MAT 449 : Problem Set 7

Due Thursday, November 8

Let G be a topological group and (π, V) be a unitary representation of G. A *matrix coefficient* of π is a function $G \to \mathbb{C}$ of the form $x \mapsto \langle \pi(x)(v), w \rangle$, with $v, w \in V$. Note that these functions are all continuous. We say that the matrix coefficient is *diagonal* if $v = w$; a diagonal matrix coefficient is a function of positive type by proposition III.2.4, and we call it a function of positive type associated to π. We say that a function of positive type is *normalized* if it is of the form $x \mapsto \langle \pi(x)(v), v \rangle$ with $\|v\| = 1$. We denote by $\mathcal{P}(\pi)$ the set of functions of positive type associated to π.

Remember also that, if G is locally compact (and μ is a left Haar measure on G), then the *left regular representation* π_L is the representation of G on $L^2(G) := L^2(G, \mu)$ given by $\pi_L(x)(f) = L_x f$, for $x \in G$ and $f \in L^2(G)$. In this problem, we'll just call π_L the regular representation of G.

1. Let (π_1, V_1), (π_2, V_2) be unitary representations of G.
   a) (3) Show that the algebraic tensor product $V_1 \otimes_{\mathbb{C}} V_2$ has a Hermitian inner product, uniquely determined by $\langle v_1 \otimes v_2, w_1 \otimes w_2 \rangle = \langle v_1, w_1 \rangle \langle v_2, w_2 \rangle$.
   b) (2) We denote the completion of $V_1 \otimes_C V_2$ for this inner form by $V_1 \hat{\otimes}_{\mathbb{C}} V_2$. Show that the formula $(x, v_1 \otimes v_2) \mapsto \pi_1(x)(v_1) \otimes \pi_2(x)(v_2)$ defines a unitary representation of G on $V_1 \hat{\otimes}_{\mathbb{C}} V_2$. (This is called the tensor product representation and usually denoted by $\pi_1 \otimes \pi_2$.)
   c) (2) If V_1 and V_2 are finite-dimensional, show that, for every $x \in G$, we have
   $$\mathrm{Tr}(\pi_1 \otimes \pi_2(x)) = \mathrm{Tr}(\pi_1(x)) \,\mathrm{Tr}(\pi_2(x)).$$

Solution.

a) As pure tensors span $V_1 \otimes_{\mathbb{C}} V_2$, there is at most one sesquilinear form B on $V_1 \otimes_{\mathbb{C}} V_2$ such that $B(v_1 \otimes v_2, w_1 \otimes w_2) = \langle v_1, w_1 \rangle \langle v_2, w_2 \rangle$. Let's show that such a form exists. Let $w_1 \in V_1$ and $w_2 \in V_2$. Then the map on $V_1 \times V_2 \to \mathbb{C}$, $(v_1, v_2) \mapsto \langle v_1, w_1 \rangle \langle v_2, w_2 \rangle$ is a bilinear form, hence it corresponds to a unique linear form on $V_1 \otimes_{\mathbb{C}} V_2$, say B_{w_1, w_2}. Next, the map on $V_1 \times V_2$ sending (w_1, w_2) to the antilinear form $v \mapsto \overline{B_{w_1, w_2}(v)}$ is bilinear, so it corresponds to a unique linear functional T on $V_1 \otimes_{\mathbb{C}} V_2$. Finally, the map $B : (V_1 \otimes_{\mathbb{C}} V_2) \times (V_1 \otimes_{\mathbb{C}} V_2) \to \mathbb{C}$ sending (v, w) to $\overline{T(w)(v)}$ is linear in v and antilinear in w, so it is a sesquilinear form, and it sends pure tensors where we want by definition.

   Now we show that B is Hermitian, i.e. that $B(w, v) = \overline{B(v, w)}$ for all $v, w \in V_1 \otimes_{\mathbb{C}} V_2$. As B is sesquilinear, it suffices to check this property for v and w pure tensors, but then it follows immediately from the analogous property of the inner products of V_1 and V_2.

Finally, we show that B is definite positive. Let $v \in V_1 \otimes_{\mathbb{C}} V_2$, and write $v = \sum_{i=1}^n v_{1,i} \otimes v_{2,i}$, $v_{1,i} \in V_1$ and $v_{2,i} \in V_2$. Then v is in $V_1' \otimes_{\mathbb{C}} V_2'$, where $V_1' = \mathrm{Span}(v_{1,1}, \dots, v_{1,n})$ and $V_2' = \mathrm{Span}(v_{2,1}, \dots, v_{2,n})$. So we may replace V_1 and V_2 by V_1' and V_2', and so may assume that V_1 and V_2 are finite-dimensional. If this is the case, let $(e_1, \dots, e_r)$ (resp. $(e_1', \dots, e_s')$) be an orthonormal basis of V_1 (resp. V_2). Then $(e_i \otimes e_j')_{1 \leq i \leq r, 1 \leq j \leq s}$ is a basis of $V_1 \otimes_{\mathbb{C}} V_2$, and it is clear from the definition of the Hermitian form B on $V_1 \otimes_{\mathbb{C}} V_2$ that it is an orthonormal basis for B . But the existence of an orthonomal basis forces the form to be positive definite (if $v = \sum_{i=1}^r \sum_{j=1}^s a_{ij} e_i \otimes e_j'$, then $B(v, v) = \sum_{i,j} |a_{ij}|^2$).

b) First note that, if $v_1 \in V_1$ and $v_2 \in V_2$, then we have $\|v_1 \otimes v_2\| = \|v_1\| \|v_2\|$.

Let $x \in G$. Then the map $V_1 \times V_2 \to V_1 \otimes_{\mathbb{C}} V_2$ sending (v_1, v_2) to $\pi_1(x)(v_1) \otimes \pi_2(x)(v_2)$ is bilinear, so it induces a $\mathbb{C}$-linear map $\pi_1 \otimes \pi_2(x)$ from $V_1 \otimes_{\mathbb{C}} V_2$ to itself. We show that this map is an isometry (hence continuous). Let $(e_i)_{i \in I}$ (resp. $(f_j)_{j \in J}$) be a Hilbert basis of V_1 (resp. V_2). If $v_1 \in V_1$ and $v_2 \in V_2$, we can write $v_1 = \sum_{i \in I} a_i e_i$ and $v_2 = \sum_{j \in J} b_j f_j$, and then, by the remark above, the series $\sum_{i,j} a_i b_i e_i \otimes f_j$ converges to $v_1 \otimes v_2$ in $V_1 \otimes_{\mathbb{C}} V_2$. As every element of $V_1 \otimes_{\mathbb{C}} V_2$ is a finite sum of elements of the form $v_1 \otimes v_2$, this proves that every element v of $V_1 \otimes_{\mathbb{C}} V_2$ can be written as the limit of a convergent series $\sum_{i \in I, j \in J} a_i b_j e_i \otimes f_j$, with $a_i, b_j \in \mathbb{C}$. Then $\pi_1 \otimes \pi_2(x)(v) = \sum_{i,j} a_i b_j \pi_1(x)(e_i) \otimes \pi_2(x)(f_j)$. As the families $(e_i \otimes f_j)$ and $(\pi_1(x)(e_i) \otimes \pi_2(x)(f_j))$ are both orthogonal in $V_1 \otimes_{\mathbb{C}} V_2$, we get $\|v\|^2 = \sum_{i,j} |a_i|^2 |b_j|^2 = \|\pi_1 \otimes \pi_2(x)(v)\|^2$.

As the map $\pi_1 \otimes \pi_2(x)$ is continuous, it extends to a continuous endormophism of $V_1 \widehat{\otimes}_{\mathbb{C}} V_2$, which is also an isometry and will still be denoted by $\pi_1 \otimes \pi_2(x)$.

If y is another element of G, the endomorphisms $\pi_1 \otimes \pi_2(xy)$ and $(\pi_1 \otimes \pi_2(x)) \circ (\pi_1 \otimes \pi_2(y))$ of $V_1 \widehat{\otimes}_{\mathbb{C}} V_2$ are equal on pure tensors, hence they are equal because pure tensors generate a dense subspace of $V_1 \widehat{\otimes}_{\mathbb{C}} V_2$.

To check that this defines a unitary representation of G on $V_1 \widehat{\otimes}_{\mathbb{C}} V_2$, we still need to check that, for every $v \in V_1 \widehat{\otimes}_{\mathbb{C}} V_2$, the map $G \to V_1 \widehat{\otimes}_{\mathbb{C}} V_2$, $x \mapsto \pi_1 \otimes \pi_2(x)(v)$ is continuous. This is true for v a pure tensor : if $v = v_1 \otimes v_2$, then, for $x, y \in G$, we have

$$\begin{aligned}
\|(\pi_1 \otimes \pi_2(x) - \pi_1 \otimes \pi_2(y))(v)\| &\leq \|\pi_1(x)(v_1) \otimes (\pi_2(x) - \pi_2(y))(v_2)\| \\
&\quad + \|(\pi_1(x) - \pi_1(y))(v_1) \otimes \pi_2(y)(v_2)\| \\
&= \|v_1\| \|(\pi_2(x) - \pi_2(y))(v_2)\| + \|(\pi_1(x) - \pi_1(y))(v_1)\| \|v_2\|,
\end{aligned}$$

which implies the result. So it is still true for a finite sum of pure tensors, and then a standard shows that it is true for every element of $V_1 \widehat{\otimes}_{\mathbb{C}} V_2$.

c) Let $(e_1, \dots, e_n)$ (resp. $(f_1, \dots, f_j)$) be an orthonormal basis of V_1 (resp. V_2). Then $(e_i \otimes f_j)_{1 \leq i \leq n, 1 \leq j \leq m}$ is an orthonormal basis of $V_1 \otimes_{\mathbb{C}} V_2$. Let $x \in G$. Then

$$\mathrm{Tr}(\pi_1(x)) = \sum_{i=1}^n \langle \pi_1(x)(e_i), e_i \rangle$$

and

$$\mathrm{Tr}(\pi_2(x)) = \sum_{j=1}^m \langle \pi_2(x)(f_j), f_j \rangle,$$

so

$$\begin{aligned}
\operatorname{Tr}(\pi_1 \otimes \pi_2(x)) &= \sum_{i=1}^{n} \sum_{j=1}^{m} \langle \pi_1 \otimes \pi_2(x)(e_i \otimes f_j), e_i \otimes f_j \rangle \\
&= \left(\sum_{i=1}^{n} \langle \pi_1(x)(e_i), e_i \rangle \right) \left(\sum_{j=1}^{m} \langle \pi_2(x)(f_j), f_j \rangle \right) \\
&= \operatorname{Tr}(\pi_1(x)) \operatorname{Tr}(\pi_2(x)).
\end{aligned}$$

□

2. Let G be a discrete group, and let $\varphi = \mathbb{1}_{\{1\}}$.
   a) (1) Show that φ is a function of positive type on G.
   b) (2) Show that V_φ is equivalent to the regular representation of G.
   c) (2, extra credit) If $G \neq \{1\}$, show that the representations V_φ and $V_\varphi \widehat{\otimes}_{\mathbb{C}} V_\varphi$ are not equivalent. (You can use problem 12.)

Solution.

a) The counting measure μ is a left Haar measure on G, so we use this measure. For every $f \in L^1(G)$, we have $\int_G f \varphi d\mu = f(1)$. So

$$\int_G (f^* * f) \varphi d\mu = (f^* * f)(1) = \sum_{y \in G} \overline{f(y^{-1})} f(y^{-1}) \in \mathbb{R}_{\geq 0}.$$

b) For all $f, g \in L^1(G)$, we have

$$\begin{aligned}
\langle f, g \rangle_\varphi &= \int_G (g^* * f) \varphi d\mu \\
&= (g^* * f)(1) \\
&= \sum_{y \in G} \overline{g(y^{-1})} f(y^{-1}) \\
&= \langle f, g \rangle_{L^2(G)}.
\end{aligned}$$

So the kernel of $\langle .,. \rangle_\varphi$ is equal to $\{0\}$, and the Hilbert space V_φ is the completion of $L^1(G)$ for the norm $\|.\|_2$, that is, $L^2(G)$. The action of G on V_φ is the extension by continuity of its action by left translations on $L^1(G)$, so we get the action of G by left translations on $L^2(G)$.

c) I couldn't solve this question in general. It is easy for dimension reasons if G is finite, and not too hard if G is uncountable : By problem 12, the representation $V_\varphi \widehat{\otimes}_{\mathbb{C}} V_\varphi$ is equivalent to $W := \overline{\bigoplus_{i \in G} V_\varphi}$. Suppose that we have an isomorphism $T : V_\varphi \to W$. Then we can write $T(\delta_1)$ as a convergent infinite sum $\sum_{i \in G} f_i$, with $f_i \in V_\varphi$. As δ_1 is a cyclic vector in V_φ, we see easily that each f_i must be a cyclic vector in V_φ, hence nonzero. But the sum can only converge if at most countably many of the f_i are nonzero, so G must be countable.

□

3. Let G be a locally compact group.

a) (1) If $f, g \in \mathcal{C}_c(G)$, show that $f * g \in \mathcal{C}_c(G)$.

b) (3) Show that every matrix coefficient of the regular representation of G vanishes at ∞.

c) (2) Suppose that G is not compact. If (π, V) is a finite-dimensional unitary representation of G, show that it has a matrix coefficient that does not vanish at ∞.

d) (1) If G is not compact, show that its regular representation has no finite-dimensional subrepresentation.

Solution.

a) First, we know that $f * g$ exists, because f and g are in $L^1(G)$. If $x, x' \in G$, then

$$\begin{aligned} |f * g(x) - f * g(x')| &= \left| \int_G f(y)(g(y^{-1}x) - g(y^{-1}x'))dy \right| \\ &\leq \|f\|_1 \operatorname{supp}_{y \in \operatorname{supp}(f)} |g(y^{-1}x) - g(y^{-1}x')|. \end{aligned}$$

As g is right uniformly continuous (see proposition I.1.12 of the notes), this tends to 0 as x' tends to x, so $f * g$ is continuous.

Let $x \in G$ such that $f * g(x) \neq 0$. We have

$$f * g(x) = \int_G f(y) g(y^{-1}x) dy,$$

so there exists $y \in \operatorname{supp}(f)$ such that $y^{-1}x \in \operatorname{supp}(g)$. In other words, $x \in \operatorname{supp}(f)\operatorname{supp}(g)$. As both $\operatorname{supp}(f)$ and $\operatorname{supp}(g)$ are compact, their product $\operatorname{supp}(f)\operatorname{supp}(g)$ is also compact, so $f * g$ has compact support.

b) Remember that, if $f \in L^2(G)$, we define $\widetilde{f} : G \to \mathbb{C}$ by $\widetilde{f}(x) = \overline{f(x^{-1})}$.

Every matrix coefficient of the left regular representation of G is of the form

$$x \mapsto \langle L_{x^{-1}} f, g \rangle_{L^2(G)},$$

with $f, g \in L^2(G)$. We have (see proposition III.2.4(iii) of the notes)

$$\begin{aligned} \langle L_{x^{-1}} f, g \rangle_{L^2(G)} &= \int_G f(x^{-1}y) \overline{g(y)} dy \\ &= \int_G \overline{\widetilde{f}(y^{-1}x)} \overline{g(y)} dy \\ &= \overline{g} * \overline{\widetilde{f}}(x). \end{aligned}$$

Moreover, if $f', g' \in L^2(G)$, then we have (using the Cauchy-Schwarz inequality)

$$\begin{aligned} |\langle L_{x^{-1}} f, g \rangle_{L^2(G)} - \langle L_{x^{-1}} f', g \rangle_{L^2(G)}| &\leq \int_G |g(y)| |f(x^{-1}y - f'(x^{-1}y)| dy \\ &\leq \|g\|_2 \|L_{x^{-1}}(f - f')\|_2 \\ &= \|g\|_2 \|f - f'\|_2 \end{aligned}$$

and

$$\begin{aligned} |\langle L_{x^{-1}} f, g \rangle_{L^2(G)} - \langle L_{x^{-1}} f, g' \rangle_{L^2(G)}| &\leq \int_G |g(y) - g'(y)| |f(x^{-1}y| dy \\ &\leq \|g - g'\|_2 \|L_{x^{-1}} f\|_2 \\ &= \|g - g'\|_2 \|f'\|_2. \end{aligned}$$

Suppose that $f, g \in \mathcal{C}_c(G)$. Then $\overline{g}, \overline{\widetilde{f}} \in \mathcal{C}_c(G)$, so, by question (a), $\overline{g} * \overline{\widetilde{f}} \in \mathcal{C}_c(G)$, and in particular this function vanishes at ∞.

In the general case, let $\varepsilon > 0$ and let $f', g' \in \mathcal{C}_c(G)$ such that $\|f - f'\|_2 \leq \varepsilon$ and $\|g - g'\|_2 \leq \varepsilon$. Then, by the two inequality above, we have, for every $x \in G$,

$$|\overline{g} * \overline{\widetilde{f}}(x) - \overline{g}' * \overline{\widetilde{f}}'(x)| \leq \varepsilon(\|f\|_2 + \|g\|_2).$$

But we have just seen that $\overline{g}' * \overline{\widetilde{f}}'$ has compact support, so there exists a compact subset K of G such that, for every $x \notin K$¡ we have

$$|\overline{g} * \overline{\widetilde{f}}(x)| \leq \varepsilon(\|f\|_2 + \|g\|_2).$$

This shows that the matrix coefficient $\overline{g} * \overline{\widetilde{f}}$ vanishes at ∞.

c) Let $(e_1, \ldots, e_n)$ be an orthonormal basis of V. For every $i \in \{1, \ldots, n\}$, let f_i be the matrix coefficient $x \mapsto \langle \pi(x)(e_1), e_i \rangle$. Then we have, for every $x \in G$,

$$\sum_{i=1}^{n} |f_i(x)|^2 = \sum_{i=1}^{n} |\langle \pi(x)(e_1), e_i \rangle|^2 = \|\pi(x)(e_1)\|^2 = 1.$$

This shows that at least one of the f_i does not vanish at ∞.

d) This follows directly from (c) and (d).

□

4. (extra credit, 3) Let V be a locally convex topological $\mathbb{C}$-vector space, K be a compact convex subset of V, and $F \subset K$ be such that K is the closure of the convex hull of F. Show that every extremal point of K is in the closure of F. (This is known as *Milman's theorem.*)

Solution. If $0 \in X$ is an open convex subset of V, then we have $\overline{X} \subset 2X$. Indeed, if $p : V \to \mathbb{R}_{\geq 0}$ be the gauge of X (see lemma B.3.8 of the notes), then $X = \{v \in V | p(v) < 1\}$, so

$$\overline{X} \subset \{v \in V | p(v) \leq 1\} \subset \{v \in V | p(v) < 2\} = 2X.$$

Let v be an extremal of K, and suppose that $v \notin \overline{F}$. Then we can find a convex neighborhood X of 0 in V such that $X = -X$ and $(v + X) \cap \overline{F} = \varnothing$. Replacing X by $\frac{1}{2}X$, we may assume that we have $(v + \overline{X}) \cap \overline{F} = \varnothing$.

As $\overline{F}$ is compact (as a closed subset of K), we can find $x_1, \ldots, x_n \in F$ such that $\overline{F} \subset \bigcup_{i=1}^{n}(x_i + X)$. For every $i \in \{1, \ldots, n\}$, let K_i be the closure of the convex hull of $\overline{F} \cap (x_i + X)$; this is a compact convex subset of V (it is compact because it is closed in K). As K is the closure of the convex hull of $\overline{F}$, we have $K \supset K_1 \cup \ldots \cup K_n$, so K contains the convex hull L of $K_1 \cup \ldots \cup K_n$. Let's show that $K = L$. As $L \supset F$ and L is convex, it suffices to show that L is compact. Let

$$S = \{(x_1, \ldots, x_n) \in [0, 1]^n | x_1 + \ldots + x_n = 1\},$$

and consider the function

$$f : S \times K_1 \times \ldots \times K_n \to L$$

sending $((x_1, \ldots, x_n), v_1, \ldots, v_n)$ to $\sum_{i=1}^{m} x_i v_i$. This map is continuous, so its image is compact. If we show that this image is convex, then it will equal to L by definition of L, and we will be done. So let $a = ((x_1, \dots, x_n), v_1, \dots, v_n)$ and $a' = ((x'_1, \dots, x'_n), v'_1, \dots, v'_n)$ be elements of $S \times K_1 \times \dots\dots K_n$ and $t \in [0, 1]$. Then

$$tf(a) + (1-t)f(a') = \sum_{i=1}^{n} (tx_i v_i + (1-t)x'_i v'_i)$$

Let $i \in \{1, \dots, n\}$. If $tx_i + (1-t)x_i \neq 0$, we set $y_i = tx_i + (1-t)x_i$ and $w_i = \frac{1}{y_i}(tx_i v_i + (1-t)x'_i v'_i)$. Otherwise, we set $w_i = v_i$ and $y_i = 0$. Then we have $w_i \in K_i$ for every i because K_i is convex, $y_i \geq 0$ for every i, and

$$\sum_{i=1}^{n} y_i = t \sum_{i=1}^{n} x_i + (1-t) \sum_{i=1}^{n} x'_i = 1.$$

So

$$tf(a) + (1-t)f(a') = f((y_1 \dots, y_n), w_1, \dots, w_n)$$

is in the image of f, and we are done.

Now we derive a contradiction. As $K = L$, we can write $v = \sum_{i=1}^{n} t_i v_i$, with $(t_1, \dots, t_n) \in S$ and $v_i \in K_i$ for every i. As v is extremal in K, there exists $i \in \{1, \dots, n\}$ such that $v = v_i$. But then $v \in K_i \subset (x_i + \overline{X})$ (because K_i is contained in the closure of the convex hull of $x_i + X$, and this is $x_i + \overline{X}$ because X is convex). As $x_i \in F$ and $X = -X$, this implies that $x_i \in (v + \overline{V}) \cap F$, contradicting the choice of X.

□

We will see soon that, if G is compact, then the regular representation of G contains all the irreducible representations of G (which are all finite-dimensional); in fact, it is the closure of the direct sum of all its irreducible subrepresentations. On the other hand, if G is abelian, then its regular representation is the direct integral of all the irreducible representations of G (which are all 1-dimensional), even though it does not contain any of them if G is not compact. We will not rigorously define direct integrals here, but we will introduce a weaker definition of containment, for which irreducible representations of an abelian locally compact group are contained in the regular representation, and start studying it.

Let (π, V) and (π', V') be unitary representations of G. We say that π is *weakly contained* in π', and write $\pi \prec \pi'$, if $\mathcal{P}(\pi)$ is contained in the closure of the set of finite sums of elements of $\mathcal{P}(\pi')$ for the topology of convergence on compact subsets of G. In other words, $\pi \prec \pi'$ if, for every $v \in V$, for every $K \subset G$ compact and every $c > 0$, there exist $v'_1, \dots, v'_n \in V'$ such that

$$\sup_{x \in K} |\langle \pi(x)(v), v \rangle - \sum_{i=1}^{n} \langle \pi'(x)(v'_i), v'_i \rangle| < c.$$

5. Let (π, V) and (π', V') be unitary representations of G. Let $C \subset V$ such that $\mathrm{Span}(\pi(x)(v),\ x \in G,\ v \in C)$ is dense in V.[1] Suppose that every function $x \mapsto \langle \pi(x)(v), v \rangle$, for $v \in C$, is in the closure of the set of finite sums of elements of $\mathcal{P}(\pi')$ (still for the topology of convergence on compact subsets of G). The goal of this problem is to show that this implies $\pi \prec \pi'$.

   Let X be the set of $v \in V$ such that $x \mapsto \langle \pi(x)(v), v \rangle$ is in the closure of the set of finite sums of elements of $\mathcal{P}(\pi')$ (for the same topology as above).

   a) (1) Show that X is stable by all the $\pi(x)$, $x \in G$, and under scalar multiplication.

[1] For example, if V is cyclic, C could just contain a cyclic vector for V.

b) (1) If $v \in X$ and $x_1, x_2 \in G$, show that $\pi(x_1)(v) + \pi(x_2)(v) \in X$.

c) (1) Show that X is closed in V.

d) (1) If $v \in X$, show that the smallest closed G-invariant subspace of V containing v is contained in X.

e) Let $v_1, v_2 \in X$, and let W_1 (resp. W_2) be the smallest closed G-invariant subspace of V containing v_1 (resp. v_2). Let $W = \overline{W_1 + W_2}$, and denote by $T : W \to W_1^\perp$ the orthogonal projection, where we take the orthogonal complement of W_1 in W.

   i. (1) Show that T is G-equivariant and that $T(W_2)$ is dense in $W_1^\perp$.

   ii. (2) Show that $W_1^\perp \subset X$.

   iii. (1) Show that $v_1 + v_2 \in X$. (Hint : Use $T(v_1 + v_2)$ and $(v_1 + v_2) - T(v_1 + v_2)$.)

f) (1) Show that $\pi \prec \pi'$.

Solution.

a) For every $v \in V$ (resp. $v \in V'$), we write φ_v for the matrix coefficient $x \mapsto \langle \pi(x)(v), v\rangle$ (resp. $x \mapsto \langle \pi'(x)(v), v\rangle$). We also write $\sum \mathcal{P}(\pi')$ for the set of finite sums of elements of $\mathcal{P}(\pi')$.

   Let $v \in X$, let $y \in G$ and let $\lambda \in \mathbb{C}$. We want to show that $\pi(y)(v)$ and λv are in X, that is, that $\varphi_{\pi(y)(v)}$ and $\varphi_{\lambda v}$ are in $\overline{\sum \mathcal{P}(\pi')}$. If $\lambda = 0$, the conclusion is obvious for λv (note that 0 is a matrix coefficient of every representation of G), so we may assume that $\lambda \neq 0$. Let K be a compact subset of G $\varepsilon > 0$. Choose $v_1', \dots, v_n'$ such that $\sup_{x \in K \cup y^{-1}Ky} |\varphi_v(x) - \sum_{i=1}^n \varphi_{v_i'}(x)| \leq \min(\varepsilon, |\lambda|^{-2}\varepsilon)$. Then, for every $x \in K$, we have

$$\begin{aligned}
|\varphi_{\pi(y)v}(x) - \sum_{i=1}^n \varphi_{\pi'(y)(v_i')}(x)| &= |\langle \pi(xy)(v), \pi(y)(v)\rangle - \sum_{i=1}^n \langle \pi'(xy)(v_i'), \pi'(y)(v_i')\rangle| \\
&= |\langle \pi(y^{-1}xy)(v), v\rangle - \sum_{i=1}^n \langle \pi'(y^{-1}xy)(v_i'), v_i'\rangle| \\
&= |\varphi_v(y^{-1}xy) - \sum_{i=1}^n \varphi_{v_i'}(y^{-1}xy)| \\
&\leq \varepsilon
\end{aligned}$$

   and

$$|\varphi_{\lambda v}(x) - \sum_{i=1}^n \varphi_{\lambda v_i'}(x)| = |\lambda|^2 |\varphi_v(x) - \sum_{i=1}^n \varphi_{v_i'}(x)| \leq \varepsilon.$$

   So $\varphi_{\pi(y)(v)}$ and $\varphi_{\lambda v}$ are in the closure of $\mathcal{P}(\pi')$.

b) Let $v \in X$ and let $x_1, x_2 \in G$. For every $y \in G$, we have

$$\begin{aligned}
\varphi_{\pi(x_1)(v)+\pi(x_2)(v)}(y) &= \langle \pi(y)(\pi(x_1)(v) + \pi(x_2)(v)), \pi(x_1)(v) + \pi(x_2)(v)\rangle \\
&= \langle \pi(x_1^{-1}yx_1)(v), v\rangle + \langle \pi(x_2^{-1}yx_1)(v), v\rangle + \langle \pi(x_1^{-1}yx_2)(v), v\rangle \\
&\quad + \langle \pi(x_2^{-1}yx_2)(v), v\rangle.
\end{aligned}$$

   In other words,

$$\varphi_{\pi(x_1)(v)+\pi(x_2)(v)} = \sum_{i,j=1}^2 L_{x_i} R_{x_j} \varphi_v.$$

Let K be a compact subset of G and let $\varepsilon > 0$. Choose $v'_1, \dots, v'_n \in V'$ such that

$$\sup_{x \in \bigcup_{i,j=1}^2 x_i^{-1} K x_j} |\varphi_v(x) - \sum_{i=1}^n \varphi_{v'_i}(x)| \leq \varepsilon.$$

Then, by the calculation above (and its analogue for the functions $\varphi_{v'_i}$), we have, for every $x \in K$,

$$|\varphi_{\pi(x_1)(v)+\pi(x_2)(v)}(x) - \sum_{i=1}^n \varphi_{\pi(x_1)(v_i)+\pi(x_2)(v_i)}(x)| \leq \varepsilon.$$

This shows that $\varphi_{\pi(x_1)(v)+\pi(x_2)(v)}$ is in the closure of $\mathcal{P}(\pi')$, that is, that $\pi(x_1)(v) + \pi(x_2)(v) \in X$.

c) It suffices to show that the map $V \to \mathcal{P}(\pi)$, $v \mapsto \varphi_v$ is continuous if we use the topology of compact convergence on $\mathcal{P}(\pi)$. Let $v, v' \in V$. Then, for every $x \in G$,

$$\begin{aligned}
|\varphi_v(x) - \varphi_{v'}(x)| &= |\langle \pi(x)(v), v\rangle - \langle \pi(x)(v'), v'\rangle \\
&\leq |\langle \pi(x)(v - v'), v\rangle| + |\langle \pi(x)(v'), v - v'\rangle| \\
&\leq \|v - v'\|\|v\| + \|v'\|\|v - v'\|.
\end{aligned}$$

So the map $v \mapsto \varphi_v$ is continuous even for the topology on $\mathcal{P}(\pi)$ given by $\|.\|_\infty$.

d) Let $v \in X$. By (a) and (b), for every $n \geq 1$ and all $\lambda_1, \dots, \lambda_n \in \mathbb{C}$ and $x_1, \dots, x_n$, we have $\sum_{i=1}^n \lambda_i \pi(x_i)(v) \in X$. So the smallest G-invariant subspace of V containing v (i.e. $\sum_{x \in G} \pi(x)(\mathbb{C}v)$) is contained in X. The conclusion now follows from (c).

e)
   i. As W_1 is G-invariant, the operator T is G-equivariant by lemma I.3.4.3 of the notes. As $W = \overline{W_1 + W_2}$, the image of $W_1 + W_2$ by T is dense in $\operatorname{Im}(T) = W_1^\perp$. As $\operatorname{Ker}(T) = W_1$, we have $T(W_1 + W_2) = T(W_2)$, so $T(W_2)$ is dense in $W_1^\perp$.

   ii. As $W = W_1 \oplus W_1^\perp$, we deduce that $T(W_1^\perp \cap W_2) = W_1^\perp \cap W_2$ is dense in $W_1^\perp$. As $W_2 \subset X$, question (c) implies that $W_1^\perp \subset X$.

   iii. We set $v = T(v_1 + v_2)$ and $w = v_1 + v_2 - v$. Then $v \in W_1^\perp \subset X$ and $w \in \operatorname{Ker}(T) = W_1 \subset X$, so $v, w \in X$. On other hand, for every $x \in G$, we have

$$\begin{aligned}
\varphi_{v_1+v_2}(x) &= \langle \pi(x)(v_1 + v_2), v_1 + v_2\rangle \\
&= \pi(x)(v + w), v + w\rangle \\
&= \varphi_v(x) + \varphi_w(x).
\end{aligned}$$

   As $\overline{\mathcal{P}(\pi')}$ is stable by sums, this implies that $v_1 + v_2 \in X$.

f) By (a), (c) and (e), the set X is closed G-invariant subspace of V, so it is equal to V by the hypothesis on C. This means that $\pi \prec \pi'$.

□

6. Let (π, V) and (π', V') be two unitary representations of G such that $\pi \prec \pi'$. Let C be the closure in the weak* topology on $L^\infty(G)$ of the convex hull of the set of normalized functions of positive type associated to π'.

   a) (1) Show that every normalized function of positive type associated to π is in C.

   b) (3) If π is irreducible, show that every normalized function of positive type associated to π is a limit in the topology of convergence on compact subsets of G of normalized functions of positive type associated to π'. (Hint : problem 4.)

c) (2) If π is the trivial representation of G, show that, for every compact subset K of G and every $c > 0$, there exists $v' \in V'$ such that $\|v'\| = 1$ and that

$$\sup_{x \in K} \|\pi'(x)(v') - v'\| < c.$$

d) (1) Conversely, suppose that, for every compact subset K of G and every $c > 0$, there exists $v' \in V'$ such that $\|v'\| = 1$ and that

$$\sup_{x \in K} \|\pi'(x)(v') - v'\| < c.$$

Show that the trivial representation is weakly contained in π'.

Solution.

a) Let φ be a normalized function of positive type associated to π. Let $f \in L^1(G)$ and $\varepsilon > 0$. We want to find a convex combination ψ of normalized functions of positive type associated to π' such that $\left|\int_G f(\varphi - \psi)d\mu\right| \leq \varepsilon$. Pick $\delta > 0$; we will see later how small it needs to be. Let $K \ni 1$ be a compact subset of G such that $\int_{G-K} |f|d\mu \leq \delta$. As $\pi \prec \pi'$, we can find $v_1, \ldots, v_n \in V'$ such that $\sup_{x \in K} |\varphi(x) - \sum_{i=1}^n \varphi_{v_i}(x)| \leq \delta$. In particular, evaluating at 1, we get $|1 - \sum_{i=1}^n \|v_i\|^2| \leq \delta$. Let $c_i = \|v\|_i^2$, $c = c_1 + \ldots + c_n$, $\varphi_i = \frac{1}{c_i}\varphi_{v_i} = \varphi_{\frac{1}{\|v_i\|}\varphi_{v_i}}$ and $\psi = \frac{1}{c}\sum_{i=1}^n \varphi_{v_i} = \frac{1}{c}\sum_{i=1}^n c_i\varphi_i$. Then $\varphi_1, \ldots, \varphi_n$ are normalized functions of positive type associated to π', and ψ is a convex combination of $\varphi_1, \ldots, \varphi_n$. In particular, $\|\psi\|_\infty \leq 1 = \|\varphi\|_\infty$.

For every $x \in K$, we have

$$\begin{aligned}
|\varphi(x) - \psi(x)| &\leq |\varphi(x) - \sum_{i=1}^n \varphi_{v_i}(x)| + |1 - c||\psi(x)| \\
&\leq 2\delta.
\end{aligned}$$

So

$$\begin{aligned}
\left|\int_G f(\varphi - \psi)d\mu\right| &\leq \sup_{x \in K} |\varphi(x) - \psi(x)| \int_K |f|d\mu + \sup_{x \in G-K} |\varphi(x) - \psi(x)| \int_{G-K} |f|d\mu \\
&\leq 2\delta\|f\|_1 + 2\delta.
\end{aligned}$$

We can make this $\leq \varepsilon$ by taking δ small enough.

b) Let F be the set of normalized functions of positive type associated to π', and let K be the weak* closure of its convex hull. Then F is contained in the convex set $\mathcal{P}_1$ of all normalized functions of positive type on G, so $K \subset \mathcal{P}_1$. Let φ be a normalized function of positive type associated to π. By question (a), we have $\varphi \in K$. By theorem III.3.2 of the notes, the function φ is extremal in $\mathcal{P}_1$, hence also in K. By problem 4, this implies that φ is in the closure of F in the weak* topology. But F and φ are in $\mathcal{P}_1$, and the weak* topology on $\mathcal{P}_1$ coincides with the topology of convergence on compact subsets of G (by Raikov's theorem, i.e. theorem III.4.3 of the notes), so φ is also in the closure of F in the topology of convergence on compact subsets of G.

c) As π is the trivial representation, the only normalized function of positive type associated to π is the constant function 1. By question (c), there exists $v' \in V'$ such that $\|v'\| = 1$ and

$$\sup_{x \in K} |1 - \langle \pi'(x)(v'), v' \rangle| \leq c^2/3.$$

Let $x \in G$. Then

$$\|\pi'(x)(v')-v'\|^2 = \|\pi'(x)(v')\|^2+\|v'\|^2-2\,\mathrm{Re}(\langle \pi'(x)(v'), v'\rangle) \leq 2|1-\langle \pi'(x)(v'), v'\rangle| \leq 2c^2/3,$$

so

$$\sup_{x\in K} \|\pi'(x)(v') - v'\| < c.$$

d) Let π be the trivial representation of G. Then $\mathcal{P}(\pi)$ is the set of nonnegative constant functions, so, to show that $\pi \prec \pi'$, it suffices to show that the constant function 1 is a limit of finite sums of functions of $\mathcal{P}(\pi')$ (in the topology of convergence on compact subsets of G). Let K be a compact subset of G and $c > 0$. Choose $v' \in V'$ such that $\|v'\| = 1$ and $\sup_{x\in K} \|\pi'(x)(v') - v'\| < c$, and define φ' by $\varphi'(x) = \langle \pi'(x)(v'), v'\rangle$. Then, for every $x \in K$, we have

$$|1 - \varphi'(x)| = |\langle v', v'\rangle - \langle \pi'(x)(v'), v'\rangle| = |\langle v' - \pi'(x)(v'), v'\rangle| \leq \|v' - \pi'(x)(v')\| < c.$$

□

7. (3) Let G be a finitely generated discrete group, and let S be a finite set of generators for G. Show that the trivial representation of G is weakly contained in the regular representation of G if and only, for every $\varepsilon > 0$, there exists $f \in L^2(G)$ such that

$$\sup_{x\in S} \|L_x f - f\|_2 < \varepsilon \|f\|_2.$$

Solution. We use the criterion of 6(c) and 6(d), that says that the trivial representation of G is weakly contained in the regular representation if and only if, for every compact (i.e. finite) subset K of G and every $\varepsilon > 0$, there exists $f \in L^2(G)$ such that $\|f\|_2 = 1$ and

$$\sup_{x\in K} \|L_x f - f\|_2 < \varepsilon.$$

First, as S is finite, we see immediately that, if the trivial representation is contained in the regular representation, then the condition of the statement is satisfied.

Conversely, suppose that the condition of the statement is satisfied. Let K be a finite subset of G, and let $\varepsilon > 0$. Let $T = S \cup S^{-1} \cup \{1\}$. We have $G = \bigcup_{n\geq 1} T^n$ because S generates G, and this is an increasing union. As K is finite, there exists $n \geq 1$ such that T^n. By assumption, we can find $f \in L^2(G)$ such that $\|f\|_2 = 1$ and

$$\sup_{x\in S} \|L_x f - f\|_2 \leq \frac{1}{n}\varepsilon.$$

We want to show that

$$\sup_{x\in K} \|L_x f - f\|_2 \leq \varepsilon.$$

It suffices to show it for $\sup_{x\in T^n}$. Let $x \in T^n$, and write $x = x_1 \dots x_n$, with $x_1, \dots, x_n \in T$. We show by induction on $i \in \{1, \dots, n\}$ that $\|L_{x_1\dots x_i} f - f\|_2 \leq \frac{i}{n}\varepsilon$. If $i = 1$, we want to show that $\|L_{x_1} f - f\|_2 \leq \frac{1}{n}\varepsilon$. This is true by the choice of f if $x_1 \in S$, it is obvious if $x_1 = 1$, and, if $x_1 \in S^{-1}$, it follows from the fact that

$$\|L_{x_1} f - f\|_2 = \|f - L_{x_1^{-1}} f\|_2.$$

Now suppose the result known for $i \in \{1, \dots, n-1\}$, and let's prove it for $i+1$. We have

$$\begin{aligned}\|L_{x_1 \dots x_{i+1}} f - f\|_2 &\leq \|L_{x_1 \dots x_i}(L_{x_{i+1}} f - f)\|_2 + \|L_{x_1 \dots x_i} f - f\|_2 \\ &= \|L_{x_{i+1}} f - f\|_2 + \|L_{x_1 \dots x_i} f - f\|_2 \\ &\leq \tfrac{i}{n}\varepsilon + \tfrac{1}{n}\varepsilon = \tfrac{i+1}{n}\varepsilon.\end{aligned}$$

□

8. (2) Let $G = \mathbb{Z}$. Show that the trivial representation of G is weakly contained in the regular representation of G.

Solution. We apply the result of problem 7, with $S = \{1\}$. So, for every $\varepsilon > 0$, we must find $f \in L^2(\mathbb{Z})$ such that $\|f\|_2 = 1$ and $\|L_1 f - f\|_2 \leq \varepsilon$. The first condition says that $\sum_{n \in \mathbb{Z}} |f(n)|^2 = 1$, and the second condition that $\sum_{n \in \mathbb{Z}} |f(n-1) - f(n)|^2 \leq \varepsilon^2$. Let $N \in \mathbb{Z}_{\geq 0}$, and consider the function $g_N = \mathbb{1}_{[0,N]} \in L^2(\mathbb{Z})$. Then $\|g_N\|_2^2 = N+1$, and $\sum_{n \in \mathbb{Z}} |g(n-1) - g(n)|^2 = 2$. So, if $f_N = \frac{1}{\sqrt{N+1}}$, we have $\|f\|_2 = 1$ and $\|L_1 f - f\|_2 = \frac{\sqrt{2}}{\sqrt{N+1}}$. Taking N big enough, we see that f_N has the desired properties.

□

9. Let $G = \mathbb{R}$.
   a) (2) Show that the trivial representation of G is weakly contained in the regular representation of G.
   b) (1) Show that every irreducible unitary representation of G is weakly contained in the regular representation of G. [2]

Solution.

a) If $a, b \in \mathbb{R}$ are such that $a < b$, let $f = (b-a)^{-1/2} \mathbb{1}_{[a,b]}$. Then $f \in L^2(\mathbb{R})$ and we have $\|f\|_2 = 1$. Moreover, for every $t \in \mathbb{R}$, we have $L_t f = (b-a)^{-1/2} \mathbb{1}_{[a+t,b+t]}$, so

$$\|L_t f - f\|_2^2 \leq \frac{2|t|}{b-a}.$$

Let K be a compact subset of $\mathbb{R}$, and let $\varepsilon > 0$. If we choose $a, b \in \mathbb{R}$ such that $b - a \geq 2\varepsilon^{-2} \sup_{t \in K} |t|$, then the construction above gives a $f \in L^2(\mathbb{R})$ such that $\|f\|_2 = 1$ and $\sup_{x \in K} \|L_t f - f\|_2 \leq \varepsilon$. By 6(d), the trivial representation of $\mathbb{R}$ is contained in its regular representation.

b) As $\mathbb{R}$ is abelian, every irreducible unitary representation is 1-dimensional by Schur's lemma. Let $\chi : \mathbb{R} \to S^1$ be such a representation. Let K be a compact subset of $\mathbb{R}$ and $\varepsilon > 0$. By (a), there exists $f \in L^2(\mathbb{R})$ such that $\|f\|_2 = 1$ and $\sup_{t \in K} \|L_t f - f\|_2 \leq \varepsilon$. Let $g = \overline{\chi} f$. Then, for every $t \in \mathbb{R}$, we have

$$\langle L_t g, g \rangle_{L^2(\mathbb{R})} = \int_{\mathbb{R}} g(x-t)\overline{g(x)} dx = \chi(t) \langle L_t f, f \rangle_{L^2(\mathbb{R})},$$

hence

$$|\chi(t) - \langle L_t g, g \rangle_{L^2(\mathbb{R})}| = |1 - \langle L_t f, f \rangle_{L^2(\mathbb{R})}| = |\langle f - L_t f, f \rangle_{L^2(\mathbb{R})}| \leq \|L_t f - f\|_2.$$

So

$$\sup_{t \in K} |\chi(t) - \langle L_t g, g \rangle_{L^2(\mathbb{R})}| \leq \varepsilon.$$

This implies the desired result by 6(d).

[2] We will see later that this is true for every abelian locally compact group.

□

10. (extra credit, 4) Let G be the free (nonabelian) group on two generators, with the discrete topology. Show that the trivial representation of G is not weakly contained in the regular representation of G.

Solution. Let $a, b \in G$ be the two generators of G, and let $S = \{1, a, b, a^{-1}, b^{-1}\}$. We have $G = \bigcup_{n\geq 1} S^n$, and this is an increasing union. Suppose that the trivial representation of G is weakly contained in the regular representation. Then, by 6(c), for every $n \geq 1$, there exists $f_n \in L^2(G)$ such that $\|f_n\|_2 = 1$ and

$$\sup_{x\in S_n} \|L_x f_n - f_n\|_2 \leq \tfrac{1}{n}.$$

Let $g_n = |f_n|^2$. Then $g_n \in L^1(G)$, $\|g_n\|_1 = 1$, and, for every $x \in S_n$, the Cauchy-Schwarz inequality gives

$$\|L_x g_n - g_n\|_1 \leq \|L_x f_n - f_n\|_2 \|L_x f_n + f_n\|_2 \leq \frac{2}{n}.$$

For every $n \geq 1$, we define a continuous linear functional Λ_n on $L^\infty(G)$ by $\Lambda_n(\varphi) = \sum_{x\in G} g_n(x)\varphi(x)$. Then $\|\Lambda_n\|_{op} = \|g_n\|_1 = 1$, so, by the Banach-Alaoglu theorem, there is a subsequence $(\Lambda_{n_k})_{k\geq 0}$ of $(\Lambda_n)_{n\geq 1}$ that converges for teh weak* topology on $\mathrm{Hom}(L^\infty(G), \mathbb{C})$. Let Λ be its limit. Let $\varphi \in L^\infty(G)$. We have

$$\Lambda(\varphi) = \lim_{k\to+\infty} \Lambda_{n_k}(\varphi).$$

Let $y \in G$. There exists $n \geq 1$ such that $y^{-1} \in S_n$. Then, if k is such that $n_k \geq n$, we have

$$\begin{aligned}
|\Lambda_{n_k}(L_y\varphi) - \Lambda_{n_k}(\varphi)| &= |\sum_{x\in G} L_{y^{-1}} g_{n_k}(x)\varphi(x) - \sum_{x\in G} g_{n_k}(x)\varphi(x)| \\
&\leq \|L_{y^{-1}} g_{n_k} - g_{n_k}\|_1 \|\varphi\|_\infty \\
&\leq \frac{2}{n_k}\|\varphi\|_\infty.
\end{aligned}$$

Taking the limit as $k \to +\infty$, we see that $\Lambda(L_y\varphi) = \Lambda(\varphi)$. As, note that $\Lambda(1) = 1$, and that $\Lambda(\varphi) \geq 0$ if φ takes nonnegative values.

Remember that every element of G can be written in a unique way as a reduced word in a, b, a^{-1} and b^{-1}. Let A be the set of elements of G whose reduced expression begins with a nonzero power of a. The, for every $x \in G$, if $x \notin A$, we have $a^{-1}x \in A$ and then $x \in aA$. In other words, $G = A \cup aA$, so $\mathbb{1}_A + \mathbb{1}_{aA} - \mathbb{1}_G$ takes nonnegative values, hence

$$\Lambda(\mathbb{1}_A) = \frac{1}{2}(\Lambda(\mathbb{1}_A) + \Lambda(\mathbb{1}_{aA})) \geq \frac{1}{2}\Lambda(\mathbb{1}_G) = \frac{1}{2}.$$

On the other hand, the group G is the disjoint union of the subset $b^n A$, $n \in \mathbb{Z}$, so we have in particular

$$1 = \Lambda(\mathbb{1}_G) \geq \Lambda(\mathbb{1}_A) + \Lambda(\mathbb{1}_{bA}) + \Lambda(\mathbb{1}_{b^2A}) = 3\Lambda(\mathbb{1}_A),$$

that is, $\Lambda(\mathbb{1}_A) \leq \frac{1}{3}$. So we get a contradiction.

□

11. (2) If $\pi_1, \pi_2, \pi_1', \pi_2'$ are unitary representations of G such that $\pi_1 \prec \pi_1'$ and $\pi_2' \prec \pi_2$, show that $\pi_1 \otimes \pi_2 \prec \pi_1' \otimes \pi_2'$.

Solution. We use the same notation φ_v for functions of positive type as in the solution of problem 5. For $i = 1, 2$, we denote by V_i (resp. V_i') the space of π_i (resp. π_i').

If $v_1 \in V_1$ and $v_2 \in V_2$, then, by definition of the inner product on $V_1 \otimes_{\mathbb{C}} V_2$, we have $\varphi_{v_1 \otimes v_2} = \varphi_{v_1}\varphi_{v_2}$. There is similar result for pure tensors in $V_1' \otimes_{\mathbb{C}} V_2'$. So $\varphi_{v_1 \otimes v_2}$ is in $\overline{\mathcal{P}(\pi_1' \otimes \pi_2')}$. As $\overline{\mathcal{P}(\pi_1' \otimes \pi_2')}$ is stable by finite sums, and as every element of $V_1 \otimes_{\mathbb{C}} V_2$ can be written as a finite sum of an orthogonal family of pure tensors (see the proof of 1(a)), this implies that $\varphi_v \in \overline{\mathcal{P}(\pi_1' \otimes \pi_2')}$ for every $v \in V_1 \otimes_{\mathbb{C}} V_2$. Finally, we have proved in 5(c) that the map $v \mapsto \varphi_v$ is continuous, and $V_1 \otimes_{\mathbb{C}} V_2$ is dense in $V_1 \widehat{\otimes}_{\mathbb{C}} V_2$, so $\varphi_v \in \overline{\mathcal{P}(\pi_1' \otimes \pi_2')}$ for every $v \in V_1 \widehat{\otimes}_{\mathbb{C}} V_2$.

□

12. Suppose that G is discrete. For every $x \in G$, we denote by $\delta_x \in L^2(G)$ the characteristic function of $\{x\}$.

Let (π, V) be a unitary representation of G, and let (π_0, V) be the trivial representation of G on V (i.e. $\pi_0(x) = \mathrm{id}_V$ for every $x \in G$).

a) (3) Show that the formula $v \otimes f \mapsto \sum_{x \in G} f(x)(\pi(x)^{-1}(v)) \otimes \delta_x$ gives a well-defined and continuous $\mathbb{C}$-linear transformation from $V \widehat{\otimes}_{\mathbb{C}} L^2(G)$ to itself.

b) (2) Show that the representations $\pi \otimes \pi_L$ and $\pi_0 \otimes \pi_L$ are equivalent (remember that π_L is the left regular representation of G).

Solution.

a) First, the map $V \times L^2(G) \to V \otimes_{\mathbb{C}} L^2(G)$, $(v, f) \mapsto \sum_{x \in G} f(x)(\pi(x)^{-1}(v)) \otimes \delta_x$ is bilinear, so it defines a linear map $\alpha : V \otimes_{\mathbb{C}} L^2(G) \to V \otimes_{\mathbb{C}} L^2(G)$. For every $v, v' \in V$ and $f, f' \in L^2(G)$, we have (observing that the family $(v_x \otimes \delta_x)_{x \in G}$ is orthogonal for every family $(v_x)_{x \in G}$ of elements of V)

$$\begin{aligned}\langle \alpha(v \otimes f), \alpha(v' \otimes f') \rangle &= \sum_{x \in G} f(x)\overline{f'(x)}\langle \pi(x)^{-1}(v), \pi(x)^{-1}(v') \rangle \\ &= \sum_{x \in G} f(x)\overline{f'(x)}\langle v, v' \rangle \\ &= \langle v \otimes f, v' \otimes f' \rangle.\end{aligned}$$

Using the fact that every element of $V \otimes_{\mathbb{C}} L^2(G)$ can be written as a finite sum or pairwise orthogonal pure tensors (see the proof of 1(a)), this implies that $\|\alpha(v)\| = \|v\|$ for every $v \in V \otimes_{\mathbb{C}} L^2(G)$. In particular, α is continuous, so it extends to a continuous endomorphism of $V \widehat{\otimes}_{\mathbb{C}} L^2(G)$, which is still an isometry.

b) We still call α the endomorphism of $V \widehat{\otimes}_{\mathbb{C}} L^2(G)$ constructed in (a). We show that it is a G-equivariant map from $\pi \otimes \pi_L$ to $\pi_0 \otimes \pi_L$. As pure tensors generates a dense subspace of $V \widehat{\otimes}_{\mathbb{C}} L^2(G)$, it suffices to check the G-equivariance on them. So let $v \in V$ and $f \in L^2(G)$, and let $x \in G$. We have

$$\alpha(\pi \otimes \pi_L(x)(v \otimes f)) = \alpha(\pi(x)(v) \otimes L_x f) = \sum_{y \in G} f(x^{-1}y)\pi(y^{-1}x)(v) \otimes \delta_y.$$

On the other hand,

$$\begin{aligned}\pi_0 \otimes \pi_L(x)(\alpha(v \otimes f)) &= \pi_0 \otimes \pi_L(x) \left(\sum_{y \in G} f(y)\pi(y)^{-1}(v) \otimes \delta_y \right) \\ &= \sum_{y \in G} f(y)\pi(y)^{-1}(v) \otimes L_x \delta_y \\ &= \sum_{y \in G} f(y)\pi(y)^{-1}(v) \otimes \delta_{xy} \\ &= \sum_{y \in G} f(x^{-1}z)\pi(z^{-1}x)(v) \otimes \delta_z \\ &= \alpha(\pi \otimes \pi_L(x)(v \otimes f)).\end{aligned}$$

We still need to check that α is an isomorphism of vector spaces. This follows from the fact that is has an inverse β, given by the formula $\beta(v \otimes f) = \sum_{x \in G} f(x)\pi(x)(v) \otimes \delta_x$. (We can check as in (a) that β is well-defined and continuous, and then we can check on pure tensors that it is the inverse of α, which is an easy verification.)

Note that the isomorphism betweem $\pi \otimes \pi_L$ and $\pi_0 \otimes \pi_L$ is an isometry, so these representations have the same functions of positive type.

□

13. (extra credit, 5) Generalize the result of 12(b) to non-discrete locally compact groups.

Solution. Let (π, V) be a unitary representation of G. We write V_0 for V with the trivial action of G.

First we define a Hilbert space $L^2(G, V_0)$ with a unitary action of G. (This is also often denoted by $\mathrm{Ind}_{\{1\}}^G V_0$.) Consider the space $\mathcal{C}_c(G, V_0)$ of continuous functions with compact support from G to V_0, with the norm $\|.\|_\infty$ defined by $\|f\|_\infty = \sup_{x \in G} \|f(x)\|$. We make G act on this space by $(x, f) \mapsto L_x f$, for $x \in G$ and $f \in \mathcal{C}_c(G, V_0)$. Looking at proposition I.1.12 of the notes, we see that its proof generalizes to functions from G to V_0 and show that every element of $\mathcal{C}_c(G, V_0)$ is left and right uniformly continuous. In particular, for every $f \in \mathcal{C}_c(G, V_0)$, the map $G \to \mathcal{C}_c(G, V_0)$, $x \mapsto L_x f$ is continuous.

Now we define a Hermitian sesquilinear form on $\mathcal{C}_c(G, V_0)$ by

$$\langle f, g \rangle = \int_G \langle f(x), g(x) \rangle_{V_0} dx.$$

It is easy to see that this is an inner form, and that the action of G on $\mathcal{C}_c(G, V_0)$ preserves this inner form and is continuous in the first variable $x \in G$ for the topology on $\mathcal{C}_c(G, V_0)$ defined by the associated norm. We denote by $L^2(G, V_0)$ the completion of $\mathcal{C}_c(G, V_0)$ for $\langle ., . \rangle$. This is a Hilbert space, and we show as in the case $V_0 = \mathbb{C}$ that the action of G on $\mathcal{C}_c(G, V_0)$ extends to a unitary action of G on $L^2(G, V_0)$.

We now construct a G-equivariant isometry $V \widehat{\otimes}_{\mathbb{C}} L^2(G) \to L^2(G, V_0)$. Consider the map $V \times \mathcal{C}_c(G) \to L^2(G, V_0)$ sending (v, f) to the function $x \mapsto f(x)\pi(x^{-1})(v)$. This is a bilinear map, so it induces a $\mathbb{C}$-linear operator $\alpha : V \otimes_{\mathbb{C}} \mathcal{C}_c(G) \to L^2(G, V_0)$. We check that α is G-equivariant. It suffices to check it on pure tensors, because they generate $V \otimes_{\mathbb{C}} \mathcal{C}_c(G)$. If $y \in G$, $v \in V$ and $f \in \mathcal{C}_c(G)$, then, for every $x \in G$,

$$\begin{aligned}\alpha(\pi(y)(v) \otimes L_y f)(y) &= f(y^{-1}x)\pi(xy^{-1}v) \\ &= L_y(\alpha(v \otimes f))(x).\end{aligned}$$

We also check that α preserves the inner forms. As before, by bilinearity, it suffices to check it on pure tensors. Let $v, w \in V$ and $f, g \in \mathcal{C}_c(G)$. Then

$$\begin{aligned}\langle \alpha(v \otimes f), \alpha(w \otimes g) \rangle &= \int_G \langle f(x)\pi(x)^{-1}(v), g(x)\pi(x)^{-1}(w) \rangle_{V_0} dx \\ &= \int_G f(x)\overline{g(x)} \langle v, w \rangle_{V_0} dx \\ &= \langle f, g \rangle_{L^2(G)} \langle v, w \rangle_{V_0}.\end{aligned}$$

This implies that α is an isometry, hence that it extends by continuity to an isometry $V\widehat{\otimes}_{\mathbb{C}}L^2(G) \to L^2(G, V_0)$ (we use the fact that $\mathcal{C}_c(G)$ is dense in $L^2(G)$), which is still G-equivariant.

We define a G-equivariant isometry $\alpha' : V_0\widehat{\otimes}_{\mathbb{C}}L^2(G) \to L^2(G, V_0)$ in a way similar to α, but, for $v \in V_0$ and $f \in \mathcal{C}_c(G)$, we take $\alpha'(v \otimes f)$ to be the function $x \mapsto f(x)v$. The proof that this does define the deisred G-equivariant isometry is the same as in the case of α.

Finally, we show that α and α' are isomorphisms. We already know that they are injective and have closed image because they are isometries, so we just need to show that they have dense image.

Let $(e_i)_{i\in I}$ be a Hilbert basis of V_0. Consider the subspace W of $L^2(G, V_0)$ whose elements are continuous functions with compact support $f : G \to V_0$ such that there exists $J \subset I$ finite with $f(G) \subset \mathrm{Span}(e_j, j \in J)$. Let's show that W is dense in $L^2(G, V_0)$. It suffices to show that W is dense in $\mathcal{C}_c(G, V_0)$. Let $f \in \mathcal{C}_c(G, V_0)$. As f has compact support, the subset $f(G)$ of V_0 is compact. Let $\varepsilon > 0$. For every $x \in K$, there exists a finite subset J of I such that the closed ball centered at x and of radius ε intersects $\mathrm{Span}(e_j, j \in J)$. As K is compact, it can be covered by a finite number of these balls, so we can find s finite subset J of I such that the distance between x and $\mathrm{Span}(e_j, j \in J)$ is $\leq \varepsilon$ for every $x \in K$. In other words, if π_J is the orthogonal projection on $\mathrm{Span}(e_j, j \in J)$, then $\|\pi_J(x) - x\| \leq \varepsilon$ for every $x \in K$. Then $\pi_J \circ f \in W$, and $\|f - \pi_J \circ f\|_\infty \leq \varepsilon$, so $\|f - \pi_J \circ f\|_2 \leq \mathrm{vol}(\mathrm{supp}\, f)\varepsilon$. This shows that W is dense in $\mathcal{C}_c(G, V_0)$ for both topologies on $\mathcal{C}_c(G, V_0)$ (the one induced by $\|.\|_\infty$ and the one induced by $\|.\|_2$; only the second one is relevant here). To finish, it suffices to show that W is contained in the images of α and α'. Let $f \in W$. We can find a finite subset J of I such that $f(G) \subset \mathrm{Span}(e_j, j \in J)$, and then we have $f(x) = \sum_{j\in J} f_j(x)e_j$, with the f_j in $\mathcal{C}_c(G)$. (Just take coordinates in the orthonormal basis $(e_j)_{j\in J}$ of $\mathrm{Span}(e_j, j \in J)$). In particular, $f = \alpha'(\sum_{j\in J} e_j \otimes f_j)$, so $f \in \mathrm{Im}(\alpha')$. This shows that α' is an isomorphism.

For α, we consider instead the subspace W' of $f \in \mathcal{C}_c(G, V_0)$ such that there exists $J \subset I$ finite such that, for every $x \in G$, the vector $\pi(x)(f(x))$ is in $\mathrm{Span}(e_j, j \in J)$. We show as before that W' is dense in $\mathcal{C}_c(G, V_0)$ (for both $\|.\|_\infty$ and $\|.\|_2$) : Let $f \in \mathcal{C}_c(G, V_0)$ and $\varepsilon > 0$. As f has compact support, the subset $\{\pi(x)(f(x)),\ x \in G\}$ of V_0 is compact, so we can find a finite subset J of I such that, for every $x \in G$, the distance between $\pi(x)(f(x))$ and $\mathrm{Span}(e_j, j \in J)$ is at most ε. Let π_J be the orthogonal projection on $\mathrm{Span}(e_j, j \in J)$, and define $g \in W'$ by $g(x) = \pi(x)^{-1} \circ \pi_J \circ \pi(x)(f(x))$. For every $x \in G$,

$$\|g(x) - f(x)\| = \|\pi(x)(g(x) - f(x))\| = \|\pi_J(\pi(x)(f(x))) - \pi(x)(f(x))\| \leq \varepsilon,$$

so $\|g - f\|_\infty \leq \varepsilon$ and $\|g - f\|_2 \leq \mathrm{vol}(\mathrm{supp}\, f)\varepsilon$. Finally, we show that W' is contained in the image of α. Let $f \in W'$, and define $g \in \mathcal{C}_c(G, V_0)$ by $g(x) = \pi(x)(f(x))$. Choose a finite subset J of I such that $g(G) \subset \mathrm{Span}(e_j, j \in J)$, and write $g = \sum_{j\in J} g_j e_j$, with $g_j \in \mathcal{C}_c(G)$. Then, for every $x \in G$, we have

$$f(x) = \sum_{j\in J} g_j(x)\pi(x)^{-1}(e_j).$$

In other words, we have $f = \alpha(\sum_{j\in J} e_j \otimes g_j)$.

$\square$

14. (2) Show that the following are equivalent :
    (i) The trivial representation of G is weakly contained in π_L.
    (ii) Every unitary representation of G is weakly contained in π_L.

Solution.

The fact that (ii) implies (i) is obvious. So let's show that (i) implies (ii). Let (π, V) be a unitary representation of G, let π_0 be the trivial representation of G on V, and let $\mathbb{1}$ be the trivial representation of G on $\mathbb{C}$. We know that $\mathbb{1} \prec \pi_L$, so, by problems 11 and 12, we have $\pi \simeq \pi \otimes \mathbb{1} \prec \pi \otimes \pi_L \simeq \pi_0 \otimes \pi_L$.

As in the solution of problem 5, for every unitary representation π' of G, we denote by $\sum \mathcal{P}(\pi')$ the set of finite sums of functions of positive type associated to π. Let's show that $\overline{\sum \mathcal{P}(\pi_L)} = \overline{\sum \mathcal{P}(\pi_0 \otimes \pi_L)}$, which will finish the proof, because we already know that $\mathcal{P}(\pi) \subset \overline{\sum \mathcal{P}(\pi_0 \otimes \pi_L)}$.

As π_L is a subrepresentation of $\pi_0 \otimes \pi_L$ (for every $v \in V - \{0\}$, the subspace $\mathbb{C}v \otimes L^2(G)$ of $V \widehat{\otimes}_{\mathbb{C}} L^2(G)$ is G-invariant and equivalent to the representation π_L by the map $v \otimes f \mapsto f$), we have $\mathcal{P}(\pi_L) \subset \mathcal{P}(\pi_0 \otimes \pi_L)$, so $\overline{\sum \mathcal{P}(\pi_L)} \subset \overline{\sum \mathcal{P}(\pi_0 \otimes \pi_L)}$. Conversely, let $(e_i)_{i\in I}$ be an orthonormal basis of V, and let $v \in V \widehat{\otimes}_{\mathbb{C}} L^2(G)$. Then we can write $v = \sum_{i\in I} e_i \otimes f_i$, where the sum converges in $V \widehat{\otimes}_{\mathbb{C}} L^2(G)$ (i.e. $\sum_{i\in I} \|f_i\|^2$ converges). Then, for every $x \in G$, we have

$$\langle \pi_0 \otimes \pi_L(x)(v), v\rangle = \sum_{i\in I} \langle L_x f_i, f_i\rangle_{L^2(G)},$$

so the function $x \mapsto \langle \pi_0 \otimes \pi_L(x)(v), v\rangle$ is in $\overline{\sum \mathcal{P}(\pi_L)}$.

$\square$