MAT 449 : Problem Set 3

Due Thursday, October 4

Van Dantzig's theorem

[1]

1. (extra credit,3) In this problem, X is a compact Hausdorff totally disconnected topological space. (Remember that "totally disconnected" means that the only nonempty connected subsets of X are the singletons.)

   Let $x \in X$, and let A be the intersection of all the open and closed subsets of X containing x. Show that $A = \{x\}$. (Hint : This is equivalent to showing that A is connected. And remember also that disjoint closed sets can be separated by open sets in any compact Hausdorff space.)

   Solution. Note that A is closed in X. Suppose that we have $A = A_1 \cup A_2$, with A_1 and A_2 closed in X and disjoint and $x \in A_1$. As any compact Hausdorff space is normal, we can find open subsets $U_1 \supset A_1$ and $U_2 \supset A_2$ of X such that $U_1 \cap U_2 = \varnothing$. Let's find a closed and open neighborhood V of x such that $V \cap \partial U_2 = \varnothing$. For every $y \in \partial U_2$, as $y \notin A$, we can find a closed and open neighborhood V_y of x such that $y \notin V_y$. Note that the $X - V_y$, $y \in \partial U_2$, form a family of open subsets of X covering ∂U_2; as ∂U_2 is compact, this family has a finite subfamily that still covers ∂U_2, say $(X - V_{y_1}, \dots, X - V_{y_n})$. Let $V = V_{y_1} \cap \dots \cap V_{y_n}$; then V is still open and closed, $x \in V$ and $V \cap \partial U_2 = \varnothing$. The last property implies that $B := V - U_2$ is also equal to $V - \overline{U}_2$, so it still open and closed. Also, we have $x \in B$ (because $x \notin U_2$), and $A_2 \cap B = \varnothing$. But A must be contained in B by definition, so $A_2 = \varnothing$. This proves that A is connected, hence a singleton, hence equal to $\{x\}$.

   □

2. (extra credit) In this problem, G is a locally compact totally disconnected topological group.

   a) (1) Show that the unit of G has a compact open neighborhood K.
   b) (2) Show that there exists an open subgroup G' of G contained in K. (Hint : Any open subset of G will generate an open subgroup. Choose your open subset wisely.)
   c) (1) Show that the compact open subgroups of G form a basis of neighborhoods of 1 in G.
   d) (2) Let G be the group $\mathbf{GL}_n(\mathbb{Q}_p)$ of problem 4 of problem set 1. Find a basis of neighborhoods of 1 in G that is composed of compact open subgroups.

   Solution.

[1] Shamelessly lifted from Terry Tao's blog.

a) Let V be a compact neighborhood of 1. Then ∂V is also compact and doesn't contain 1. By problem 1, for every $y \in \partial V$, there exists an open and closed neighborhood B_y of 1 such that $B_y \cap \partial V = \varnothing$. As $\partial V \subset \bigcup_{y \in \partial V}(X - B_y)$ and the $X - B_y$ are open, there exist $y_1, \dots, y_n \in \partial V$ such that $\partial V \cap B = \varnothing$, with $B = \bigcap_{i=1}^n B_{y_i}$. Note that B is still open and closed, and that $1 \in B$. Also, as $\partial V \cap B = \varnothing$, we have $B \cap V = B \cap \overset{\circ}{V}$, and so $K := B \cap V$ is open and compact (because it is closed in V) and contains 1.

b) Let U be an open symmetric neighborhood of 1 such that $UK \subset K$, and let G' be the subgroup of G generated by U. Let's show that G' is an open compact subgroup of G and that $G' \subset K$. First we show that G' is open. Let $g \in G'$; then $gU \subset G'$ and gU is open in G, so G' contains a neighborhood of G. As every open subgroup of a topological group is also closed, we also get that G' is closed. So, to show that is compact, it suffices to show that it is contained in K. Note that, as U is symmetric and contains 1, we have $G' = \bigcup_{n \geq 1} U^n$. As $U \subset K$ (because $1 \in K$) and $UK \subset K$, an easy induction shows that $U^n \subset K$ for every $n \geq 1$. So $G' \subset K$.

c) The argument in the solution of question (a) actually shows that every compact neighborhood of 1 contains an open compact neighborhood of 1, and then question (b) implies that it also contains a compact open subgroup of G. Hence, as G is locally compact, every neighborhood of 1 in G contains a compact open subgroup of G.

d) Let's choose a norm on $M_n(\mathbb{Q}_p)$ that induces the product topology. For example, the norm $\|.\|$ defined by

$$\|(a_{ij})_{1 \leq i,j \leq n} - (b_{ij})_{1 \leq i,j \leq n}\| = \sup_{1 \leq i,j \leq n} |a_{ij} - b_{ij}|_p$$

works. For every integer $m \geq 1$, let $K_m = I_n + p^m M_n(\mathbb{Z}_p)$. With our choice of norm, this is just the open ball of center I_n and radius p^{-m+1} in $M_n(\mathbb{Q}_p)$ (and also the closed ball of center I_n and radius p^{-m}). In particular, the sets K_m, for $m \geq 1$, form a family of open neighborhoods of I_n in $M_n(\mathbb{Q}_p)$, and hence the sets K_m for $m >> 0$ form a family of open neighborhoods of I_n in $\mathbf{GL}_n(\mathbb{Q}_p)$ (because $\mathbf{GL}_n(\mathbb{Q}_p)$ is open in $M_n(\mathbb{Q}_p)$, as the preimage by the continuous map det of the open subset $\mathbb{Q}_p^\times$ of $\mathbb{Q}_p$).

Note also that K_m is homeomorphic to $M_n(\mathbb{Z}_p) \simeq \mathbb{Z}_p^{n^2}$ (by the map $I_n + p^m X \mapsto X$), so it is also compact.

At this point, we have our basis of neighborhoods consisting of compact open subgroups. We can actually be more precise and show that $K_m \subset \mathbf{GL}_n(\mathbb{Q}_p)$ for every $m \geq 1$ (and not just for m big enough), which just means that $K_1 \subset \mathbf{GL}_n(\mathbb{Q}_p)$. In fact, we even have $K_1 \subset \mathbf{GL}_n(\mathbb{Z}_p)$. Indeed, it is clear that $K_1 \subset M_n(\mathbb{Z}_p)$. Moreover, if $X \in K_1$, then it is easy to see that $\det(X) \in 1 + p\mathbb{Z}_p \subset \mathbb{Q}_p$, which implies that $|\det(X)|_p = 1$ (by question 4(a) of problem set 1), hence that $\det(X)^{-1}$ is also in $\mathbb{Z}_p$, i.e., that $\det(X) \in \mathbb{Z}_p^\times$.

□

3. a) (1) Let G be a compact subgroup of $\mathbf{GL}_n(\mathbb{C})$. Show that there exists $x \in \mathbf{GL}_n(\mathbb{C})$ such that $xGx^{-1} \subset \mathbf{U}(n)$.

b) (3) Put your favorite norm on $M_n(\mathbb{C})$ (they are all equivalent anyway). Show that there exists $c > 0$ such that the only subgroup of $\mathbf{GL}_n(\mathbb{C})$ included in the ball $\{x \in \mathbf{GL}_n(\mathbb{C}) | \|x - I_n\| < c\}$ is the trivial group.

c) (2) Show that, for every continuous representation of $\mathbf{GL}_n(\mathbb{Q}_p)$ on a finite-dimensional $\mathbb{C}$-vector space, there exists an integer $m \geq 0$ such that the subgroup $I_n + p^m M_n(\mathbb{Z}_p)$ of $\mathbf{GL}_n(\mathbb{Q}_p)$ acts trivially.

Solution.

a) Consider the representation ρ of G on $\mathbb{C}^n$ given by the inclusion $G \subset \mathbf{GL}_n(\mathbb{C})$. We have seen in class (theorem 3.2.8 of the notes) that there exists a Hermitian inner product on $\mathbb{C}^n$ for which this representation is unitary. Let A be the matrix of this Hermitian inner product in the canonical basis of $\mathbb{C}^n$. Then A is a Hermitian positive matrix, so we can write it $A = B^*B$ with $B \in \mathbf{GL}_n(\mathbb{C})$. (This is an easy consequence of the spectral theorem. As A is Hermitian, we have a unitary matrix P and a diagonal matrix D such that $A = P^*DP$. As A is positive, the diagonal entries of D are positive real numbers, so we can write $D = C^2$ with C another diagonal matrix with positive diagonal entries. Take $B = P^*CP$, then B is Hermitian positive and $A = B^2 = B^*B$.)

   The fact that ρ is unitary for A means that $X^*AX = A$ for every $X \in G$. As $A = B^*B$, this is equivalent to $(BXB^{-1})^*(BXB^{-1}) = I_n$. So $BGB^{-1} \subset \mathbf{U}(n)$.

b) Let G be a subgroup of $\mathbf{GL}_n(\mathbb{C})$ contained in a ball of the form $\{x \in \mathbf{GL}_n(\mathbb{C}) | \|x - I_n\| < c\}$. Then the closed subgroup $\overline{G}$ is contained in the closed ball $\{x \in \mathbf{GL}_n(\mathbb{C}) | \|x - I_n\| \leq c\}$, so it is compact, so it is contained in a subgroup of the form $PU(n)P^{-1}$ by question (a). In particular, every element of G is diagonalizable and has all its eigenvalues of modulus 1.

   Fix any norm on $\mathbb{C}^n$, and consider the corresponding operator norm $\|.\|$ on $M_n(\mathbb{C})$. We will use this norm. Note that, if $X \in M_n(\mathbb{C})$ and if λ is an eigenvalue of X, then we have a norm 1 vector $v \in \mathbb{C}^n$ such that $Xv = \lambda v$, hence $\|X\| \geq |\lambda|$. Now let's show that every subgroup of $\mathbf{GL}_n(\mathbb{C})$ included in the open ball $B := \{x \in \mathbf{GL}_n(\mathbb{C}) | \|x - I_n\| < \sqrt{2}\}$ is trivial. Let G be such a subgroup, and let $X \in G$. Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of X. We just saw that $|\lambda_1| = \ldots = |\lambda_n| = 1$. Suppose that we have a $r \in \{1, \ldots, n\}$ such that λ_r is not equal to 1, then we can write $\lambda_r = e^{i\theta}$ with $-\pi/2 < \theta < \pi/2$, because $|\lambda_r - 1| \leq \|X - I_n\| < \sqrt{2}$; but then, if we choose an integer $m \geq 1$ such that $\pi/2 \leq m|\theta| \leq \pi$, we'll have $\|X^m - I_n\| \geq |\lambda_r^m - 1| \geq \sqrt{2}$, which contradicts the fact that $X^m \in G$. So we must have $\lambda_1 = \ldots = \lambda_n = 1$, which means that $X = I_n$.

c) We have seen in the solution of question 2(d) that $K_m := I_n + p^m M_n(\mathbb{Z}_p)$ is indeed a subgroup of $\mathbf{GL}_n(\mathbb{Q}_p)$. (Note that problem 3 does not actually ask you to (re)prove this fact.) We have also put a norm on $M_n(\mathbb{Q}_p)$ such that K_m is the open ball with center I_n and radius p^{-m+1}.

   Let $\rho : \mathbf{GL}_n(\mathbb{Q}_p) \to \mathbf{GL}(V)$ be a continuous representation of $\mathbf{GL}_n(\mathbb{Q}_p)$ on a finite-dimensional vector space V. By proposition 3.5.1 of the notes, the morphism ρ is continuous. Let U be an open neighborhood of id_V in $\mathbf{GL}(V)$ such that the only subgroup of $\mathbf{GL}(V)$ contained in U is $\{1\}$ (this exists by question (b)). Then $\rho^{-1}(U)$ is an open neighborhood of I_n in $\mathbf{GL}_n(\mathbb{Q}_p)$, so it contains K_m for $m >> 0$. But K_m is a subgroup of $\mathbf{GL}_n(\mathbb{Q}_p)$, so $\rho(K_m)$ is a subgroup of $\mathbf{GL}(V)$, so $\rho(K_m) = \{1\}$ as soon as $\rho(K_m) \subset U$.

□

Haar measure on $\mathrm{SU}(2)$

4. Let $G = \mathbf{SU}(2)$.

   a) (2) Show that every element of G is of the form $\begin{pmatrix} a & -\overline{b} \\ b & \overline{a} \end{pmatrix}$, with $a, b \in \mathbb{C}$ and $|a|^2 + |b|^2 = 1$.

If we identify $\mathbb{C}$ and $\mathbb{R}^2$ in the usual way, the previous question gives a homeomorphism α between $\mathbf{SU}(2)$ and S^3 (the unit sphere in $\mathbb{R}^4$).

b) (1) If $g \in \mathbf{SU}(2)$, show that left translation by g on $\mathbf{SU}(2)$ corresponds by α to the restriction to S^3 of the action of an element of $\mathbf{SO}(4)$ on $\mathbb{R}^4$ (i.e. there exists $A \in \mathbf{SO}(4)$ such that, for every $h \in \mathbf{SU}(2)$, we have $gh = A\alpha(h)$).

c) (2) Let μ be the usual spherical measure on S^3; that is, if λ is Lebesgue measure on $\mathbb{R}^4$, we have by definition, for every Borel subset E of S^3,

$$\mu(E) = \tfrac{2}{\pi^2}\lambda(\{tx, t \in [0,1], x \in E\})$$

(note that the volume of the unit ball in $\mathbb{R}^4$ is $\frac{\pi^2}{2}$).

Show that the inverse image by α of μ is a left and right Haar measure on $\mathbf{SU}(2)$.

d) (2) We use the following (hyperspherical) coordinates on S^3 : if $(x_1, x_2, x_3, x_4) \in S^3$, we write

$$\begin{cases} x_1 = \cos\theta \\ x_2 = \sin\theta\cos\psi \\ x_3 = \sin\theta\sin\psi\cos\phi \\ x_4 = \sin\theta\sin\psi\sin\phi \end{cases}$$

with $0 \le \theta \le \pi$, $0 \le \psi \le \pi$ and $0 \le \phi \le 2\pi$. Show that, for every $f \in \mathcal{C}_c(S^3)$, we have $\int_{S^3} f d\mu =$

$$\frac{1}{2\pi^2}\int_0^\pi\int_0^\pi\int_0^{2\pi} f(\cos\theta, \sin\theta\cos\psi, \sin\theta\sin\psi\cos\phi, \sin\theta\sin\psi\sin\phi)\sin^2\theta\sin\psi d\theta d\psi d\phi.$$

(Feel free to use a computer to calculate any big determinants.)

Solution.

a) It is clear that every matrix as in the statement is in $\mathbf{SU}(2)$. Let's show the converse. Let $A = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \in M_2(\mathbb{C})$. Then $A \in \mathbf{U}(2)$ if and only if $A^*A = I_2$, which means that the two column vectors of A are orthogonal and norm 1 for the usual Hermitian inner product on $\mathbb{C}^2$. As the orthogonal of a line in $\mathbb{C}^2$ is one-dimensional, it implies that there exists $\lambda \in \mathbb{C}^\times$ such that $\begin{pmatrix} c \\ d \end{pmatrix} = \lambda \begin{pmatrix} -\overline{b} \\ \overline{a} \end{pmatrix}$. The condition on the norm of the columns gives $a\overline{a} + b\overline{b} = \lambda\overline{\lambda}(a\overline{a} + b\overline{b}) = 1$, and the condition that $\det(A) = 1$ gives $\lambda(a\overline{a} + b\overline{b}) = 1$. So we get $\lambda = 1$, as desired.

b) Let V be the space of matrices of the form $\begin{pmatrix} a & -\overline{b} \\ b & \overline{a} \end{pmatrix}$, with $a, b \in \mathbb{C}$. Then α extends to a $\mathbb{C}$-linear isomorphism from V to $\mathbb{C}^2$, hence to a $\mathbb{R}$-linear isomorphism from V to $\mathbb{R}^4$, sending $\begin{pmatrix} a & -\overline{b} \\ b & \overline{a} \end{pmatrix}$ to $(\mathrm{Re}(a), \mathrm{Im}(a), \mathrm{Re}(b), \mathrm{Im}(b))$. If $A \in \mathbf{SU}(2)$, then the action by left multiplication of A on V is the usual action of A on $\mathbb{C}^2$, so it corresponds to a linear automorphism of $\mathbb{R}^4$ which preserves the usual Euclidian norm, i.e. is in $\mathbf{O}(4)$. Also, the determinant of this action is just $\det(A) = 1$, so the corresponding automorphism of $\mathbb{R}^4$ is in $\mathbf{SO}(4)$.

c) First, note that μ is a regular Borel measure on S^3 (a subset E of S^3 is a Borel subset if and only $\{tx, t \in [0,1], x \in E\}$ is a Borel subset of $\mathbb{R}^4$, it is compact if and only if $\{tx, t \in [0,1], x \in E\}$ is compact and openif and only if $\{tx, t \in (0,1], x \in E\}$ (which has the same measure as $\{tx, t \in [0,1], x \in E\}$) is open).

By the change of variables formula in $\mathbb{R}^4$, the measure μ is invariant by the action of $\mathbf{SO}(4)$ on S^3. By question (b), its inverse image by α is invariant by left translations on $\mathbf{SU}(2)$, hence a left Haar measure. But the group $\mathbf{SU}(2)$ is compact, so every left Haar measure is also a right Haar measure.

d) Let B^4 be the closed unit ball in $\mathbb{R}^4$. Let $f \in \mathcal{C}_c(S^3)$. We define a function $g \in L^1(B^4)$ by

$$g(r\cos\theta, r\sin\theta\cos\psi, r\sin\theta\sin\psi\cos\phi, r\sin\theta\sin\psi\sin\phi) =$$

$$f(\cos\theta, \sin\theta\cos\psi, \sin\theta\sin\psi\cos\phi, \sin\theta\sin\psi\sin\phi)$$

for $0 \leq r \leq 1$. (Note : g might not be well-defined at 0, but it doesn't matter because $\{0\}$ has volume 0.) Then, by definition of μ, we have $\int_{S^3} f d\mu = \frac{2}{\pi^2}\int_{B^4} g d\lambda$. We can calculate this last integral using the change of variables formula (and avoiding the set where this change of variables is not bijective, which is of volume 0 anyway). If β is the map sending (r,θ,φ,ψ) to $(r\cos\theta, r\sin\theta\cos\psi, r\sin\theta\sin\psi\cos\phi, r\sin\theta\sin\psi\sin\phi)$, then we have

$$D\beta(r,\theta,\varphi,\psi) = r^3(\sin\theta)^2\sin\psi,$$

so $\int_{B^4} g d\lambda$ is equal to

$$\int_0^1\int_0^\pi\int_0^\pi\int_0^{2\pi} f(\cos\theta, \sin\theta\cos\psi, \sin\theta\sin\psi\cos\phi, \sin\theta\sin\psi\sin\phi) r^3\sin^2\theta\sin\psi dr d\theta d\psi d\phi =$$

$$\frac{1}{4}\int_0^\pi\int_0^\pi\int_0^{2\pi} f(\cos\theta, \sin\theta\cos\psi, \sin\theta\sin\psi\cos\phi, \sin\theta\sin\psi\sin\phi)\sin^2\theta\sin\psi d\theta d\psi d\phi.$$

We get the result by multiplying by $\frac{2}{\pi^2}$.

$\square$

The dual of a locally compact abelian group

5. Let G be an abelian topological group. We write $\widehat{G}$ for the set of continuous group morphisms $G \to S^1$.

   As the product of two continuous morphisms from G is S^1 is also a continuous morphism from G to S^1 (because S^1 is commutative), the set $\widehat{G}$ has a natural group structure. We put the topology of compact convergence on $\widehat{G}$; that is, if $\chi \in \widehat{G}$, then a basis of neighborhoods of χ is given by $\{\psi \in \widehat{G} | \sup_{x\in K}|\chi(x) - \psi(x)| < c\}$, for all compact subsets K of G and all $c > 0$.

   a) (1) Show that $\widehat{G}$ is a topological group.

   b) Suppose that $G = \mathbb{R}$.

      i. (2) Let $\rho : G \to \mathbf{GL}_n(\mathbb{C})$ be a continuous group morphism. Show that there exists a unique $A \in M_n(\mathbb{C})$ such that, for every $t \in \mathbb{R}$, $\rho(t) = \exp(tA)$. (There are several ways to do this. One way is to notice that, if the conclusion is true, then $c'(0)$ must exist and be equal to A, and to work backwards from there.)

      ii. (2) Show that the image of ρ is contained in $\mathbf{U}(n)$ if and only if $A^* = -A$.

      iii. (2) Show that the map $\mathbb{R} \to \widehat{G}$ sending $x \in \mathbb{R}$ to the group morphism $G \to S^1$, $t \mapsto e^{ixt}$ is an isomorphism of topological groups (i.e. a group isomorphism that is also a homeomorphism).

   c) (1) Show that there is an isomorphism of topological groups $\widehat{S^1} \simeq \mathbb{Z}$ that sends id_{S^1} to 1.

d) (1) What is the topological group $\widehat{\mathbb{Z}}$?

e) Suppose that $G = \mathbb{Q}_p$ (cf. problem 4 of problem set 1). We define a map $\chi_1 : \mathbb{Q}_p \to S^1$ in the following way : If $x \in \mathbb{Q}_p$, we can write $x = \sum_{n=-\infty}^{+\infty} c_n p^n$, with $0 \leq c_n \leq p-1$ and $c_n = 0$ for n small enough, and this uniquely determines the c_n (see problem 4(i) of problem set 1). We set

$$\chi_1(x) = \exp\left(2\pi i \sum_{n=-\infty}^{-1} c_n p^n\right).$$

   i. (3) Show that $\chi_1 : \mathbb{Q}_p \to S^1$ is a continuous group morphism and that $\mathrm{Ker}(\chi_1) = \mathbb{Z}_p$.

   ii. (2) For every $y \in \mathbb{Q}_p$, we define $\chi_y : \mathbb{Q}_p \to S^1$ by $\chi_y(x) = \chi(xy)$. Show that this is also a continuous group morphism, and find its kernel.

   iii. (1) Let $\chi \in \widehat{\mathbb{Q}_p}$. Show that there exists $k \in \mathbb{Z}$ such that $\chi = 1$ on $\{x \in \mathbb{Q}_p | |x|_p \leq p^{-k}\}$.

   iv. (2) Let $\chi \in \widehat{\mathbb{Q}_p}$ such that $\chi(1) = 1$ and $\chi(p^{-1}) \neq 1$. Show that there exists a sequence of integers $(c_r)_{r\geq 0}$ such that $1 \leq c_0 \leq p-1$ and $0 \leq c_r \leq p-1$ for $r \geq 1$ and that, for every $k \in \mathbb{Z}_{\geq 1}$,

$$\chi(p^{-k}) = \exp\left(2\pi i \sum_{r=1}^{k} c_{k-r} p^{-r}\right).$$

   v. (1) Let $\chi \in \widehat{\mathbb{Q}_p}$ such that $\chi(1) = 1$ and $\chi(p^{-1}) \neq 1$. Show that there exists $y \in \mathbb{Q}_p$ such that $|y|_p = 1$ and $\chi = \chi_y$.

   vi. (3) Show that the map $\mathbb{Q}_p \to \widehat{\mathbb{Q}_p}$, $y \mapsto \chi_y$ is an isomorphism of topological groups.

   vii. (extra credit, 4) Show that $\chi_{y|\mathbb{Z}_p} = \chi_{y'|\mathbb{Z}_p}$ if and only $y - y' \in \mathbb{Z}_p$, and that the map $y \mapsto \chi_y$ induces an isomorphism of topological groups $\mathbb{Q}_p/\mathbb{Z}_p \xrightarrow{\sim} \widehat{\mathbb{Z}_p}$, where $\mathbb{Q}_p/\mathbb{Z}_p$ has the discrete topology.

Solution.

a) We need to check the group operations of $\widehat{G}$ are continuous. Let's start with multiplication. Let $\chi_1, \chi_2 \in \widehat{G}$, and choose a neighborhood U of $\chi_1\chi_2$ of the form $\{\psi \in \widehat{G} | \sup_{x\in K} |\chi(x) - \psi(x)| < c\}$, with $K \subset G$ compact and $c > 0$. We need to find neighborhoods U_1 of χ_1 and U_2 of χ_2 such that $U_1 U_2 \subset U$. Take

$$U_i = \{\psi \in \widehat{G} | \sup_{x\in K} |\chi_i(x) - \psi(x)| < c/2\}$$

for $i = 1, 2$. Let $\psi_1 \in U_1$ and $\psi_2 \in U_2$. Then, if $x \in K$, we have

$$\begin{aligned}|(\psi_1\psi_2)(x) - (\chi_1\chi_2)(x)| &= |\psi_1(x)(\psi_2(x) - \chi_2(x)) + \chi_2(x)(\psi_1(x) - \chi_1(x))| \\ &\leq |\psi_1(x)||\psi_2(x) - \chi_2(x)| + |\chi_2(x)||\psi_1(x) - \chi_1(x)| \\ &= |\psi_2(x) - \chi_2(x)| + |\psi_1(x) - \chi_1(x)| \text{ (because } \psi_1 \text{ and } \chi_2 \text{ are unitary)} \\ &< c.\end{aligned}$$

So $\psi_1\psi_2 \in U$.

The proof for inversion is similar. Let $\chi \in \widehat{G}$, and choose a neighborhood U of χ^{-1} of the form $\{\psi \in \widehat{G} | \sup_{x\in K} |\chi^{-1}(x) - \psi(x)| < c\}$, with $K \subset G$ compact and

$c > 0$. We need to find a neighborhood V of χ such that $V^{-1} \subset U$. Take $V = \{\psi \in \widehat{G} | \sup_{x \in K} |\chi(x) - \psi(x)| < c\}$. Let $\psi \in V$. Then, for every $x \in K$, we have

$$|\psi^{-1}(x) - \chi^{-1}(x)| = |\psi^{-1}(x)||\chi^{-1}||\chi(x) - \psi(x)| = |\chi(x) - \psi(x)| < c.$$

So $\psi^{-1} \in U$.

b) i. Choose a norm $\|.\|$ on $M_n(\mathbb{C})$. As $\mathbf{GL}_n(\mathbb{C})$ is open in $M_n(\mathbb{C})$, we can choose a nonempty open ball B center of I_n such that $B \subset \mathbf{GL}_n(\mathbb{C})$. We only care about the fact that B is a convex subset of $M_n(\mathbb{C})$. As ρ is continuous and $\rho(0) = I_n$, we can find $c > 0$ such that $\rho([0, c]) \subset B$. Then

$$\int_0^1 \rho(cx)dx = c \int_0^c \rho(x)dx \in B,$$

so $X := \int_0^c \rho(x)dx \in \mathbf{GL}_n(\mathbb{C})$. For every $t \in \mathbb{R}$, we have

$$X\rho(t) = \int_0^c \rho(x+t)dx = \int_t^{t+c} \rho(x)dx.$$

In particular, ρ is continuously differentiable, and

$$\rho'(t) = X^{-1}(\rho(t+c) - \rho(t)) = X^{-1}(\rho(c) - I_n)\rho(t).$$

The only solution of this differential equation satisfying the initial condition $\rho(0) = I_n$ is $\rho(t) = \exp(tA)$, with $A = X^{-1}(\rho(c) - I_n)$. Finally, the matrix is uniquely determined by ρ, because we must have $A = \rho'(0)$.

ii. If $A^* = -A$, then, for every $t \in \mathbb{R}$,

$$\rho(t)\rho(t)^* = \exp(tA)\exp(tA^*) = \exp(t(A + A^*)) = \exp(0) = I_n$$

(we use the fact that tA and tA^* commute to get the equality $\exp(tA)\exp(tA^*) = \exp(tA + tA^*)$), so $\rho(t) \in \mathbf{U}(n)$.

Conversely, suppose that $\rho(\mathbb{R}) \subset \mathbf{U}(n)$. Note that $A = \lim_{t\to 0} \frac{1}{t}(\rho(t) - I_n)$, so $A^* = \lim_{t \to 0} \frac{1}{t}(\rho(t)^* - I_n)$. As

$$\rho(t)^* - I_n = \rho(t)^{-1} - I_n = -\rho(t)^{-1}(\rho(t) - I_n)$$

and $\rho(t)^{-1} \to I_n$ as $t \to 0$, this implies that $A^* = -A$.

iii. Let's denote by α the map $\mathbb{R} \to \widehat{G}$ of the statement.

We have seen in (i) and (ii) that every continuous group morphism $\rho : \mathbb{R} \to S^1$ is of the form $\rho(t) = e^{zt}$, for a unique $z \in \mathbb{C}$ such that $\overline{z} = -z$; that last condition means that $z = ix$ for some $x \in \mathbb{R}$. This means that α is bijective. It is also easy to see that α is a morphism of groups, so we just need to show that α is a homeomorphism.

We first show that α is continuous. Let $x \in \mathbb{R}$, and consider a neighborhood U of $\alpha(x)$ of the form $\{\rho \in \widehat{G} | \forall t \in K,\ |\alpha(x)(t) - \rho(t)| < c\}$, where $K \subset \mathbb{R}$ is a compact subset and $c > 0$. Then, for every $y, t \in \mathbb{R}$, we have

$$|\alpha(x)(t) - \alpha(y)(t)|^2 = |e^{ixt} - e^{iyt}|^2 = |1 - e^{it(x-y)}|^2 = (1 - \cos(t(x-y)))^2 + (\sin(t(x-y)))^2.$$

Choose $\varepsilon > 0$ such that, for every $t \in K$ and $z \in (-\varepsilon, \varepsilon)$, we have $(1 - \cos(tz))^2 + (\sin(tz))^2 < c^2$. Then, if $|x - y| < \varepsilon$, we have $\alpha(y) \in U$.

Now we show that α is open. Let $x \in \mathbb{R}$, and choose a neighborhood V of x of the form $(x-\varepsilon, x+\varepsilon)$, with ε (these form a basis of neighborhoods). We want to show that $\alpha(V)$ contains a neighborhood of $\alpha(x)$. Choose $\delta > 0$ such that the functions $t \mapsto \sin(t)$ and $t \mapsto 1-\cos(t)$ are both increasing on $[0, 2\delta\varepsilon]$, and let

$$U = \{\rho \in \widehat{G} | \forall t \in K,\ |\alpha(x)(t) - \rho(t)| < c\},$$

where $K = [-\delta, \delta]$ and $c = \left(\sup_{t\in[0,\varepsilon\delta/2]}(1-\cos(t))^2 + (\sin(t))^2\right)^{1/2}$ (note that this is also the sup on $[-\varepsilon\delta/2, \varepsilon\delta/2]$, because the function we are bounding is even). Let $y \in \mathbb{R}$ such that $|x-y| \geq \varepsilon$. We want to show that $\alpha(y) \notin U$. We can find $t \in K$ such that $\varepsilon\delta \leq t(x-y) \leq 2\varepsilon\delta$. Then we have

$$|\alpha(x)(t) - \alpha(y)(t)| = \left((1-\cos(t(x-y)))^2 + (\sin(t(x-y)))^2\right)^{1/2} > c,$$

by the choice of δ and c. So $\alpha(y) \notin U$.

c) Note that we have an isomorphism of topological groups $\mathbb{R}/2\pi\mathbb{Z} \overset{\sim}{\to} S^1$ given by $t \mapsto e^{it}$. So we get an isomorphism of groups

$$\widehat{S^1} \simeq \{\rho \in \widehat{\mathbb{R}} | \rho(2\pi\mathbb{Z}) = \{1\}\} \simeq \{x \in \mathbb{R} | \forall t \in 2\pi\mathbb{Z},\ e^{ixt} = 1\} = \mathbb{Z}$$

(where the second isomorphism comes from question (b)). It remains to show that this is an isomorphism of topological groups, i.e. that $\widehat{S^1}$ is discrete. If you have read ahead, you know that this is a particular case of question 6(e) (and I don't know a simpler proof in the case of S^1).

d) As $\mathbb{Z}$ is discrete, a continuous group morphism from $\mathbb{Z}$ to S^1 is just a group morphism from $\mathbb{Z}$ to S^1. As $\mathbb{Z}$ is the free abelian group generated by $1 \in \mathbb{Z}$, the map $\rho \mapsto \rho(1)$ is an isomorphism between the set of group morphisms $\mathbb{Z} \to S^1$ and S^1. So, as a group, $\widehat{\mathbb{Z}}$ is isomorphic to S^1. Let's denote this isomorphism by $\beta : S^1 \to \widehat{\mathbb{Z}}$ (so β sends $z \in S^1$ to the morphism $\mathbb{Z} \to S^1$, $n \mapsto z^n$). If we show that β is continuous, then it will automatically be a homeomorphism because S^1 is compact. But the compact subsets of $\mathbb{Z}$ are its finite subsets, so the continuity of β follows immediately from the continuity of the maps $S^1 \to S^1$, $z \mapsto z^n$.

e) i. Let $x, x' \in \mathbb{Q}_p$, and write $x = \sum_{n=-\infty}^{+\infty} c_n p^n$ $x' = \sum_{n=-\infty}^{+\infty} c'_n p^n$ (with the same conditions on the c_n and c'_n as in the statement). Then, by 4(h) in problem set 1, we have, for every $N \in \mathbb{Z}$, $|x-x'|_p \leq p^{-N}$ if $c_n = c'_n$ for every $n \leq N-1$. In particular, $\chi_1(x) = \chi_1(x')$ if $|x-x'|_p \leq 1$, so χ_1 is continuous and sends every $x \in \mathbb{Z}_p$ to $1 = \chi_1(0)$.

We still need to show that χ_1 is a morphism of groups. Let G' be the subgroup of $\mathbb{Q}_p$ whose elements are the $x \in \mathbb{Q}_p$ that can be writte $x = \sum_{n=-\infty}^{+\infty} a_n p^n$, with $a_n \in \mathbb{Z}$ and $a_n = 0$ for $|n|$ big enough. This is a dense subgroup (because $\sum_{n=-\infty}^{+\infty} c_n p^n$ is the limit as $N \to +\infty$ of $\sum_{n=-\infty}^{N} c_n p^n$), and it is contained in $\mathbb{Q}$. As we know that χ_1 is continuous, it suffices to prove that $\chi_1(x+y) = \chi_1(x)\chi_1(y)$ $x, y \in G'$. But note that, if $x \in G'$, then $\chi_1(x) = \exp(2\pi i x)$, where we see x as an element of $\mathbb{Q}$. This implies the result.

Finally, we have to show that $\mathrm{Ker}(\chi_1) = \mathbb{Z}_p$. We have already seen that $\mathbb{Z}_p \subset \mathrm{Ker}(\chi_1)$. Conversely, let $x \in \mathbb{Q}_p$, and write $x = \sum_{n=-\infty}^{+\infty} c_n p^n$ as above. Suppose that $x \notin \mathbb{Z}_p$, then there exists $m < 0$ such that $c_m \neq 0$. Choose such a m. We have

$$p^{-m} \leq c_m p^{-m} < \sum_{n=-\infty}^{-1} c_n p^n \leq (p-1) \sum_{r \geq 1} p^{-r} = 1$$

(the second inequality is strict because the c_n are 0 for n small enough). So $\sum_{n=-\infty}^{-1} c_n p^n \in (0,1)$, and $\chi_1(x) = \exp(2\pi i \sum_{n=-\infty}^{-1} c_n p^n) \neq 1$.

ii. The map χ_y is a continuous group morphism because it is the composition of the continuous group morphisms χ_1 and $m_y : \mathbb{Q}_p \to \mathbb{Q}_p,\ x \mapsto xy$. An element $x \in \mathbb{Q}_p$ is in the kernel of χ_y if and only $xy \in \mathrm{Ker}(\chi_1) = \mathbb{Z}_p$. So, if $y = 0$, we have $\mathrm{Ker}(\chi_y) = \mathbb{Q}_p$, and if $y \neq 0$, we have

$$\mathrm{Ker}(\chi_y) = y^{-1}\mathbb{Z}_p = |y|_p\mathbb{Z}_p = \{x \in \mathbb{Q}_p | |x|_p \leq |y|_p^{-1}\}.$$

iii. Choose a neighborhood U of 1 in $\mathbb{C}^\times$ such that the only subgroup of $\mathbb{C}^\times$ contained in U is the trivial group. (See 3(b).) Then $\chi^{-1}(U \cap S^1)$ is a neighborhood of 1 in $\mathbb{Q}_p$, so there exists $k \in \mathbb{Z}$ such that $\chi^{-1}(U \cap S^1) \supset \{x \in \mathbb{Q}_p | |x|_p \leq p^k\}$. But as $\{x \in \mathbb{Q}_p | |x|_p \leq p^k\}$ is a subgroup of $\mathbb{Q}_p$, its image by χ is a subgroup of S^1 contained in U, hence is equal to $\{1\}$.

iv. Write, for every integer $r \geq 0$, $z_r = \chi(p^{-r})$. Then $z_r \in S^1$ and, for every $r \geq 0$, we have

$$z_{r+1}^p = \chi(p^{-r-1})^p = \chi(p^{-r}) = z_r.$$

We will construct the integers c_r by induction on $r \geq 0$. Note first that $z_1 \neq 1 = z_0$ by hypothesis, so we can find $c_0 \in \{1, \dots, p-1\}$ such that $z_1 = \exp(2\pi i c_0 p^{-1})$. Suppose that we have found $c_0, \dots, c_{r-1}$ (with $r \geq 1$) such that, for $1 \leq s \leq r$, we have

$$\chi(p^{-s}) = z_s = \exp(2\pi i \sum_{k=1}^{s} c_{s-k} p^{-k}).$$

We have to find $c_r \in \{0, \dots, p-1\}$ such that

$$z_{r+1} = \exp(2\pi i \sum_{k=1}^{r+1} c_{r+1-k} p^{-k}) = \exp(2\pi i p^{-(r+1)} \sum_{s=0}^{r} c_s p^s).$$

As $z_{r+1}^p = z_r$, we have

$$\left(z_{r+1} \exp(2\pi i p^{-r-1} \sum_{s=0}^{r-1} c_s p^s) \right)^p = 1,$$

so there exists $c_r \in \{0, \dots, p-1\}$ such that

$$z_{r+1} \exp(2\pi i p^{-r-1} \sum_{s=0}^{r-1} c_s p^s) = \exp(2\pi i p^{-1} c_r),$$

i.e.

$$z_{r+1} = \exp(2\pi i p^{-(r+1)} \sum_{s=0}^{r} c_s p^s).$$

v. Let $(c_r)_{r\geq 0}$ be as in (iv), and set $y = \sum_{r=0}^{+\infty} c_r p^r$. As $c_0 \in \{1, \dots, p-1\}$, we have $|y|_p = 1$. Also, for every $r \geq 1$, we have

$$\chi(p^{-r}) = \exp(2\pi i p^{-(r+1)} \sum_{k=-r}^{-1} c_{r+k} p^k) = \chi_1(p^{-r}y) = \chi_y(1),$$

because

$$p^{-r}y = \sum_{s\geq 0} c_s p^{r-s} = \sum_{n=-r}^{+\infty} c_{r+n} p^n.$$

On the other hand, if $r \geq 0$, then

$$\chi(p^r) = \chi(1)^{p^r} = 1 = \chi_y(p^r).$$

As χ and χ_y are continuous morphisms of groups, and as the family $(p^r)_{r\in\mathbb{Z}}$ generates a dense subgroup of $\mathbb{Q}_p$, this implies that $\chi = \chi_y$.

vi. Let us denote the map $\mathbb{Q}_p \to \widehat{\mathbb{Q}_p}$, $y \mapsto \chi_y$ by α. It is easy to see that α is a morphism of groups (this follows immediately from the fact that χ_1 is a morphism of groups and the distributivity of multiplication on $\mathbb{Q}_p$.)

We first show that $\mathrm{Ker}(\alpha) = \{0\}$. Let $y \in \mathbb{Q}_p - \{0\}$. Then we have $y = \sum_{n=m}^{+\infty} c_n p^n$ with $m \in \mathbb{Z}$, $0 \leq c_n \leq p-1$ and $c_m \geq 1$. So

$$p^{-m-1}y = c_m p^{-1} + \sum_{n\geq 0} c_{n+m+1}p^n,$$

and $\chi_y(p^{-m-1}) = \exp(2\pi i p^{-1}c_m) \neq 1$. This shows that $y \notin \mathrm{Ker}(\alpha)$.

Now we show that α is surjective. Let $\chi \in \widehat{\mathbb{Q}_p}$. If $\chi = 1$, then $\chi = \chi_0$, so we assume that $\chi \neq 1$. By (iii), there exists $k \in \mathbb{Z}$ such that $\chi = 1$ on $\{x \in \mathbb{Q}_p | |x|_p \leq p^{-k}\}$. Choose k minimal for this property (this is possible because otherwise χ would be 1 on all of $\mathbb{Q}_p$, which contradicts our hypothesis that $\chi \neq 1$). Then there exists $a \in \mathbb{Q}_p$ such that $|a|_p = p^{-k+1}$ and $\chi(a) \neq 1$. Define $\psi \in \widehat{\mathbb{Q}_p}$ by $\psi(x) = \chi(pax)$. Then $\psi(p^{-1}) = \chi(a) \neq 1$ and $\psi(1) = \chi(pa) = 1$ (because $|pa|_p = p^{-k}$). By (v), there exists $y \in \mathbb{Z}_p$ such that $\psi = \chi_y$. In other words, for every $x \in \mathbb{Q}_p$,

$$\chi(x) = \psi(p^{-1}a^{-1}x) = \chi_1(p^{-1}a^{-1}yx),$$

i.e. $\chi = \alpha(p^{-1}a^{-1}y)$.

We show that α is continuous. Let $y \in \mathbb{Q}_p$, and choose a neighborhood U of $\alpha(y)$ of the form

$$U = \{\chi \in \widehat{\mathbb{Q}_p} | \forall x \in K,\ |\chi(x) - \chi_y(x)| < c\},$$

where K is a compact subset of $\mathbb{Q}_p$ and $c > 0$. We are looking for a neighborhood V of y in $\mathbb{Q}_p$ such that $\alpha(V) \subset U$.

As $\mathbb{Q}_p = \bigcup_{k\in\mathbb{Z}} p^k\mathbb{Z}_p$, we may assume that $K = p^N\mathbb{Z}_p$ for some $N \in \mathbb{Z}$. We know that χ_1 is constant on the cosets of $\mathbb{Z}_p$ in $\mathbb{Q}_p$, so, if $x \in p^N\mathbb{Z}_p$, then χ_x is constant on the cosets of $p^{-N}\mathbb{Z}_p$ in $\mathbb{Q}_p$. Hence, if $y' \in y + p^{-N}\mathbb{Z}_p$, then, for every $x \in K = p^N\mathbb{Z}_p$,

$$|\chi_{y'}(x) - \chi_y(x)| = |\chi_x(y') - \chi_x(y)| = 0 < c.$$

In other words, $\alpha(y + p^{-N}\mathbb{Z}_p) \subset U$.

Finally, we show that α is open. Let $y \in \mathbb{Q}_p$, and let V be a neighborhood of y. We may assume that V is of the form $y + p^N\mathbb{Z}_p = \{y' \in \mathbb{Q}_p | |y' - y|_p \leq p^{-N}\}$ for some $N \in \mathbb{Z}$. We want to show that $\alpha(V)$ contains a neighborhood of $\alpha(y)$. As α is a morphism of groups, we may assume that $y = 0$. Let

$$U = \{\chi \in \widehat{\mathbb{Q}_p} | \forall x \in p^{-N}\mathbb{Z}_p,\ |\chi(x) - \chi_y(x)| < c\},$$

where $c = \min_{1\leq r\leq p-1} |1 - e^{2\pi i r p^{-1}}|$, and let's show that $\alpha(p^N\mathbb{Z}_p) \supset U$. Let $y' \notin p^N\mathbb{Z}_p$, we want to show that $\chi_{y'} \notin U$. We write $y' = \sum_{n=m}^{+\infty} c_n p^n$ with

$c_n \in \{0, \dots, p-1\}$ for every $n \geq m$ and $c_m \geq 1$. Then the hypothesis on y' says that $m < N$. Let $x = p^{-m-1}$. Then $x \in p^{-N}\mathbb{Z}_p$, and

$$\chi_{y'}(x) = \chi_1(xy') = \exp(2\pi i p^{-1} c_m),$$

so $|\chi_{y'}(x) - 1| \geq c$ and $\chi_{y'} \notin U$.

vii. As the map $y \mapsto \chi_y$ is a morphism of groups, the first statement is equivalent to the fact that $\chi_{y|\mathbb{Z}_p} = 1$ if and only if $y \in \mathbb{Z}_p$. We know that $\mathrm{Ker}(\chi_1) = \mathbb{Z}_p$, so $\mathrm{Ker}(\chi_y) \supset \mathbb{Z}_p$ for every $y \in \mathbb{Z}_p$. Conversely, let $y \in \mathbb{Q}_p - \mathbb{Z}_p$. Then $|y|_p > 1$, so $|y|_p \geq p$, so $|py|_p \geq 1$, and $p^{-1}y^{-1} \in \mathbb{Z}_p$. As $\chi_y(p^{-1}y^{-1}) = \chi_1(p^{-1}) = \exp(2\pi i p^{-1}) \neq 1$, $\mathrm{Ker}(\chi_y) \not\supset \mathbb{Z}_p$.

So the map $y \mapsto \chi_y$ induces an injective morphism of groups from $\mathbb{Q}_p/\mathbb{Z}_p$ to $\widehat{\mathbb{Z}_p}$. We know (or will soon know) that $\widehat{\mathbb{Z}_p}$ is discrete by 6(e), so it just remains to show that every element of $\widehat{\mathbb{Z}_p}$ is of the form $\chi_{y|\mathbb{Z}_p}$ for some $y \in \mathbb{Q}_p$.

Let $\chi \in \widehat{\mathbb{Z}_p}$. As in (iii), we can find $k \in \mathbb{N}$ such that $\mathrm{Ker}(\chi) \supset p^k\mathbb{Z}_p$. Let $z = \chi(1)$. Then $z^{p^k} = \chi(p^k) = 1$, so we can find $c \in \{0, \dots, p^k - 1\}$ such that $z = e^{2\pi i c p^{-k}}$. Write $c = \sum_{r=0}^{k-1} c_r p^r$, with $c_r \in \{0, \dots, p-1\}$. Then

$$\chi(1) = \exp(2\pi i \sum_{r=0}^{k-1} c_r p^{r-k}) = \exp(2\pi i \sum_{n=-k}^{-1} c_{k+n} p^n).$$

Let $y = \sum_{n=-k}^{-1} c_{k+n} p^n$. Then $\chi(1) = \chi_y(1)$. As $\chi_{y|\mathbb{Z}_p}$ and χ are continuous group morphisms on $\mathbb{Z}_p$, and as 1 generates a dense subgroup of $\mathbb{Z}_p$, this implies that $\chi = \chi_{y|\mathbb{Z}_p}$.

□

6. We use the notation of the previous problem, and we suppose that G is an abelian locally compact group and fix a Haar measure μ on G.

   Remember that we have an isomorphism $L^\infty(G) \to L^1(G)^\vee := \mathrm{Hom}(L^1(G), \mathbb{C})$ sending $f \in L^\infty(G)$ to the bounded operator $g \mapsto \int_G fg d\mu$ on $L^1(G)$. (This does not use the fact that G is an abelian group.) So we can consider the weak* topology (or topology of pointwise convergence) on $L^\infty(G)$: for $f \in L^\infty(G)$, a basis of neighborhoods of f is given by the sets $U_{g_1,\dots,g_n,c} = \{f' \in L^\infty(G) || \int_G (f - f')g_i d\mu| < c,\ 1 \leq i \leq n\}$, for $n \in \mathbb{Z}_{\geq 1}$, $g_1, \dots, g_n \in L^1(G)$ and $c > 0$.

   a) (2+2 extra credit) Show that $\widehat{G} \subset L^\infty(G)$, and that the topology of $\widehat{G}$ is induced by the weak* topology of $L^\infty(G)$.
   b) (2) Show that the subset $\widehat{G} \cup \{0\}$ of $L^\infty(G)$ is closed for the weak* topology. (Hint : Identify it to the set of representations of the Banach $*$-algebra $L^1(G)$ on $\mathbb{C}$.)
   c) (1) Show that $\widehat{G}$ is a locally compact topological group. (Hint : Alaoglu's theorem.)
   d) (2) If G is discrete, show that $\widehat{G}$ is compact.
   e) (2) If G is compact, show that $\widehat{G}$ is discrete.

   Solution.

   a) Dan : This turned out to be harder than I expected. You can give them the 2 points if they get the easy direction (= the first one below) right, and 2 extra credit if they can do the other direction.

An element of $\widehat{G}$ is a continuous function from G to S^1, hence a continuous bounded function on $\widehat{G}$, hence an element of $L^\infty(G)$. Now we have to show that the two topologies on $\widehat{G}$ coincide.

Let $\chi \in \widehat{G}$. First, let $f_1, \dots, f_n \in L^1(G)$, and let $c > 0$. This defines a weak* open neighborhood

$$U = \{\psi \in \widehat{G} | \forall i \in \{1, \dots, n\},\ |\int_G \chi f_i d\mu - \int_G \psi f_i d\mu| < c\}$$

of χ. We want to find an open neighborhood V of χ for the topology of compact convergence such that $V \subset U$. Let $\varepsilon > 0$. Choose a compact subset K of G such that $\int_{G-K} |f_i| d\mu < \varepsilon$ for every $i \in \{1, \dots, n\}$ (this is possible by inner regularity of μ). Let

$$V = \{\psi \in \widehat{G} | \forall x \in K,\ |\chi(x) - \psi(x)| < \varepsilon\}.$$

Then, if $\psi \in V$ and $i \in \{1, \dots, n\}$, we have

$$\begin{aligned} |\int_G \chi f_i d\mu - \int_G \psi f_i d\mu| &\leq \int_K |f_i(x)||\chi(x) - \psi(x)| d\mu(x) + \int_{G-K} |f_i(x)||\chi(x) - \psi(x)| dx \\ &\leq \varepsilon \int_K |f_i(x)| d\mu(x) + 2 \int_{G-K} |f_i(x)| d\mu(x) \\ &\leq \varepsilon(\|f_i\|_1 + 2) \end{aligned}$$

So, if we take ε small enough, we'll get $V \subset U$.

Now we prove the converse. Let $\chi \in \widehat{G}$, let K be a compact subset of G and let $c > 0$. We consider the neighborhood

$$V = \{\psi \in \widehat{G} | \forall x \in K,\ |\chi(x) - \psi(x)| < \varepsilon\}$$

of χ in the topology of compact convergence. We have to find a weak* neighborhood included in it. Let $\eta > 0$ (to be fiddled with later), and choose a compact neighborhood A of 1 such that, for every $y \in A$, we have $|\chi(y) - 1| < \eta$. Let $f = \mathbb{1}_A$; this is in $L^1(G)$ because A is compact. Note that, for every $x \in G$,

$$\begin{aligned} |\mu(A)\chi(x) - f * \chi(x)| &= \left| \int_A (\chi(x) - \chi(y^{-1}x)) dy \right| \\ &\leq \int_A |1 - \overline{\chi(y)}| dy \\ &\leq \eta \mu(A). \end{aligned}$$

Now we try to find a weak* neighborhood of χ in $\widehat{G}$ whose elements ψ will satisfy a similar inequality, but for $x \in K$. Note that, if $\psi \in \widehat{G}$ and $x \in G$, then

$$\begin{aligned} f * \psi(x) &= \int_A \chi(y^{-1}x) dy \\ &= \chi(x) \int_A \overline{\chi(y)} dy \\ &= \int_G \psi(y^{-1}) f(xy) dy \\ &= \int_G \overline{\psi(y)} L_{x^{-1}} f(y) dy \end{aligned}$$

(we use that G is commutative and that ψ is a morphism of groups from G to S^1). Now remember that the map $G \to L^1(G)$, $x \mapsto L_{x^{-1}} f$ is continuous (proposition

3.1.13 of the notes). As K, we can find $x_1, \dots, x_n$ such that, for every $x \in K$, there exists $i \in \{1, \dots, n\}$ with $\|L_{x^{-1}}f - L_{x_i^{-1}}f\|_1 < \eta\mu(A)$. Consider the following weak* neighborhood of χ :

$$U = \{\psi \in \widehat{G} | \forall x \in \{1, x_1^{-1}, \dots, x_n^{-1}\}, \left| \int_G \overline{\chi(y)} L_x f(y) dy - \int_G \overline{\psi(y)} L_x f(y) dy \right| < \eta\mu(A)\}.$$

Let $\psi \in U$. First, we have, for every $x \in G$,

$$\begin{aligned} |\mu(A)\psi(x) - f * \psi(x)| &= \left| \int_A (\psi(x) - \psi(y^{-1}x) dy \right| \\ &= \left| \int_A (1 - \overline{\psi(y)}) dy \right| \\ &\leq \left| \int_A (1 - \overline{\chi(y)}) dy \right| + \left| \int_A (\overline{\chi(y)} - \overline{\psi(y)}) dy \right| \\ &\leq 2\eta\mu(A). \end{aligned}$$

Second, we want to bound $|f * \chi(x) - f * \psi(x)|$ for $x \in K$. So fix $x \in K$. Let $i \in \{1, \dots, n\}$ be such that $\|L_{x^{-1}}f - L_{x_i^{-1}}f\|_1 \leq \eta\mu(A)$. Then :

$$\begin{aligned} |f * \chi(x) - f * \psi(x)| &= \left| \int_G (\overline{\chi(y)} - \overline{\psi(y)}) L_{x^{-1}} f(y) dy \right| \\ &\leq \left| \int_G (\overline{\chi(y)} - \overline{\psi(y)}) L_{x_i^{-1}} f(y) dy \right| \\ &+ \left| \int_G (\overline{\chi(y)} - \overline{\psi(y)}) (L_{x_i^{-1}} f(y) - L_{x^{-1}} f(y)) dy \right| \\ &< \eta\mu(A) + 2 \int_G |L_{x_i^{-1}} f(y) - L_{x^{-1}} f(y)| dy \\ &\leq 3\eta\mu(A). \end{aligned}$$

Putting everything together, we get, for $x \in K$,

$$|\mu(A)\chi(x) - \mu(A)\psi(x)| < 6\eta\mu(A),$$

i.e. $|\chi(x) - \psi(x)| < 6\eta$. Choosing η at the beginning such that $6\eta \leq c$, we get $U \subset V$, as desired.

b) We have seen in class that $\widehat{G} \subset L^\infty(G) \simeq L^1(G)^\vee$ is the set of nondegenerate representations of the Banach $*$-algebra $L^1(G)$. Let $\pi : L^1(G) \to \mathbb{C}$ be a representation of $L^1(G)$ on $\mathbb{C}$, and assume that it is not nondegenerate. Then there exists $v \in \mathbb{C} - \{0\}$ such that $\pi(f)v = 0$ for every $f \in L^1(G)$. But this implies that $\pi = 0$. So we see that $\widehat{G} \cup \{0\} \subset L^\infty(G)$ is indeed the set of representation of $L^1(G)$ on $\mathbb{C}$. But the conditions saying that a bounded linear functional $\Lambda : L^1(G) \to \mathbb{C}$ is a representation are all closed conditions in the weak* topology (because they all assert that the values of Λ at some points of $L^1(G)$ are equal), so the set of representations of $L^1(G)$ is a weak* closed subset of $L^\infty(G)$.

c) Alaoglu's theorem says that the closed unit ball of $L^\infty(G)$ (for the operator norm coming from $\|.\|_1$, which is just $\|.\|_\infty$) is compact Hausdorff for the weak* topology. But $\widehat{G} \cup \{0\}$ is clearly included in this closed unit ball (this is easy even if we don't know that the operator norm is $\|.\|_\infty$), so is compact Hausdorff for the weak* topology. Hence its open subset $\widehat{G}$ is locally compact for the weak* topology, and we have seen in (i) that the weak* topology on $\widehat{G}$ is equal to the topology of compact convergence, so we are done.

d) Consider the map $\alpha : \widehat{G} \to (S^1)^G$ sending χ to the family $(\chi(x))_{x\in G}$. This is obviously injective. As G is discrete, its compact subsets are exactly its finite subsets, so the topology of compact convergence is exactly the topology induced by the product topology on $(S^1)^G$. Also, by Tychonoff's theorem, $(S^1)^G$ is compact Hausdorff. So, to get teh result, we only need to show that the image of α is closed in $(S^1)^G$. But the image of α is the intersection of the subsets

$$\{(a_x)_{x\in G} \in (S^1)^G | a_{x_0}a_{y_0} = a_{x_0y_0}\}$$

for all $x_0, y_0 \in G$, and each of these subsets is closed, so $\mathrm{Im}(\alpha)$ is closed.

e) Suppose that G is compact. Then the topology of $\widehat{G}$ is the topology of uniform convergence (induced by the norm $\|.\|_\infty$). To show that $\widehat{G}$ is discrete, it suffices to show that its subset $\{1\}$ is open (because $\widehat{G}$ is a topological group). Let $c > 0$ be such that the only subgroup of $\mathbb{C}^\times$ included in $\{z \in \mathbb{C}^\times || 1 - z| < c\}$ is the trivial group (see 3(b)). Let $U = \{\chi \in \widehat{G} | \|\chi - 1\|_\infty < c\}$. This is an open neighborhood of 1 in $\widehat{G}$. On the other hand, if $\chi \in U$, we have $\chi(G) \subset \{z \in \mathbb{C}^\times || 1 - z| < c\}$; as $\chi(G)$ is a subgroup of $\mathbb{C}^\times$, this means that $\chi(G) = \{1\}$, i.e. $\chi = 1$, and so $U = \{1\}$.

□

Remark We have seen in (a) that the topology of compact convergence and the weak* topology coincide on $\widehat{G}$. This is not the case for $\{0\} \cup \widehat{G}$.

For example, take $G = \mathbb{R}$ and consider the elements $\chi_y : x \mapsto e^{ixy}$ of $\widehat{G}$. I claim that the family $(\chi_y)_{y\in\mathbb{R}}$ converges weakly to 0 when $|y| \to +\infty$. (Obviously, it does not converge to 0 for the topology of compact convergence; in fact, it has no limit in this topology.) Remember the this statement means that, for every $f \in L^1(\mathbb{R})$, we have

$$\lim_{|y|\to+\infty} \int_{\mathbb{R}} f(x)e^{ixy}dx = 0.$$

Suppose first that f is the characteristic function of a compact interval $[a, b]$. Then

$$\int_{\mathbb{R}} f(x)e^{ixy}dy = \frac{1}{y}(e^{iby} - e^{iay}) \xrightarrow[|y|\to+\infty]{} 0.$$

So, if f is a (finite) linear combination of characteristic functions of compact intervals, the conclusion still holds. Now let f be any element of $L^1(\mathbb{R})$, and let $\varepsilon > 0$. We can find a linear combination g of characteristic functions of compact intervals g such that $\|f - g\|_1 \le \varepsilon$. By what we just saw, we can also find $A \in \mathbb{R}$ such that $|\int_{\mathbb{R}} g(x)e^{ixy}dy| \le \varepsilon$ for $|x| \ge A$. Then, if $|y| \ge A$, we have

$$\begin{aligned} \left|\int_{\mathbb{R}} f(x)e^{ixy}dx\right| &\le \left|\int_{\mathbb{R}} g(x)e^{ixy}dx\right| + \left|\int_{\mathbb{R}} (f(x) - g(x))e^{ixy}dx\right| \\ &\le \varepsilon + \int_{\mathbb{R}} |f(x) - g(x)|dx \\ &\le 2\varepsilon. \end{aligned}$$

So $\int_{\mathbb{R}} f(x)e^{ixy}dx$ converges to 0 as $|y| \to +\infty$.