MAT 449 : Problem Set 4

Due Thursday, October 11

Vector-valued integrals

Note : You are allowed to use without proof the following results :

- The Hahn-Banach theorem.
- The fact that every continuous linear functional on a Hilbert space V is of the form $\langle ., v\rangle$, with $v \in V$.
- Hölder's inequality.
- The fact that, if (X, μ) is a measure space, and if $1 \leq p < +\infty$ and $1 < q \leq +\infty$ are such that $p^{-1} + q^{-1} = 1$, then the map $L^p(X, \mu) \to \mathrm{Hom}(L^q(X, \mu), \mathbb{C})$, $f \mapsto (g \mapsto \int_X fg d\mu)$ is an isomorphism that preserves the norm (L^p norm on the left, operator norm on the right). [1]

1. Let (X, μ) be a measure space and V be a Banach space. We write $V^\vee$ for $\mathrm{Hom}(V, \mathbb{C})$. We say that a function $f : X \to V$ is *weakly integrable* if, for every $T \in V^\vee$, the function $T \circ f : X \to \mathbb{C}$ is in $L^1(X, \mu)$. If f is weakly integrable and if there exists an element v of V such that $T(v) = \int_X T \circ f(x) d\mu(x)$ for every $T \in V^\vee$, we say that v is the *integral* of f on X and write $v = \int_X f(x) d\mu(x) = \int_X f d\mu$.

   a) (1) Show that the integral of f is unique if it exists.

   b) (2) Let W be another Banach space and $u \in \mathrm{Hom}(V, W)$. If $f : X \to V$ is weakly integrable and has an integral v, show that $u \circ f : X \to W$ is weakly integrable and has an integral, which is equal to $u(v)$.

   c) (1, extra credit) Give an example of a weakly intergrable function that doesn't have an integral.

   Solution.

   a) By the Hahn-Banach theorem, for every $v \in V$, there exists $T \in \mathrm{Hom}(V, \mathbb{C})$ such that $T(v) = \|v\|$ and $\|T\|_{op} \leq 1$. In particular, an element v of V is zero if and only if $T(v) = 0$ for every $T \in \mathrm{Hom}(V, \mathbb{C})$, or, in other words, two elements $v, w \in V$ are equal if and only if $T(v) = T(w)$ for every $T \in \mathrm{Hom}(V, \mathbb{C})$. This implies that the integral of f is unique if it exists.

[1] Technical note : This is not true in general for $p = 1$, $q = +\infty$ if μ is not σ-finite, but it can be salvaged for a regular Borel measure on a locally compact Hausdorff space by slightly modifying the definition of L^∞. You can ignore this.

b) We first show that $u \circ f$ is weakly integrable. Let $T \in \mathrm{Hom}(W, \mathbb{C})$. Then $T \circ u \in \mathrm{Hom}(V, \mathbb{C})$, so the function $T \circ u \circ f : X \to \mathbb{C}$ is integrable.

Now suppose that f has an integral v. Then, for every $T \in \mathrm{Hom}(W, \mathbb{C})$, we have $T \circ \mathrm{Hom}(V, \mathbb{C})$, so $\int_X T \circ u \circ f d\mu = T \circ u(v)$. This means that $u(v)$ is the integral of $u \circ f$.

c) Let $X = \mathbb{N}$ with the counting measure μ, and

$$V = c_0(\mathbb{N}) := \{(x_n)_{n\geq 0} \in \mathbb{C}^{\mathbb{N}} | \lim_{n\to+\infty} x_n = 0\}.$$

We will use the fact that $\ell^1(\mathbb{N})$ is the continuous dual of $c_0(\mathbb{N})$, via the map $\ell^1(\mathbb{N}) \times c_0(\mathbb{N}) \to \mathbb{C}$, $((x_n), (y_n)) \mapsto \sum_{n\geq 0} x_n y_n$, and that the continuous dual of $\ell^1(\mathbb{N})$ is $\ell^\infty(\mathbb{N})$ (by a similar map). The map from $c_0(\mathbb{N})$ into its bidual is the usual embedding $c_0(\mathbb{N}) \subset \ell^\infty(\mathbb{N})$.

We define $f : X \to V$ by $f(n) = \mathbb{1}_{\{n\}}$. Then, for every $(x_n)_{n\geq 0} \in \ell^1(\mathbb{N})$, if $T : c_0(\mathbb{N}) \to \mathbb{C}$ is the corresponding linear functional, we have

$$\int_X T(f(x)) d\mu(x) = \sum_{n\geq 0} x_n,$$

which converges because $(x_n)_{n\geq 0}$ is in $\ell^1(\mathbb{N})$. Hence f is weakly integrable. But f does not have an integral (at least in $c_0(\mathbb{N})$), because the continuous linear functional it defines on $\ell^1(\mathbb{N})$ is representable by an element of $\ell^\infty(\mathbb{N})$ which is not in $c_0(\mathbb{N})$ (the constant sequence 1). As evaluating on points of $\ell^1(\mathbb{N})$ separates the elements of $\ell^\infty(\mathbb{N})$, there cannot be any element of $c_0(\mathbb{N})$ giving the same linear functional on $\ell^1(\mathbb{N})$.

$\square$

2. In this problem, X is a locally compact Hausdorff space and μ is a regular Borel measure on X. Let V be a Banach space, and let $f : X \to V$ be a continuous function with compact support.

   a) (1) Show that f is weakly integrable.

   b) (1) If $\mu(\mathrm{supp}\, f) = 0$, show that $\int_X f d\mu$ exists and is equal 0.

   The goal if this problem is to show that:

   (i) f has an integral v;

   (ii) $\|v\| \leq \int_X \|f(x)\| d\mu(x)$;

   (iii) if $\mu(\mathrm{supp}\, f) \neq 0$, then $\mu(\mathrm{supp}\, f)^{-1} v$ is in the closure of the convex hull of $f(X)$.

   By question (b), we may (and will) assume that $\mu(\mathrm{supp}\, f) \neq 0$.

   c) (1) Show that we may assume that $X = \mathrm{supp}\, f$ (in particular, X is compact) and that $\mu(X) = 1$.

   From now on, we assume that X is compact and that $\mu(X) = 1$.

   d) (2) Let $T_1, \ldots, T_n : V \to \mathbb{R}$ be bounded $\mathbb{R}$-linear functionals (we see V as a $\mathbb{R}$-vector space in the obvious way), and define $a_1, \ldots, a_n \in \mathbb{R}$ by $a_i = \int_X T_i \circ f d\mu$. Show that $(a_1, \ldots, a_n)$ is in the convex hull of the compact subset $((T_1, \ldots, T_n) \circ f)(X)$ of $\mathbb{R}^n$. (Hint : What happens if it is not ?)

Let K be the closure of the convex hull of $f(X)$. This is a compact subset of V by problem 6. For every finite subset Ω of $\mathrm{Hom}_{\mathbb{R}}(V,\mathbb{R})$ (the space of bounded $\mathbb{R}$-linear functionals from V to $\mathbb{R}$), we denote by I_Ω the set of $v \in K$ such that, for every $T \in \Omega$, we have $T(v) = \int_X T \circ f d\mu$.

e) (1) Show that I_Ω is compact for every $\Omega \subset \mathrm{Hom}_{\mathbb{R}}(V,\mathbb{R})$.

f) (1) Show that I_Ω is nonempty if Ω is finite.

g) (1) Show that the integral of f exists and is in K.

h) (2) Show that :

$$\|\int_X f d\mu\| \leq \int_X \|f(x)\| d\mu(x).$$

(Hint : Hahn-Banach.)

Solution.

a) Let $T \in \mathrm{Hom}(V,\mathbb{C})$. Then $T \circ f : X \to \mathbb{C}$ is a continuous, and its support is contained in $\mathrm{supp}(f)$, hence compact. Hence $T \circ f$ is integrable.

b) Suppose that $\mu(\mathrm{supp}\, f) = 0$. If $T \in \mathrm{Hom}(V,\mathbb{C})$, then $T \circ f : X \to \mathbb{C}$ is continuous and $\mu(\mathrm{supp}(T \circ f)) = 0$, so $\int_X T \circ f(f) d\mu = 0$. This shows that 0 is the integral of f.

c) Suppose that we know the conclusion if $X = \mathrm{supp}\, f$ and $\mu(X) = 1$. Let $f : X \to V$ be continuous with compact support. We have already seen that we may assume $\mu(\mathrm{supp}\, f) \neq 0$, so let's do that. Let $X' = \mathrm{supp}\, f$, and consider the measure μ' on X' that is $\mu(\mathrm{supp}\, f)^{-1}$ times the restriction of μ. By our assumption, $\int_{X'} f_{|X'} d\mu'$ exists, let's call it v, we have $\|v\| \leq \int_{X'} \|f(x)\| d\mu'(x)$ and v is in the closure of the convex hull of $f(X')$.

Let's show that $w := \mu(\mathrm{supp}\, f)v$ is the integral of f. Note that $\mu(\mathrm{supp}\, f)^{-1} w$ is in the convex hull of $f(X') = f(X)$ and that

$$\|w\| \leq \mu(\mathrm{supp}\, f) \int_{X'} |f(x')| d\mu'(x) = \int_X |f(x)| d\mu(x),$$

so this proves the conclusion for f.

Let $T \in \mathrm{Hom}(V,\mathbb{C})$. Then $\mathrm{supp}(T \circ f) \subset X'$, so

$$\int_X T \circ f(x) d\mu(x) = \mu(\mathrm{supp}\, f) \int_{X'} T \circ f(x) d\mu'(x) = w.$$

So $w = \int_X f d\mu$.

d) Let $L = ((T_1, \dots, T_n) \circ f)(X)$. Suppose that $(a_1, \dots, a_n)$ is not in the convex hull of L. Then, by the hyperplane separation theorem, there exists a linear functional $\lambda : \mathbb{R}^n \to \mathbb{R}$ and $c > 0$ such that $\lambda(a_1, \dots, a_n) \geq c + \lambda(v)$, for every $v \in L$. Let $\lambda_1, \dots, \lambda_n \in \mathbb{R}$ be the images by λ of the vectors of the canonical basis of $\mathbb{R}^n$. Then we have, for every $x \in X$,

$$\sum_{i=1}^n \lambda_i a_i = \lambda(a_1, \dots, a_n) \geq c + \lambda \circ (T_1, \dots, T_n) \circ f(x) = c + \sum_{i=1}^n \lambda_i T_i(f(x)).$$

Taking the integral over X (and using $\mu(X) = 1$) gives

$$\sum_{i=1}^n \lambda_i a_i \geq c + \sum_{i=1}^n \lambda_i \int_X T_i \circ f(x) d\mu(x) = c + \sum_{i=1}^n \lambda_i a_i,$$

a contradiction.

e) For every $T \in \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{R})$, the set of $v \in K$ such that $T(v) = \int_X T \circ f d\mu$ is a closed subset of K. As the set I_Ω is an intersection of sets of this form, it is also a closed subset of K, hence compact because K is compact.

f) If Ω is finite, write $\Omega = \{T_1, \dots, T_n\}$ and $T_\Omega = (T_1, \dots, T_n) : V \to \mathbb{R}^n$. We have seen in question (d) that $a := \int_X T_\Omega(f(x))d\mu(x)$ is in the convex hull of $T_\Omega(f(X))$, so there exists $v \in V$ such that v is in the convex hull of $f(X)$ (hence in K) and $T_\Omega(v) = a = \int_V T_\Omega(f(x))d\mu(x)$. The second condition says exactly that $v \in I_\Omega$.

g) The subsets $(I_{\{T\}})_{T \in \mathrm{Hom}_{\mathbb{R}}(V,\mathbb{R})}$ of K have the finite intersection property by question (f). As K is compact, this implies that $\cap_{T \in \mathrm{Hom}_{\mathbb{R}}(V,\mathbb{R})} I_{\{T\}}$ is nonempty. Choose a vector v in it. Let $T \in \mathrm{Hom}(V, \mathbb{C})$. As $\mathrm{Re}(T)$ and $\mathrm{Im}(T)$ are in $\mathrm{Hom}_{\mathbb{R}}(V, \mathbb{R})$, we have

$$\begin{aligned} T(v) &= \mathrm{Re}(T(v)) + i\,\mathrm{Im}(T(v)) \\ &= \int_X \mathrm{Re}(T(f(x)))d\mu(x) + i \int_X \mathrm{Im}(T(f(x)))d\mu(x) \\ &= \int_X T(f(x))d\mu(x), \end{aligned}$$

so $v = \int_X f d\mu$. Also, $v \in K$ because all the $I_{\{T\}}$ are contained in K by definition.

h) By the Hahn-Banach theorem, there exists $T \in \mathrm{Hom}(V, \mathbb{C})$ such that $T(v) = \|v\|$ and $\|T\|_{op} \le 1$. Then

$$\|v\| = |T(v)| = |\int_X T(f(x))d\mu(x)| \le \int_X |T(f(x))|d\mu(x) \le \int_X \|f(x)\|d\mu(x).$$

□

3. In this problem, X is a locally compact Hausdorff space and μ is a regular Borel measure on X. Let V be a Banach space, let $f : X \to \mathbb{C}$ be a function in $L^1(X, \mu)$, and let $G : X \to V$ be a bounded continuous function.

   The goal of this problem is to show that :

   (i) the function $fG : X \to V$ has an integral v;

   (ii) $\|v\| \le (\sup_{x \in X} \|G(x)\|) \left(\int_X |f(x)|d\mu(x)\right)$;

   (iii) $v \in \overline{\mathrm{Span}(G(X))}$.

   a) (1) Show that fG is weakly integrable.

   b) (1) Let $(f_n)_{n \ge 0}$ be a sequence of functions of $\mathcal{C}_c(X)$ that converges to f in $L^1(X, \mu)$. Show that $\int f_n G d\mu$ exists for each $n \ge 0$, and that $(\int_X f_n G d\mu)_{n \ge 0}$ is a Cauchy sequence.

   c) (2) Prove assertions (i), (ii) and (iii) above.

   Solution.

   a) Let $T \in \mathrm{Hom}(V, \mathbb{C})$. Then, for every $x \in X$,

   $$|T(f(x)G(x))| \le |f(x)||T(G(x))| \le |f(x)|\|T\|_{op}\|G(x)\| \le |f(x)|\|T\|_{op} \sup_{y \in X} \|G(y)\|.$$

   As $\sup_{y \in X} \|G(y)\| < +\infty$ and $f \in L^1(X, mu)$, the function $T \circ (fG)$ is integrable. So fG is weakly integrable.

b) For every $h \in \mathcal{C}_c(X)$, the function $hG : X \to V$ is continuous and has support contained in $\operatorname{supp}(h)$, hence compact. By problem 2, this function is integrable, and we have

$$\| \int_X (hG)(x) d\mu(x) \| \leq \int_X |h(x)| \|G(x)\| d\mu(x) \leq \sup_{y \in X} \|G(y)\| \|h\|_1.$$

Applying this to f_n shows that $f_n G$ is integrable, and applying it to $f_n - f_m$ shows that

$$\| \int_X (f_n G) d\mu(x) - \int_X (f_m G) d\mu \| \leq \sup_{y \in X} \|G(y)\| \|f_n - f_m\|_1 \xrightarrow[n,m \to +\infty]{} 0$$

because $(f_n)_{n \geq 0}$ converges in $L^1(X, \mu)$.

c) As V is complete, the Cauchy sequence $(\int_X (f_n G) d\mu)$ has a limit in V, that we'll call v. For every $T \in \operatorname{Hom}(V, \mathbb{C})$, we have

$$T(v) = \lim_{n \to +\infty} T \left(\int_X (f_n G) d\mu \right) = \lim_{n \to +\infty} \int_X T(f_n(x) G(x)) d\mu(x).$$

As in (a), we have

$$\| \int_X T(f_n(x) G(x)) d\mu(x) - \int_X T(f(x) G(x)) d\mu(x) \| \leq \sup_{y \in X} \|G(y)\| \|T\|_{op} \|f_n - f\|_1,$$

so this converges to 0 as $n \to +\infty$, and we get

$$T(v) = \int_X T(f(x) G(x)) d\mu(x).$$

This shows that x is the integral of fG.

Moreover, by problem 2, $\int_X (f_n G) d\mu$ is in the closure of the span $G(X)$ for every $n \geq 0$. As v is the limit of these vectors, it is also in the close of $\operatorname{Span}(G(X))$.

Finally, to show the bound on $\|v\|$, we could use the Hahn-Banach theorem and the property characterising v as in question 2(h), or use the fact that

$$\| \int_X (f_n G) d\mu \| \leq \int_X |f_n(x)| \|G(x)\| d\mu(x) \leq \sup_{x \in X} \|G(x)\| \|f_n\|_1$$

for every $n \geq 0$ and that this sequence of integrals converges to v.

$\square$

4. Let G be a locally compact group, μ be a left Haar measure on G, and $L^1(G) = L^1(G, \mu)$. Let $f, g \in L^1(G)$.

   a) (1) Show that the function $G \to L^1(G)$, $y \mapsto f(y) L_y g$ is weakly integrable and has an integral.

   b) (2) Show that

$$f * g = \int_G f(y) L_y g d\mu(y).$$

Solution.

a) Note that the function $G \to L^1(G)$, $y \mapsto L_y g$ is continuous and that $\sup_{y \in G} \|L_y g\|_1 = \|g\|_1 < +\infty$. So the conclusion follows from problem 3.

b) Let $F = \int_G f(y) L_y g d\mu(y) \in L^1(G)$. By definition of the intergral, for every $h \in L^\infty(G)$, we have

$$\int_G h(x)F(x)d\mu(x) = \int_{G\times G} h(x)f(y)g(y^{-1}x)d\mu(y)d\mu(x) = \int_G h(x)(f*g)(x)d\mu(x).$$

As $L^\infty(G)$ is the continuous dual of $L^1(G)$, we have $f * g = F$ by question 1(a).

$\square$

5. Let G be a locally compact group, μ be a left Haar measure on G, and $L^1(G) = L^1(G, \mu)$. Let π be a unitary representation of G on a Hilbert space V, and let $f \in L^1(G)$.

We would like to define a continuous endomorphism $\pi(f)$ of V by setting $\pi(f) = \int_G f(x)\pi(x)d\mu(x)$, but we cannot apply problem 3 because the map $x \mapsto pi(x)$ is not continuous in general.

So we define $\pi(f)$ as we did in class, but with more details : Let $v \in V$. Then the function $G \to V$, $x \mapsto \pi(x)(v)$ is continuous, and it is bounded by $\|v\|$ because all the $\pi(x)$ are unitary, so problem 3 implies that $\int_G f(x)\pi(x)(v)dx$ exists, and that its norm is bounded by $\int_G |f(x)| \|\pi(x)(v)\| d\mu(x) = \|f\|_1 \|v\|$ (again using the fact that $\pi(x)$ is unitary for every $x \in G$). So we define a map $\pi(f) : V \to V$ by sending v to $\int_G f(x)\pi(x)(v)d\mu(x)$. It is easy to see that $\pi(f)$ is $\mathbb{C}$-linear, and we have just seen that $\|\pi(f)(v)\| \le \|f\|_1 \|v\|$ for every $v \in V$, which means that $\pi(f)$ is bounded and that $\|\pi(f)\|_{op} \le \|f\|_1$.

Let $v, w \in V$. Then $T \mapsto \langle T(v), w\rangle$ is a continuous linear functional on $\mathrm{End}(V)$, and we have

$$\langle \pi(f)(v), w\rangle = \langle \int_G f(x)\pi(x)(v)d\mu(x), w\rangle.$$

By question 2(b), this is equal to $\int_G f(x)\langle \pi(x)(v), w\rangle d\mu(x)$. Finally, it is easy to see that $\int_G f(x)\pi(x)d\mu(x)$ is linear in f. (If you want to check it rigorously, you can the use continuous linear functionals on $\mathrm{End}(V)$ defined in the previous paragraph, and then it's a straightforward calculation.) Also, we saw above that $\left\| \int_X f(x)\pi(x)d\mu(x) \right\| \le \|f\|_1$, so the linear map $L^1(G) \to \mathrm{End}(V)$ is continuous and its operator norm is ≤ 1.

Unfortunately, the linear functionals $T \mapsto \langle T(v), w\rangle$ do separate points on $\mathrm{End}(V)$, but they don't generate a dense subspace of $\mathrm{Hom}(\mathrm{End}(V), \mathbb{C})$ if V is infinite-dimensional, so the calculation above is not enough to prove that $\pi(f)$ is the integral of the function $G \to \mathrm{End}(V)$, $x \mapsto \pi(x)$. (In fact, I am not sure whether this is true or not.)

Let's denote the span of the functionals $T \mapsto \langle T(v), w\rangle$, $v, w \in V$ by MC (for "matrix coefficients"). I will try to prove that MC is not dense in $\mathrm{Hom}(\mathrm{End}(V), \mathbb{C})$. Let $\mathrm{End}(V)_c$ be the closure in $\mathrm{End}(V)$ of the space of operators of finite rank.[2] Then $\mathrm{End}(V)_c \subsetneq \mathrm{End}(V)$ if V is infinite-dimensional (see problem 6 of problem set 5). So, by the Hahn-Banach theorem, we can finite a nonzero bounded linear functional Λ on $\mathrm{End}(V)$ such that $\Lambda(\mathrm{End}(V)_c) = 0$. Let's show that Λ cannot be in $\overline{MC}$.

Choose a Hilbert basis $(e_i)_{i\in I}$. Then I claim that $\overline{MC}$ is exactly the space of Λ of the form $\Lambda(T) = \sum_{i,j\in I} a_{ij}\langle T(e_i), e_j\rangle$ with $a_{ij} \in \mathbb{C}$ and $\sum_{i,j} |a_{ij}|^2 < +\infty$. First, every linear functional of this form is in $\overline{MC}$, and the space of functionals of this form is closed. So we just need to prove that every functional $T \mapsto \langle T(v), w\rangle$ is of this form. Let $v, w \in V$, and write $v = \sum_{i\in I} a_i e_i$ and $w = \sum_{i\in I} b_i e_i$. We have $\sum_{i\in I} |a_i|^2 < +\infty$ and $\sum_{i\in I} |b_i|^2 < +\infty$, so

$$\sum_{i,j\in I} |a_i b_j|^2 = \left(\sum_{i\in I} |a_i|^2\right)\left(\sum_{j\in I} |b_j|^2\right) < +\infty.$$

[2]This is the space of compact operators, hence the notation.

As $\langle T(v), w\rangle = \sum_{i,j\in I} a_i \bar{b}_j \langle T(e_i), e_j\rangle$, we get the result.

Now suppose that $\Lambda \in \overline{MC}$, and write $\Lambda(T) = \sum_{i,j\in I} a_{ij}\langle T(e_i), e_j\rangle$ as above. For every $i \in I$ and every $j \in J$, we can find $T \in \mathrm{End}(V)$ such that $T(e_{i'}) = 0$ unless $i' = i$, and $T(e_i) = e_j$ (just take T defined by $T(v) = \langle v, e_i\rangle e_j$); then $T \in \mathrm{End}(V)_c$ because $\dim(\mathrm{Im}(T)) = 1$, so $0 = \Lambda(T) = a_{ij}$. This implies that $a_{ij} = 0$ for all $i, j \in I$, which contradicts the fact that $\Lambda \neq 0$.

6. (extra credit) Let V be a normed vector space. A subset A is V is called *totally bounded* if, for every neighborhood U of 0 in V, there is a finite set F such that $A \subset F + U$.

   a) (1) Show that the convex hull of a finite subset of V is compact.

   b) (1) Show that every compact subset of V is totally bounded.

   c) (1) If $A \subset V$ is totally bounded, show that $\overline{A}$ is totally bounded.

   d) (2) If A is a totally bounded subset of V, show that its convex hull is totally bounded. (Hint : Open balls are convex.)

   e) (2) If V is complete and A is totally bounded, show that $\overline{A}$ is compact.

   f) (1) If V is complete and $K \subset V$ is compact, show that the closure of the convex hull of K is compact.

   Solution.

   a) Let F be a finite subset subset of V. Then its convex hull is contained in $\mathrm{Span}(F)$, which is finite-dimensional. In a finite-dimensional vector space, the convex hull of any compact set is compact, so the convex hull of the finite set F is compact.

   b) Let K be compact subset of V, and let U be a neighborhood of 0 in V. We may assume that U is open. Then $K \subset \bigcup_{x\in K}(x + U)$. As K is compact, there exists a finite subset F of K such that $K \subset \bigcup_{x\in F}(x + U) = F + U$.

   c) Let $A \subset V$ be a totally bounded subset, and let U be a neighborhood of 0 in V. We may assume that U is an open ball centered at 0 and of positive radius, say c. Let U' be the open ball centered at 0 of radius $c/2$. As A is totally bounded, there exists a finite set F such that $A \subset F + U'$. As F is finite, the set $F + \overline{U'}$ is closed, so it contains $\overline{A}$. But $U \supset \overline{U'}$, so $\overline{A} \subset F + U$.

   d) Let U be a neighborhood of 0 in V. Choose a convex open neighborhood U' of 0 (a ball for example) such that $U' + U' \subset U$, and let F be a finite set such that $A \subset F + U'$. Let K be the convex hull of F, then $A \subset K + U'$. As K and U' are convex, so is $K + U'$, so the convex hull of A is contained in $K + U'$. On the other hand, the set K is compact by question (a), hence totally bounded by question (b), so there exists a finite set F' such that $K \subset F' + U'$, hence $K + U' \subset F' + U' + U' \subset F' + U$. So we have found a finite set F' such that the convex hull of A is contained in $F' + U$.

   e) Write $K = \overline{A}$. For every $x \in V$ and $c > 0$, let $B(x, c)$ be the closed ball of radius c center at x.

      Let $(U_i)_{i\in I}$ be a family of open subsets of K such that $K \subset \bigcup_{i\in I} U_i$, and assume that no finite subfamily of $(U_i)_{i\in I}$ covers K. We know that K is totally bounded by question (c). We will construct by induction on n a decreasing sequence $(K_n)_{n\geq 1}$ of nonempty closed subsets of K such that K_n is contained in a ball of radius $1/n$ and K_n cannot be covered by a finite subfamily of $(U_i)_{i\in I}$.

      First, as K is totally bounded, there exists a finite set F such that $K \subset F + B(0, 1) = \bigcup_{x\in F} B(x, 1)$. We choose $x \in F$ such that $K \cap B(x, 1)$ is nonempty and cannot be covered by a finite number of the U_i, and take $K_1 = K \cap B(x, 1)$.

Now suppose that we have constructed $K_1, \dots, K_n$, with $n \geq 1$. Then, as K_n is totally bounded (as a subset of K), there exists a finite set F such that $K_n \subset F + B(0, (n+1)^{-1})$. Again, as K_n cannot be covered by a finite number of the U_i, there must exist $x \in F$ such that $K_n \cap B(x, (n+1)^{-1})$ is nonempty and can also not be covered by a finite number of the U_i, and we take $K_{n+1} = K_n \cap B(x, (n+1)^{-1})$.

Choose $x_n \in K_n$ for every $n \geq 1$. By the condition that K_n is contained in a ball of radius $1/n$, the sequence $(x_n)_{n \geq 0}$ is a Cauchy sequence. As V is complete, $(x_n)_{n \geq 1}$ has a limit, say x. As $x \in K$, there exists $i \in I$ such that U_i. But then $B(x, c) \subset U_i$ for $c > 0$ small enough, so $K_n \subset U_i$ for n big enough, which contradicts the properties of K_n.

f) By question (d), the convex hull of K is totally bounded, so its closure is compact by question (e).

□

7. (extra credit) Let (X, μ) and (Y, ν) be measure spaces, which we will take σ-finite to simplify. [3] Let $p \in (1, +\infty)$. [4] Let $\varphi : X \times Y \to \mathbb{C}$ be a measurable function. We assume that

$$\int_Y \left(\int_X |\varphi(x, y)|^p d\mu(x) \right)^{1/p} d\nu(y) < \infty.$$

a) (1) Show that the function $\varphi(., y)$ is in $L^p(X, \mu)$ for almost every $y \in Y$.

b) (2) Let Y' be a measurable subset of Y such that $\nu(Y - Y') = 0$ and $\varphi(., y) \in L^p(X, \mu)$ for every $y \in Y'$. Show that the function $Y' \to L^p(X, \mu)$, $y \mapsto \varphi(., y)$ is weakly integrable.

c) (2) Show that the integral $h \in L^p(X, \mu)$ of the function of (b) exists, and that we have

$$h(x) = \int_Y \varphi(x, y) d\nu(y)$$

for almost all $x \in X$.

d) (1) Show Minkowski's inequality :

$$\left(\int_X \left| \int_Y \varphi(x, y) d\nu(y) \right|^p d\mu(x) \right)^{1/p} \leq \int_Y \left(\int_X |\varphi(x, y)|^p d\mu(x) \right)^{1/p} d\nu(y).$$

Solution.

a) The function $y \mapsto \left(\int_X |\varphi(x, y)|^p d\mu(x) \right)^{1/p}$ is integrable by hypothesis, so it must take finite values for almost all $y \in Y$, which means that $\int_X |\varphi(x, y)|^p d\mu(x) < +\infty$ for almost every $y \in Y$.

b) Let $q \in (1, +\infty)$ be such that $\frac{1}{p} + \frac{1}{q}$. We want to check that, for every $f \in L^q(X, \mu)$, the integral $\int_{Y'} \int_X f(x) \varphi(x, y) d\mu(x) d\mu(y)$ converges. Let $f \in^q (X, \mu)$. By Hölder's inequality, for every $y \in Y'$, $\int_X f(x) \varphi(x, y) d\mu(x)$ converges absolutely, and

$$\int_X |f(x) \varphi(x, y)| d\mu(x) \leq \|f\|_q \|\varphi(., y)\|_p.$$

As $\int_{Y'} \|\varphi(., y)\|_p d\nu(y)$ converges by hypothesis, this gives the convergence of $\int_{Y'} \int_X f(x) \varphi(x, y) d\mu(x) d\mu(y)$, and even its absolute convergence.

[3] There is a way to extend the results to not necessarily σ-compact locally compact groups with their Haar measures.

[4] Minkowski's inequality is still true for $p = 1$, but it follows immediately from the Fubini-Torelli theorem in that case.

c) We have in question (b) that $\int_X \int_Y |f(x)\varphi(x,y)|d\mu(x)d\nu(y) < +\infty$ for every $f \in L^1(X,\mu)$. By Fubini's theorem, this implies that, for every $f \in L^q(X,\mu)$, $\int_Y |f(x)\varphi(x,y)|d\nu(y) = |f(x)| \int_Y |\varphi(x,y)|d\nu(y) < +\infty$ for almost all $x \in X$. As f is arbitrary (and μ is σ-finite), we get that $\int_Y |\varphi(x,y)|d\nu(y) < +\infty$ for almost all $x \in X$, say for $x \in X'$ with $\mu(X - X') = 0$.

We define a function $h : X' \to \mathbb{C}$ by $h(x) = \int_Y \varphi(x,y)d\nu(y)$. We want to show that this is the integral of $y \mapsto \varphi(.,y)$. If $f \in L^q(X,\mu)$, we have

$$\int_X f(x)h(x)d\mu(x) = \int_X f(x)\varphi(x,y)d\mu(x)d\mu(y),$$

so, using Hölder's inequality as in question (b),

$$\left|\int_X f(x)h(x)d\mu(x)\right| \leq \int_{X\times Y} |f(x)\varphi(x,y)|d\mu(x) \leq C\|f\|_q,$$

where

$$C = \int_Y \left(\int_X |\varphi(x,y)|^p\right)^{1/p} d\nu(y).$$

This shows that $f \mapsto \int_X f(x)h(x)d\mu(x)$ is a bounded linear functional on $L^q(X,\mu)$, and that its operator norm is bounded by C. As the continuous dual of $L^q(X,\mu)$ is $L^p(X,\mu)$, we must have $h \in L^p(X,\mu)$ and $\|h\|_p \leq C$. The first property, together with the formula for $\int_X f(x)h(x)d\mu(x)$, says that h is indeed the integral of $y \mapsto \varphi(.,y)$.

d) The second property of h proved above is exactly Minkowski's inequality.

□