MAT 449 : Problem Set 7

Due Thursday, November 8

Let G be a topological group and (π, V) be a unitary representation of G. A *matrix coefficient* of π is a function $G \to \mathbb{C}$ of the form $x \mapsto \langle \pi(x)(v), w \rangle$, with $v, w \in V$. Note that these functions are all continuous. We say that the matrix coefficient is *diagonal* if $v = w$; a diagonal matrix coefficient is a function of positive type by proposition III.2.4, and we call it a function of positive type associated to π. We say that a function of positive type is *normalized* if it is of the form $x \mapsto \langle \pi(x)(v), v \rangle$ with $\|v\| = 1$. We denote by $\mathcal{P}(\pi)$ the set of functions of positive type associated to π.

Remember also that, if G is locally compact (and μ is a left Haar measure on G), then the *left regular representation* π_L is the representation of G on $L^2(G) := L^2(G, \mu)$ given by $\pi_L(x)(f) = L_x f$, for $x \in G$ and $f \in L^2(G)$. In this problem, we'll just call π_L the regular representation of G.

1. Let (π_1, V_1), (π_2, V_2) be unitary representations of G.
   a) (3) Show that the algebraic tensor product $V_1 \otimes_{\mathbb{C}} V_2$ has a Hermitian inner product, uniquely determined by $\langle v_1 \otimes v_2, w_1 \otimes w_2 \rangle = \langle v_1, w_1 \rangle \langle v_2, w_2 \rangle$.
   b) (2) We denote the completion of $V_1 \otimes_C V_2$ for this inner form by $V_1 \hat{\otimes}_{\mathbb{C}} V_2$. Show that the formula $(x, v_1 \otimes v_2) \mapsto \pi_1(x)(v_1) \otimes \pi_2(x)(v_2)$ defines a unitary representation of G on $V_1 \hat{\otimes}_{\mathbb{C}} V_2$. (This is called the tensor product representation and usually denoted by $\pi_1 \otimes \pi_2$.)
   c) (2) If V_1 and V_2 are finite-dimensional, show that, for every $x \in G$, we have
   $$\operatorname{Tr}(\pi_1 \otimes \pi_2(x)) = \operatorname{Tr}(\pi_1(x)) \operatorname{Tr}(\pi_2(x)).$$

2. Let G be a discrete group, and let $\varphi = \mathbb{1}_{\{1\}}$.
   a) (1) Show that φ is a function of positive type on G.
   b) (2) Show that V_φ is equivalent to the regular representation of G.
   c) (2, extra credit) If $G \neq \{1\}$, show that the representations V_φ and $V_\varphi \widehat{\otimes}_{\mathbb{C}} V_\varphi$ are not equivalent. (You can use problem 12.)

3. Let G be a locally compact group.
   a) (1) If $f, g \in \mathcal{C}_c(G)$, show that $f * g \in \mathcal{C}_c(G)$.
   b) (3) Show that every matrix coefficient of the regular representation of G vanishes at ∞.
   c) (2) Suppose that G is not compact. If (π, V) is a finite-dimensional unitary representation of G, show that it has a matrix coefficient that does not vanish at ∞.
   d) (1) If G is not compact, show that its regular representation has no finite-dimensional subrepresentation.

4. (extra credit, 3) Let V be a locally convex topological $\mathbb{C}$-vector space, K be a compact convex subset of V, and $F \subset K$ be such that K is the closure of the convex hull of F. Show that every extremal point of K is in the closure of F. (This is known as *Milman's theorem.*)

We will see soon that, if G is compact, then the regular representation of G contains all the irreducible representations of G (which are all finite-dimensional); in fact, it is the closure of the direct sum of all its irreducible subrepresentations. On the other hand, if G is abelian, then its regular representation is the direct integral of all the irreducible representations of G (which are all 1-dimensional), even though it does not contain any of them if G is not compact. We will not rigorously define direct integrals here, but we will introduce a weaker definition of containment, for which irreducible representations of an abelian locally compact group are contained in the regular representation, and start studying it.

Let (π, V) and (π', V') be unitary representations of G. We say that π is *weakly contained* in π', and write $\pi \prec \pi'$, if $\mathcal{P}(\pi)$ is contained in the closure of the set of finite sums of elements of $\mathcal{P}(\pi')$ for the topology of convergence on compact subsets of G. In other words, $\pi \prec \pi'$ if, for every $v \in V$, for every $K \subset G$ compact and every $c > 0$, there exist $v'_1, \dots, v'_n \in V'$ such that

$$\sup_{x \in K} |\langle \pi(x)(v), v \rangle - \sum_{i=1}^{n} \langle \pi'(x)(v'_i), v'_i \rangle| < c.$$

5. Let (π, V) and (π', V') be unitary representations of G. Let $C \subset V$ such that $\mathrm{Span}(\pi(x)(v),\ x \in G,\ v \in C)$ is dense in V.[1] Suppose that every function $x \mapsto \langle \pi(x)(v), v \rangle$, for $v \in C$, is in the closure of the set of finite sums of elements of $\mathcal{P}(\pi')$ (still for the topology of convergence on compact subsets of G). The goal of this problem is to show that this implies $\pi \prec \pi'$.

   Let X be the set of $v \in V$ such that $x \mapsto \langle \pi(x)(v), v \rangle$ is in the closure of the set of finite sums of elements of $\mathcal{P}(\pi')$ (for the same topology as above).

   a) (1) Show that X is stable by all the $\pi(x)$, $x \in G$, and under scalar multiplication.
   b) (1) If $v \in X$ and $x_1, x_2 \in G$, show that $\pi(x_1)(v) + \pi(x_2)(v) \in X$.
   c) (1) Show that X is closed in V.
   d) (1) If $v \in X$, show that the smallest closed G-invariant subspace of V containing v is contained in X.
   e) Let $v_1, v_2 \in X$, and let W_1 (resp. W_2) be the smallest closed G-invariant subspace of V containing v_1 (resp. v_2). Let $W = \overline{W_1 + W_2}$, and denote by $T : W \to W_1^\perp$ the orthogonal projection, where we take the orthogonal complement of W_1 in W.
      i. (1) Show that T is G-equivariant and that $T(W_2)$ is dense in $W_1^\perp$.
      ii. (2) Show that $W_1^\perp \subset X$.
      iii. (1) Show that $v_1 + v_2 \in X$. (Hint : Use $T(v_1 + v_2)$ and $(v_1 + v_2) - T(v_1 + v_2)$.)
   f) (1) Show that $\pi \prec \pi'$.

6. Let (π, V) and (π', V') be two unitary representations of G such that $\pi \prec \pi'$. Let C be the closure in the weak* topology on $L^\infty(G)$ of the convex hull of the set of normalized functions of positive type associated to π'.

   a) (1) Show that every normalized function of positive type associated to π is in C.

[1] For example, if V is cyclic, C could just contain a cyclic vector for V.

b) (3) If π is irreducible, show that every normalized function of positive type associated to π is a limit in the topology of convergence on compact subsets of G of normalized functions of positive type associated to π'. (Hint : problem 4.)

c) (2) If π is the trivial representation of G, show that, for every compact subset K of G and every $c > 0$, there exists $v' \in V'$ such that $\|v'\| = 1$ and that

$$\sup_{x \in K} \|\pi'(x)(v') - v'\| < c.$$

d) (1) Conversely, suppose that, for every compact subset K of G and every $c > 0$, there exists $v' \in V'$ such that $\|v'\| = 1$ and that

$$\sup_{x \in K} \|\pi'(x)(v') - v'\| < c.$$

Show that the trivial representation is weakly contained in π'.

7. (3) Let G be a finitely generated discrete group, and let S be a finite set of generators for G. Show that the trivial representation of G is weakly contained in the regular representation of G if and only, for every $\varepsilon > 0$, there exists $f \in L^2(G)$ such that

$$\sup_{x \in S} \|L_x f - f\|_2 < \varepsilon \|f\|_2.$$

8. (2) Let $G = \mathbb{Z}$. Show that the trivial representation of G is weakly contained in the regular representation of G.

9. Let $G = \mathbb{R}$.

   a) (2) Show that the trivial representation of G is weakly contained in the regular representation of G.

   b) (1) Show that every irreducible unitary representation of G is weakly contained in the regular representation of G. [2]

10. (extra credit, 4) Let G be the free (nonabelian) group on two generators, with the discrete topology. Show that the trivial representation of G is not weakly contained in the regular representation of G.

11. (2) If $\pi_1, \pi_2, \pi_1', \pi_2'$ are unitary representations of G such that $\pi_1 \prec \pi_1'$ and $\pi_2' \prec \pi_2$, show that $\pi_1 \otimes \pi_2 \prec \pi_1' \otimes \pi_2'$.

12. Suppose that G is discrete. For every $x \in G$, we denote by $\delta_x \in L^2(G)$ the characteristic function of $\{x\}$.

    Let (π, V) be a unitary representation of G, and let (π_0, V) be the trivial representation of G on V (i.e. $\pi_0(x) = \mathrm{id}_V$ for every $x \in G$).

    a) (3) Show that the formula $v \otimes f \mapsto \sum_{x \in G} f(x)(\pi(x)^{-1}(v)) \otimes \delta_x$ gives a well-defined and continuous $\mathbb{C}$-linear transformation from $V \widehat{\otimes}_{\mathbb{C}} L^2(G)$ to itself.

    b) (2) Show that the representations $\pi \otimes \pi_L$ and $\pi_0 \otimes \pi_L$ are equivalent (remember that π_L is the left regular representation of G).

13. (extra credit, 5) Generalize the result of 12(b) to non-discrete locally compact groups.

14. (2) Show that the following are equivalent :

    (i) The trivial representation of G is weakly contained in π_L.

    (ii) Every unitary representation of G is weakly contained in π_L.

[2] We will see later that this is true for every abelian locally compact group.