MAT 449 : Problem Set 3

Due Thursday, October 4

1. (extra credit,3) In this problem, X is a compact Hausdorff totally disconnected topological space. (Remember that "totally disconnected" means that the only nonempty connected subsets of X are the singletons.)

   Let $x \in X$, and let A be the intersection of all the open and closed subsets of X containing x. Show that $A = \{x\}$. (Hint : This is equivalent to showing that A is connected. And remember also that disjoint closed sets can be separated by open sets in any compact Hausdorff space.)

2. (extra credit) In this problem, G is a locally compact totally disconnected topological group.

   a) (1) Show that the unit of G has a compact open neighborhood K.

   b) (2) Show that there exists an open subgroup G' of G contained in K. (Hint : Any open subset of G will generate an open subgroup. Choose your open subset wisely.)

   c) (1) Show that the compact open subgroups of G form a basis of neighborhoods of 1 in G.

   d) (2) Let G be the group $\mathbf{GL}_n(\mathbb{Q}_p)$ of problem 4 of problem set 1. Find a basis of neighborhoods of 1 in G that is composed of compact open subgroups.

3. a) (1) Let G be a compact subgroup of $\mathbf{GL}_n(\mathbb{C})$. Show that there exists $x \in \mathbf{GL}_n(\mathbb{C})$ such that $xGx^{-1} \subset \mathbf{U}(n)$.

   b) (3) Put your favorite norm on $M_n(\mathbb{C})$ (they are all equivalent anyway). Show that there exists $c > 0$ such that the only subgroup of $\mathbf{GL}_n(\mathbb{C})$ included in the ball $\{x \in \mathbf{GL}_n(\mathbb{C}) | \|x - I_n\| < c\}$ is the trivial group.

   c) (2) Show that, for every continuous representation of $\mathbf{GL}_n(\mathbb{Q}_p)$ on a finite-dimensional $\mathbb{C}$-vector space, there exists an integer $m \geq 0$ such that the subgroup $I_n + p^m M_n(\mathbb{Z}_p)$ of $\mathbf{GL}_n(\mathbb{Q}_p)$ acts trivially.

Haar measure on $\mathbf{SU}(2)$

4. Let $G = \mathbf{SU}(2)$.

   a) (2) Show that every element of G is of the form $\begin{pmatrix} a & -\overline{b} \\ b & \overline{a} \end{pmatrix}$, with $a, b \in \mathbb{C}$ and $|a|^2 + |b|^2 = 1$.

   If we identify $\mathbb{C}$ and $\mathbb{R}^2$ in the usual way, the previous question gives a homeomorphism α between $\mathbf{SU}(2)$ and S^3 (the unit sphere in $\mathbb{R}^4$).

   b) (1) If $g \in \mathbf{SU}(2)$, show that left translation by g on $\mathbf{SU}(2)$ corresponds by α to the restriction to S^3 of the action of an element of $\mathbf{SO}(4)$ on $\mathbb{R}^4$ (i.e. there exists $A \in \mathbf{SO}(4)$ such that, for every $h \in \mathbf{SU}(2)$, we have $gh = A\alpha(h)$).

c) (2) Let μ be the usual spherical measure on S^3; that is, if λ is Lebesgue measure on $\mathbb{R}^4$, we have by definition, for every Borel subset E of S^3,

$$\mu(E) = \tfrac{2}{\pi^2}\lambda(\{tx, t \in [0,1], x \in E\})$$

(note that the volume of the unit ball in $\mathbb{R}^4$ is $\frac{\pi^2}{2}$).

Show that the inverse image by α of μ is a left and right Haar measure on $\mathbf{SU}(2)$.

d) (2) We use the following (hyperspherical) coordinates on S^3 : if $(x_1, x_2, x_3, x_4) \in S^3$, we write

$$\begin{cases} x_1 = \cos\theta \\ x_2 = \sin\theta\cos\psi \\ x_3 = \sin\theta\sin\psi\cos\phi \\ x_4 = \sin\theta\sin\psi\sin\phi \end{cases}$$

with $0 \leq \theta \leq \pi$, $0 \leq \psi \leq \pi$ and $0 \leq \phi \leq 2\pi$. Show that, for every $f \in \mathcal{C}_c(S^3)$, we have $\int_{S^3} f d\mu =$

$$\frac{1}{2\pi^2}\int_0^\pi\int_0^\pi\int_0^{2\pi} f(\cos\theta, \sin\theta\cos\psi, \sin\theta\sin\psi\cos\phi, \sin\theta\sin\psi\sin\phi)\sin^2\theta\sin\psi d\theta d\psi d\phi.$$

(Feel free to use a computer to calculate any big determinants.)

The dual of a locally compact abelian group

5. Let G be an abelian topological group. We write $\widehat{G}$ for the set of continuous group morphisms $G \to S^1$.

   As the product of two continuous morphisms from G is S^1 is also a continuous morphism from G to S^1 (because S^1 is commutative), the set $\widehat{G}$ has a natural group structure. We put the topology of compact convergence on $\widehat{G}$; that is, if $\chi \in \widehat{G}$, then a basis of neighborhoods of χ is given by $\{\psi \in \widehat{G} | \sup_{x\in K}|\chi(x) - \psi(x)| < c\}$, for all compact subsets K of G and all $c > 0$.

   a) (1) Show that $\widehat{G}$ is a topological group.

   b) Suppose that $G = \mathbb{R}$.

      i. (2) Let $\rho : G \to \mathbf{GL}_n(\mathbb{C})$ be a continuous group morphism. Show that there exists a unique $A \in M_n(\mathbb{C})$ such that, for every $t \in \mathbb{R}$, $\rho(t) = \exp(tA)$. (There are several ways to do this. One way is to notice that, if the conclusion is true, then $c'(0)$ must exist and be equal to A, and to work backwards from there.)

      ii. (2) Show that the image of ρ is contained in $\mathbf{U}(n)$ if and only if $A^* = -A$.

      iii. (2) Show that the map $\mathbb{R} \to \widehat{G}$ sending $x \in \mathbb{R}$ to the group morphism $G \to S^1$, $t \mapsto e^{ixt}$ is an isomorphism of topological groups (i.e. a group isomorphism that is also a homeomorphism).

   c) (1) Show that there is an isomorphism of topological groups $\widehat{S^1} \simeq \mathbb{Z}$ that sends id_{S^1} to 1.

   d) (1) What is the topological group $\widehat{\mathbb{Z}}$?

   e) Suppose that $G = \mathbb{Q}_p$ (cf. problem 4 of problem set 1). We define a map $\chi_1 : \mathbb{Q}_p \to S^1$ in the following way : If $x \in \mathbb{Q}_p$, we can write $x = \sum_{n=-\infty}^{+\infty} c_n p^n$, with $0 \leq c_n \leq p-1$ and $c_n = 0$ for n small enough, and this uniquely determines the c_n (see problem 4(i) of problem set 1). We set

$$\chi_1(x) = \exp\left(2\pi i \sum_{n=-\infty}^{-1} c_n p^n\right).$$

i. (3) Show that $\chi_1 : \mathbb{Q}_p \to S^1$ is a continuous group morphism and that $\mathrm{Ker}(\chi_1) = \mathbb{Z}_p$.

ii. (2) For every $y \in \mathbb{Q}_p$, we define $\chi_y : \mathbb{Q}_p \to S^1$ by $\chi_y(x) = \chi(xy)$. Show that this is also a continuous group morphism, and find its kernel.

iii. (1) Let $\chi \in \widehat{\mathbb{Q}_p}$. Show that there exists $k \in \mathbb{Z}$ such that $\chi = 1$ on $\{x \in \mathbb{Q}_p | |x|_p \leq p^{-k}\}$.

iv. (2) Let $\chi \in \widehat{\mathbb{Q}_p}$ such that $\chi(1) = 1$ and $\chi(p^{-1}) \neq 1$. Show that there exists a sequence of integers $(c_r)_{r \geq 0}$ such that $1 \leq c_0 \leq p-1$ and $0 \leq c_r \leq p-1$ for $r \geq 1$ and that, for every $k \in \mathbb{Z}_{\geq 1}$,

$$\chi(p^{-k}) = \exp\left(2\pi i \sum_{r=1}^{k} c_{k-r} p^{-r}\right).$$

v. (1) Let $\chi \in \widehat{\mathbb{Q}_p}$ such that $\chi(1) = 1$ and $\chi(p^{-1}) \neq 1$. Show that there exists $y \in \mathbb{Q}_p$ such that $|y|_p = 1$ and $\chi = \chi_y$.

vi. (3) Show that the map $\mathbb{Q}_p \to \widehat{\mathbb{Q}_p}$, $y \mapsto \chi_y$ is an isomorphism of topological groups.

vii. (extra credit, 4) Show that $\chi_{y|\mathbb{Z}_p} = \chi_{y'|\mathbb{Z}_p}$ if and only $y - y' \in \mathbb{Z}_p$, and that the map $y \mapsto \chi_y$ induces an isomorphism of topological groups $\mathbb{Q}_p/\mathbb{Z}_p \xrightarrow{\sim} \widehat{\mathbb{Z}_p}$, where $\mathbb{Q}_p/\mathbb{Z}_p$ has the discrete topology.

6. We use the notation of the previous problem, and we suppose that G is an abelian locally compact group and fix a Haar measure μ on G.

   Remember that we have an isomorphism $L^\infty(G) \to L^1(G)^\vee := \mathrm{Hom}(L^1(G), \mathbb{C})$ sending $f \in L^\infty(G)$ to the bounded operator $g \mapsto \int_G fg d\mu$ on $L^1(G)$. (This does not use the fact that G is an abelian group.) So we can consider the weak* topology (or topology of pointwise convergence) on $L^\infty(G)$: for $f \in L^\infty(G)$, a basis of neighborhoods of f is given by the sets $U_{g_1,\ldots,g_n,c} = \{f' \in L^\infty(G) || \int_G (f - f') g_i d\mu| < c,\ 1 \leq i \leq n\}$, for $n \in \mathbb{Z}_{\geq 1}$, $g_1, \ldots, g_n \in L^1(G)$ and $c > 0$.

   a) (2+2 extra credit) Show that $\widehat{G} \subset L^\infty(G)$, and that the topology of $\widehat{G}$ is induced by the weak* topology of $L^\infty(G)$.

   b) (2) Show that the subset $\widehat{G} \cup \{0\}$ of $L^\infty(G)$ is closed for the weak* topology. (Hint : Identify it to the set of representations of the Banach $*$-algebra $L^1(G)$ on $\mathbb{C}$.)

   c) (1) Show that $\widehat{G}$ is a locally compact topological group. (Hint : Alaoglu's theorem.)

   d) (2) If G is discrete, show that $\widehat{G}$ is compact.

   e) (2) If G is compact, show that $\widehat{G}$ is discrete.