$, if $U_1, \ldots, U_n$ are weak* neighborhoods of 0 in $\mathrm{Hom}(L^\infty(G), \mathbb{C})$, then there exists $g \in L^1(G)_{1,+}$ such that $M_{f_i * g} - M_g$ is in U_i for $i \in \{1, \ldots, n\}$.

If $f \in L^1(G)_{1,+}$, the map $c_f : \Lambda \mapsto \Lambda(f * (.))$ from $\mathrm{Hom}(L^\infty(G), \mathbb{C})$ to itself is weak* continuous (because $\varphi \mapsto f * \varphi$ is continuous on $L^\infty(G)$ by 3(a)). Moreover, if $\Lambda = M_g$ with $g \in L^1(G)$, then, for every $\varphi \in L^\infty(G)$, we have

$$\begin{aligned}
(c_{f'}\Lambda)(\varphi) &= \int_G g(y)(f' * \varphi)(y)dy \\
&= \int_{G \times G} g(y)\Delta(x)^{-1} f(x^{-1})\varphi(x^{-1}y)dxdy \\
&= \int_{G \times G} \Delta(x)^{-1} f(x^{-1}) g(xz)\varphi(z)dxdz \\
&= \int_G (f * g)\varphi d\mu \quad \text{(see proposition I.4.1.3 of the notes)},
\end{aligned}$$

where $f' \in L^1(G)$ is defined by $f'(x) = \Delta(x)^{-1} f(x^{-1})$. In other words, $c_{f'} M_g = M_{f*g}$.

Fix n, $f_1, \ldots, f_n$ and $U_1, \ldots, U_n$ as above. Then $U := U_1 \cap \ldots \cap U_n$ is a weak* neighborhood of 0 in $\mathrm{Hom}(L^\infty(G), \mathbb{C})$. Choose another weak* neighborhood V of 0 such that $V = -V$ and $V + V \subset U$.

Then $\widetilde{m} + V$ is a weak* neighborhood of $\widetilde{m}$, so, by 1(c) and the previous paragraph, there exists $g \in L^1(G)_{1,+}$ such that $M_g - \widetilde{m}$ and all the $M_{f_i * g} - c_{f_i'}\widetilde{m}$, $1 \leq i \leq n$, are in V. Note that $c_{f_i'}\widetilde{m} = \widetilde{m}$ by (e), so, for $1 \leq i \leq n$, we have

$$M_{f_i * g} - M_g = (M_{f_i * g} - \widetilde{m}) + (\widetilde{m} - M_g) \in V - V \subset U_i.$$

g) Note that Σ is a convex subset of E. Let $\overline{\Sigma}$ be the closure of Σ for the strong topology. If $0 \notin \overline{\Sigma}$, then, by the Hahn-Banach theorem (second geometric version), there exists a strongly continuous $\mathbb{R}$-linear functional $\Lambda' : E \to \mathbb{R}$ such that $0 = \Lambda'(0) > \sup_{x \in \Sigma} \Lambda'(x)$. As in the solution of 1(c) and 1(d), we can write $\Lambda' = \mathrm{Re}\,\Lambda$, for $\Lambda : E \to \mathbb{C}$ a strongly continuous $\mathbb{C}$-linear functional (defined by $\Lambda(x) = \Lambda'(x) + \frac{1}{i}\Lambda'(ix)$).

Now an important remark is that, as we are using the product topology on E, the direct sum $\bigoplus_{f \in L^1(G)_{1,+}} L^1(G)$ is dense in E.

For every $f_0 \in L^1(G)_{1,+}$, consider the linear functional $\Lambda_{f_0} : L^1(G) \to \mathbb{C}$ that is the composition of Λ and of the inclusion of the factor indexed by f_0 in $\prod_{f \in L^1(G)_{1,+}} L^1(G) = E$. This is a continuous linear functional on $L^1(G)$, so there exists a unique $\varphi_{f_0} \in L^\infty(G)$ such that Λ_{f_0} is integration against φ_{f_0}.

Now consider an increasing family $(X_n)_{n \geq 0}$ of subsets of $L^1(G)_{1,+}$ such that $L^1(G)_{1,+} = \bigcup_{n \geq 0} X_n$. For every $x = (g_f)_{f \in L^1(G)_{1,+}} \in E$, the sequence $((g_f)_{f \in X_n})_{n \geq 0}$ converges to x in the strong topology, so

$$\Lambda(x) = \lim_{n \to +\infty} \Lambda((g_f)_{f \in X_n}) = \lim_{n \to +\infty} \sum_{f \in X_n} \int_G g_f \varphi_f d\mu = \sum_{f \in L^1(G)_{1,+}} \int_G g_f \varphi_f d\mu.$$

As the sum converges for any $(g_f) \in E$, we must have $\varphi_f = 0$ for all but a finite number of $f \in L^1(G)_{1,+}$.

But then, if we consider any real number c such that $0 > c > \sup_{x \in \Sigma} \mathrm{Re}(\Lambda(x))$, the set $\{x \in E | \mathrm{Re}(\Lambda(x)) \leq c\}$ is weakly closed in E, hence contains the weak closure of Σ, hence contains 0 by (f), contradiction.

h) For every $x \in Q$, let U_x be a neighborhood of x in G such that, for $y \in U_x$, we have $\|L_y f - L_x f\|_1 \leq \varepsilon/2$. (See proposition I.3.1.13 of the notes.) As Q is compact, we can find $x_1, \dots, x_n \in Q$ such that $Q \subset \bigcup_{i=1}^n U_{x_i}$. By question (f), there exists $g \in L^1(G)_{1,+}$ such that, for every $i \in \{1, \dots, n\}$, we have $\|(L_{x_i} f) * g - g\|_1 \leq \varepsilon/2$.

   Let's show that this g works. Let $x \in Q$. Then there exists $i \in \{1, \dots, n\}$ such that $x \in U_{x_i}$, and we have

$$\begin{aligned}\|(L_x f) * g - g\|_1 &\leq \|(L_x f) * g - (L_{x_i} f) * g\|_1 + \|(L_{x_i} f) * g - g\|_1 \\ &\leq \|L_x f - L_{x_i} f\|_1 \|g\|_1 + \|(L_{x_i} f) * g - g\|_1 \\ &\leq \varepsilon.\end{aligned}$$

   (The second equality uses proposition I.4.1.2 of the notes.)

i) Let $h = f * g$. We have $h \in L^1(G)_{1,+}$ (see the solution of question (e)), and, as $1 \in Q$, for every $x \in Q$,

$$\|L_x h - h\|_1 \leq \|L_x h - g\|_1 + \|g - h\|_1 \leq \|(L_x f) * g - g\|_1 + \|(L_1 f) * g - g\|_1 \leq 2\varepsilon.$$

j) Suppose that G is σ-compact, and write $G = \subset_{n \geq 0}$, where each Q_n is a compact subset of G containing 1. For every $n \in \mathbb{Z}_{\geq 0}$, we can find by (h) a function $h_n \in L^1(G)_{1,+}$ such that $\sup_{x \in Q_n} \|L_x h_n - h_n\|_1 \leq 2^{-n}$. The sequence $(M_{h_n})_{n \geq 0}$ of elements of the weak* compact subset of means on $L^\infty(G)$ (we have seen in 1(c) that this set is weak* compact) has a convergent subsequence, so we may assume that it is convergent. Let $M = \lim_{n \geq 0} M_{h_n}$. We show that M is left invariant. Let $x \in G$. Then $x^{-1} \in Q_n$ for $n >> 0$, so, for every $\varphi \in L^\infty(G)$,

$$\begin{aligned}M(L_x \varphi) &= \lim_{n \to +\infty} M_{h_n}(L_x \varphi) \\ &= \lim_{n \to +\infty} \int_G h_n(y) \varphi(x^{-1} y) dy \\ &= \lim_{n \to +\infty} \int_G h_n(xy) \varphi(y) dy \\ &= \lim_{n \to +\infty} \int_G L_{x^{-1}} h_n \varphi d\mu,\end{aligned}$$

   and

$$\begin{aligned}|M(L_x \varphi) - M(\varphi)| &= \lim_{n \to +\infty} \left| \int_G (L_{x^{-1}} h_n - h_n) \varphi d\mu \right| \\ &\leq \lim_{n \to +\infty} \|L_{x^{-1}} h_n - h_n\|_1 \|\varphi\|_\infty \\ &= 0,\end{aligned}$$

   that is, $M(L_x \varphi) = M(\varphi)$.

   Assume that G is not σ-compact. Then we write $G = \bigcup_{Q \in \mathcal{Q}\{} Q$, where Q is a family of compact subsets of G such that, if $Q_1, Q_2 \in \mathcal{Q}$, then $Q_1 \cup Q_2 \in \mathcal{Q}$. That is, $\mathcal{Q}$ is a directed set for the order relation given by inclusion. For every $Q \in \mathcal{Q}$, we can find by (i) a function $h_Q \in L^1(G)_{1,+}$ such that $\sup_{x \in Q} \|L_x h_Q - h_Q\|_1 \leq (1 + \mu(Q))^{-1}$. If G is not compact, then $\mu(G) = +\infty$, so $\lim_{Q \in \mathcal{Q}} (1 + \mu(Q))^{-1} = 0$. Let M be a weak* limit point of $(M_{h_Q})_{Q \in \mathcal{Q}}$, which exists because the set of means on $L^\infty(G)$ is weak* compact. Then we see exactly as above that M is left invariant.

$\square$

5. Let G be a locally compact group. Remember problem 6 of problem set 7.

   a) (3) If G is amenable, show that the trivial representation is weakly contained in the left regular representation of G. (Hint : For all $a, b \in \mathbb{R}_{\geq 0}$, we have $|a-b|^2 \leq |a^2-b^2|$.)

   b) (3) If the trivial representation is weakly contained in the left regular representation of G, show that G is amenable. (Hint : For all $a, b \in \mathbb{C}$, prove that $||a|^2 - |b|^2| \leq |a+b||a-b|$.)

Solution.

a) Let's first prove the inequality in the hint. Let $a, b \in \mathbb{R}_{\geq 0}$. We may assume that $a \geq b$. Then

$$|a-b|^2 = (a-b)^2 = a^2 + b^2 - 2ab \leq a^2 + b^2 - 2b^2 = a^2 - b^2 = |a^2 - b^2|.$$

Suppose that G is amenable. Let K be a compact subset of G and let $c > 0$. By 4(i), there exists $h \in L^1(G)_{1,+}$ such that $\sup_{x \in K} \|L_x h - h\|_1 < c^2$. Let $f = \sqrt{h}$. Then, by the inequality above, for every $x \in K$, we have

$$\begin{aligned}
\|L_x f - f\|_2^2 &= \int_G |f(x^{-1}y) - f(y)|^2 dy \\
&\leq \int_G |h(x^{-1}y) - h(y)| dy \\
&= \|L_x h - h\|_1 \\
&< c^2,
\end{aligned}$$

so $\|L_x f - f\|_2 < c$.

By 6(d) of problem set 7, this implies that the trivial representation of G is weakly contained in the regular representation.

b) We check that the result of 4(i) holds, i.e. that, for every compact subset Q of G and every $\varepsilon > 0$, there exists $h \in L^1(G)_{1,+}$ such that $\sup_{x \in Q} \|L_x h - h\|_1 \leq \varepsilon$. Indeed, we have seen in 4(j) that this implies the existence of a left invariant mean on $L^\infty(G)$.

Let Q be a compact subset of G and $\varepsilon > 0$. By 6(c) of problem set 7, there exists $f \in L^2(G)$ such that $\|f\|_2 = 1$ and $\sup_{x \in Q} \|L_x f - f\|_2 \leq \varepsilon/2$. Let $h = |f|^2$. Then $\|h\|_1 = \|f\|_2^2 = 1$, so $h \in L^1(G)_{1,+}$. Note that, for all $a, b \in \mathbb{C}$, we have $|a^2 - b^2| \geq ||a|^2 - |b|^2|$ by the triangle inequality, so

$$|a+b|^2 |a-b|^2 = (a^2 - b^2)(\overline{a}^2 - \overline{b}^2) = |a^2 - b^2|^2 \geq ||a|^2 - |b|^2|^2.$$

Now, if $x \in Q$, we get

$$\begin{aligned}
\|L_x h - h\|_1 &= \int_G ||L_x f(y)|^2 - |f(y)|^2| dy \\
&\leq \int_G (|L_x f(y) + f(y)|)|L_x f(y) - f(y)| dy \\
&\leq \|L_x f - f\|_2 \|L_x + f\|_2 \quad \text{(Cauchy-Schwarz)} \\
&\leq \varepsilon
\end{aligned}$$

(because $\|L_x f + f\|_2 \leq \|L_x f\|_2 + \|f\|_2 = 2\|f\|_2 = 2$).

$\square$

6. Let G be an abelian locally compact group. The goal of this problem is to show that G has the fixed point property (hence is amenable).

   Let V be a locally convex topological vector space, K be a nonempty compact convex subset of V and $G \times K \to K$, $(x, v) \mapsto x \cdot v$ be an affine action of G on K.

   For every $n \in \mathbb{Z}_{\geq 0}$ and every $x \in G$, we define a continuous affine map $A_n(x) : K \to K$ by

$$A_n(x)(v) = \frac{1}{n+1} \sum_{i=0}^{n} x^i \cdot v.$$

   Let $\mathcal{G}$ be the semigroup of continuous affine maps $K \to K$ generated by all the $A_n(x)$, for $n \geq 0$ and $x \in G$. (That is, the semigroup whose elements are finite compositions of morphisms $A_n(x)$, where the semigroup operation is the composition of maps $K \to K$.)

   a) (2) Let $v \in \bigcap_{\gamma \in \mathcal{G}} \gamma(K)$. Show that v is a fixed point of the action of G. (Hint : For every continuous linear functional Λ on V and every $x \in G$, show that $\Lambda(v) = \Lambda(x \cdot v)$.)

   b) (2) For all $\gamma_1, \ldots, \gamma_n \in \mathcal{G}$, show that $\bigcap_{i=1}^{n} \gamma_i(K) \neq \varnothing$.

   c) (1) Show that G has a fixed point in K.

Solution.

   a) Let $x \in G$. Let Λ be a continuous linear functional on V. As K is compact, $C := \sup_{w \in K} |\Lambda(w)| < +\infty$. If $n \geq 0$, we have $x \in A_n(x)(K)$, so there exists $w \in K$ such that $v = A_n(x)(w)$. As the action of G is affine, this implies that

$$x \cdot v = \frac{1}{n+1} \sum_{i=0}^{n} x^{i+1} \cdot w,$$

   so $v - x \cdot v = \frac{1}{n+1}(w - x^{n+1} \cdot w)$, so $|\Lambda(v - x \cdot v)|\frac{2C}{n+1}$. As this is true for every $n \geq 0$, we have $|\Lambda(v) - \Lambda(x \cdot v)|$, i.e. $\Lambda(v) = \Lambda(x \cdot v)$. As continuous linear functional on V separate points, we finally get $x \cdot v = v$.

   b) Note that, if $x, y \in G$ and $n, m \in \mathbb{Z}_{\geq 0}$, then, for every $v \in K$,

$$\begin{aligned}
A_n(x) \circ A_m(y)(v) &= \frac{1}{n+1} \sum_{i=1}^{n} x^i \cdot A_m(y)(v) \\
&= \frac{1}{(n+1)(m+1)} \sum_{i=0}^{n} \sum_{j=0}^{m} x^i \cdot (y^j \cdot v) \\
&= \frac{1}{(n+1)(m+1)} \sum_{i=0}^{n} \sum_{j=0}^{m} y^j \cdot (x^i \cdot v) \quad \text{(because } G \text{ is commutative)} \\
&= A_m(y) \circ A_n(x)(v).
\end{aligned}$$

   This implies that the semigroup $\mathcal{G}$ is commutative.

   Now let $\gamma_1, \ldots, \gamma_n \in \mathcal{G}$. Then, for every $i \in \{1, \ldots, n\}$,

$$\gamma_i(K) \supset \gamma_i(\gamma_1 \circ \ldots \circ \gamma_{i-1} \circ \ldots \circ \gamma_n(K)) = \gamma_1 \circ \ldots \circ \gamma_n(K).$$

   So

$$\bigcap_{i=1}^{n} \gamma_i(K) \supset \gamma_1 \circ \ldots \circ \gamma_n(K) \neq \varnothing.$$

c) As K is compact and each $\gamma \in \mathcal{G}$ is continuous, the subset $\gamma(K)$ of K is compact, hence closed in K, for every $\gamma \in \mathcal{G}$. By (b), the family $(\gamma(K))_{\gamma\in\mathcal{G}}$ has the finite intersection property. By compactness of K, we have $\bigcap_{\gamma\in\mathcal{G}} \gamma(K) \neq \varnothing$. By (a), any point of this intersection is a fixed point of G on K.

$\square$

7. (extra credit)

a) (3) Let G be a group acting on a set X. Suppose that we have subgroups G_1, G_2 of G and subsets X_1, X_2 of X such that :
   - The sets X_1 and X_2 are not empty, and $X_1 \neq X_2$;
   - For every $x \in G_1 - \{1\}$, we have $x \cdot X_1 \subset X_2$;
   - For every $x \in G_2 - \{1\}$, we have $x \cdot X_2 \subset X_1$;
   - The cardinality of G_2 is at least 3.

   Show that we cannot have an equality $1 = h_1 \dots h_n$ with h_i in $G_1 - \{1\}$ for i odd, h_i in $G_2 - \{1\}$ for i even and $n \geq 1$.

b) (3) Let $a_1, a_2 \in \mathbb{C}$ such that $|a_1| \geq 2$ and $|a_2| \geq 2$. Define $x, y \in \mathbf{SL}_2(\mathbb{C})$ by

$$x = \begin{pmatrix} 1 & a_1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad y = \begin{pmatrix} 1 & 0 \\ a_2 & 1 \end{pmatrix}.$$

   Show that the subgroup of $\mathbf{SL}_2(\mathbb{C})$ generated by x and y is isomorphic to the free group on two generators. (Hint : Let $\mathbf{SL}_2(\mathbb{C})$ act on $\mathbb{C}^2$ in the usual way. Look at the subsets $\{(z_1, z_2) \in \mathbb{C}^2 | |z_1| > |z_2|\}$ and $\{(z_1, z_2) \in \mathbb{C}^2 | |z_1| < |z_2|\}$.)

c) (3) Let $G = \mathbf{SL}_2(\mathbb{R})$ with the discrete topology. Show that G is not amenable.

Solution.

a) If $G_1 = \{1\}$, the result if obvious. So we may assume $G_1 \neq \{1\}$.

   Suppose that we have $1 = h_1 \dots h_n$ with h_i in $G_1 - \{1\}$ for i odd, h_i in $G_2 - \{1\}$ for i even and $n \geq 1$.

   We first assume that n is even. As $|G_2| \geq 3$, we can find $h \in G_2 - \{1\}$ such that $h \neq h_n$. Note that $1 = hh^{-1} = hh_1 \dots (h_n h^{-1})$, with $h_n h^{-1} \in G_2 - \{1\}$. Let $g \in G_1 - \{1\}$. We also have $1 = gg^{-1} = ghh_1 \dots (h_n h^{-1})g^{-1}$. So, for every $x \in X_2$, we have

$$x = hh_1 \dots h_{n-1}(h_n h^{-1})(x) \in X_1,$$

   hence $X_2 \subset X_1$. On the other hand, for every $y \in X_1$, we get

$$y = ghh_1 \dots (h_n h^{-1})g^{-1}(y) \in X_2,$$

   so $X_1 \subset X_2$. This contradicts the fact that $X_1 \neq X_2$.

   Now suppose that n is odd. Let $h \in G_2 - \{1\}$. Then $1 = hh^{-1} = hh_1 \dots h_n h^{-1}$. So, for every $x \in X_2$, we have

$$x = hh^{-1} = hh_1 \dots h_n h^{-1}(x) \in X_1,$$

   hence $X_2 \subset X_1$. On the other hand, for every $y \in X_1$, we have

$$y = h_1 \dots h_n(y) \in X_2,$$

   so $X_1 \subset X_2$. Again, this contradicts the fact that $X_1 \neq X_2$.

b) We want to apply question (a) with $X = \mathbb{C}^2$, $X_1 = \{(z_1, z_2) \in \mathbb{C}^2 | |z_1| < |z_2|\}$, $X_2 = \{(z_1, z_2) \in \mathbb{C}^2 | |z_1| > |z_2|\}$, $G_1 = \langle x \rangle$ and $G_2 = \langle y \rangle$. We have to check that these subsets and subgroups satisfy the conditions of (a).

   Let $g \in G_1 - \{1\}$ and $(z_1, z_2) \in X_1$. We have $g = x^n$, with $n \in \mathbb{Z} - \{0\}$, so $g = \begin{pmatrix} 1 & na_1 \\ 0 & 1 \end{pmatrix}$, and $g \cdot (z_1, z_2) = (z_1 + na_1 z_2, z_2)$. Hence

$$|z_1 + na_1 z_2| \geq |n||a_1||z_2| - |z_1| \geq 2|z_2| - |z_1| > |z_2|,$$

   that is, $g \cdot (z_1, z_2) \in X_2$. (We have used the fact that $|n| \geq 1$.) The proof that $g \cdot (z_1, z_2) \in X_1$ for $g \in G_2 - \{1\}$ and $(z_1, z_2) \in X_2$ is similar.

   Let G be the subgroup of $\mathbf{SL}_2(\mathbb{C})$ generated by x and y, and let F be the free group on two generators a and b. We have a surjective morphisms of groups $\varphi : F \to G$ sending an element $a^{n_1} b^{m_1} \dots a^{n_r} b^{m_r}$ of F (with $r \geq 0$ and $n_1, m_1, \dots, n_r, m_r \in \mathbb{Z}$) to $x^{n_1} y^{m_1} \dots x^{n_r} y^{m_r} \in G$. We want to check that φ is injective. This means that its kernel is trivial, i.e. that it sends reduced words in F to nontrivial elements of G. But this property is exactly the conclusion of (a).

c) Suppose that G is amenable. Then, by problem 5, the trivial representation $\mathbb{1}$ of G on $\mathbb{C}$ is contained in its regular representation π_L. Let H be a subgroup of G. It follows immediately from the definition of weak containment that the representation $\mathbb{1}_{|H}$ of H (which is just its trivial representation) is weakly contained in $\pi_{L|H}$. Let π be the regular representation of H, and let's show that $\pi_{L|H}$ is weakly contained in π. This will imply that the trivial representation of H is contained in its regular representation.

   Let $(x_i)_{i \in I}$ be a system of representatives of the quotient $H \backslash G$; we have $G = \coprod_{i \in I} Hx_i$. Let φ be a function of positive type associated to $\pi_{L|H}$. This means that we have $f \in L^2(G)$ such that, for every $x \in H$,

$$\varphi(x) = \langle L_x f, f \rangle_{L^2(G)}.$$

   For every $i \in I$, let $f_i = f_{|Hx_i} \in L^2(G)$. Then the series $\sum_{i \in I} f_i$ converges to f in $L^2(G)$, and, if $i \neq j$, then $\langle L_x f_i, f_i \rangle_{L^2(G)} = 0$ for every $x \in H$ (because $L_x f_i$ and f_j have disjoint supports). In particular, $\|f\|_2^2 = \sum_{i \in I} \|f_i\|_2^2$. So, for every $x \in H$,

$$\varphi(x) = \sum_{i \in I} \langle L_x f_i, f_i \rangle_{L^2(G)},$$

   and this sums converges uniformly on $x \in H$ (because $|\langle L_x f_i, f_i \rangle_{L^2(G)}| \leq \|f_i\|_2^2$). For every $i \in I$, we define $g_i \in L^2(H)$ by $g_i(y) = f_i(yx_i)$. Then $\langle L_x g_i, g_i \rangle_{L^2(H)} = \langle L_x f_i, f_i \rangle_{L^2(G)}$ for every $x \in H$. So we have written φ as a limit of finite sums of functions of positive type associated to the regular representation of H, which is what we wanted.

   In summary, we have shown that, if G is amenable, then, for every subgroup H of G, the trivial representation of H is contained in its regular representation (i.e. H is also amenable). Note that we only used the fact that G is discrete so far.

   Now if $G = \mathbf{SL}_2(\mathbb{R})$, question (b) says that G has a subgroup H isomorphic to the free group on two generators (just take $a_1, a_2 \in \mathbb{R}$ in (b)). Then the result above contradicts problem 10 of problem set 7.

$\square$