MAT 449 : Problem Set 6

Due Sunday, October 28

If G is a group, we say that a representation (π, V) of G is *faithful* if $\pi : G \to \mathbf{GL}(V)$ is injective.

1. Let $G = \mathbf{SU}(2)$. The group G acts on $\mathbb{C}^2$ via the inclusion $G \subset \mathbf{GL}_2(\mathbb{C})$, and we just denote this action by $(g, (z_1, z_2)) \mapsto g(z_1, z_2)$. (This is called the *standard representation* of G.)

   For every integer $n \geq 1$, let V_n be the space of polynomials $P \in \mathbb{C}[t_1, t_2]$ that are homogeneous of degree n (i.e. $P(t_1, t_2) = \sum_{r=0}^{n} a_r t_1^r t_2^{n-r}$, with $a_0, \ldots, a_n \in \mathbb{C}$).

   a) (3) If $P \in V_n$ and $g \in G$, show that the function $\mathbb{C}^2 \to \mathbb{C}$, $(z_1, z_2) \mapsto P(g^{-1}(z_1, z_2))$ is still given by a polynomial in V_n, and that this defines a continuous representation of G on V_n.

   b) (3) Show that the representation V_n of G is irreducible for every $n \geq 0$.

   c) (2) For which values of n is the representation V_n faithful ?

   Remark : We will see later that every irreducible unitary representation of $\mathbf{SU}(2)$ is isomorphic to one of the V_n.

2. Let (π, V) be a finite-dimensional unitary representation of $G := \mathbf{SL}_2(\mathbb{R})$. We want to show that V is trivial (i.e. $\pi(x) = \mathrm{id}_V$ for every $x \in G$).

   a) (2) Consider the morphism of groups $\alpha : \mathbb{R} \to G$ sending $t \in \mathbb{R}$ to the matrix $\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$.

      Show that there exist a basis $\mathcal{B}$ of V and $y_1, \ldots, y_n \in \mathbb{R}$, where $n = \dim V$, such that, for every $t \in \mathbb{R}$, the endormophism $\pi(\alpha(t))$ is diagonal in $\mathcal{B}$ with diagonal entries $e^{ity_1}, \ldots, e^{ity_n}$.

   b) (3) Show that $\pi(\alpha(t)) = \mathrm{id}_V$ for every $t \in \mathbb{R}$. (Hint : If $u \in \mathbb{R}^\times$ and $x = \begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix}$, consider the action of $x\alpha(t)x^{-1}$ on V.)

   c) (2) Show that $\pi(x) = \mathrm{id}_V$ for every $x \in G$.

   d) (2, extra credit) If $n \geq 3$, show that every finite-dimensional unitary representation of $\mathbf{SL}_n(\mathbb{R})$ is trivial.

The preceding problem shows that $\mathbf{SL}_2(\mathbb{R})$ has no faithful finite-dimensional unitary representation. But at least $\mathbf{SL}_2(\mathbb{R})$ has faithful continuous finite-dimensional representations, for example the one given by the inclusion $\mathbf{SL}_2(\mathbb{R}) \subset \mathbf{GL}_2(\mathbb{C})$.

We can also ask if there exist locally compact groups that don't have faithful irreducible unitary representations at all. The answer is "yes".

3. (3) Show that, if (π, V) is an irreducible unitary representation of $\mathbf{GL}_n(\mathbb{Z}_p)$, then there exists $m \geq 1$ such that $\pi(I_n + p^m M_n(\mathbb{Z}_p)) = \{1\}$.

4. (extra credit, 3) More generally, show that, if G is a profinite group (i.e. a projective limit of finite discrete groups, see problem 3 of problem set 1), then G has a faithful irreducible unitary representation only if G is finite.

We will now see how to find discrete groups that have no faithful finite-dimensional representations at all, over any field.

Let Γ be a (discrete) group. We say that Γ is *residually finite* if, for every $x \in \Gamma - \{1\}$, there exists a normal subgroup Δ of Γ such that Γ/Δ is finite (we say that Δ is of *finite index* in Γ) and that the image of x in Γ/Δ is not trivial.

The goal of the following two problems is to prove that, if k is a field and $\Gamma \subset \mathbf{GL}_n(k)$ is a finitely generated subgroup, then Γ is residually finite. [1]

5. Let R be a finitely generated $\mathbb{Z}$-algebra that is also a domain. We fix an integer $n \geq 1$. For every ideal I of R, we set

$$\Gamma(I) = \mathrm{Ker}(\mathbf{GL}_n(R) \to \mathbf{GL}_n(R/I)).$$

   a) (3) Show that R is a field if and only if R is finite.
   b) (2) If $\mathfrak{m}$ is a maximal ideal of R, show that $\Gamma(\mathfrak{m})$ is a normal subgroup of finite index in $\mathbf{GL}_n(R)$.
   c) (3) Show that the intersection of all the maximal ideals of R is 0. (Hint : We may assume that R is not a field. If $a \in R - \{0\}$, show that the localization $R[1/a]$ is not a field, take a maximal ideal in $R[1/a]$, and intersect it with R.)
   d) (1) Show that $\mathbf{GL}_n(R)$ is residually finite.

6. Let k be a field, and let Γ be a finitely generated subgroup of $\mathbf{GL}_n(k)$.
   a) (2) Show that there exists a finitely generated $\mathbb{Z}$-subalgebra R of k such that $\Gamma \subset \mathbf{GL}_n(R)$.
   b) (1) Show that Γ is residually finite.

Of course, the result of the previous problem would not be very interesting if we could not give any example of a finitely generated non residually finite group. So let's do that.

7. Let Γ be the quotient of the free group on the generators a and b by the relation $a^{-1}b^2a = b^3$. In this problem, we will assume that $b_1 := a^{-1}ba$ and b do not commute in Γ, and deduce that Γ is not residually finite.

   Let $u : \Gamma \to \Gamma'$ be a morphism of groups, with Γ' finite.
   a) (2) Let n be the order of $u(a)$ in Γ'. Show that the order of $u(b)$ divides $3^n - 2^n$.
   b) (2) Show that there exists an integer $N \geq 0$ such that $u(b_1) = u(b_1^2)^N$. (Note that the order of $u(b)$ is prime to both 2 and 3.)
   c) (1) Show that $u(b_1)$ and $u(b)$ commute.
   d) (1) Show that Γ is not residually finite.

[1] In fact, we can use similar ideas to show that, if $\mathrm{char}(k) = 0$, such a Γ has to be virtually residually p-finite (i.e. it has a finite index subgroup Γ' such that, for every $x \in \Gamma' - \{1\}$, there exists a finite index normal subgroup $\Delta \not\ni x$ of Γ' such that Γ'/Δ is a p-group) for almost every prime number p, but the only proof I know uses the Noether normalization theorem.

8. (extra credit) Let Γ be the quotient of the free group on the generators a and b by the relation $a^{-1}b^2a = b^3$. The goal of this problem is to show that $b_1 := a^{-1}ba$ and b do not commute in Γ, i.e. that $b_1 b b_1^{-1} b^{-1}$ is not trivial in Γ. [2]

   Let F be the free group on the generators a and b. Remember that elements of F are reduced words in the letters a, a^{-1}, b, b^{-1}. (A reduced words is a word that contains no redundant pair aa^{-1}, $a^{-1}a$, bb^{-1} or $b^{-1}b$.) We write an element of F as $a^{n_1}b^{m_1}\dots a^{n_r}b^{m_r}$, with $n_1, m_1, \dots, n_r, m_r \in \mathbb{Z}$ and $m_1, n_2, m_2, \dots, n_{r-1}, m_{r-1}, n_r \neq 0$.

   Let Ω be the set of reduced words of the form $b^{r_1}a^{s_1}\dots b^{r_m}a^{s_m}b^r$, with :

   (i) $m \in \mathbb{Z}_{\geq 0}$ and $r_i, s_i, r \in \mathbb{Z}$;

   (ii) $s_i \neq 0$ for every $i \in \{1, \dots, m\}$;

   (iii) $r_i \neq 0$ for every $i \in \{2, \dots, m\}$;

   (iv) for every $i \in \{1, \dots, m\}$, if $s_i > 0$, then $0 \leq r_i \leq 1$;

   (v) for every $i \in \{1, \dots, m\}$, if $s_i < 0$, then $0 \leq r_i \leq 2$.

   By definition of Γ, we have a surjective group morphism $F \to \Gamma$, that we will denote by φ.

   a) (2) Show that $\varphi(\Omega) = \Gamma$.

   b) (1) For every $w \in \Omega$ and every $s \in \{a, a^{-1}, b, b^{-1}\}$, find a word $w' \in \Omega$ such that $\varphi(w') = \varphi(ws)$. We will denote this w' by $w \cdot s$ in what follows.

   c) (1) For every $w \in \Omega$ and every $s \in \{a, a^{-1}, b, b^{-1}\}$, show that $(w \cdot s) \cdot s^{-1} = w$.

   d) (1) Show that $(w, s) \mapsto w \cdot s$ extends to a right action of Γ on Ω.

   e) (1) Show that φ induces a bijection $\Omega \xrightarrow{\sim} \Gamma$.

   f) (1) Show that $b_1 b b_1^{-1} b^{-1} \neq 1$ in Γ.

[2] The easiest way to show this would to find a finite-dimensional representation of Γ on which $b_1 b b_1^{-1} b$ acts non-trivially, but we can't. Still, some variant of this idea will work.