MAT 449 : Problem Set 2

Due Thursday, September 27

More examples of Haar measures

1. Let G be a locally compact group, and let H be a closed subgroup of G. We write π for the quotient map from G to G/H. We denote by Δ_G (resp. Δ_H) the modular function of G (resp. H), and we assume that $\Delta_{G|H} = \Delta_H$. We fix left Haar measures μ_G and μ_H on G and H.

   a) (1) Show that, for every compact subset K' of G/H, there exists a compact subset K of G such $\pi(K) = K'$.

   b) (1) Let $f \in L^1(G)$. Show that the function $G \to \mathbb{C}$, $x \mapsto \int_H f(xh)d\mu_H(h)$ is invariant by right translations by elements of H. Hence it defines a function $G/H \to \mathbb{C}$, that we will denote by f^H.

   c) (2) If $f \in \mathcal{C}_c(G)$, show that $f^H \in \mathcal{C}_c(G/H)$.

   d) (2) Show that the map $\mathcal{C}_c(G) \to \mathcal{C}_c(G/H)$, $f \mapsto f^H$ is surjective. (Hint : You may use the fact that, for every compact subset K of G, there exists a function $\varphi \in \mathcal{C}_c^+(G)$ such that $\varphi(x) > 0$ for every $x \in K$.)

   e) (2) If $f \in \mathcal{C}_c(G)$ is such that $f^H = 0$, show that $\int_G f(x)d\mu_G(x) = 0$. (Hint : use a function in $\mathcal{C}_c(G/H)$ that is equal to 1 on $\pi(\mathrm{supp}(f))$, and proposition 2.12 of the notes. (Sorry.))

   f) (2) Show that there exists a unique regular Borel measure $\mu_{G/H}$ on G/H that is invariant by left translations by elements of G and such that, for every $f \in \mathcal{C}_c(G)$, we have $\int_G f(x)d\mu_G(x) = \int_{G/H} f^H(y)d\mu_{G/H}(y)$.

   g) (1) If P is a closed subgroup of G such that π induces a homeomorphism $P \xrightarrow{\sim} G/H$, show that the inverse image of $\mu_{G/H}$ by this homeomorphism is a left Haar measure on P.

   h) (2) If P is a closed subgroup of G such that the map $P \times H \to G$, $(p, h) \mapsto ph$ is a homeomorphism, and if $d\mu_P$ is a left Haar measure on P, show that the linear functional $\mathcal{C}_c(G) \to \mathbb{C}$, $f \mapsto \int_H \int_P f(ph)d\mu_P(p)d\mu_H(h)$ defines a left Haar measure on G.

2. Let G be a locally compact group. Let A and N be two closed subgroups of G such that $A \times N \to G$, $(a, n) \mapsto an$ is a homeomorphism and that A normalizes N (i.e. for every $a \in A$ and $n \in N$, we have $ana^{-1} \in N$).

   a) (2) If μ_A and μ_N are left Haar measures on A and N, show that the linear functional $\mathcal{C}_c(G) \to \mathbb{C}$, $f \mapsto \int_A \int_N f(an)d\mu_A(a)d\mu_N(n)$ defines a left Haar measure on G.

   b) (1) Let $a \in A$. Show that there exists $\alpha(a) \in \mathbb{R}_{>0}$ such that, for every $f \in \mathcal{C}_c(N)$, we have
   $$\int_N f(ana^{-1})d\mu_N(n) = \alpha(a) \int_N f(n)d\mu_N(n).$$

c) (1) If Δ_G, Δ_A and Δ_N are the modular functions of G, A and N respectively, show that $\Delta_G(an) = \alpha(a)\Delta_A(a)\Delta_N(n)$ if $a \in A$ and $n \in N$.

3. Let $G = \mathbf{SL}_n(\mathbb{R})$, $H = \mathbf{SO}(n)$, and let $P \subset G$ be the subgroup of upper triangular matrices with positive entries on the diagonal (and determinant 1).

   a) (4) Show that the map $P \times H \to G$, $(p, h) \mapsto ph$ is a homeomorphism. (Hint : Gram-Schmidt.)

   b) (3) Give a formula for a left Haar measure on P similar to the formula in problem 6(d) of problem set 1.

   c) (4) Calculate the modular function of P.

   d) (2) Show that G is unimodular. (There are several ways to do this.)

   e) (2) If $n = 2$, show that $\mathbf{SO}(n) \simeq S^1$ (the circle group), and give a left Haar measure on G.

4. (Remember problems 4, 5,6,8 of problem set 1.) We denote by dx a Haar measure on the additive group $\mathbb{Q}_p$. We also denote by dx (resp. dA) the product measure on $\mathbb{Q}_p^n$ (resp. $M_n(\mathbb{Q}_p) \simeq \mathbb{Q}_p^{n^2}$); note that it is a Haar measure for the corresponding additive group.

   a) (2) Show that, for every $f \in L^1(\mathbb{Q}_p)$ and every $a \in \mathbb{Q}_p^\times$, $b \in \mathbb{Q}_p$, we have

   $$\int_{\mathbb{Q}_p} f(x)dx = |a|_p \int_{\mathbb{Q}_p} f(ax + b)dx.$$

   b) (3) Let $n \geq 1$. Show that, if $f \in L^1(\mathbb{Q}_p^n)$, $A \in \mathbf{GL}_n(\mathbb{Q}_p)$ and $b \in \mathbb{Q}_p^n$, we have

   $$\int_{\mathbb{Q}_p^n} f(x)dx = |\det(A)|_p \int_{\mathbb{Q}_p^n} f(Ax + b)dx.$$

   c) (2) Show that $|\det(A)|_p^{-n} dA$ is a left and right Haar measure on $\mathbf{GL}_n(\mathbb{Q}_p)$.

   d) (3) Let B be the group of upper triangular matrices in $\mathbf{GL}_n(\mathbb{Q}_p)$. Find a left Haar measure on B and calculate the modular function of B.

5. (extra credit) The goal of this problem is to give a formula for a Haar measure on $\mathbf{SO}(n)$. (We could do something similar for the unitary group $\mathbf{U}(n)$.)

   a) (1) For $X \in M_n(\mathbb{R})$, we set $\Phi(X) = (I_n - X)(I_n + X)^{-1}$. Show that this is well-defined if -1 is not an eigenvalue of X, and that we have $\Phi(\Phi(X)) = X$ whenever this makes sense.

   b) (2) We denote by A_n the $\mathbb{R}$-vector space of $n \times n$ antisymmetric matrices (i.e. of $X \in M_n(\mathbb{R})$ such that $X^T = -X$) and by U the set of elements of $\mathbf{SO}(n)$ that don't have -1 as an eigenvalue. Show that U is an open dense subset of $\mathbf{SO}(n)$, and that Φ induces a homeomorphism $A_n \overset{\sim}{\to} U$.

   c) (2) Let $X \in A_n$. Show that there exist open dense subsets V and W of A_n such that the formula $\Phi(L_X Y) = \Phi(X)\Phi(Y)$ defines a diffeomorphism $L_X : V \overset{\sim}{\to} W$, and that $0 \in V$.

   d) Let dX be Lebesgue measure on A_n. For every $X \in A_n$ and every $Y \in A_n$ on which L_X is defined, we denote by $L'_X(Y)$ the differential at Y of L_X. It is a linear transformation from A_n to A_n such that, for every $H \in A_n$,

   $$L_X(Y + tH) = L_X(Y) + tL'_X(Y)(H) + o(t).$$

Fix $X \in A_n$. We want to compute $\det(L'_X(0))$. Remember that $L'_X(0)$ is a linear endomorphism of A_n, and note that $A_n \otimes_R \mathbb{C}$ is the space of antisymmetric matrices in $M_n(\mathbb{C})$.

i. (1) Show that $\det(L'_X(0))$ is well-defined and nonzero.

ii. (1) Show that we have

$$L'_X(0)(H) = (I_n - X)H(I_n + X),$$

for every $H \in A_n$.

iii. (1) Show that X has a basis of (complex) eigenvectors $(v_1, \dots, v_n)$ such that the corresponding eigenvalues are of the form $i\lambda_1, \dots, i\lambda_n$, with $\lambda_1, \dots, \lambda_n \in \mathbb{R}$.

iv. (1) For $j, k \in \{1, \dots, n\}$, we set $Y_{jk} = v_j v_k^T - v_k v_j^T$. Show that $Y_{jk} \in A_n \otimes_{\mathbb{R}} \mathbb{C}$, and that it is an eigenvector for $L'_X(0)$, with corresponding eigenvalue $(1 - i\lambda_j)(1 - i\lambda_k)$.

v. (1) Show that $(Y_{jk})_{1 \leq j < k \leq n}$ is a basis of $A_n \otimes_{\mathbb{R}} \mathbb{C}$.

vi. (1) Show that $\det(L'_X(0)) = \det(I_n - iX)^{n-1}$.

e) (3) Show that the linear functional sending $f \in \mathcal{C}_c(\mathbf{SO}(n))$ to

$$\int_{A_n} f(\Phi(X)) \frac{1}{|\det L'_X(0)|} dX$$

defines a left Haar measure on $\mathbf{SO}(n)$. (Hint : Note that $(L_X \circ L_Y)(0) = L_X(Y)$, and use the chain rule.)