MAT 449 : Problem Set 2

Due Thursday, September 27

More examples of Haar measures

1. Let G be a locally compact group, and let H be a closed subgroup of G. We write π for the quotient map from G to G/H. We denote by Δ_G (resp. Δ_H) the modular function of G (resp. H), and we assume that $\Delta_{G|H} = \Delta_H$. We fix left Haar measures μ_G and μ_H on G and H.

   a) (1) Show that, for every compact subset K' of G/H, there exists a compact subset K of G such $\pi(K) = K'$.

   b) (1) Let $f \in L^1(G)$. Show that the function $G \to \mathbb{C}$, $x \mapsto \int_H f(xh)d\mu_H(h)$ is invariant by right translations by elements of H. Hence it defines a function $G/H \to \mathbb{C}$, that we will denote by f^H.

   c) (2) If $f \in \mathcal{C}_c(G)$, show that $f^H \in \mathcal{C}_c(G/H)$.

   d) (2) Show that the map $\mathcal{C}_c(G) \to \mathcal{C}_c(G/H)$, $f \mapsto f^H$ is surjective. (Hint : You may use the fact that, for every compact subset K of G, there exists a function $\varphi \in \mathcal{C}_c^+(G)$ such that $\varphi(x) > 0$ for every $x \in K$.)

   e) (2) If $f \in \mathcal{C}_c(G)$ is such that $f^H = 0$, show that $\int_G f(x)d\mu_G(x) = 0$. (Hint : use a function in $\mathcal{C}_c(G/H)$ that is equal to 1 on $\pi(\mathrm{supp}(f))$, and proposition 2.12 of the notes. (Sorry.))

   f) (2) Show that there exists a unique regular Borel measure $\mu_{G/H}$ on G/H that is invariant by left translations by elements of G and such that, for every $f \in \mathcal{C}_c(G)$, we have $\int_G f(x)d\mu_G(x) = \int_{G/H} f^H(y)d\mu_{G/H}(y)$.

   g) (1) If P is a closed subgroup of G such that π induces a homeomorphism $P \overset{\sim}{\to} G/H$, show that the inverse image of $\mu_{G/H}$ by this homeomorphism is a left Haar measure on P.

   h) (2) If P is a closed subgroup of G such that the map $P \times H \to G$, $(p, h) \mapsto ph$ is a homeomorphism, and if $d\mu_P$ is a left Haar measure on P, show that the linear functional $\mathcal{C}_c(G) \to \mathbb{C}$, $f \mapsto \int_H \int_P f(ph)d\mu_P(p)d\mu_H(h)$ defines a left Haar measure on G.

   Solution.

   a) Let V be a compact neighborhood of 1 in G/H. Then $\pi(V)$ is a compact neighborhood of $\pi(1)$ in G/H. We have $K' \subset \bigcup_{x\in\pi^{-1}(K')} \pi(xV)$. As K' is compact, we can find $x_1, \dots, x_n$ such that $K' \subset \bigcup_{i=1}^n \pi(x_i V)$. Let $K = \pi^{-1}(K') \cap (\bigcup_{i=1}^n x_i V)$. Then K is a closed subset of the compact set $\bigcup_{i=1}^n x_i V$, hence it is compact, and we have $\pi(K) = K'$.

b) Let $x \in H$. Then, for every $g \in G$, we have

$$\int_H f(gxh)d\mu_H(h) = \int_H f(gh)d\mu_H(h)$$

by the left invariance of μ_H.

c) We need to show that f^H is continuous and that it has compact support.

Fix a symmetric compact neighborhood V_0 of 1, and note that $A := \operatorname{supp} f \cup V_0(\operatorname{supp} f)$ is compact. Let $\varepsilon > 0$. As f is left uniformly continuous, there exists a neighborhood $V \subset V_0$ of 1 such that, for every $x \in G$ and every $y \in V$, we have $|f(yx) - f(x)| \leq \varepsilon$. Then, for every $x \in G$ and every $y \in V$, we have

$$|f^H(\pi(yx)) - f^H(x)| = |\int_H (f(yxh) - f(xh))d\mu_H(h)| \leq \varepsilon\mu_H(x^{-1}A \cap H),$$

because $f(yxh) = f(xh) = 0$ unless $y \in (x^{-1}\operatorname{supp} f) \cup (x^{-1}y^{-1}\operatorname{supp} f) \subset x^{-1}A$. As $x^{-1}A \cap H$ is compact, it has finite measure, and the calculation above implies that f^H is continuous at the point $\pi(x)$.

Now we show that f^H has compact support. By definition of f^H, we have $f^H(\pi(x)) = 0$ if $x \notin KH$. So the support of f^H is contained in $\pi(KH) = \pi(K)$, hence it is compact.

d) Let $g \in \mathcal{C}_c(G/H)$, and let K' be its support. By question (a), there exists a compact subset K of G such that $\pi(K) = K'$. Let $\varphi \in \mathcal{C}_c^+(G)$ be such that $\varphi(x) > 0$ for every $x \in K$. We show that $\varphi^H(y) > 0$ for every $y \in K'$. Let $y \in K'$, write $y = \pi(x)$ with $x \in K$. As $\varphi(x) > 0$ and φ is continuous, we can find an open neighborhood V of 1 in G and a $c \in \mathbb{R}_{>0}$ such that $\varphi(x') \geq c$ for every $x' \in xV$. In particular,

$$\varphi^H(y) = \int_H \varphi(xh)d\mu_H(h) \geq \int_{H \cap V} \varphi(xh)d\mu_H(h) \geq c \cdot \mu_H(U \cap H) > 0$$

(as $U \cap H$ is a nonempty open subset of H, we have $\mu_H(U \cap H) > 0$).

We define a function $F : G \to \mathbb{C}$ in the following way :

$$F(x) = \begin{cases} \frac{g(\pi(x))}{\varphi^H(\pi(x))} & \text{if } \varphi^H(\pi(x)) > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Note that F is continuous on the open subsets $U_1 = \{x \in G | \varphi^H(\pi(x)) > 0\}$ and $U_2 = G - \operatorname{supp}(g \circ \pi)$ (on the second subset, it is identically zero). As $U_1 \supset \pi^{-1}(K')$ and $\pi^{-1}(K') = \operatorname{supp}(g \circ \pi)$, we have $U_1 \cup U_2 = G$, the function F is continuous on G. Finally, we take $f = F\varphi$. Then $f \in \mathcal{C}_c(G)$, and we just need to show that $f^H = g$.

Let $x \in G$. If $\varphi^H(\pi(x)) = 0$, then $f(xh) = 0$ for every $h \in H$, so $f^H(\pi(x)) = 0$. We have seen that φ^H takes positive values on $K' = \operatorname{supp}(g)$, so we also have $x \notin \operatorname{supp}(g)$, i.e., $g(x) = 0 = f^H(x)$. Now assume that $\varphi^H(\pi(x)) > 0$. Note that the function $H \to \mathbb{C}$, $h \mapsto F(xh)$ is constant. So

$$f^H(\pi(x)) = F(x) \int_H \varphi(xh)d\mu_H(h) = \frac{g(\pi(x))}{\varphi^H(\pi(x))}\varphi^H(\pi(x)) = g(\pi(x)).$$

Finally, note that $f \in \mathcal{C}_c^+(G)$ if $g \in \mathcal{C}_c^+(G/H)$, and that we also proved along the way that $f^H \in \mathcal{C}_c^+(G/H)$ if $f \in \mathcal{C}_c^+(G)$ (we proved this for φ).

e) Let $\psi \in \mathcal{C}_c(G/H)$ be such that $\psi(y) = 1$ for every $y \in \pi(\operatorname{supp} f)$. By question (d), there exists $\varphi \in \mathcal{C}_c(G)$ such that $\varphi^H = \psi$. We have

$$\begin{aligned}
\int_G f(x) d\mu_G(x) &= \int_G f(x)\varphi^H(\pi(x)) d\mu_G(x) \\
&= \int_{G\times H} f(x)\varphi(xh) d\mu_G(x) d\mu_H(h) \\
&= \int_H (\int_G f(x)\varphi(xh) d\mu_G(x)) d\mu_H(h) \\
&= \int_H (\Delta_G(h)^{-1} \int_G f(xh^{-1})\varphi(x) d\mu_G(x)) d\mu_H(h) \\
&= \int_G \varphi(x)(\int_H \Delta_H(h)^{-1} f(xh^{-1}) d\mu_H(h)) d\mu_G(x) \\
&= \int_G \varphi(x)(\int_H f(xh) d\mu_H(h)) d\mu_G(x) \quad \text{(by proposition 2.12 of the notes)} \\
&= 0 \quad \text{(because } f^H = 0\text{)}.
\end{aligned}$$

f) By question (e), the positive linear function $\mathcal{C}_c(G) \to \mathbb{C}$, $f \mapsto \int_G f d\mu_G$ factors through the linear map $\mathcal{C}_c(G) \to \mathcal{C}_c(G/H)$, $f \mapsto f^H$. By question (d) (and the remark at the end of its solution), it defines a positive linear functional $\mathcal{C}_c(G/H) \to \mathbb{C}$. By the Riesz representation theorem, this comes from a regular Borel measure $\mu_{G/H}$ on G/H. Unravelling the definition, we get, for every $f \in \mathcal{C}_c(G)$,

$$\int_G f d\mu_G = \int_{G/H} f^H d\mu_{G/H}.$$

By the left invariance of μ_G and question (d), we have, if $f \in \mathcal{C}_c(G/H)$ and $x \in G$,

$$\int_{G/H} f(xy) d\mu_{G/H}(y) = \int_{G/H} f(y) d\mu_{G/H}.$$

Using the uniqueness part of the Riesz representation theorem (as we did in class), we see that $\mu_{G/H}(xE) = \mu_{G/H}(E)$ for every Borel subset E of G/H.

g) Let ν be the inverse image of $\mu_{G/H}$ by the homeomorphism $\alpha : P \xrightarrow{\sim} G/H$. It is a regular Borel measure because α is a homeomorphism. Also, note that $\alpha(xy) = x\alpha(y)$ for every $x \in P$ (this is obvious on the definition of α). As $\mu_{G/H}$ is invariant by left translations by elements of P, so is ν.

h) The hypothesis implies that π induces a homeomorphism $P \xrightarrow{\sim} G/H$, hence we get a left Haar measure ν on P as in question (g). By the uniqueness of left Haar measures, we have $\mu_P = c\nu$ for some $c \in \mathbb{R}_{>0}$. Hence, for every $f \in \mathcal{C}_c(G)$,

$$\int_H \int_P f(ph) d\mu_P(p) d\mu_H(h) = c\int_P (\int_H f(ph) d\mu_H(h)) d\nu(p) =$$

$$c\int_{G/H} f^H(y) d\mu_{G/H}(y) = c\int_G f(x) d\mu_G(x).$$

So the functional $f \mapsto \int_H \int_P f(ph) d\mu_P(p) d\mu_H(h)$ is positive and corresponds to the left Haar measure $c\mu_G$ on G.

$\square$

2. Let G be a locally compact group. Let A and N be two closed subgroups of G such that $A \times N \to G$, $(a, n) \mapsto an$ is a homeomorphism and that A normalizes N (i.e. for every $a \in A$ and $n \in N$, we have $ana^{-1} \in N$).

   a) (2) If μ_A and μ_N are left Haar measures on A and N, show that the linear functional $\mathcal{C}_c(G) \to \mathbb{C}$, $f \mapsto \int_A \int_N f(an) d\mu_A(a) d\mu_N(n)$ defines a left Haar measure on G.

   b) (1) Let $a \in A$. Show that there exists $\alpha(a) \in \mathbb{R}_{>0}$ such that, for every $f \in \mathcal{C}_c(N)$, we have
$$\int_N f(ana^{-1}) d\mu_N(n) = \alpha(a) \int_N f(n) d\mu_N(n).$$

   c) (1) If Δ_G, Δ_A and Δ_N are the modular functions of G, A and N respectively, show that $\Delta_G(an) = \alpha(a)\Delta_A(a)\Delta_N(n)$ if $a \in A$ and $n \in N$.

Solution.

a) The setup is very similar to that of problem 1 (with for example N playing the role of H), with the difference that we don't make any assumption on the modular functions. Still, the results questions (a)-(d) of problem 1 still stay true, since their proof doesn't use the assumption on the modular functions. In particular, we get a surjective linear transformation $f \mapsto f^N$ from $\mathcal{C}_c(G)$ to $\mathcal{C}_c(G/N) \simeq \mathcal{C}_c(A)$, and it sends $\mathcal{C}_c^+(G)$ onto $\mathcal{C}_c^+(A)$. The linear functions of the statement sends $f \in \mathcal{C}_c(G)$ to $\int_A f^N(a) d\mu_A(a)$, so it is positive, and the Riesz representation theorem says that there is a unique regular Borel measure μ_G on G such that, for every $f \in \mathcal{C}_c(G)$, we have
$$\int_G f d\mu_G = \int_A \int_N f(an) d\mu_A(a) d\mu_N(n).$$
As μ_A is a left Haar measure on A, the formula above implies that $\int_G L_a f d\mu_G = \int_G f d\mu_G$ for every $f \in \mathcal{C}_c(G)$ and every $a \in A$. We show that μ_G is left invariant by N. Let $x \in N$ and $f \in \mathcal{C}_c(G)$. Then we have
$$\begin{aligned} \int_G L_x f d\mu_G &= \int_A \int_N f(xan) d\mu_A(a) d\mu_N(n) = \int_A (\int_N f(a(a^{-1}xa)n) d\mu_N(n)) d\mu_A(a) \\ &= \int_A (\int_N f(an) d\mu_N(n)) d\mu_A(a) \quad \text{because } a^{-1}xa \in N \text{ and } \mu_N \text{ is left invariant} \\ &= \int_G f d\mu_G. \end{aligned}$$
As $G = AN$, this implies that $\int_G L_x g d\mu_G = \int_G f d\mu_G$ for every $x \in G$ and every $f \in \mathcal{C}_c(G)$. By proposition 2.6 of the notes, μ_G is a left Haar measure on G.

b) Note that the map $N \to N$, $n \mapsto a^{-1}na$ is a homeomorphism. Hence the formula $E \mapsto \mu_N(a^{-1}Ea)$ defines a regular Borel measure on N, which we denote by ν. If E is a Borel subset and $n \in N$, then
$$\nu(nE) = \mu(a^{-1}nEa) = \mu((a^{-1}na)a^{-1}Ea) = \mu(a^{-1}Ea) = \nu(E).$$
Hence ν is a left Haar measure on N, and so there exists $\alpha(a) \in \mathbb{R}_{>0}$ such that $\nu = \alpha(a)\mu_N$. Now, if E is Borel subset of N and $f = \mathbb{1}_E$, the function $n \mapsto f(ana^{-1})$ is the characteristic function of $a^{-1}Ea$, so
$$\int_N f(ana^{-1}) d\mu_N(n) = \mu(a^{-1}Ea) = \alpha(a)\mu(E) = \alpha(a) \int_N f d\mu_N.$$
This extends in the usual way to all the functions $f \in L^1(N)$, and in particular to $f \in \mathcal{C}_c(N)$.

c) Let $a \in A$ and $n \in N$, and fix $f \in \mathcal{C}_c^+(G)$. Then we have

$$\begin{aligned}
\Delta_G(an)^{-1} \int_G f d\mu_G = \int_G R_{an}(f) d\mu_G &= \int_A \int_N f(bman) d\mu_A(b) d\mu_N(m) \\
&= \int_A (\int_N f(ba(a^{-1}ma)n) d\mu_N(m)) d\mu_A(b) \\
&= \alpha(a)^{-1} \int_A (\int_N f(bamn) d\mu_N(m)) d\mu_A(b) \text{ by question (b)} \\
&= \alpha(a)^{-1} \Delta_N(n)^{-1} \int_A (\int_N f(bam) d\mu_N(m)) d\mu_A(b) \text{ by definition of } \Delta_N \\
&= \alpha(a)^{-1} \Delta(n)^{-1} \int_N (\int_A f(bam) d\mu_A(b)) d\mu_N(m) \\
&= \alpha(a)^{-1} \Delta_N(n)^{-1} \Delta_A(a)^{-1} \int_N (\int_A f(bm) d\mu_A(b)) d\mu_N(m) \text{ by definition of } \Delta_A \\
& \alpha(a)^{-1} \Delta_N(n)^{-1} \Delta_A(a)^{-1} \int_G f d\mu_G.
\end{aligned}$$

As $\int_G f d\mu_G > 0$, this implies that $\Delta_G(an) = \alpha(a)\Delta_A(a)\Delta_N(n)$.

$\square$

3. Let $G = \mathbf{SL}_n(\mathbb{R})$, $H = \mathbf{SO}(n)$, and let $P \subset G$ be the subgroup of upper triangular matrices with positive entries on the diagonal (and determinant 1).

   a) (4) Show that the map $P \times H \to G$, $(p, h) \mapsto ph$ is a homeomorphism. (Hint : Gram-Schmidt.)

   b) (3) Give a formula for a left Haar measure on P similar to the formula in problem 6(d) of problem set 1.

   c) (4) Calculate the modular function of P.

   d) (2) Show that G is unimodular. (There are several ways to do this.)

   e) (2) If $n = 2$, show that $\mathbf{SO}(n) \simeq S^1$ (the circle group), and give a left Haar measure on G.

Solution.

a) In this problem, we denote the usual Euclidian inner product on $\mathbb{R}^n$ by $\langle ., . \rangle$, and the associated norm by $\|.\|$.

   We denote the map $P \times H \to G$ of the statement by α. This map is continuous because $\mathbf{SL}_n(\mathbb{R})$ is a topological group. We first show that it is injective. Suppose that we have $p, p' \in P$ and $h, h' \in H$ such that $ph = p'h'$. Then $p^{-1}p' = h(h')^{-1} \in P \cap H$ is a special orthogonal matrix that is upper triangular with positive entries on the diagonal. Such a matrix has to be the identity. Indeed, let $(v_1, \dots, v_n)$ be its columns, and let $(e_1, \dots, e_n)$ be the canonical basis of $\mathbb{R}^n$. We want to show that $(v_1, \dots, v_n) = (e_1, \dots, e_n)$. As v_1 is a norm 1 vector and a positive multiple of e_1, we must have $v_1 = e_1$. As the vectors $v_2, \dots, v_n$ are orthogonal to v_1, their first entries are all 0. So v_2 is a positive multiple of e_2; as v_2 is norm 1, we must have $v_2 = e_2$. Now the vectors $v_3, \dots, v_n$ are orthogonal to v_2, so their second entries are zero, so v_3 is a positive multiple of e_3 etc.

   Now remember the Gram-Schmidt orthonormalization process. If $(v_1, \dots, v_n)$ is a basis of $\mathbb{R}^n$, it produces an orthogonal basis $(w_1, \dots, w_n)$ and an orthonormal basis $(u_1, \dots, u_n)$ in the following way :

- $w_1 = v_1$ and $u_1 = \frac{1}{\|w_1\|} w_1$;
- For $1 \le k \le n-1$, $u_{k+1} = \frac{1}{\|w_{k+1}\|} w_{k+1}$, where $w_{k+1} = v_{k+1} - \sum_{i=1}^{k} \frac{\langle w_i, v_{k+1} \rangle}{\langle w_i, w_i \rangle} w_i$.

In particular, if A (resp. B, resp. C) is the matrix with columns $(v_1, \dots, v_n)$ (resp. $(w_1, \dots, w_n)$, resp. $(u_1, \dots, u_n)$), then we have $B = AN$ and $C = AND$, where N is an upper triangular matrix with ones on the diagonal and D is the diagonal matrix with diagonal entries $(\|w_1\|^{-1}, \dots, \|w_n\|^{-1})$. Note also that the entries of N and of D are continuous functions of $v_1, \dots, v_n$, hence also the entries of B and C, and that C is an orthogonal matrix. If $x \in \mathbf{SL}_n(\mathbb{R})$, applying this process to the columns of x, we get an orthogonal matrix h and a matrix $p \in P$, both depending continuously on x, such that $h = xp$, i.e. $x = hp^{-1}$. Also, $\det(h) = \det(xp) = \det(p) > 0$, so h is actually in $\mathbf{SO}(n)$. As $g \mapsto g^{-1}$ is a continuous function on $\mathbf{GL}_n(\mathbb{R})$ (hence on its subgroup P), we have constructed a continuous map $\beta : G \to P \times H$ such that $\alpha \circ \beta = \mathrm{id}_G$. In particular, the map α is surjective, so it si bijective. Then its inverse must be β, and we know that β is continuous. So α is a homeomorphism.

b) Note that P is an open subset of the $\mathbb{R}$-vector space V of upper triangular matrices in $M_n(\mathbb{R})$. Moreover, for every $p \in P$, left translation by p on P is the restriction of the linear endomorphism $T_p : V \to V$, $x \mapsto px$. So we can apply problem 5 of problem set 1 to define a Haar measure on P as $|\det(T_p)|^{-1} d_V(p)$, where d_V is Lebesgue measure on V.

We still need to calculate $\det(T_p)$ for $p \in P$. Let $p \in P$, and let $a_1, \dots, a_n$ be its diagonal entries. Let $(e_1, \dots, e_n)$ be the canonical basis of $\mathbb{R}^n$ as before, and let $V_i = \mathrm{Span}(e_1, \dots, e_i) \subset \mathbb{R}^n$ for $1 \le i \le n$. Note that the action of $p \in \mathbf{GL}_n(\mathbb{R})$ preserves the subspace $V_1, \dots, V_n$, and that ther determinant of the endormophism of V_i induced by p is $a_1 \dots a_i$. By decomposing V using the columns of the matrices (as in the solution of problem 6(c) of problem set 1), we get an isomorphism $V \simeq V_1 \oplus V_2 \oplus \dots \oplus V_n$ such that the endomorphism T_p corresponds to the action of p on each V_i. So we get

$$\det(T_p) = \prod_{i=1}^{n} \prod_{r=1}^{i} a_r = a_1^n a_2^{n-2} \dots a_{n-1}^2 a_n = \prod_{i=1}^{n} a_i^{n+1-i}.$$

c) Daniel : I'm not even sure of my own signs here, so don't take points off for a sign mistakes.

We will use problem 2, with $G = P$, N the group of unipotent upper triangular matrices (i.e. of upper triangular matrices with ones on the diagonal) and A the group of diagonal matrices with positive diagonal entries. Let $\alpha : A \times N \to P$ be the map defined by $\alpha(a, n) = an$. Let's show that α is a homeomorphism. The map α is obviously continuous, and it is injective because $N \cap A = \{1\}$. Let $x \in P$, and let $a \in A$ be the matrix with the same diagonal entries as x. Then $n := a^{-1}x$ is in N, and $\alpha(a, n) = x$. Hence α is bijective. Moreover, the matrix a depends continuously on x, hence so does n, so the inverse of α is continuous, and finally α is a homeomorphism.

We want to apply question 2(c). For this, we need to calculate the modular functions of A and N and the function $\alpha : A \to \mathbb{R}_{>0}$.

First, as A is commutative, we have $\Delta_A = 1$.

For N, there are several ways to proceed. For example, you may notice that N is obviously homeomorphic (as a topological space only) to the $\mathbb{R}$-vector space W of upper triangular matrices in $M_n(\mathbb{R})$ with zeroes on the diagonal. (Just forget the diagonal terms of the matrices.) Moreover, for $n \in N$, left translation by n on N corresponds to the linear endomorphism U_n of W given by $U_n(X) = nX$, for $X \in W$. Note that W is a subspace of the space V of the previous question, and that U_n is the restriction of T_n. So we can use the same method as in the previous question to calculate $\det(U_n)$, and we get $\det(U_n) = 1$. Hence Lebesgue measure on W is a left Haar measure on N. We can redo everything using right translations instead of left translations, and we get that Lebesgue measure on W is also a right Haar measure on N. This means that N is unimodular, so $\Delta_N = 1$.

Finally, we need to calculate the function α. Remember that it is defined by

$$\int_N f(ana^{-1})dn = \alpha(a) \int_N f(n)dn$$

for every $f \in \mathcal{C}_c(N)$, where dn is Lebesgue measure on W (which we have just seen is a Haar measure on N). Note that $c_a : X \mapsto aXa^{-1}$ is a linear endomrophism of W, so we can calculate $\int_N f(ana^{-1})dn$ using the change of variables formula once we know $\det(c_a)$. We get $\det(c_a) \int_N f(ana^{-1})dn = \int_N f(n)dn$, hence $\alpha(a) = \det(c_a)^{-1}$. But is is easy to see that, if the diagonal entries of a are $(a_1, \dots, a_n)$, then

$$\det(c_a) = a_1^{n-1} a_2^{n-3} \dots a_n^{1-n} = \prod_{i=1}^{n} a_i^{n-2i+1}.$$

Hence finally, for $p \in P$,

$$\Delta_P(p) = a_1^{1-n} a_2^{3-n} \dots a_n^{n-1} = \prod_{i=1}^{n} a_i^{2i-n-1},$$

where $a_1, \dots, a_n$ are the diagonal entries of p.

d) If you know (or know how to prove) that $\mathbf{SL}_n(\mathbb{R})$ is equal to its commutator subgroup, then this isn easy. Here is another way : Let $\mathbf{GL}_n(\mathbb{R})^+$ be the group of $n \times n$ matrices with positive determinant. This is an open subgroup of $\mathbf{GL}_n(\mathbb{R})$ (it's the inverse image of $\mathbb{R}_{>0}$ by the continuous group morphism $\det : \mathbf{GL}_n(\mathbb{R}) \to \mathbb{R}^\times$), so, if μ is a Haar measure on $\mathbf{GL}_n(\mathbb{R})$ (remember that $\mathbf{GL}_n(\mathbb{R})$ is unimodular by problem 6(c) of problem set 1), its restriction to $\mathbf{GL}_n(\mathbb{R})^+$ is a nonzero regular Borel measure, and it is obviously a left and right Haar measure on $\mathbf{GL}_n(R)^+$. Now note that we have an isomorphism of topological groups $\mathbb{R}_{>0} \times \mathbf{SL}_n(\mathbb{R}) \to \mathbf{GL}_n(\mathbb{R})^+$, $(\lambda, x) \mapsto \lambda x$ (whose inverse is given by $x \mapsto (\det(x)^{1/n}, \det(x)^{-1/n}x)$), so we can apply problem 2 with $G = \mathbf{GL}_n(\mathbb{R})^+$, $A = \mathbb{R}_{>0} I_n$ and $N = \mathbf{SL}_n(\mathbb{R})$. As A and N commute, we have $\alpha = 1$. We know that A is unimodular because it is commutative, and we have just seen that $\mathbf{GL}_n(\mathbb{R})^+$ is unimodular, hence 2(c) implies that $\mathbf{SL}_n(\mathbb{R})$ is also unimodular.

e) It is well-known that the group of rotations in $\mathbb{R}^2$ (i.e. $\mathbf{SO}(2)$) is isomorphic to the circle group S^1. The isomorphism sends $e^{2i\pi\theta} \in S^1$ to the matrix $\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$. Also, we have seen in class that we can define a Haar measure on S^1 by the linear functional sending $f \in \mathcal{C}_c(S^1)$ to $\int_0^1 f(e^{2i\pi\theta})d\theta$, where $d\theta$ is Lebesgue measure on $\mathbb{R}$.

The point of this, of course, is that problem 1 now allows you to define a Haar measure on $\mathbf{SL}_2(\mathbb{R})$. To treat the case of $\mathbf{SL}_n(\mathbb{R})$, we need a Haar measure on $\mathbf{SO}(n)$. An example of such a measure is given in problem 6.

□

4. (Remember problems 4, 5,6,8 of problem set 1.) We denote by dx a Haar measure on the additive group $\mathbb{Q}_p$. We also denote by dx (resp. dA) the product measure on $\mathbb{Q}_p^n$ (resp. $M_n(\mathbb{Q}_p) \simeq \mathbb{Q}_p^{n^2}$); note that it is a Haar measure for the corresponding additive group.

   a) (2) Show that, for every $f \in L^1(\mathbb{Q}_p)$ and every $a \in \mathbb{Q}_p^\times$, $b \in \mathbb{Q}_p$, we have
$$\int_{\mathbb{Q}_p} f(x)dx = |a|_p \int_{\mathbb{Q}_p} f(ax+b)dx.$$

   b) (3) Let $n \geq 1$. Show that, if $f \in L^1(\mathbb{Q}_p^n)$, $A \in \mathbf{GL}_n(\mathbb{Q}_p)$ and $b \in \mathbb{Q}_p^n$, we have
$$\int_{\mathbb{Q}_p^n} f(x)dx = |\det(A)|_p \int_{\mathbb{Q}_p^n} f(Ax+b)dx.$$

   c) (2) Show that $|\det(A)|_p^{-n}dA$ is a left and right Haar measure on $\mathbf{GL}_n(\mathbb{Q}_p)$.

   d) (3) Let B be the group of upper triangular matrices in $\mathbf{GL}_n(\mathbb{Q}_p)$. Find a left Haar measure on B and calculate the modular function of B.

Solution.

a) First, using the invariance by translation of dx, we see that
$$\int_{\mathbb{Q}_p} f(ax+b)dx = \int_{\mathbb{Q}_p} f(ax)dx$$
for every $f \in L^1(\mathbb{Q}_p)$ and $a, b \in \mathbb{Q}_p$.

Let $a \in \mathbb{Q}_p^\times$. We use the notation of problem 8 of problem set 1. If $x \in \mathbb{Q}_p$ and $m \in \mathbb{Z}$, then
$$aB(x,p^m) = \{ay \text{ with } |x-y|_p \leq p^m\} = \{y \in \mathbb{Q}_p | |ax-y|_p \leq |a|_p p^m\} = B(ax, |a|_p p^m),$$
and so, by 8(a) of problem set 1, $\mathrm{vol}(aB(x,p^m)) = |a|_p \mathrm{vol}(B(x,p^m))$. Using question (b) of the same problem, we get $\mathrm{vol}(aE) = |a|_p \mathrm{vol}(E)$ for every Borel subset E of $\mathbb{Q}_p$. Suppose that $f = \mathbb{1}_E$, with E a Borel subset of $\mathbb{Q}_p$. Then
$$\int_{\mathbb{Q}_p} f(ax)dx = \mathrm{vol}(a^{-1}E) = |a|_p^{-1} \int_{\mathbb{Q}_p} f(x)dx,$$
so we get the desired result for this function f. The result now follows for every $f \in L^1(\mathbb{Q}_p)$ by linearity and continuity of the integral.

b) Using the translation invariance of dx as in question (a), we see that it suffices to prove the result in the case $b = 0$. Let $A \in \mathbf{GL}_n(\mathbb{Q}_P)$. First note that $A = A_1A_2$ and if we know the result for A_1 and A_2, then we know it for A; indeed, for every $f \in L^1(\mathbb{Q}_p^n)$, we'll have
$$\int_{\mathbb{Q}_p^n} f(x)dx = |\det(A_1)|_p \int_{\mathbb{Q}_p^n} f(A_1x)dx =$$
$$|\det(A_1)|_p |\det(A_2)|_p \int_{\mathbb{Q}_p^n} f(A_1(A_2x))dx = |\det(A)|_p \int_{\mathbb{Q}_p^n} f(Ax)dx.$$
The Gauss algorithm (for solving systems of linear equations) says that we can make A upper triangular by elementary row operations (with correspond to multiplying on the left by a lower triangular matrix) and permutations of rows (with correspond

to multiplying on the left by a permutation matrix). So, by the observation above, it suffices to prove the result for upper and lower triangular matrices and for permutation matrices.

Suppose first that A is a permutation matrix. So there exists a permutation $\sigma \in \mathfrak{S}_n$ such that, for every $x = (x_1 \dots, x_n) \in \mathbb{Q}_p^n$, $Ax = (x_{\sigma(1)}, \dots, x_{\sigma(n)})$. As dx is the product of identical measures on the n factors $\mathbb{Q}_p$ of $\mathbb{Q}_p^n$, we have, for every $f \in L^1(\mathbb{Q}_p)$, $\int_{\mathbb{Q}_p^n} f(Ax)dx = \int_{\mathbb{Q}_p^n} f(x)dx$. The result now follows from the fact that $\det(A) = \pm + 1$.

Suppose that A is upper triangular, and write $A = (a_{ij})_{1, \leq i,j \leq n}$. Let $f \in L^1(\mathbb{Q}_p)$. Then

$$\int_{\mathbb{Q}_p^n} f(A(x_1, \dots, x_n)) =$$

$$\int_{\mathbb{Q}_p} \dots \int_{\mathbb{Q}_p} f(a_{11}x_1 + \dots + a_{1n}x_n, \dots, a_{n-1,n-1}x_{n-1} + a_{n-1,n}x_n, a_{nn}x_n) dx_n dx_{n-1} \dots dx_1.$$

Using question (a), we see that this last integral is equal to

$$|a_{11}|_p^{-1} \dots |a_{n-1,n-1}|_p^{-1} |a_{nn}|_p^{-1} \int_{\mathbb{Q}_p^n} f(x)dx = |\det(A)|_p^{-1} \int_{\mathbb{Q}_p^n} f(x)dx.$$

The case of lower triangular matrices is similar (just put the dx_i reverse order).

c) Once we have the change of variables formula of question (b), we can replace $\mathbb{R}$ by $\mathbb{Q}_p$ in problems 5 and 6 of problem set 1 and all the results will stay true, with exactly the same proofs. (Except 6(b), which doesn't make sense for $\mathbb{Q}_p$.) In particular, we get that $|\det(A)|_p^{-n} dA$ is a left and right Haar measure on $\mathbf{GL}_n(\mathbb{Q}_p)$.

d) Again, we can just apply the proofs of questions (b) and (c) of problem 3 (and the analogue for $\mathbb{Q}_p$ of problem 5 of problem set 1) to get the result. Assuming that there is no sign mistake in problem 3, a left Haar measure on B is $\prod_{i=1}^n |a_{ii}|_p^{i-n-1} dA$, where dA is the product measure on the $\mathbb{Q}_p$-vector space $V \simeq \mathbb{Q}_p^{n(n+1)/2}$ of upper triangular matrices and the a_{ij} are the entries of the matrix. And the modular function of B is given by

$$\Delta(A) = \prod_{i=1}^n |a_{ii}|_p^{2i-n-1}.$$

□

5. (extra credit) The goal of this problem is to give a formula for a Haar measure on $\mathbf{SO}(n)$. (We could do something similar for the unitary group $\mathbf{U}(n)$.)

a) (1) For $X \in M_n(\mathbb{R})$, we set $\Phi(X) = (I_n - X)(I_n + X)^{-1}$. Show that this is well-defined if -1 is not an eigenvalue of X, and that we have $\Phi(\Phi(X)) = X$ whenever this makes sense.

b) (2) We denote by A_n the $\mathbb{R}$-vector space of $n \times n$ antisymmetric matrices (i.e. of $X \in M_n(\mathbb{R})$ such that $X^T = -X$) and by U the set of elements of $\mathbf{SO}(n)$ that don't have -1 as an eigenvalue. Show that U is an open dense subset of $\mathbf{SO}(n)$, and that Φ induces a homeomorphism $A_n \overset{\sim}{\to} U$.

c) (2) Let $X \in A_n$. Show that there exist open dense subsets V and W of A_n such that the formula $\Phi(L_X Y) = \Phi(X)\Phi(Y)$ defines a diffeomorphism $L_X : V \overset{\sim}{\to} W$, and that $0 \in V$.

d) Let dX be Lebesgue measure on A_n. For every $X \in A_n$ and every $Y \in A_n$ on which L_X is defined, we denote by $L'_X(Y)$ the differential at Y of L_X. It is a linear transformation from A_n to A_n such that, for every $H \in A_n$,

$$L_X(Y + tH) = L_X(Y) + tL'_X(Y)(H) + o(t).$$

Fix $X \in A_n$. We want to compute $\det(L'_X(0))$. Remember that $L'_X(0)$ is a linear endomorphism of A_n, and note that $A_n \otimes_R \mathbb{C}$ is the space of antisymmetric matrices in $M_n(\mathbb{C})$.

i. (1) Show that $\det(L'_X(0))$ is well-defined and nonzero.

ii. (1) Show that we have

$$L'_X(0)(H) = (I_n - X)H(I_n + X),$$

for every $H \in A_n$.

iii. (1) Show that X has a basis of (complex) eigenvectors $(v_1, \dots, v_n)$ such that the corresponding eigenvalues are of the form $i\lambda_1, \dots, i\lambda_n$, with $\lambda_1, \dots, \lambda_n \in \mathbb{R}$.

iv. (1) For $j, k \in \{1, \dots, n\}$, we set $Y_{jk} = v_j v_k^T - v_k v_j^T$. Show that $Y_{jk} \in A_n \otimes_{\mathbb{R}} \mathbb{C}$, and that it is an eigenvector for $L'_X(0)$, with corresponding eigenvalue $(1 - i\lambda_j)(1 - i\lambda_k)$.

v. (1) Show that $(Y_{jk})_{1 \le j < k \le n}$ is a basis of $A_n \otimes_{\mathbb{R}} \mathbb{C}$.

vi. (1) Show that $\det(L'_X(0)) = \det(I_n - iX)^{n-1}$.

e) (3) Show that the linear functional sending $f \in \mathcal{C}_c(\mathbf{SO}(n))$ to

$$\int_{A_n} f(\Phi(X)) \frac{1}{|\det L'_X(0)|} dX$$

defines a left Haar measure on $\mathbf{SO}(n)$. (Hint : Note that $(L_X \circ L_Y)(0) = L_X(Y)$, and use the chain rule.)

Solution.

a) If $X \in M_n(\mathbb{R})$, then -1 is not an eigenvalue of X if and only if $I_n + X$ is invertible, i.e. if and only if the formula defining $\Phi(X)$ makes sense. So the set of definition of Φ is the open set defined by the equation $\det(I_n + X) \neq 0$. Note also that $I_n - X$ and $I_n + X$ commute, so $I_n - X$ and $(I_n + X)^{-1}$ commute (if the second is defined), so we also have $\Phi(X) = (I_n + X)^{-1}(I_n - X)$.

Let $X \in M_n(\mathbb{R})$ such that $\Phi(X)$ is defined. Then we have

$$I_n + \Phi(X) = ((I_n + X) + (I_n - X))(I_n + X)^{-1} = 2(I_n + X)^{-1}$$

and

$$I_n - \Phi(X) = ((I_n + X) - (I_n - X))(I_n + X)^{-1} = 2X(I_n + X)^{-1}.$$

In particular, $I_n + \Phi(X)$ is invertible, so $\Phi(\Phi(X))$ makes sense, and we have

$$\Phi(\Phi(X)) = (I_n - \Phi(X))(I_n + \Phi(X))^{-1} = 2X(I_n + X)^{-1}(2(I_n + X)^{-1})^{-1} = X.$$

b) Let $g \in \mathbf{SO}(n)$. Then we can find $P \in \mathbf{GL}_n(\mathbb{R})$ such that

$$PgP^{-1} = \begin{pmatrix} r_1 & 0 & \dots & 0 \\ 0 & r_2 & 0 & 0 \\ \vdots & 0 & \ddots & 0 \\ 0 & \dots & 0 & r_m \end{pmatrix},$$

where :

- if n is even, then $m = n/2$ and $r_1, \dots, r_m$ are 2×2 matrices of the form $\begin{pmatrix} \cos\theta_i & \sin\theta_i \\ -\sin\theta_i & \cos\theta_i \end{pmatrix}$, with $\theta_i \in [0, 2\pi)$;
- if n is odd, then $m = (n+1)/2$, the matrix r_m is the 1×1 matrix 1 and $r_1, \dots, r_{m-1}$ are 2×2 matrices of the form $\begin{pmatrix} \cos\theta_i & \sin\theta_i \\ -\sin\theta_i & \cos\theta_i \end{pmatrix}$, with $\theta_i \in [0, 2\pi)$.

In both cases, -1 is an eigenvalue of g if and only if one at least one of the θ_i is equal to π. So, by varying the θ_i, we can find a sequence of elements of $\mathbf{SO}(n)$ that converge to g and don't have -1 as an eigenvalue. This proves that U is dense in $\mathbf{SO}(n)$.

Next, as antisymmetric matrices have only imaginary eigenvalues, the function Φ is defined on A_n. Note also that it is clear on the definition of Φ that Φ is continuous on its open set of definition. By the second part of question (a), Φ is injective and, to show that Φ is a homeomorphism from A_n to U, it suffices to show that is a bijection from A_n to U (because then its inverse will be Φ). So we just need to show that $\Phi(A_n) = U$. Using again the fact that $\Phi(\Phi(X)) = X$ whenever this makes sense, we see that it suffices to prove that $\Phi(A_n) \subset U$ and $\Phi(U) \subset A_n$.

Let $X \in A_n$. Then $X^T = -X$, so $\Phi(X)^T = (I_n + X^T)^{-1}(I_n - X^T) = (I_n - X)^{-1}(I_n + X)$, and hence $\Phi(X)^T\Phi(X) = I_n$, which means that $\Phi(X) \in \mathbf{O}(n)$. As Φ is continuous and A_n is connected, $\Phi(A_n)$ is connected. But $I_n = \Phi(0) \in \Phi(A_n)$, so $\Phi(A_n)$ is contained in $\mathbf{SO}(n)$.

Let $X \in \mathbf{SO}(n)$ such that -1 is not an eigenvalue of X. Then $X^T = X^{-1}$, so

$$\Phi(X)^T = (I_n - X^T)(I_n + X^T)^{-1} = (I_n - X^{-1})(I_n + X^{-1})^{-1} = (X - I_n)(X + I_n)^{-1} = -\Phi(X).$$

So we have $\Phi(U) \subset A_n$.

c) Fix $X \in A_n$. Note that the formula $\Phi(L_X Y) = \Phi(X)\Phi(Y)$ can also be written $L_X Y = \Phi(\Phi(X)\Phi(Y))$, by (a).

For $Y \in A_n$, $\Phi(X)\Phi(Y)$ has animage by Φ (which will automatically be in A_n by (b)) if and only if $\Phi(Y) \in \Phi(X)^{-1}U$. So we can take $V = \Phi(U \cap (\Phi(X)^{-1}U))$; this is dense in A_n because $U \cap (\Phi(X)^{-1}U)$ is dense in $\mathbf{SO}(n)$ by (b). Then the image of V by the map $L_X : Y \mapsto \Phi(\Phi(X)\Phi(Y))$ is $W := \Phi((\Phi(X)U) \cap U)$.

The map $L_X : V \to W$ is continuous and surjective. In fact, as Φ is infinitely differentiable (it is given by rational functions in the entries of its arguments, by the formula saying that A^{-1} is $\det(A)^{-1}$ times the transpose of its cofactor matrix, for every $A \in \mathbf{GL}_n(\mathbb{R})$), the map L_X is also infinitely differentiable.

Let $X' = \Phi(\Phi(X)^{-1}) \in A_n$. Then we get as above a continuous and surjective map $L_{X'} : W \to V$, defined by the formula $L_{X'}(Y) = \Phi(\Phi(X)^{-1}\Phi(Y))$. The maps L_X and $L_{X'}$ are inverses of each other, and in particular they are both diffeomorphisms.

Finally, if $Y = 0$, then $\Phi(Y) = I_n$. So $\Phi(Y) \in U$, and we also have $\Phi(Y) \in \Phi(X)^{-1}U$, because $\Phi(X)\Phi(Y) = \Phi(X) \in U$. This shows that $0 \in V$.

d) i. Let $X \in A_n$. As L_X is defined at the point 0, the differential $L'_X(0)$ makes sense; also, as L_X is a diffeomorphism, $\det(L'_X(0)) \neq 0$.

ii. Note that, for $Y \in A_n$,

$$(I_n + X + Y)^{-1} = (I_n + X)^{-1}(I_n + (I_n + X)^{-1}Y) = (I_n + X)^{-1}(I_n - Y(I_n + X)^{-1} + o(Y)),$$

hence

$$\begin{aligned}\Phi(X+Y) &= (I_n - X - Y)(I_n + X + Y)^{-1} \\ &= ((I_n - X) - Y)(I_n + X)^{-1}(I_n - Y(I_n + X)^{-1} + o(Y)) \\ &= \Phi(X) - \Phi(X)Y(I_n + X)^{-1} - Y(I_n + X)^{-1} + o(Y).\end{aligned}$$

In particular (taking $X = 0$), we have

$$\Phi(Y) = I_n - 2Y + o(Y).$$

So

$$\Phi(X)\Phi(Y) = \Phi(X) - 2\Phi(X)Y + o(Y),$$

and

$$L_X(Y) = \Phi(\Phi(X)\Phi(Y)) = \Phi(\Phi(X) - 2\Phi(X)Y + o(Y)) =$$

$$\Phi(\Phi(X)) - \Phi(\Phi(X))(-2\Phi(X)Y)(I_n + \Phi(X))^{-1} - (-2\Phi(X)Y)(I_n + \Phi(X))^{-1} + o(Y).$$

Using $\Phi(\Phi(X)) = X$ and $I_n + \Phi(X) = 2(I_n + X)^{-1}$ (see (a)), we can simplify this last expression to

$$\begin{aligned}X + X\Phi(X)Y(I_n + X)^{-1} + \Phi(X)Y(I_n + X) + o(Y) &= X + (I_n + X)\Phi(X)Y(I_n + Y) + o(Y) \\ &= X + (I_n - X)Y(I_n + X) + o(Y).\end{aligned}$$

But then the conclusion that $L'_X(0)(Y) = (I_n - X)Y(I_n + X)$ follows immediately from the definition of the differential.

iii. As X is antisymmetric and has real entries, it is normal, so the spectral theorem says that X is diagonalizable in an orthonormal basis of $\mathbb{C}^n$; in other words, there exists a unita matrix P such that PXP^{-1} is diagonal. We have already used the fact that the eigenvalues of X are imaginary, but it is easy to recheck it quickly : we have $X^* = -X$ and $P^* = P^{-1}$, and $(PXP^{-1})^* = (P^*)^{-1}X^*P^* = -PXP^{-1}$. As PXP^{-1} is diagonal, this means that its diagonal entries (which are the eigenvalues of X) are all imaginary.

iv. It follows directly from the definition of Y_{jk} that $Y_{jk}^T = -Y_{jk}$, so $Y_{jk} \in A_n \otimes_{\mathbb{R}} \mathbb{C}$. Furthermore, by (ii), we have

$$\begin{aligned}L'_X(0)(Y_{ij}) &= (I_n - X)Y_{ij}(I_n + X) \\ &= (I_n - X)(v_j v_k^T)(I_n - X^T) - (I_n - X)(v_k v_j^T)(I_n - X^T) \\ &= (1 - i\lambda_j)(v_j v_k^T)(1 - i\lambda_k) - (1 - i\lambda_k)(v_k v_j^T)(1 - i\lambda_j \\ &= (1 - i\lambda_j)(1 - i\lambda_k)Y_{ij}.\end{aligned}$$

v. As $(v_1, \dots, v_n)$ is a basis of $\mathbb{C}^n$, the matrices $v_j v_k^T$, for $1 \le j, k \le n$, form a basis of $M_n(\mathbb{C})$. So the matrices $Y_{jk} = (v_j v_k^T) - (v_j v_k^T)^T$, for $1 \le j, k \le n$, generate $A_n \otimes_{\mathbb{R}} \mathbb{C}$. Note that $Y_{jj} = 0$ and $Y_{kj} = -Y_{jk}$, so $A_n \otimes_{\mathbb{R}} \mathbb{C}$ is actually spanned by the matrices Y_{jk}, for $1 \le j < k \le n$. As there are $n(n-1)/2$ such matrices and $\dim_{\mathbb{C}}(A_n \otimes_{\mathbb{R}} \mathbb{C}) = \dim_{\mathbb{R}}(A_n) = n(n-1)/2$, they form a basis of $A_n \otimes_{\mathbb{R}} \mathbb{C}$.

vi. By (iv) and (v), we have

$$\det(L'_X(0)) = \prod_{1 \le j < k \le n} (1 - i\lambda_j)(1 - i\lambda_k) = \prod_{r=1}^{n} (1 - i\lambda_r)^{n-1}$$

(because each $1 - i\lambda_r$ appears $n - 1$ times in the first big product : $(n - r)$ times as the first factor $(1 - i\lambda_j)$, and $(r - 1)$ times as the second factor $(1 - i\lambda_k)$).

To get the result, we just need to note that the eigenvalues of $I_n - iX$ are $1 - i\lambda_1, \dots, 1 - i\lambda_n$, so that

$$\det(I_n - X) = \prod_{r=1}^{n}(1 - i\lambda_r).$$

e) Let us denote this functional by Λ. First, by question (e), the function $X \mapsto \frac{1}{|\det(L'_X(0))|}$ is defined everywhere on A_n and continuous, so the integral defining Λ makes sense.

We need to check that Λ is positive and invariant by left translations. We first check the positivity. Let $f \in \mathcal{C}_c^+(\mathbf{SO}(n))$. Then we can find $\varepsilon > 0$ and a nonempty open subset Ω of $\mathbf{SO}(n)$ such that $f_{|\Omega} \geq \varepsilon$. As U is open dense in $\mathbf{SO}(n)$, its intersection with Ω is open and nonempty, so $\Phi(U \cap \Omega)$ is open and nonempty in A_n, and we have

$$\Lambda(f) \geq \varepsilon \int_{\Phi(U\cap\Omega)} \frac{1}{|\det(L'_X(0))|} dX > 0$$

(because the function $X \mapsto \frac{1}{|\det(L'_X(0))|}$ is continuous and positive on $\Phi(U \cap \Omega)$).

Now we check the left invariance. Fix $f \in \mathcal{C}_c(\mathbf{SO}(n))$. Let $g \in U$. Then $\Lambda(L_g f) = \int_{A_n} f(g^{-1}\Phi(Y)) \frac{1}{|\det(L'_Y(0))|} dY$. Choose $X, X' \in A_n$ such that $\Phi(X) = g^{-1}$ and $\Phi(X') = g$. Then

$$\begin{aligned}
\Lambda(L_g f) &= \int_{A_n} f(\Phi(X)\Phi(Y)) \frac{1}{|\det(L'_Y(0))|} dY \\
&= \int_V f(\Phi(X)\Phi(Y)) \frac{1}{|\det(L'_Y(0))|} dY \text{ (because } \mathrm{vol}(A_n - V) = 0) \\
&= \int_V f(\Phi(L_X Y)) \frac{1}{|\det(L'_Y(0))|} dY.
\end{aligned}$$

Now note that, if $Y \in V$, then so does $L_Y(0) = Y$, so $L_X(Y) = L_X \circ L_Y(0) = L_{L_X Y}(0)$ makes sense, and we have by the chain rule

$$L'_{L_X Y}(0) = L'_X(Y) \circ L'_Y(0),$$

hence in particular

$$\frac{1}{|\det(L'_Y(0))|} = \frac{|\det(L'_X(Y))|}{|\det(L'_{L_X Y}(0))|}.$$

This implies that

$$\Lambda_g(f) = \int_V f(\Phi(L_X Y)) \frac{|\det(L'_X(Y))|}{|\det(L'_{L_X Y}(0))|} dY.$$

Using the substitution $Z = L_X Y$, we see that this is equal to

$$\int_W f(\Phi(Z)) \frac{1}{|\det(L'_Z(0))|} dZ.$$

As $\mathrm{vol}(A_N - W) = 0$, the last integral is equal to $\int_{A_n} f(\Phi(Z)) \frac{1}{|\det(L'_Z(0))|} dZ$, i.e. to $\Lambda(f)$.

So we have shown that the function $\mathbf{SO}(n) \to \mathbb{C}$, $g \mapsto \Lambda(L_g f)$ is constant on the open dense subset U. As this function is continuous (it is the composition of the continuous function $\mathbf{SO}(n) \to \mathcal{C}_c(\mathbf{SO}(n))$, $g \mapsto L_g f$ and of the continuous linear function $\Lambda : \mathcal{C}_c(\mathbf{SO}(n)) \to \mathbb{C}$), it is constant on the whole $\mathbf{SO}(n)$, which means that $\Lambda(L_g f) = \Lambda(f)$ for every $g \in \mathbf{SO}(n)$.

$\square$