MAT 449 : Problem Set 5

Due Thursday, October 18

1. a) (2) Let V be a finite-dimensional $\mathbb{C}$-vector space such that $\dim_{\mathbb{C}}(V) \geq 2$. Show that $\sigma(\mathrm{End}(V)) = \varnothing$.

   b) (2) Let V be an infinite-dimensional Hilbert space. Show that $\sigma(\mathrm{End}(V)) = \varnothing$.

   (Hint : Look at nilpotent endomorphisms.)

   Solution.

   a) We may assume that $V = \mathbb{C}^n$, so that $\mathrm{End}(V) = M_n(\mathbb{C})$. Let $\varphi : M_n(\mathbb{C}) \to \mathbb{C}$ be a multiplicative linear functional. We want to prove that $\varphi = 0$. Let $(E_{ij})_{1 \leq i,j \leq n}$ be the canonical basis of $M_n(\mathbb{C})$ (so E_{ij} is the matrix with all entry 0, except for a 1 at the (i, j)-entry). Then $E_{ij}E_{kl}$ is equal to 0 unless $j = k$, and $E_{ij}E_{jl} = E_{il}$. In particular, if $i \neq j$, then $E_{ij}^2 = 0$, hence $0 = \varphi(E_{ij}^2) = \varphi(E_{ij})^2$, and $\varphi(E_{ij}) = 0$. Also, for every $i \in \{1, \dots, n\}$, if we choose j such that $j \neq i$ (this is possible because $n \geq 2$), then $E_{ii} = E_{ij}E_{ji}$, so $\varphi(E_{ii}) = \varphi(E_{ij})\varphi)E_{ji}) = 0$. To sum up, we have shown that φ is 0 on a basis of $M_n(\mathbb{C})$, so $\varphi = 0$.

   b) Let $\varphi : \mathrm{End}(V) \to \mathbb{C}$ be a multiplicative linear functional. As in (a), as the key is to note that, if $T \in \mathrm{End}(V)$ is such that $T^2 = 0$, then we have $\varphi(T)^2 = 0$, hence $\varphi(T) = 0$. Now choose two closed subspaces V_1 and V_2 such that $V = V_1 \oplus V_2$ and that V_1 and V_2 are isomorphic. (This is possible because V is infinite-dimensional. For example, choose a Hilbert basis $(e_i)_{i \in I}$ of V. As I is infinite, we can find $I_1, I_2 \subset I$ such that $I = I_1 \sqcup I_2$ and that there exists a bijection between I_1 and I_2. Take $V_r = \overline{\bigoplus_{i \in I_r} \mathbb{C}e_i}$, for $r = 1, 2$.)

   Choose isomorphisms $U_1 : V_1 \xrightarrow{\sim} V_2$ and $U_2 : V_2 \xrightarrow{\sim} V_1$. Let $T_1 \in \mathrm{End}(V)$ be defined by $T_1(v + w) = U_1(v)$ if $v \in V_1$ and $w \in V_2$, and $T_2 \in \mathrm{End}(V)$ be defined by $T_2(v + w) = U_2(w)$ if $v \in V_1$ and $w \in V_2$. Then $T_1^2 = T_2^2 = 0$, so $\varphi(T_1) = \varphi(T_2) = 0$, and also $\varphi(T_1 + T_2) = 0$. But $T := T_1 + T_2$ is an automorphism of V, so, for every $T' \in \mathrm{End}(V)$, we have $T' = T(T^{-1}T')$, hence $\varphi(T') = \varphi(T)\varphi(T^{-1}T') = 0$.

   □

2. Let V be a finite-dimensional Hilbert space. The goal of this problem is to relate the spectral theorem of the notes (theorem II.4.1) with the usual finite-dimensional spectral theorem (which says that a normal endomorphism of V is diagonalizable in an orthonormal basis).

   Remember that, if R is a commutative ring, we say that $x \in R$ is *nilpotent* if there exists an integer $n \geq 1$ such that $x^n = 0$, and we say that R is *reduced* if the only nilpotent element of R is 0.

   a) (2) Show that the usual finite-dimensional spectral theorem (as stated above) implies theorem II.4.1 for V.

b) (2) Let $T \in \text{End}(V)$, and let A be the unital subalgebra of $\text{End}(V)$ generated by T (i.e. the space of polynomials in T). Show that T is diagonalizable if and only if A is reduced.

c) (3) Let A be a commutative unital subalgebra of $\text{End}(V)$. If A is reduced, show that there exist subspaces $V_1, \ldots, V_r$ of V, uniquely determined up to ordering, such that $V = \bigoplus_{i=1}^r V_i$ and that

$$A = \{T \in \text{End}(V) | \forall i \in \{1, \ldots, r\},\ T(V_i) \subset V_i \text{ and } T_{|V_i} \in \mathbb{C} \cdot \text{id}_{V_i}\}.$$

d) (2) Let A be as in question (c). Show that A is stable by the map $T \mapsto T^*$ if and only the V_i are pairwise orthogonal.

e) (1) Show that theorem II.4.1 implies the usual finite-dimensional spectral theorem (as stated above).

Solution.

a) Let $T \in \text{End}(V)$ be a normal endomorphism. By the finite-dimensional spectral theorem, we can find an orthonormal basis $(e_1, \ldots, e_n)$ of V and $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ such that $T(e_i) = \lambda_i e_i$ for every $i \in \{1, \ldots, n\}$. As the basis is orthonormal, we also have $T^*(e_i) = \overline{\lambda}_i e_i$ for every $i \in \{1, \ldots, n\}$. After rearranging the e_i, we may also assume that we have $1 \leq n_0 \leq \ldots \leq n_r = n + 1$ such that $\lambda_i = \lambda_j$ if there exists $s \in \{0, \ldots, r-1\}$ with $n_s \leq i, j \leq n_{s+1} - 1$ and $\lambda_i \neq \lambda_j$ otherwise.

In particular, we may assume that $V = \mathbb{C}^n$ and that T is the diagonal matrix with diagonal entries $\lambda_1, \ldots, \lambda_n$, with the same conditions on the λ_i. I claim that A_T is the subalgebra of diagonal matrices in $M_n(\mathbb{C})$ with diagonal entries $x_1, \ldots, x_n$ satisfying : $x_i = x_j$ if there exists $s \in \{0, \ldots, r-1\}$ with $n_s \leq i, j \leq n_{s+1} - 1$. First, this does define a subalgebra of $M_n(\mathbb{C})$. It is also clear that every matrix in A_T is of this form, because A_T is generated (as an algebra) by I_n, T and T^*, and all three of these matrices satisfy the condition defining A_T. Finally, let $X \in A_T$, and let $x_1, \ldots, x_n$ be its diagonal entries. By Lagrange interpolation, there exists a polynomial $P \in \mathbb{C}[t]$ such that $P(\lambda_{n_s}) = x_{n_s}$ for $s \in \{0, \ldots, r-1\}$, and then $P(T)$ is the diagonal matrix with entries $x_1, \ldots, x_n$, i.e. X.

b) Let $P \in \mathbb{C}[t]$ be the minimal polynomial of T. Then $\mathbb{C}[t] \to M_n(\mathbb{C})$, $f(t) \mapsto f(T)$ is a morphism of $\mathbb{C}$-algebra with image A and kernel the ideal generated by P, by definition of the minimal polynomial. So $A \simeq \mathbb{C}[t]/(P)$. If we write $P(t) = \prod_{i=}^r (t - a_i)^{n_i}$ with $a_1, \ldots, a_r \in \mathbb{C}$ pairwise distinct and $n_1, \ldots, n_r \geq 1$, then, by the Chinese remainder theorem, $A \simeq \prod_{i=1}^n \mathbb{C}[t]/(t - a_i)^{n_i}$. So A is reduced if and only if all the n_i are equal to 1, i.e., if and only if P has only simple roots, which is equivalent to the fact that T is diagonalizable.

c) if $T \in A$, then the unital subalgebra of $\text{End}(V)$ generated by T is contained in A, and in particular it is reduced; by question (b), this implies that T is diagonalizable. As A by a finite number of elements (because it is a finite-dimensional $\mathbb{C}$-vector space), and these are diagonalizable and commute with each other, we can find a basis $(e_1, \ldots, e_n)$ in which every element of A is diagonal. For $i \in \{1, \ldots, n\}$, define $\varphi_i : A \to \mathbb{C}$ by $T(e_i) = \varphi_i(T) e_i$, for $T \in A$. Then $\varphi_1, \ldots, \varphi_n$ are multiplicative functionals on A. After reordering the e_i, we may assume that we have $1 \leq n_0 \leq \ldots \leq n_r = n+1$ such that $\varphi_i = \varphi_j$ if there exists $s \in \{0, \ldots, r-1\}$ with $n_s \leq i, j \leq n_{s+1} - 1$ and $\varphi_i \neq \varphi_j$ otherwise.

Note that all the φ_i are nonzero (because they send I_n to 1), so they are surjective. I claim that $\varphi_{n_0}, \varphi_{n_1} \ldots, \varphi_{n_{r-1}}$ are linearly independent (as function $A \to \mathbb{C}$). This is a classical result, but let's prove it quickly. Suppose that it is not true, and choose a nontrivial relation of linear dependence $\sum_{i=0}^{r-1} a_i\varphi_{n_i} = 0$, with $a_i \in \mathbb{C}$, such that the number of nonzero a_i is minimal. There are at least two nonzero a_i, so, up to reordering, we may assume that $a_0, a_1 \neq 0$. Choose $x_0 \in A$ such that $\varphi_{n_1}(x_0) \neq \varphi_{n_1}(x_0)$. Then, for every $x \in A$,

$$0 = \varphi_{n_0}(x_0)\sum_{i=0}^{r-1} a_i\varphi_{n_i}(x) - \sum_{i=0}^{r-1} a_i\varphi_{n_i}(x_0x) = \sum_{i=1}^{r-1} a_i(\varphi_{n_0}(x_0) - \varphi_{n_i}(x_0))\varphi_{n_i}(x).$$

So $\sum_{i=1}^{r-1} a_i(\varphi_{n_0}(x_0) - \varphi_{n_i}(x_0))\varphi_{n_i} = 0$, with $a_1(\varphi_{n_1}(x_0) - \varphi_{n_1}(x_0)) \neq 0$. So we have another nontrivial relation of linear dependence among the φ_{n_i}, and it has fewer nonzero coefficients than the first one, which is a contradiction.

Now that we know that $\varphi_{n_0}, \varphi_{n_1} \dots, \varphi_{n_{r-1}}$ are linearly independent, we also know that $(\varphi_{n_0}, \varphi_{n_1} \dots, \varphi_{n_{r-1}}) : A \to \mathbb{C}^r$ is surjective. For $i \in \{1, \dots, r\}$, let $V_i = \mathrm{Span}(e_{n_{i-1}}, \dots, e_{-1+n_i}) \subset V$. Then, if T is in A, T acts as a multiple of id on each V_i, and the surjectivity of $(\varphi_{n_0}, \varphi_{n_1} \dots, \varphi_{n_{r-1}}) : A \to \mathbb{C}^r$ implies that the converse is true. (If $(a_1, \dots, a_r) \in \mathbb{C}^r$, choose $T \in A$ such that $\varphi_{n_i} = a_{i+1}$ for $0 \leq i \leq r-1$. Then T acts on each V_i by multiplication by $\varphi_{n_{i+1}}(T) = a_i$).

Finally, let's show that $V_1, \dots, V_r$ are uniquely determined. Let $V = V_1' \oplus \dots \oplus V_s'$ be another decomposition satisfying the same property. Choose $a_1, \dots, a_r \in \mathbb{C}$ pairwise distinct, and let $T \in A$ such that $T_{|V_i} = a_i \mathrm{id}_{V_i}$ for every i. Then $V_1, \dots, V_r$ are the eigenspaces of T, and T acts by a multiple of identity on each V_j', so we must have a partition $I_1, \dots, I_s$ of $\{1, \dots, r\}$ such that $V_j' = \bigoplus_{i \in I_j} V_i$ for every $j \in \{1, \dots, s\}$. But the roles of the V_i and the V_j' are symmetric, so we have a similar property with V_i and V_j' exchanged. This implies that $r = s$ and that $V_1', \dots, V_r'$ are equal to $V_1, \dots, V_r$ up to reordering.

d) For $i \in \{1, \dots, r\}$, we define a linear endomorphism $\pi_i : V \to V$ by $\pi_i(v_1 \dots + v_r) = v_i$ if $v_j \in V_j$ for $j \in \{1, \dots, r\}$. Then $\mathrm{Im}(\pi_i) = V_i$, $\mathrm{Ker}(\pi_i) = \bigoplus_{j \neq i} V_j$ and $\pi_1 + \dots + \pi_r = \mathrm{id}_V$. Note that A is exactly the subalgebra $\{\sum_{i=1}^r \lambda_i\pi_i,\ \lambda_1, \dots, \lambda_r \in \mathbb{C}\}$ of $\mathrm{End}(T)$.

If $V_1, \dots, V_r$ are pairwise orthogonal, then $\pi_1, \dots, \pi_r$ are orthogonal projections, so they are self-adjoint, and so A is stable by $T \mapsto T^*$.

Conversely, suppose that A is stable by $T \mapsto T^*$. If $v \in V$ and $w \in V_1^\perp$, then

$$0 = \langle \pi_1(v), w \rangle = \langle v, \pi_1^*(w) \rangle.$$

This implies that $V_1^\perp \subset \mathrm{Ker}(\pi_1^*)$. As $\mathrm{rank}(\pi_1^*) = \mathrm{rank}(\pi_1) = \dim(V_1)$, we actually have $\mathrm{Ker}(\pi_1^*) = V_1^\perp$. But $\pi_1^* \in A$, so every eigenspace of π_1^* is a sum of V_i's, so there exists $I \subset \{1, \dots, r\}$ such that $\mathrm{Ker}(\pi_1^*) = \bigoplus_{i \in I} V_i$. As $V_1^\perp \cap V_1 = \{0\}$, the set I cannot contain 1. But then the only way that $\ker(\pi_1^*)$ can have dimension $\dim(V) - n_1$ is if $I = \{2, \dots, r\}$. Finally, we have shown that

$$V_1^\perp = \mathrm{Ker}(\pi_1^*) = V_2 \oplus \dots \oplus V_r.$$

Repeating this procedure with the other π_i's, we see that, for every $i \in \{a, \dots, r\}$,

$$V_i^\perp = \bigoplus_{j \neq i} V_j.$$

e) Let $T \in \mathrm{End}(V)$, and let $\Phi : \mathcal{C}(\sigma(T)) \xrightarrow{\sim} A_T$ be as in theorem II.4.1. In particular, A_T is a commutative reduced subalgebra of $\mathrm{End}(V)$ (because $\mathcal{C}(\sigma(T))$ is reduced), and it is stable by $*$ (by definition), so, by (c) and (d), we have a decomposition

$V = V_1 \oplus \ldots \oplus V_r$ of V into pairwise orthogonal subspaces such that every element of A_T preserves this decomposition and acts as a scalar on each V_i. If we choose an orthonormal basis for each V_i and put these together, we'll get an orthonormal basis on V in which each element of A_T is diagonal. Now just remember that $T \in A_T$.

$\square$

3. In this problem, you are not allowed to use any of the results from section II.3 and II.4.

Let X be a locally compact Hausdorff topological space. Let $\overline{X}$ be the Alexandroff compactification of X. This means that $\overline{X} = X \cup \{\infty\}$, and that the open sets of $\overline{X}$ are the open subsets and the sets of the form $(X - K) \cup \{\infty\}$, where K is a compact subset of X.

a) (3) Show that $\overline{X}$ is a compact Hausdorff topological space, that X is open in $\overline{X}$, and that X is dense in $\overline{X}$ if and only if X is not compact.

b) (2) Show that $\mathcal{C}(\overline{X})$ is isomorphic to the Banach $*$-algebra that you get by adjoining a unit to $\mathcal{C}_0(X)$. (Don't forget to compare the topologies.)

c) (2) If X is compact, show that every proper ideal of $\mathcal{C}(X)$ is contained in one of the ideals $\mathfrak{m}_x = \{f \in \mathcal{C}(X) | f(x) = 0\}$, $x \in X$.

d) (3) In general, show that the map $X \to \sigma(\mathcal{C}_0(X))$, $x \mapsto (\varphi_x : f \mapsto f(x))$ is a homeomorphism.

Let A be a commutative Banach algebra. If I is an ideal of A, we set

$$V(I) = \{x \in \sigma(A) | \forall f \in A, \widehat{f}(x) = 0\}.$$

If N is a subset of $\sigma(A)$, we set

$$I(N) = \{f \in A | \forall x \in N, \widehat{f}(x) = 0\}.$$

e) (extra credit, 3) Suppose that X is compact. Show that, for every closed ideal I of $\mathcal{C}(X)$ and every closed subset N of $\sigma(\mathcal{C}(X)) \simeq X$, we have

$$I(V(I)) = I \text{ and } V(I(N)) = N.$$

(The result is still true without the assumption that X is compact (just use $\mathcal{C}_0(X)$ everywhere), I just didn't want to type the proof.)

Solution.

a) First we show that the definition does give a topology on $\overline{X}$. Let $(U_i)_{i \in I}$ be a family of open subsets of $\overline{X}$. Then we can write $I = I' \sqcup I''$, with $U_i \subset X$ open and $i \in I'$ and $U_i = \overline{X} - K_i$ with K_i compact if $i \in I''$. We have

$$\bigcup_{i \in I} U_i = \left(\bigcup_{i \in I'} U_i \right) \cup \left(\overline{X} - \bigcap_{i \in I''} K_i \right).$$

If I'' is empty, this is an open subset of X, hence an open subset of $\overline{X}$. Otherwise, this is the complement on the compact subset $\bigcap_{i \in I''} K_i - \bigcup_{i \in I'} U_i$ of X, so it is again an open subset of $\overline{X}$. On the other hand, we have

$$\bigcap_{i \in I} U_i = \left(\bigcap_{i \in I'} U_i \right) \cap \left(\overline{X} - \bigcup_{i \in I''} K_i \right).$$

Suppose that I is finite. Then, if $I'' = \varnothing$, the set $\bigcap_{i\in I} U_i$ is the open subset $\bigcap_{i\in I'} U_i$ of X, hence it is an open subset of $\overline{X}$. Otherwise, it is the complement of the compact subset $\bigcup_{i\in I''} K_i - \bigcap_{i\in I'} U_i$ of X, hence it is again an open subset of $\overline{X}$.

Let's show that $\overline{X}$ is Hausdorff. Let $x, y \in \overline{X}$ such that $x \neq y$. We want to find disjoint open neighborhoods of x and y. If $x, y \in X$, then there exists open subsets U and V of X such that $x \in U$, $y \in V$ and $U \cap V = \varnothing$. These sets are still open in $\overline{X}$, so we are done. If one of x or y is ∞, we may assume that it is x. As X is locally compact, we can find a compact subset K of X and an open subset V of X such that $y \in V \subset K$. Then $U := \overline{X} - K$ is an open subset of $\overline{X}$ containing $x = \infty$, and we have $U \cap V = \varnothing$.

Let's show that $\overline{X}$ is compact. Let $(U_i)_{i\in I}$ be a family of open subsets of $\overline{X}$ such that $\overline{X} = \bigcup_{i\in I} U_i$. Let $i_0 \in I$ be such that $\infty \in U_{i_0}$, and write $K = \overline{X} - U_{i_0}$. This is a compact subset of X, and it is covered by the open subsets $U_i \cap X$, $i \in I - \{i_0\}$. So there exists a finite subset J of $I - \{i_0\}$ such that $K \subset \bigcup_{i\in J} U_i$, and then we have $\overline{X} = \bigcup_{i\in J\cup\{i_0\}} U_i$.

The set X is open in $\overline{X}$ by definition of the topology of $\overline{X}$.

Suppose that X is not compact. Then, if U is an open neighborhood of ∞ in $\overline{X}$, the compact subset $\overline{X} - U$ of X cannot be equal to X, which means that $U \cap X \neq \varnothing$. So ∞ is in the closure of X in $\overline{X}$. Conversely, suppose that X is compact. Then $\{\infty\} = \overline{X} - X$ is an open subset of $\overline{X}$, so ∞ is an isolated point of $\overline{X}$.

b) Let A be the Banach $*$-algebra that you get by adjoining a unit to $\mathcal{C}_0(X)$. We have $A = \mathcal{C}_0(X) \oplus \mathbb{C}e$, with $\|f + \lambda e\| = \|f\|_\infty + |\lambda|$ and $(f + \lambda e)^* = \overline{f} + \overline{\lambda} e$ (for $f \in \mathcal{C}_0(X)$ and $\lambda \in \mathbb{C}$).

Note that we can extend every $f \in \mathcal{C}_0(X)$ to a continuous function f on $\overline{X}$ by setting $f(\infty) = 0$. (The condition that f is 0 at infinity exactly says that the extended function is continuous, by definition of the topology on $\overline{X}$.) This gives an injective $\mathbb{C}$-algebra map $\mathcal{C}_0(X) \to \mathcal{C}(\overline{X})$. So we get a map $\alpha : A \to \mathcal{C}(\overline{X})$ sending $f + \lambda e$ to $f + \lambda$, where the second "λ" is the constant function on $\overline{X}$. This α is a morphism of $\mathbb{C}$-algebras by definition of the multiplication on A, and it is a $*$-homomorphism by definition of $*$ on A. Also, α is bounded, because, if $f \in \mathcal{C}_0(X)$ and $\lambda \in \mathbb{C}$, we have

$$\|f + \lambda\|_\infty \leq \|f\|_\infty + |\lambda| = \|f + \lambda e\|.$$

Finally, note that α is surjective, because it has an inverse sending $f \in \mathcal{C}(\overline{X})$ to $(f_{|X} - \lambda) + \lambda e$. By the open mapping theorem (also known as the Banach-Schauder theorem), the inverse of α is also bounded, so α is a homeomorphism.

c) Let I be an ideal of $\mathcal{C}(X)$, and suppose that I is not contained in any $\mathfrak{m}_x$. Then, for every $x \in X$, we can find $f_x \in I$ such that $f_x(x) \neq 0$; as f_x is continuous, we can also find an open neighborhood U_x of x such that $f_x(y) \neq 0$ for every $y \in U_x$. We have $X = \bigcup_{x\in X} U_x$ and X is compact, so there exist $x_1, \ldots, x_n \in X$ such that $X = \bigcup_{i=1}^n U_{x_i}$. Let $f = \sum_{i=1}^n |f_{x_i}|^2 = \sum_{i=1}^n \overline{f_{x_i}} f_{x_i}$. Then $f \in I$ because I is an ideal, and f doesn't vanish on X; indeed, if $x \in X$, we can find $i \in \{1, \ldots, x\}$ such that $x \in U_{x_i}$, and then $f(x) \geq |f_{x_i}(x)|^2 > 0$. So the function $g : x \mapsto f(x)^{-1}$ exists and is continuous on X, and we have $gf = 1$, which implies that $1 \in I$, hence that $I = \mathcal{C}(X)$.

d) Let's call this map α. First we show that α is injective. If $x, y \in X$ are such that $x \neq y$, then there exists $f \in \mathcal{C}_0(X)$ such that $f(x) \neq f(y)$ (by Urysohn's lemma), so $\varphi_x \neq \varphi_y$.

Let's show that α is surjective. Let $\varphi : \mathcal{C}_0(X) \to \mathbb{C}$ be a multiplicative functional. We can extend it to a multiplicative functional $\widetilde{\varphi}$ on $\mathcal{C}_0(X)_e$, and we have seen in

(b) that $\mathcal{C}_0(X)_e$ is isomorphic to $\mathcal{C}(\overline{X})$. Let $I = \mathrm{Ker}(\widetilde{\varphi})$. This is a maximal ideal of $\mathcal{C}(\overline{X})$, hence, by (c), there exists $x \in X$ such that $I \subset \mathfrak{m}_x$, and we must have $I = \mathfrak{m}_x$ because I is maximal. Also, note that the isomorphism $\mathcal{C}_0(X)_e \simeq \mathcal{C}(\overline{X})$ constructed in (b) identities $\mathcal{C}_0(X)$ to $\mathfrak{m}_\infty$. Hence, as φ is not 0 on $\mathcal{C}_0(X)$,w e cannot have $x = \infty$, so $x \in X$, and we have $\mathrm{Ker}(\varphi) = \{f \in \mathcal{C}_0(X) | f(x) = 0\} = \mathrm{Ker}(\varphi_x)$. As in the proof of theorem II.2.10 of the notes, this easily implies that $\varphi = \varphi_x$.

The map α is continuous by definition of the topology on $\sigma(\mathcal{C}_0(X))$. If X is compact, this implies that α is a homeomorphism. In general, the analogue of α for the Alexandroff compactification $\overline{X}$ of X is a homeomorphism because $\overline{X}$ is compact, and its restriction to X is α (if we identify $\mathcal{C}_0(X)$ to a subalgebra of $\mathcal{C}(\overline{X})$ as in (b)), so α is open, and we are done.

e) Note that, if N is a closed subset of X, then $I(N) = \bigcap_{x \in N} \mathfrak{m}_x$, so $I(N)$ is an ideal of $\mathcal{C}(X)$.

Let I be a closed ideal of $\mathcal{C}(X)$, and let $N = N(I)$. For every $x \in N$ and every $f \in I$, we have $f(x) = 0$ by definition of $N(I)$. So $I \subset \bigcap_{x \in N} \mathfrak{m}_x = I(N)$. Conversely, let $f \in \bigcap_{x \in N} \mathfrak{m}_x$; we want to show that $f \in I$. By assumption, $f(x) = 0$ for every $x \in N$, so $\mathrm{supp}(f) \cap N = \varnothing$. For every $y \in \mathrm{supp}(f)$, choose $f_y \in I$ such that $f_y(y) \neq 0$; as f_y is continuous, we can find an open subset $U_y \ni y$ of X such that $f_y(z) \neq 0$ for every $z \in U_y$. We have $\mathrm{supp}(f) \subset \bigcup_{y \in \mathrm{supp}(f)} U_y$ and $\mathrm{supp}(f)$ is compact, so we can find $y_1, \dots, y_n \in \mathrm{supp}(f)$ such that $\mathrm{supp}(f) \subset \bigcup_{i=1}^n U_{y_i}$. Let $g = \sum_{i=1}^n |f_{y_i}|^2$. Then $g \in I$, and $g(y) > 0$ for every $y \in \mathrm{supp}(f)$. Define a function $h : X \to \mathbb{C}$ by

$$h(x) = \begin{cases} f(x)g(x)^{-1} & \text{if } x \in \mathrm{supp}(f) \\ 0 & \text{otherwise.} \end{cases}$$

Let $U = \{x \in X | g(x) \neq 0\}$ and $V = X - \mathrm{supp}(f)$. Then U and V are open subsets of X and $X = U \cup V$. On U, the function h is equal to fg^{-1}, hence continuous; on V, it is equal to 0, hence also continuous. So $h \in \mathcal{C}(X)$, and we have $f = gh$ by definition of h. As $g \in I$, this shows that $f \in I$, as desired.

Now let N be a closed subset of X, and let $I = I(N)$. For every $x \in N$ and every $f \in I$, we have $f(x) = 0$ by definition of $I(N)$, so $N \subset V(I)$. Conversely, if $x \notin N$, then, by Urysohn's lemma, we can find $f \in \mathcal{C}(X)$ such that $f_{|N} = 0$ and $f(x) \neq 0$. Then $f \in I$ by definition of $I(N)$, so $x \notin V(I)$.

□

4. Consider the Banach $*$-algebra $\ell^1(\mathbb{Z})$ (i.e. $L^1(G)$ for the discrete group $G = \mathbb{Z}$, with the convolution product and the involution defined in class). We write elements of $\ell^1(\mathbb{Z})$ as sequences $a = (a_n)_{n \in \mathbb{Z}}$ in $\mathbb{C}^{\mathbb{Z}}$.

   a) (1) Show that $\ell^1(\mathbb{Z})$ is not a C^*-algebra.

   b) (3) Show that there is a homeomorphism $\sigma(\ell^1(\mathbb{Z})) \xrightarrow{\sim} S^1$ such that the Gelfand transform of $a = (a_n)_{n \in \mathbb{Z}}$ is the function $S^1 \to \mathbb{C}$, $e^{i\theta} \mapsto \sum_{n=-\infty}^{+\infty} a_n e^{in\theta}$. [1]

   c) (extra credit, 3) More generally, if G is a commutative locally compact group, show that the map $\widehat{G} \to \sigma(L^1(G))$ sending χ to the morphism $L^1(G) \to \mathbb{C}$, $f \mapsto \int_G f(x)\chi(x)dx$ is a homeomorphism. (Hint : What is the dual of $L^1(G)$?)

Solution.

[1] This means that the Gelfand transform is a $*$-homomorphism, i.e. the Banach $*$-algebra $L^1(G)$ is symmetric, even though it is not a C^*-algebra.

a) Let $a = (a_n)_{n\in\mathbb{Z}}$. Then $a^* = (\overline{a}_{-n})_{n\in\mathbb{Z}}$ (remember that $\mathbb{Z}$ is unimodular, because it is commutative (or because it is discrete)). Let $b = a^* * a$. We have, for every $n \in \mathbb{Z}$,

$$b_n = \sum_{m\in\mathbb{Z}} a^*_m a_{n-m} = \sum_{m\in\mathbb{Z}} \overline{a}_{-m} a_{n-m}.$$

Take a defined by $a_0 = i$, $a_1 = 1$, $a_2 = i$ and $a_n = 0$ for $n \in \mathbb{Z} - \{0, 1, 2\}$. Then $a^*_0 = -i$, $a^*_{-1} = 1$, $a^*_2 = -i$, and $a^*_n = 0$ if $n \in \mathbb{Z} - \{-2, -1, 0\}$. So $b_n = 0$ if $n \notin \{-2, -1, 0, 1, 2\}$, and we have

$$b_{-2} = a^*_{-2} a_0 = 1,$$

$$\begin{aligned} b_{-1} &= a^*_{-1}a_0 + a^*_{-2}a_1 = i - i = 0, \\ b_0 &= a^*_{-2}a_2 + a^*_{-1}a_1 + a^*_0 a_0 = 3, \\ b_1 &= a^*_0 a_1 + a^*_{-1}a_2 = -i + i = 0, \end{aligned}$$

and

$$b_2 = a^*_0 a_2 = 1.$$

So $|b|_1 = 5 \neq |a|_1^2 = 9$.

c) Let G be a commutative locally compact group. Let $\varphi \in \sigma(L^1(G))$. We want to show that φ comes from an element χ of $\widehat{G}$. As φ is a continuous linear functional on $L^1(G)$, there exists $\chi \in L^\infty(G)$ such that $\varphi(f) = \int_G f(x)\chi(x)dx$ for every $f \in L^1(G)$. For every $f, g \in L^1(G)$, we have

$$\begin{aligned} \varphi(f) \int_G g(y)\chi(y)dy &= \varphi(f)\varphi(g) \\ &= \varphi(g * f) \\ &= \int_{G\times G} g(y) f(y^{-1}x)\chi(x)dxdy \\ &= \int_G g(y)\varphi(L_y f)dy. \end{aligned}$$

As this is true for every $g \in L^1(G)$, the functions $\varphi(f)\chi$ and $y \mapsto \varphi(L_y f)$ (both in $L^\infty(G)$ are equal almost everywhere. Hence, if we choose $f \in L^1(G)$ such that $\varphi(f) \neq 0$, we can replace χ by $y \mapsto \varphi(f)^{-1}\varphi(L_y f)$. As the functions $\varphi : L^1(G) \to \mathbb{C}$ and $G \to L^1(G)$, $y \mapsto L_y f$ are continuous (the second by proposition I.3.1.13 in the notes), this new χ is continuous. Also, we have $\varphi(g)\chi(y) = \varphi(L_y g)$ for every $g \in L^1(G)$ and every $y \in G$.

Let $x, y \in G$. As $L_{xy} f = L_x(L_y f)$, we have

$$\begin{aligned} \varphi(L_{xy} f) &= \chi(xy)\varphi(f) \\ &= \chi(x)\varphi(L_y f) \\ &= \chi(x)\chi(y)\varphi(f), \end{aligned}$$

so $\chi(xy) = \chi(x)\chi(y)$. So $\chi \in \widehat{G}$, and we have shown that the map $\widehat{G} \to \sigma(L^1(G))$ of the problem is surjective. Note that this map is also injective, because a continuous function on G is determined by the linear functional it defines on $L^1(G)$. Also, the topology on $\sigma(L^1(G)) \subset L^\infty(G)$ is the weak* topology by definition, and we have seen in problem 6(a) of problem set 3 that this coincides with the topology on compact convergence on $\widehat{G}$, so the map $\widehat{G} \to \sigma(L^1(G))$ is a homeomorphism.

b) We know that $\widehat{\mathbb{Z}} \simeq S^1$ by 5(d) of problem set 3, so we get a homeomorphism $S^1 = \widehat{\mathbb{Z}} \xrightarrow{\sim} \sigma(\ell^1(\mathbb{Z}))$ by question c). Unpacking the formulas, we see that it sends $z \in S^1$ to the multiplicative functional $a = (a_n)_{n\in\mathbb{Z}} \mapsto \sum_{n\in\mathbb{Z}} a_n z^n$ on $\ell^1(\mathbb{Z})$, which is exactly what we wanted.

□

5. (3) Let A be a unital $\mathbb{C}$-algebra with an involutive anti-isomorphism $*$. Show that there is at most one norm on A that makes A into a C^*-algebra.

Solution. Let $\|.\|$ be a norm on A that makes A into a C^*-algebra. Let $x \in A$. Note that $(x^*x)^* = x^*x$, so x^*x is normal. By definition of a C^*-algebra and corollary II.3.9 of the notes, we have

$$\|x\| = \|x^*x\|^{1/2} = \rho(x^*x)^{1/2}.$$

But, by theorem II.1.1.13 of the notes,

$$\rho(x^*x) = \max\{|\lambda|,\ \lambda \in \mathbb{C},\ x^*x - \lambda e \notin A^\times\}.$$

This last quantity only depends on the algebra structure of A and on $*$, and it determines $\|x\|$.

□

6. The goal of this problem it to prove I.3.2.13 of the notes, i.e. the fact that every irreducible unitary representation of a compact group is finite-dimensional.

Let G be a compact group, let dx be the normalized Haar measure on G, and let (π, V) be a nonzero unitary representation of G. Fix $u \in V - \{0\}$, and define $T : V \to V$ by

$$T(v) = \int_G \langle v, \pi(x)(u)\rangle \pi(x)(u)dx.$$

a) (2) Show that T is well-defined and that $T \in \mathrm{End}(V)$.

b) (1) Show that T is G-equivariant.

c) (1) Show that $\langle T(v), v\rangle \geq 0$ for every $v \in V$.

d) (1) Show that $T \neq 0$.

e) (2) Show that T is in the closure (for $\|.\|_{op}$) of $\{T' \in \mathrm{End}(V) |\ \dim_{\mathbb{C}}(\mathrm{Im}(T')) < +\infty\}$; in other words, T is in the closure of the space of endomorphisms of finite rank. (Hint : $G \to V$, $x \mapsto \pi(x)(u)$ is uniformly continuous.)

f) (2) Let B be the closed unit ball in V. Show that $\overline{T(B)}$ is compact. (In other words, the operator T is a compact operator. Problem 5 of PS4 can help shorten the proof.)

g) (1) If V is an irreducible representation of V, show that V is finite-dimensional.

Solution.

a) We must show that the integral defining $T(v)$ converges for every $v \in V$. Let $v \in V$. Then the function $G \to V$, $\langle v, \pi(x)(u)\rangle\pi(x)(u)$ is continuous (because $x \mapsto \pi(x)(u)$ is continuous); as G is compact, the integral exists by problem 2 of problem set 4, and moreover we have

$$\|T(v)\| \leq \int_G |\langle v, \pi(x)(u)\rangle| \|\pi(x)(u)\| dx \leq \|v\|\|u\|^2.$$

The function $T : V \to V$ is $\mathbb{C}$-linear (because addition and multiplication by a scalar are continuous on V, so they commute with the integral by 1(b) of problem set 4), and the inequality above shows that T is bounded and that $\|T\|_{op} \leq \|u\|^2$.

b) Let $v \in V$ and $x \in G$.

$$\begin{aligned}
T(\pi(x)(v)) &= \int_G \langle \pi(x)(v), \pi(y)(u)\rangle \pi(y)(u)dy \\
&= \int_G \langle v, \pi(x)^* \pi(y)(u)\rangle \pi(y)(u)dy \\
&= \int_G \langle v, \pi(x^{-1}y)(u)\rangle \pi(y)(u)dy \\
&= \int_G \langle v, \pi(y)(u)\rangle \pi(x)\pi(y)(u)dy
\end{aligned}$$

(by left invariance of the Haar measure for the last equality). As $\pi(x) : V \to V$ is continuous and linear, 1(b)1 of problem set 4 implies that the last line is equal to

$$\pi(x)\left(\int_G \langle v, \pi(y)(u)\rangle \pi(y)(u)dy\right) = \pi(x)(T(v)),$$

which is what we wanted.

c) Let $v \in V$. As $\overline{\langle v, .\rangle}$ is continuous and linear on V, we have

$$\begin{aligned}
\langle T(v), v\rangle &= \int_G \langle \pi(x)(u), v\rangle \langle v, \pi(x)(u)\rangle dx \\
&= \int_G |\langle v, \pi(x)(u)\rangle|^2 dx \\
&\geq 0.
\end{aligned}$$

d) Take $v = u$. As $\langle u, \pi(x)(u)\rangle = \|u\|^2 > 0$ and $x \mapsto \langle u, \pi(x)(u)\rangle$ is a continuous function from G to $\mathbb{C}$, there exists $\varepsilon > 0$ and an open neighborhood U of 1 in G such that $|\langle u, \pi(x)(u)\rangle|^2 \geq \varepsilon$ for $x \in U$. Then, by the calculation in the proof of (c), we have

$$\langle T(u), u\rangle = \int_G |\langle u, \pi(x)(u)\rangle|^2 dx \geq \varepsilon\mu(U) > 0.$$

So $T \neq 0$.

e) Let $\varepsilon > 0$. As G is compact, the continuous function $G \to \mathbb{C}$, $x \mapsto \pi(x)(u)$ is uniformly continuous, so there exists a neighborhood U of 1 such that, for $x \in G$ and $y \in xU$, we have $\|\pi(x)(u) - \pi(y)(u)\| \leq \varepsilon$. As G is compact and the family $(xU)_{x\in G}$ covers G, we can $x_1, \dots, x_n \in G$ such that $G = \bigcup_{i=1}^n x_iU$. Choose Borel subsets $E_1, \dots, E_n$ of X such that $x_i \in E_i \subset x_iU$ for every $i \in \{1, \dots, n\}$ and $X = E_1 \sqcup \dots \sqcup E_n$ (as sets). If $x \in E_i$ and $v \in V$, then we have

$$\begin{aligned}
&\|\langle v, \pi(x)(u)\rangle \pi(x)(u) - \langle v, \pi(x_i)(u)\rangle \pi(x_i)(u)\| \\
&\qquad \leq \|\langle v, (\pi(x) - \pi(x_i))(u)\rangle \pi(x)(u)\| + \|\langle v, \pi(x_i)(u)\rangle (\pi(x) - \pi(x_i)))(u)\| \\
&\qquad \leq \|v\|\varepsilon\|u\| + \|v\|\|u\|\varepsilon = 2\varepsilon\|v\|\|u\|.
\end{aligned}$$

Define $U \in \mathrm{End}(V)$ by

$$T(v) = \sum_{i=1}^n \mu(E_i)\langle v, \pi(x_i)(u)\rangle \pi(x_i)(u) = \sum_{i=1}^n \int_{E_i} \langle v, \pi(x_i)(u)\rangle \pi(x_i)(u)dx.$$

This operator U has finite rank, because its image is contained $\mathrm{Span}(\pi(x_1)(u), \dots, \pi(x_n)(u))$. Also, by the calculation above (and problem 2 of problem set 4), for every $v \in V$, we have

$$\begin{aligned}\|T(v) - U(v)\| &\leq \sum_{i=1}^{n} \left| \int_{E_i} \langle v, \pi(x)(u)\rangle \pi(x)(u)dx - \int_{E_i} \langle v, \pi(x_i)(u)\rangle \pi(x_i)(u)dx \right| \\ &\leq \sum_{i=1}^{n} \mu(E_i) 2\varepsilon \|v\| \|u\| \\ &= 2\varepsilon \|v\| \|u\|.\end{aligned}$$

So $\|T - U\|_{op} \leq 2\varepsilon\|u\|$. As $\varepsilon > 0$ was arbitrary, this shows that T is a limit of operators of finite rank.

f) By 5(e) of problem set 4, it suffices to show that $T(B)$ is totally bounded. Let U be a neighborhoof of 0, which we may assume to be an open ball of radius $\varepsilon > 0$. We must find $x_1, \dots, x_n \in B$ such that every point of $T(B)$ is at distance $< \varepsilon$ from one of the $T(x_i)$.

By (e), we know that T is a limit of operators of finite rank, so we can find $U \in \mathrm{End}(V)$ of finite rank such that $\|T - U\|_{op} \leq \varepsilon/4$. As U has finite rank, $\overline{U(B)}$ is a closed bounded subset of the finite-dimensional space $\mathrm{Im}(U)$, so it is compact. In particular, we can find $x_1, \dots, x_n \in B$ such that, for every $y \in B$, there exists $i \in \{1, \dots, n\}$ such that $\|U(y) - U(x_i)\| < \varepsilon/2$.

Now let $y \in B$, and choose $i \in \{1, \dots, n\}$ such that $\|U(y) - U(x_i)\| < \varepsilon/2$. Then

$$\begin{aligned}\|T(y) - T(x_i)\| &\leq \|T(y) - U(y)\| + \|U(y) - U(x_i)\| + \|U(x_i) - T(x_i)\| \\ &< \|y\|\varepsilon/4 + \varepsilon/2 + \|x_i\|\varepsilon/4 \\ &\leq \varepsilon.\end{aligned}$$

(Remember that y, x_i are in the closed unit ball of V.)

g) Now we put everything together. Suppose that V is an irreducible unitary representation of G. Then the operator $T \in \mathrm{End}(V)$ that we constructed is G-equivariant, so, by Schur's lemma, there exists $\lambda \in \mathbb{C}$ such that $T = \lambda \mathrm{id}_V$. As $T \neq 0$, $\lambda \neq 0$. So $T(\lambda B)$ is the closed unit ball in V. Part (f) says that this is compact, which, by Riesz's lemma, implies that V is finite-dimensional.

□

7. (extra credit) Let A be a C^*-algebra. Then A_e is a Banach $*$-algebra, but it is not always a C^*-algebra with the norm defined by $\|x + \lambda e\| = \|x\| + |\lambda$. (See question 3(b) for an example of this phenomenon.)

We define a new norm $\|.\|'$ on A_e by :

$$\|x + \lambda e\|' = \sup\{\|xy + \lambda y\|, \ y \in A, \ \|y\| \leq 1\}.$$

We now suppose that A does *not* have a unit and that $A \neq \{0\}$.

a) (2) Show that $\|.\|'$ is a submultiplicative norm on A_e.

b) (3) Show that $\|.\|'$ agrees with $\|.\|$ on A, that A is closed in A_e and that A_e is complete for $\|.\|'$.

c) (2) Show that A_e is a C^*-algebra for the norm $\|.\|'$.

Solution.

a) Let $x_1 = y_1 + \lambda_1 e$, $x_2 = y_2 + \lambda_2 e$ be elements of A_e ($y_1, y_2 \in A$ and $\lambda_1, \lambda_2 \in \mathbb{C}$) and $c \in \mathbb{C}$. Then

$$\begin{aligned}\|x_1 + x_2\|' &= \sup\{\|y_1 y + \lambda_1 y + y_2 y + \lambda_2 y\|,\ y \in A, \|y\| = 1\}\\ &\leq \sup\{\|y_1 y + \lambda_1 y\|,\ y \in A,\ \|y\| \leq 1\} + \sup\{\|y_2 y + \lambda_2 y\|,\ y \in A,\ \|y\| \leq 1\}\\ &= \|x_1\|' + \|x_2\|',\end{aligned}$$

$$\begin{aligned}\|cx_1\| &= \sup\{\|cy_1 y + c\lambda_1 y\|,\ y \in A,\ \|y\| \leq 1\}\\ &= |c| \sup\{\|y_1 y + \lambda_1 y\|,\ y \in A,\ \|y\| \leq 1\}\\ &= |c| \|x_1\|',\end{aligned}$$

and

$$\begin{aligned}\|x_1 x_2\|' &= \sup\{\|(y_1 y_2 + \lambda_2 y_1 + \lambda_1 y_2) y + \lambda_1 \lambda_2 y\|,\ y \in A, \|y\| = 1\}\\ &= \sup\{\|y_1(y_2 y + \lambda_2 y) + \lambda_1(y_2 y + \lambda_2 y)\|,\ y \in A, \|y\| = 1\}\\ &\leq \sup\{\|y_1 + \lambda_1 e\|' \|y_2 y + \lambda_2 y\|,\ y \in A,\ \|y\| \leq 1\}\\ &= \|x_1\|' \|x_2\|'.\end{aligned}$$

To show that $\|.\|'$ is a norm on A_e, we still need to show that $\|x + \lambda e\|' \neq 0$ if $x + \lambda e \neq 0$. Suppose that $\|x + \lambda e\|' = 0$, then $xy + \lambda y = 0$ for every $y \in A$ such that $\|y\| = 1$, hence for every $y \in A$. If $x = 0$, then $\lambda = 0$. If $x \neq 0$, then, taking $y = x^*$ (and noting that $xx^* \neq 0$ because $\|xx^*\| = \|x^*\|^2 \neq 0$), we see that $\lambda \neq 0$. Let, so $\lambda^{-1} xy = y$ for every $y \in A$, i.e. $\lambda^{-1} x$ is a left unit for A. This implies that $(\lambda^{-1} y)^*$ is a right unit for A, so A has a unit, contradicting our assumption. So $x = 0$.

b) If $x \in A$, then we have

$$\|x\|' = \sup\{\|xy\|,\ y \in A,\ \|y\| = 1\} \leq \|x\|.$$

If $x = 0$, then $\|x\|' = \|x\| = 0$. Otherwise, we also have $x^* \neq 0$; taking $y = \frac{1}{\|x^*\|} x^*$, we get

$$\|x\|' \geq \frac{1}{\|x^*\|} \|xx^*\| = \|x^*\| = \|x\|.$$

Hence A is complete for $\|.\|'$, so it is closed in A_e. In particular, the quotient map $A_e \to A_e/A \simeq \mathbb{C}$, $x + \lambda e \mapsto \lambda$ is continuous.

Now we show that A_e is complete for $\|.\|'$. Let $(x_n + \lambda_n e)_{n \geq 0}$ be a Cauchy sequence in A_e, with $x_n \in A$ and $\lambda_n \in \mathbb{C}$. By the previous paragraph, the sequence $(\lambda_n)_{n \geq 0}$ is Cauchy, so the sequence $(x_n)_{n \geq 0}$ in A is also Cauchy. As the two norms coincide on A, the sequence $(x_n)_{n \geq 0}$ converges to some $x \in A$, and of course $(\lambda_n)_{n \geq 0}$ converges to some $\lambda \in \mathbb{C}$. It is now clear (using the obvious fact that $\|z + \mu e\|' \leq \|z\| + |\mu|$ for $z \in A$ and $\mu \in \mathbb{C}$) that the sequence $(x_n + \lambda_n e)_{n \geq 0}$ converges to $x + \lambda e$ in A_e.

c) Finally, we show that A_e is a C^*-algebra. Let $x \in A$ and $\lambda \in \mathbb{C}$. We want to show that $\|(x + \lambda e)^*(x + \lambda e)\|' = (\|x + \lambda e\|')^2$. We may assume that $x + \lambda e \neq 0$. Let $\varepsilon > 0$. Then we can find $y \in A$ such that $\|y\| = 1$ and

$$\|xy + \lambda y\| \geq \|x + \lambda e\|'(1 - \varepsilon).$$

Note that $xy + \lambda y = (x + \lambda e)y$ (in A_e). So

$$\begin{aligned}(1 - \varepsilon)^2 (\|x + \lambda e\|')^2 &\leq \|xy + \lambda y\|^2\\ &= \|(xy + \lambda y)^*(xy + \lambda y)\|\\ &= \|y^*(x + \lambda e)^*(x + \lambda e)y\|'\\ &\leq \|y\|^2 \|(x + \lambda e)^*(x + \lambda e)\|'\\ &= \|(x + \lambda e)^*(x + \lambda e)\|'.\end{aligned}$$

As this is true for every ε, we get

$$\|(x+\lambda e)^*(x+\lambda e)\|' \geq (\|x+\lambda e\|')^2.$$

Using the submultiplicativity of the norm, we deduce that

$$\|x+\lambda e\|' \leq \|(x+\lambda e)^*\|'.$$

As $*$ is bijective on A_e, the last inequality is actually an equality, and so we also get

$$(\|x+\lambda e\|')^2 \leq \|(x+\lambda e)^*(x+\lambda e)\|' \leq (\|x+\lambda e\|')^2,$$

which finishes the proof.

$\square$