MAT 449 : Problem Set 11

Due Sunday, December 16

The goal of this problem set is to study the Gelfand pair $(\mathfrak{S}_n, \mathfrak{S}_r \times \mathfrak{S}_{n-r})$. We will embed $\mathfrak{S}_r \times \mathfrak{S}_{n-r}$ in $\mathfrak{S}_n$ in the following way : If $\sigma \in \mathfrak{S}_r$ and $\tau \mathfrak{S}_{n-r}$, then $\sigma \times \tau \in \mathfrak{S}_n$ is given by

$$(\sigma \times \tau)(i) = \begin{cases} \sigma(i) & \text{if } 1 \leq i \leq r \\ \tau(i-r)+r & \text{if } r+1 \leq i \leq n. \end{cases}$$

If E is a finite set, we will denote by $L(E)$ the space of functions $f : E \to \mathbb{C}$, with the L^2 inner product given by $\langle f, f' \rangle = \sum_{x \in E} f(x)\overline{f'(x)}$.

1. In this problem, we fix a finite group G acting transitively (on the left) on a set E. Let $x_0 \in E$, and let $K \subset G$ be the stabilizer of x_0.
   a) (2) Show that the following conditions are equivalent :
      (i) For all $x, y \in E$, there exists $g \in G$ such that $g \cdot (x, y) = (y, x)$.
      (ii) For every $g \in G$, we have $g^{-1} \in KgK$.
   b) (1) If the conditions of (a) are satisfied, show that (G, K) is a Gelfand pair.

   We now assume that there is a metric $d : E \times E \to \mathbb{R}_{\geq 0}$, and that the group G acts by isometries. Suppose that the action of G on E is *distance-transitive*, that is : for all $(x, y), (x', y') \in E \times E$ such that $d(x, y) = d(x', y')$, there exists $g \in G$ such that $g \cdot (x, y) = (x', y')$.

   c) (1) Show that (G, K) is a Gelfand pair.
   d) (1) Show that the orbits of K on E are the spheres $\{x \in E | d(x, x_0) = j\}$, for $j \in \mathbb{R}_{\geq 0}$.
   e) (1) Let Ω_r be the set of cardinality r subsets of $\{1, \ldots, n\}$. Show that the formula $d(A, B) = r - |A \cap B|$ defines a metric on Ω_r.
   f) (2) Show that $(\mathfrak{S}_n, \mathfrak{S}_r \times \mathfrak{S}_{n-r})$ is a Gelfand pair.

   Solution.

   a) Suppose that (i) holds. Let $g \in G$. By (i), there exists $h \in G$ such that $h \cdot (K, gK) = (gK, K)$. Then the equality of the first entries gives $g^{-1}h \in K$, and the equality of the second entries gives $g^{-1}hg \in g^{-1}K$, hence $g \in (g^{-1}h)^{-1}g^{-1}K \subset Kg^{-1}K$.

      Suppose that (ii) holds. Let $g, h \in G$. By (i), we can find $k_1, k_2 \in K$ such that $g^{-1}h = k_1h^{-1}gk_2$, and then

$$\begin{aligned}(gK, hK) = g \cdot (K, g^{-1}hK) = g \cdot (K, k_1h^{-1}gK) &= gk_1^{-1} \cdot (K, h^{-1}gK) \\ &= gk_1^{-1}h^{-1} \cdot (hK, gK).\end{aligned}$$

   b) This follows from proposition V.2.5 of the notes, taking $\theta = \mathrm{id}_G$.

c) Let $x, y \in X$. Then $d(y, x) = d(x, y)$, so there exists $g \in G$ such that $g \cdot (x, y) = (y, x)$. In other words, condition (i) of (a) is satisfied, so condition (ii) is also satisfied. By (b), this implies that (G, K) is a Gelfand pair.

d) Write $S_j = \{x \in X | d(x, x_0) = j\}$, for $j \in \mathbb{Z}_{\geq 0}$. As G acts by isometries on X and K fixes x_0, the sets S_j are stable by K. To show that they are the orbits of K on X, we need to show that K acts transitively on each nonempty S_j. So let $j \geq 0$, and suppose that we have $x, y \in S_j$. Then $d(x_0, x) = d(x_0, y) = j$, so, by the hypothesis, there exists $g \in G$ such that $g \cdot (x_0, x) = (x_0, y)$. The fact that $g \cdot x_0 = x_0$ implies that $g \in K$, so x and y are in the same K-orbit.

e) We clearly have $d(A, B) = d(B, A)$ for all $A, B \in \Omega_r$. Let $A, B, C \in \Omega_r$. First, if $d(A, B) = 0$, then $|A \cap B| = r = |A| = |B|$, so $A \cap B = A$ and $A \cap B = B$, and so $A = B$. Let's prove the triangle inequality. We have

$$|A \cap B| + |B \cap C| = |(A \cap B) \cup (B \cap C)| + |A \cap B \cap C| \leq |B| + |A \cap C| = r + |A \cap C|,$$

so

$$d(A, C) = r - |A \cap C| \leq 2r - |A \cap B| - |B \cap C| = d(A, B) + d(B, C).$$

f) We make $\mathfrak{S}_n$ act by Ω_r by $\sigma \cdot A = \sigma(A)$. This action is transitive, and $\mathfrak{S}_r \times \mathfrak{S}_{n-r}$ is the stabilizer of $\{1, \ldots, r\}$. Also, it is clear that $\mathfrak{S}_n$ acts by isometries on Ω_r. So, by (c), we just need to check that the action is distance-transitive. Let $A, B, A', B' \in \Omega_r$ such that $d(A, B) = d(A', B')$, i.e. $|A \cap B| = |A' \cap B'|$. Choose a bijection $\varphi : A \xrightarrow{\sim} A'$ that sends the subset $A \cap B$ of A onto $A' \cap B'$; this is possible because $|A \cap B| = |A' \cap B'|$. Choose a bijection $\psi : B - (A \cap B) \xrightarrow{\sim} B' - (A' \cap B')$; this is also possible, because $|B - (A \cap B)| = r - |A \cap B| = |B' - (A' \cap B')|$. Putting φ and ψ together gives a bijection $A \cup B \xrightarrow{\sim} A' \cup B'$ that sends A to A' and B to B', and any extension of this to an element σ of $\mathfrak{S}_n$ will satisfy $\sigma \cdot (A, B) = (A', B')$.

□

2. Let (Ω_r, d) be the finite metric space of problem 1(e). Let N be the *diameter* of Ω_r, that is,

$$N = \max\{d(A, B),\ A, B \in \Omega_r\}.$$

For $i \in \{0, \ldots, N\}$, we define a linear operator $\Delta_i : L(\Omega_r) \to L(\Omega_r)$ by

$$\Delta_i f(A) = \sum_{B \in \Omega_r,\ d(A,B)=i} f(B),$$

for every $f \in L(\Omega_r)$. We also denote by $\mathcal{A}$ the subalgebra of $\mathrm{End}(L(\Omega_r))$ generated by $\Delta_0, \ldots, \Delta_N$.

a) (2) Show that $N = \min(r, n - r)$.

b) (2) Show that there exist integers $b_0, \ldots, b_N, c_0, c_1, \ldots, c_N$ such that, for every i and all $A, B \in \Omega_r$ such that $d(A, B) = i$, we have

$$|\{C \in \Omega_r | d(A, C) = 1 \text{ and } d(B, C) = i + 1\}| = b_i$$

and

$$|\{C \in \Omega_r | d(A, C) = 1 \text{ and } d(B, C) = i - 1\}| = c_i.$$

(Of course, $c_0 = 0$ and $c_1 = 1$.)

c) (1) Show that $c_2, \ldots, c_N > 0$.

d) (3) If $i \in \{1, \dots, N\}$, show that

$$\Delta_i \Delta_1 = b_{i-1}\Delta_{i-1} + (b_0 - b_i - c_i)\Delta_i + c_{i+1}\Delta_{i+1},$$

with the convention that $\Delta_{N+1} = 0$.

e) (1) Show that there exist polynomials $p_0, \dots, p_N \in \mathbb{R}[t]$ such that $\deg(p_i) = i$ and $\Delta_i = p(\Delta_1)$.

f) (1) Show that $\mathcal{A}$ is the subalgebra of $\mathrm{End}(L(\Omega_r))$ generated by Δ_1.

g) (1) Show that $\mathcal{A}$ is spanned as a $\mathbb{C}$-vector space by $\Delta_0, \dots, \Delta_N$.

h) (2) Show that $\dim_{\mathbb{C}} \mathcal{A} = N + 1$.

i) (1) Show that the endormorphism Δ_1 of $L(\Omega_r)$ is self-adjoint.

j) (2) Show that we have a decomposition into pairwise orthogonal subspaces $L(\Omega_r) = \bigoplus_{i=0}^{N} V_i$, where $V_0, \dots, V_N$ are the eigenspaces of Δ_1. (Hint : problem 2 of problem set 5.)

Solution.

a) Note that $d(A, B) \le r$ for all $A, B \in \Omega_r$ by definition of d, so $N \le r$. Also, for all $A, B \in \Omega_r$, we have

$$d(A, B) = r - |A \cap B| = r - (|A| + |B| - |A \cup B|) = |A \cup B| - r \le n - r,$$

so $N \le n - r$, and $N \le \min(r, n - r)$.

Now take $A = \{1, \dots, r\}$ and $B = \{n-r+1, \dots, n\}$. Then $A, B \in \Omega_r$, and $|A \cap B| = \max(0, 2r - n)$. So $N \ge d(A, B) = r - \max(0, 2r - n) = \min(r, n - r)$.

b) Fix $i \in \{0, \dots, N\}$. For all $A, B \in \Omega_r$ such that $d(A, B) = i$, let

$$X_i(A, B) = \{C \in \Omega_r | d(A, C) = 1 \text{ and } d(B, C) = i + 1\}$$

and

$$Y_i(A, B) = \{C \in \Omega_r | d(A, C) = 1 \text{ and } d(B, C) = i - 1\}.$$

If $\sigma \in \mathfrak{S}_n$ is such that $\sigma(A, B) = (A', B')$, then σ induces bijections $X_i(A, B) \xrightarrow{\sim} X_i(A', B')$ and $Y_i(A, B) \xrightarrow{\sim} Y_i(A', B')$, because $\mathfrak{S}_n$ acts on Ω_r by isometries. So the statement follows from the fact that the action of $\mathfrak{S}_n$ on Ω_r is distance-transitive, which we showed in the proof of 1(f).

c) Let $i \in \{2, \dots, N\}$. Take $A = \{1, \dots, r\}$ and $B = \{i+1, i+2, \dots, i+r\}$. (Note that $i+r \le N+r \le n$ by (a).) We need to show that there exists at least one $C \in \Omega_r$ such that $d(A, C) = 1$ (i.e. $|A \cap C| = r - 1$) and $d(B, C) = i - 1$ (i.e. $|B \cap C| = r - i + 1$). This holds for $C = \{2, 3, \dots, r + 1\}$.

d) Let $f \in L(\Omega_r)$ and $A \in \Omega_r$. Then we have

$$\Delta_i \Delta_1 f(A) = \sum_{B \in \Omega_r, d(A,B)=i} \sum_{C \in \Omega_r, d(B,C)=1} f(C).$$

Let $C \in \Omega_r$. If there exists $B \in \Omega_r$ such that $d(A, B) = i$ and $d(B, C) = 1$, then we must have $i - 1 \le d(A, C) \le i + 1$ by the triangle inequality.

Suppose that $d(A, C) = i + 1$. Then

$$\{B \in \Omega_r | d(A, B) = i \text{ and } d(B, C) = 1\} = Y_{i+1}(C, A)$$

(with the notation of the proof of (b)). Suppose that $d(A,C) = i-1$. Then

$$\{B \in \Omega_r | d(A,B) = i \text{ and } d(B,C) = 1\} = X_{i-1}(C,A).$$

Finally, suppose that $d(A,C) = i$. Consider the set

$$\{B \in \Omega_r | d(A,B) = i \text{ and } d(B,C) = 1\} \cup \{B \in \Omega_r | d(A,B) = i+1 \text{ and } d(B,C) = 1\} \\ \cup \{B \in \Omega_r | d(A,B) = i-1 \text{ and } d(B,C) = 1\}.$$

The union is clearly disjoint. We are trying to calculate the cardinality of the first set, the second set is $X_i(C,A)$ and the third set is $Y_i(C,A)$. Also, by the triangle inequality, the union is simply

$$\{B \in \Omega_r | d(B,C) = 1\} = X_0(C,C).$$

So we get

$$|\{B \in \Omega_r | d(A,B) = i \text{ and } d(B,C) = 1\}| + b_i + c_i = b_).$$

Finally, we see that

$$\begin{aligned}\Delta_i \delta_1 f(A) &= c_{i+1} \sum_{C,\ d(A,C)=i+1} f(C) b_{i-1} \sum_{C,\ d(A,C)=i-1} f(C) + (b_0 - b_i - c_i) \sum_{C,\ d(A,C)=i} f(C) \\ &= c_{i+1}\Delta_{i+1} f(A) + b_{i-1}\Delta_{i-1} f(A) + (b_0 - b_i - c_i)\Delta_i f(A).\end{aligned}$$

e) We prove the statement by induction on i. It is obvious $i = 0$ (note that $\Delta_0 = \mathrm{id}$, so we take $p_0 = 1$) and for $i = 1$ (take $p_1(t) = t$). Suppose the result known up to some $i \geq 1$, and let's prove it for $i+1$. By (c) and (d), we have

$$\Delta_{i+1} = c_{i+1}^{-1}(\Delta_i \Delta_1 - b_{i-1}\Delta_{i-1} - (b_0 - b_i - c_i)\Delta_i),$$

so $\Delta_{i+1} = p_{i+1}(\Delta_1)$, with

$$p_{i+1}(t) = c_{i+1}^{-1}(tp_i(t) - p_{i-1}(t) - (b_0 - b_i - c_i)p_i(t)).$$

It is also clear that $\deg(p_{i+1}(t)) = i+1$.

f) Let $\mathcal{A}'$ be the subalgebra of $\mathrm{End}(L(\Omega_r))$ generated by Δ_1. Then $\mathcal{A}' \subset \mathcal{A}$ by definition of $\mathcal{A}$. By (e), we have $\Delta_0, \dots, \Delta_N \in \mathcal{A}'$, and so $\mathcal{A} \subset \mathcal{A}'$.

g) We show by induction on $i \geq 0$ that $\Delta_1^i \in \mathrm{Span}(\Delta_0, \Delta_1, \dots, \Delta_i)$. (The conclusion will follow by (f).) The assertion is clear for $i = 0$ and $i = 1$. Suppose that holds up to $i \geq 1$, and let's prove it for $i+1$. By (e), there exist a nonnezero $c \in \mathbb{R}$ and $c_0, \dots, c_i \in \mathbb{R}$ such that $\Delta_{i+1} = a\Delta_1^{i+1} + \sum_{j=0}^{i} a_j \Delta_1^j$. As $\Delta_1^j \in \mathrm{Span}(\Delta_0, \dots, \Delta_j)$ for every $j \in \{0, \dots, i\}$ by the induction, we deduce that $\Delta_1^{i+1} \in \mathrm{Span}(\Delta_0, \dots, \Delta_{i+1})$.

h) We know that $\mathcal{A} = \mathrm{Span}(\Delta_0, \dots, \Delta_N)$ by (g), so we must show that the family $(\Delta_0, \dots . \Delta_N)$ is linearly independent. Let $c_0, \dots, c_N \in \mathbb{C}$. If $A, B \in \Omega_r$, and if we denote by δ_A the indicator function of $\{A\}$, then $\Delta_i \delta_A(B) \neq 0$ only if $d(A,B) = i$, and we have

$$\sum_{i=0}^{N} c_i \Delta_i \delta_A(B) = c_{d(A,B)}.$$

As there are couples $(A,B) \in \Omega_r^2$ such that $d(A,B) = i$ for every $i \in \{0, \dots, N\}$, we conclude that, if $\sum_{i=0}^{N} c_i \Delta_i = 0$, then $c_0 = \dots = c_N = 0$.

i) Let $f, g \in L(\Omega_r)$. Then

$$\begin{aligned}
\langle f, \Delta_1 g\rangle &= \sum_{A\in\Omega_r} f(A)\overline{(\Delta_1 g)(A)} \\
&= \sum_{A\in\Omega_r} f(A) \sum_{B\in\Omega_r,\ d(A,B)=1} \overline{g(B)} \\
&= \sum_{A,B\in\Omega_r,\ d(A,B)=1} f(A)\overline{g(B)} \\
&= \sum_{B\in\Omega_r} (\Delta_1 f)(B)\overline{g(B)} \\
&= \langle \Delta_1 f, g\rangle.
\end{aligned}$$

j) As Δ_1 is self-adjoint, the spectral theorem says that it is diagonalizable and that its eigenspaces are pairwise orthogonal. So the only thing we have to show is that there are $N+1$ eigenspaces. By 2(b) of problem set 5, we know that the subalgebra of $\mathrm{End}(L(\Omega_r))$ generated by Δ_1, i.e. $\mathcal{A}$ (see (f)), is reduced. By 2(c) of the same problem set, we know that the number of eigenspaces of $\mathcal{A}$, i.e. of Δ_1, is $\dim(\mathcal{A})$, and by (h), we know that $\dim(\mathcal{A}) = N+1$.

$\square$

3. We use the notation of problem 2. Note that we have an action of $G := \mathfrak{S}_n$ on Ω_r, and that the stabilizer of $\{1,\dots,r\}$ is $K := \mathfrak{S}_r \times \mathfrak{S}_{n-r}$. Let $M^{n-r,r} = L(\Omega_r)$, seen as a representation of $\mathfrak{S}_n$ via the quasi-regular representation (that is, if $g \in G$, $f \in M^{n-r,n}$ and $A \in \Omega_r$, we have $(g\cdot f)(A) = f(g^{-1}A)$).

We define $d : M^{n-r,r} \to M^{n-r+1,r-1}$ and $d^* : M^{n-r+1,r-1} \to M^{n-r,r}$ by

$$(df)(A) = \sum_{B\in\Omega_r | A\subset B} f(B)$$

and

$$(d^* f)(B) = \sum_{A\in\Omega_r | A\subset B} f(A).$$

(If $r=0$, we take $d=0$ and $d^*=0$.)

We also denote by Δ the operator Δ_1 of problem 2; that is, for every $f \in M^{n-r,r}$, the function $\Delta f \in M^{n-r,r}$ is defined by

$$(\Delta f)(A) = \sum_{B\in\Omega_r | d(A,B)=1} f(B).$$

Note that the functions d, d^* and Δ are defined for every r; we will not indicate r in the notation, it should be clear from the context.

Finally, if $a \in \mathbb{C}$ and $i \in \mathbb{Z}_{\geq 0}$, we write

$$(a)_i = a(a+1)\dots(a+i-1).$$

For example, we have $(1)_n = n!$ and $\binom{n}{k} = \frac{(n-k+1)_k}{k!}$.

a) (1) Show that $A \mapsto \{1,\dots,n\} - A$ induces a G-equivariant isomorphism $M^{r,n-r} \xrightarrow{\sim} M^{n-r,r}$.

b) (2) Show that d and Δ are G-equivariant.

c) (1) Show that d^* is the adjoint of d.

d) (2) If $f \in M^{n-r,r}$, show that

$$dd^*f = \Delta f + (n-r)f \quad \text{and} \quad d^*df = \Delta f + rf.$$

e) (2) Let $f \in M^{n-r,r}$ and $1 \leq p \leq q \leq n-r$. Show that

$$d(d^*)^q f = (d^*)^q df + q(n-2r-q+1)(d^*)^{q-1}f.$$

If moreover $df = 0$, show that

$$d^p(d^*)^q f = (q-p+1)_p(n-2r-q+1)_p(d^*)^{q-p}f.$$

Suppose that $0 \leq r \leq n/2$. If $r > 0$, set $S^{n-r,r} = \mathrm{Ker}(d : M^{n-r,r} \to M^{n-r+1,r-1})$; if $r = 0$, set $S^{n-r,r} = M^{n-r,r}$. This is a G-stable subspace of $M^{n-r,r}$.

f) (2) If $0 \leq m \leq n$ and $0 \leq r \leq \min(m, n-m)$, show that $(d^*)^{m-r} : S^{n-r,r} \to M^{n-m,m}$ is injective. (Hint : calculate $\|(d^*)^{m-r}f\|_2^2$).

g) (1) Under the hypothesis of (f), show that $(d^*)^{m-r}(S^{n-r,r})$ is contained in the eigenspace of Δ for the eigenvalue $m(n-m) - r(n-r+1)$.

h) (1) Show that the orthogonal of $S^{n-m,m}$ in $M^{n-m,m}$ is $d^*(M^{n-m+1,m-1})$, if $1 \leq m \leq n/2$.

i) (1) Show that $S^{n-r,r} \neq 0$ for every r such that $0 \leq r \leq n/2$.

j) (3) If $0 \leq m \leq n$, show that we have

$$M^{n-m,m} = \bigoplus_{r=0}^{\min(m,n-m)} (d^*)^{m-r}(S^{n-r,r}),$$

where the summands are pairwise orthogonal and are exactly the eigenspaces of Δ.

k) (1) Show that $\dim_{\mathbb{C}}(S^{n-r,n}) = \binom{n}{r} - \binom{n}{r-1}$ if $r > 0$.

l) (3) Show that the representations $S^{n-r,r}$, $0 \leq r \leq n/2$, are irreducible and pairwise inequivalent. (Hint : how many irreducible constituents does $M^{m,n-m}$ have ?)

Solution.

a) The map $A \mapsto \{1, \ldots, n\} - A$ is a bijection from Ω_r to Ω_{n-r}, and it commutes with the action of G on these two sets. The conclusion follows immediately.

b) Let $f \in M^{n-r,r}$ and $\sigma \in \mathfrak{S}_n$. If $A \in \Omega_{r-1}$, then

$$\begin{aligned}
(dL_\sigma f)(A) &= \sum_{B \in \Omega_r,\ B \supset A} f(\sigma^{-1}(B)) \\
&= \sum_{B' \in \Omega_r,\ B' \supset \sigma^{-1}(A)} f(B') \\
&= (df)(\sigma^{-1}(A)) \\
&= L_\sigma(df)(A).
\end{aligned}$$

If $A \in \Omega_r$, then

$$\begin{aligned}
(\Delta L_\sigma f)(A) &= \sum_{B \in \Omega_r,\ d(A,B)=1} f(\sigma^{-1}(B)) \\
&= \sum_{B' \in \Omega_r,\ d(\sigma^{-1}(A),B')=1} f(B') \\
&= \Delta f(\sigma^{-1}(A)) \\
&= L_\sigma(\Delta f)(A).
\end{aligned}$$

c) Let $f \in M^{n-r,r}$ and $g \in M^{n-r+1,r-1}$. Then

$$\begin{aligned}\langle df, g\rangle &= \sum_{A\in\Omega_{r-1}} df(A)\overline{g(A)} \\ &= \sum_{A\in\Omega_{r-1}, B\in\Omega_r,\ A\subset B} f(B)\overline{g(A)} \\ &= \sum_{B\in\Omega_r} f(B)\overline{d^*g(B)}.\end{aligned}$$

d) Let $f \in M^{n-r,r}$ and $A \in \Omega_r$. Then

$$\begin{aligned}dd^*f(A) &= \sum_{B\in\Omega_{r+1}, B\supset A} d^*f(B) \\ &= \sum_{B\in\Omega_{r+1},\ C\in\Omega_r\ C\subset B\supset A} f(C)\end{aligned}$$

Let $C \in \Omega_r$. If there exists $B \in \Omega_{r+1}$ such that $C \subset B \supset A$, then $d(A,C) \le 1$. If $C = A$, then

$$|\{B \in \Omega_{r+1} | C \subset B \supset A\}| = |\{B \in \Omega_{r+1} | A \subset B\}| = n - r\}.$$

If $d(A,C) = 1$, then the only element of Ω_{r+1} that contains both A and C is $A \cup C$. Finally, we get

$$\begin{aligned}dd^*f(A) &= (n-r)f(A) + \sum_{C\in\Omega_r,\ d(A,C)=1} f(C) \\ &= (n-r)f(A) + \Delta f(A).\end{aligned}$$

Similarly, we have

$$\begin{aligned}d^*df(A) &= \sum_{B\in\Omega_{r-1}, B\subset A} df(B) \\ &= \sum_{C\in\Omega_r,\ B\in\Omega_{r-1}, C\supset B\subset A} f(C).\end{aligned}$$

Let $C \in \Omega_r$. If there exists $B \in \Omega_{r-1}$ such that $C \supset B \subset A$, then $d(A,C) \le 1$. If $A = C$, then

$$|\{B \in \Omega_{r-1} | C \supset B \subset A\}| = r.$$

If $d(A,C) = 1$, then the only $B \in \Omega_{r-1}$ that is contained in both A and C is $B = A \cap C$. So we get

$$\begin{aligned}d^*df(A) &= rf(A) + \sum_{C\in\Omega_r,\ d(A,C)=1} f(C) \\ &= rf(A) + \Delta f(A).\end{aligned}$$

e) We show the first identity by induction on q. If $q = 1$, then, by (d), we have

$$dd^*f = \Delta f + (n-r)f = \Delta f + rf + (n-2r)f = d^*df + q(n-2r-q+1)(d^*)^{q-1}f$$

for every $f \in M^{n-r,r}$. Now suppose the identity known for $q \in \{1, \dots, n-r-1\}$, every s and every element of $M^{n-s,s}$, and let's show it for $q+1$. If $f \in M^{n-r,r}$, we have

$$
\begin{aligned}
d(d^*)^{q+1}f &= d(d^*)^q(d^*f) \\
&= (d^*)^q d(d^*f) + q(n-2(r+1)-q+1)(d^*)^{q-1}(d^*f) \\
&\qquad \text{(by the induction hypothesis for } d^*f \in M^{n-r-1,r+1}) \\
&= (d^*)^q(d^*df + (n-2r)f) + q(n-2r-q-1)(d^*)^q f \\
&\qquad \text{(by the case } q=1) \\
&= (d^*)^{q+1}df + (n-2r+q(n-2r-q-1))(d^*)^q f \\
&= (d^*)^{q+1}df + (q+1)(n-2r-(q+1)+1)(d^*)^q f.
\end{aligned}
$$

Now let's prove the second identity by induction on p. If $p=1$, it just reduce to the first identity (using that $df=0$). Suppose that we have proved it for some $p \in \{1, \dots, q-1\}$ (and all s and all $f \in M^{n-s,s}$ such that $df=0$), and let's prove it for $p+1$. Let $f \in M^{n-r,r}$ such that $df=0$. Then we have

$$
\begin{aligned}
d^{p+1}(d^*)^q f &= d(d^p(d^*)^q f) \\
&= d((q-p+1)_p(n-2r-q+1)_p(d^*)^{q-p}f) \\
&= (q-p+1)_p(n-2r-q+1)_p(q-p)(n-2r-(q-p)+1)(d^*)^{q-p-1}f \\
&\qquad \text{(using the first identity and the fact that } df=0) \\
&= (q-p)_{p+1}(n-2r-q+1)_{p+1}(d^*)^{q-p-1}f.
\end{aligned}
$$

f) Let $f \in S^{n-r,r}$, $f \neq 0$. Using (c) and then the second identity of (e), we see that

$$
\begin{aligned}
\langle (d^*)^{m-r}f, (d^*)^{m-r}f \rangle &= \langle d^{m-r}(d^*)^{m-r}f, f \rangle \\
&= (1)_{m-r}(n-2r-(m-r)+1)_{m-r}\langle f, f \rangle \\
&\neq 0,
\end{aligned}
$$

so $(d^*)^{m-r}f \neq 0$.

g) Let $f \in S^{n-r,r}$. Using the second formula of (d) to calculate Δ on $M^{n-m,m}$ and the second formula of (e) (with $p=1$), we get

$$
\begin{aligned}
\Delta((d^*)^{m-r}f) &= d(d^*)^{m-r+1}f - (n-m)(d^*)^{m-r}f \\
&= (m-r+1)(n-2r-(m-r+1)+1)(d^*)^{m-r}f - (n-m)(d^*)^{m-r}f \\
&= (m(n-m) - r(n-r+1))(d^*)^{m-r}f.
\end{aligned}
$$

h) This is an immediate consequence of the definition of $S^{n-m,m}$ and of (c).

i) The space $S^{n-r,r}$ is the kernel of $d : M^{n-r,r} \to M^{n-r+1,r-1}$, and $\dim(M^{n-r+1,r-1}) = \binom{n}{r-1} < \binom{n}{r} \dim(M^{n-r,r})$ because $r \leq n/2$, so d cannot be injective.

j) The subspaces $(d^*)^{m-r}(S^{n-r,r})$, for $0 \leq r \leq \min(m, n-m)$, are contained in eigenspaces of Δ for different eigenvalues by (g). They are all nonzero by (f) and (i). We know that Δ is seld-adjoint by 2(i), so these spaces are pairwise orthogonal. Also, we know that $\Delta \in \mathrm{End}(M^{n-m,m})$ has exactly $1+\min(m,n-m)$ eigenvalues by 2(a) and 2(j), so these eigenvalues have to be the numbers $m(n-m) - r(n-r+1)$, $0 \leq r \leq \min(m, n-m)$. It remains to show that

$$
M^{n-m,m} = \bigoplus_{r=0}^{\min(m,n-m)} (d^*)^{m-r}(S^{n-r,r}).
$$

We prove this by induction on m. It's obvious if $m = 0$. Suppose that we have the result for $m-1$, with $n/2 \geq m \geq 1$, and let's prove it for m. By (h), we have

$$M^{n-m,m} = S^{n-m,m} \oplus d^*(M^{n-m+1,m-1}).$$

By the induction hypothesis, we have

$$M^{n-m+1,m-1} = \bigoplus_{r=0}^{m-1} (d^*)^{m-1-r}(S^{n-r,r}).$$

The result for m follows immediately from these two facts.

Finally, we treat the case $m \geq n/2$. Let $m' = n - m$. We have seen that

$$M^{n-m',m'} = \bigoplus_{r=0}^{m'} (d^*)^{m'-r}(S^{n-r,r}).$$

By (f), this implies that $\dim(M^{n-m'}, m') = \sum_{r=0}^{m'} \dim(S^{n-r,r})$. We have also seen that

$$M^{n-m,m} \supset \bigoplus_{r=0}^{m'} (d^*)^{m-r}(S^{n-r,r}),$$

and, again by (f), this implies that $\dim(M^{n-m,m}) \geq \sum_{r=0}^{m'} \dim(S^{n-r,r}) = \dim(M^{m',n-m'})$. But $\dim(M^{n-m,m}) = \dim(M^{n-m',m'})$ (by (a)), so the inequality above is an equality, and

$$M^{n-m,m} = \bigoplus_{r=0}^{m'} (d^*)^{m-r}(S^{n-r,r}).$$

k) By (j) and (f), the map $d^* : M^{n-r+1,r-1} \to M^{n-r,r}$ is injective. By (h), this implies that

$$\dim(S^{n-r,r}) = \dim(M^{n-r,r}) - \dim(M^{n-r+1,r-1}) = \binom{n}{r} - \binom{n}{r-1}.$$

l) Let $m = \lfloor n/2 \rfloor$. As the maps d and d^* are $\mathfrak{S}_n$-equivariant (see (b) for d, and d^* is the adjoint of d by (c) so it also equivariant), the subspace $S^{n-r,r} \subset M^{n-r,r}$ is $\mathfrak{S}_n$-stable for every $r \leq n/2$, and the decomposition of (j) is a decomposition into $\mathfrak{S}_n$-subspaces. Next, we know that $(\mathfrak{S}_n, \mathfrak{S}_m \times \mathfrak{S}_{n-m})$ is a Gelfand pair by 1(f), so the corresponding quasi-regular representation, which is $M^{n-m,r}$, decomposes into a direct sum of distinct irreducible representations by theorem V.3.2.4 of the notes. By corollary V.7.2 of the notes, the number of irreducible summands in $M^{n-m,m}$ is the number of spherical functions for the Gelfand pair, which is the dimension of the space of bi-invariant functions on $\mathfrak{S}_n$ (because spherical functions form a basis for these bi-invariants functions by (iii) of the same corollary), i.e. the cardinality of $(\mathfrak{S}_m \times \mathfrak{S}_{n-m}) \setminus \mathfrak{S}_n / (\mathfrak{S}_m \times \mathfrak{S}_{n-m})$, and this is also equal to the number of orbits of $\mathfrak{S}_m \times \mathfrak{S}_{n-m}$ on Ω_m. But we have seen in 1(d) that the orbits of $\mathfrak{S}_m \times \mathfrak{S}_{n-m}$ on Ω_m are the spheres with center $A_0 := \{1, \ldots, m\}$. The possible radii for these spheres are $0, 1, \ldots, \min(m, n-m) = m$ by 2(a), and it is easy to see that all the spheres are nonempty (we already used this in the proof of 2(f)). Finally, we get that the number of irreducible constituents of $M^{n-m,m}$ is $m+1$. As the decomposition of (j) is a decomposition of $M^{n-m,m}$ int $m+1$ nonzero subrepresentations, it must be its decomposition into irreducible constituents, and so we get the conclusion. (Note that $(d^*)^{m-r}(S^{n-r,r})$ is equivalent to $S^{n-r,r}$ as a representation of $\mathfrak{S}_n$ by (f).)

$\square$

4. We keep the notation of problem 3. If $m, h \in \{0, \dots, n\}$, $A \in \Omega_m$, and $\max(0, h-m) \leq \ell \leq \min(n-m, h)$, we denote by $\sigma_{\ell,h-\ell}(A) \in L(\Omega_h)$ the characteristic function of the set $\{C \in \Omega_h | |A \cap C| = h - \ell\}$. We also write $\sigma_{-1,h+1}(A) = 0$. We fix $A \in \Omega_m$.

   a) (1) If $h = m$, show that, for every $\ell \in \{0, \dots, \min(m, n-m)\}$, the function $\sigma_{\ell,m-\ell}(A)$ is the characteristic function of the sphere $\{C \in \Omega_m | d(A, C) = \ell\}$.

   b) (2) Show that

$$d(\sigma_{\ell,h-\ell}(A)) = (n - m - \ell + 1)\sigma_{\ell-1,h-\ell}(A) + (m - h + \ell + 1)\sigma_{\ell,h-\ell-1}(A).$$

   c) (3, extra credit) If $k \leq h$ and $\max(0, k-m) \leq i \leq \min(k, n-m)$, show that

$$\frac{1}{(h-k)!}(d^*)^{h-k}\sigma_{i,k-i}(A) = \sum_{\ell=\max(i,h-m)}^{\min(h-k+i,n-m)} \binom{\ell}{i}\binom{h-\ell}{k-i}\sigma_{\ell,h-\ell}(A).$$

From now on, we take $A = \{1, \dots, m\}$.

   d) (2) If $0 \leq h \leq \min(m, n-m)$, show that the space of $\mathfrak{S}_m \times \mathfrak{S}_{n-m}$-invariant vectors in $M^{n-h,h}$ is spanned by the functions $\sigma_{\ell,h-\ell}(A)$, for $0 \leq \ell \leq h$.

   e) (2, extra credit) If $0 \leq h \leq \min(m, n-m)$, show that the space of $\mathfrak{S}_m \times \mathfrak{S}_{n-m}$-invariant vectors in $S^{n-h.h}$ is spanned by the function

$$\sum_{\ell=0}^{h} \frac{(n-m-h+1)_{h-\ell}}{(-m)_{h-\ell}}\sigma_{\ell,h-\ell}(A).$$

   f) (3, extra credit) For $0 \leq h \leq \min(m, n-m)$, let $\varphi_h \in M^{n-m,m}$ be the unique spherical function contained in the summand $(d^*)^{m-h}(S^{n-h,h})$. Show that

$$\varphi_h = \sum_{\ell=0}^{\min(m,n-m)} \varphi(n, m, h; \ell)\sigma_{\ell,m-\ell}(A),$$

   where

$$\varphi(n, m, h; \ell) = (-1)^h \frac{1}{\binom{n-m}{h}} \sum_{i=\max(0,\ell-m+h)}^{\min(\ell,h)} \binom{m-\ell}{h-i}\binom{\ell}{i}\frac{(n-m-h+1)_{h-i}}{(-m)_{h-i}}.$$

   g) (3) Fix h such that $0 \leq h \leq \min(m, n-m)$. Show that the coefficient of $\sigma_{1,m-1}(A)$ in φ_h is $1 - \dfrac{h(n-h+1)}{m(n-m)}$. (Remark : there is a way to solve this question with minimal calculations and without using any of the extra credit questions.)

Solution.

   a) The function $\sigma_{\ell,m-\ell}(A)$ is the characteristic function of the set $\{C \in \Omega_m | |A \cap C| = m - \ell\}$. As $d(A, C) = m - |A \cap C|$, this set is exactly the sphere of radius ℓ with center A.

b) Let $B \in \Omega_{h-1}$. We have

$$d(\sigma_{\ell,h-\ell}(A))(B) = \sum_{C\in\Omega_h,\ C\supset B} \sigma_{\ell,h-\ell}(A)(C)$$
$$= |\{C \in \Omega_h | C \supset B \text{ and } |A \cap C| = h-\ell\}|.$$

If there exists at least one $C \in \Omega_h$ such that $C \supset B$ and $|A \cap C| = h-\ell$, then $A \cap C \supset A \cap B$ and these two sets differ by at most one element, so $|A \cap B| \in \{h-\ell, h-\ell-1\}$.

Suppose that $|A \cap B| = h-\ell$. Then, for every $C \in \Omega_h$ such that $C \supset B$ and $|A \cap C| = h-\ell$, we must have $A \cap C = A \cap B$. We get each such C by adding an element of $\{1, \dots, n\} - (A \cup B)$ to B, so the number of possibilities for C is

$$n - |A \cup B| = n - (|A| + |B| - |A \cap B|) = n - (m + h - 1 - (h-\ell)) = n - m - \ell + 1.$$

Suppose that $|A \cap B| = h - \ell + 1$. Then, for every $C \in \Omega_h$ such that $C \supset B$ and $|A \cap C| = h-\ell$, the unique element of $C - B$ must be the element of $A \cap C - A \cap B$. So the number of possibilities for C is

$$|A - A \cap B| = m - h + \ell - 1.$$

Finally, we get

$$d(\sigma_{\ell,h-\ell}(A))(B) = (n-m-\ell-1)\sigma_{\ell-1,h-\ell}(B) + (m-h+\ell-1)\sigma_{\ell,h-\ell-1}(B),$$

as desired.

c) For every i and every $D \in \Omega_i$, denote by $\delta_D \in L(\Omega_i)$ the characteristic function of $\{D\}$. Let S be the set $\{C \in \Omega_k | |A \cap C| = k-i\}$. Then $\sigma_{i,k-i}(A)$ is the characteristic function of S. If $C \in S$, then, for every $D \in \Omega_h$, we have

$$(d^*)^{h-k}\delta_C(D) = \sum_{C\subset D_1\subset\ldots\subset D_{h-k-1}\subset D_{h-k}=D,\ D_i\in\Omega_{k+i}} 1.$$

If $C \not\subset D$, the set $\{C \subset D_1 \subset \ldots \subset D_{h-k-1} \subset D_{h-k} = D,\ D_i \in \Omega_{k+i}\}$ is empty; if $C \subset D$, this set has $(h-k)!$ elements. So we see that

$$(d^*)^{h-k}\delta_C = (h-k)! \sum_{C\subset D\in\Omega_h} \delta_D.$$

So, if $D \in \Omega_h$, the coefficient of δ_D in $\frac{1}{(h-k)!}(d^*)^{h-k}\sigma_{i,k-i}(A)$ is the cardinality of the set $\{C \in S | C \subset D\}$. Write $|A \cap D| = h-\ell$, with $0 \leq \ell \leq h$; note that we have

$$h - \ell = |A \cap D| = |A| + |D| - |A \cup D| \geq m + h - |A \cup D|$$

and $h \leq |A \cup D| \leq n$, so $h - m \leq \ell \leq n - m$, and all the nonnegative ℓ in this range can occur. We get a $C \in S$ such that $C \subset D$ by removing $(h-\ell)-(k-i)$ elements from $A \cap D$ and $\ell - i$ elements from $D - (A \cap D)$. This is only possible if $h - \ell - (k-i) \geq 0$ and $0 \leq \ell - i$, and we have $\binom{h-\ell}{(h-\ell)-(k-i)}\binom{\ell}{\ell-i}$ different possible choices. Putting all this together, we see that, if $|A \cap D| = h-\ell$, then the coefficient of δ_D in $\frac{1}{(h-k)!}(d^*)^{h-k}\sigma_{i,k-i}(A)$ is $\binom{h-\ell}{k-i}\binom{\ell}{i}$ if $\max(h-m, i) \leq \ell \leq \min(n-m, h-k+i)$ and 0 otherwise. This gives the result.

d) The statement is equivalent to the fact that the orbits $\mathfrak{S}_m \times \mathfrak{S}_{n-m}$ in Ω_h are the $S_\ell := \{C \in \Omega_h || A \cap C| = h - \ell\}$, for $0 \leq \ell \leq h$. Let's prove this fact. As $\mathfrak{S}_m \times \mathfrak{S}_{n-m}$ fixes A, the sets S_ℓ are invariant by $\mathfrak{S}_m \times \mathfrak{S}_{n-m}$, so we just need to show that $\mathfrak{S}_m \times \mathfrak{S}_{n-m}$ acts transitively on these sets. Fix $\ell \in \{0, \dots, h\}$ and take $C, C' \in S_\ell$. As $|A \cap C| = |A \cap C'|$, we can find an element $\sigma \in \mathfrak{S}_m$ that sends $A \cap C$ to $A \cap C'$. Also, we have $|\{m+1, \dots, n\} \cap C| = |\{m+1, \dots, n\} \cap C'| = \ell$, so we can find an element $\tau \in \mathfrak{S}_{n-m}$ that sends $\{m+1, \dots, n\} \cap C$ to $\{m+1, \dots, n\} \cap C'$. Then $\sigma \times \tau \in \mathfrak{S}_m \times \mathfrak{S}_{n-m}$ sends C and C'.

e) Let $f \in S^{n-h,h}$ be a $\mathfrak{S}_m \times \mathfrak{S}_{n-m}$-invariant vector. By (d), the invariance condition is equivalent to the fact that we can write $f = \sum_{\ell=0}^h a_\ell \sigma_{\ell,h-\ell}(A)$, with $a_0, \dots, a_h \in \mathbb{C}$. The fact that $f \in S^{n-h,h}$ means that $df = 0$. Using (b), we can rewrite this condition as

$$\begin{aligned}
0 &= \sum_{\ell=0}^{h} a_\ell((n-m-\ell+1)\sigma_{\ell-1,h-\ell}(A) + (m-h+\ell+1)\sigma_{\ell,h-1-\ell}(A)) \\
&= \sum_{\ell=0}^{h-1} a_{\ell+1}(n-m-\ell)\sigma_{\ell,h-\ell+1}(A) + \sum_{\ell=0}^{h} a_\ell(m-h+\ell+1)\sigma_{\ell,h-1-\ell}(A).
\end{aligned}$$

As the functions $\sigma_{\ell,h-1-\ell}(A)$, $0 \leq \ell \leq h-1$, are linearly independent (because they have disjoint supports), this equality if equivalent to the fact that

$$a_\ell = a_{\ell+1} \frac{n-m-\ell}{-m+h-\ell-1},$$

for every $\ell \in \{0, \dots, h-1\}$. A straightforward descending induction on ℓ shows that this is equivalent to

$$a_\ell = \frac{(n-m-h+1)_{h-\ell}}{(-m)_{h-\ell}} a_h$$

for every $\ell \in \{0, \dots, h\}$. This implies the desired result.

f) By (e) (and the $\mathfrak{S}_n$-equivariance of d^*), the function φ_h is a multiple of

$$\psi := \frac{1}{(m-h)!}(d^*)^{m-h}\left(\sum_{i=0}^{h} \frac{(n-m-h+1)_{h-i}}{(-m)_{h-i}} \sigma_{i,h-i}(A)\right).$$

We calculate ψ using the formula of (c). We get

$$\begin{aligned}
\psi &= \sum_{i=0}^{h} \frac{(n-m-h+1)_{h-i}}{(-m)_{h-i}} \sum_{\ell=i}^{\min(m-h+i,n-m)} \binom{\ell}{i}\binom{m-\ell}{h-i} \sigma_{\ell,m-\ell}(A) \\
&= \sum_{\ell=0}^{\min(m,n-m)} \sum_{i=\max(0,\ell-m+h)}^{\min(\ell,h)} \binom{\ell}{i}\binom{m-\ell}{h-i} \frac{(n-m-h+1)_{h-i}}{(-m)_{h-i}} \sigma_{\ell,m-\ell}(A).
\end{aligned}$$

This is almost the formula we want, except for the constant $(-1)^h \frac{1}{\binom{n-m}{h}}$ at the beginning.

The spherical function φ_h is normalized by the fact that $\varphi_h(A_0) = 1$, so we have $\varphi_h = \frac{1}{\psi(A_0)}\psi$. So to finish the proof, we just need to show that $\psi(A_0) = (-1)^h \binom{n-m}{h}$.

Note that $\sigma_{\ell,m-\ell}(A)(A_0) = 0$ unless $\ell = 0$ and $\sigma_{0,m}(A)(A_0) = 1$, so

$$\begin{aligned}\psi(A_0) &= \binom{m}{h} \frac{(n-m-h+1)_h}{(-m)_h} \\ &= \frac{m!}{h!(m-h)!} \frac{(n-m)!}{(n-m-h)!} \frac{(-1)^h (m-h)!}{m!} \\ &= (-1)^h \binom{n-m}{h}.\end{aligned}$$

g) Let's try and ignore questions (c), (d), (e) and (f).

The function φ_h is spherical, so it is constant on the $\mathfrak{S}_m \times \mathfrak{S}_{n-m}$-orbits in Ω_m, which are the spheres with center A by 1(d). By (a) and 2(a), this means that φ_h is a linear combination of the functions $\sigma_{\ell,m-\ell}(A)$, for $0 \leq \ell \leq \min(m, n-m)$. Also, by 3(g), the function φ_h is an eigenvector of Δ with eigenvalue $m(n-m) - h(n-h+1)$. Let $S = \{B \in \Omega_m | d(A, B) = 1\}$. Then $\sigma_{1,m-1}(A)$ is the characteristic function of S, and we have seen that φ_h is constant on S, the coefficient of $\sigma_{1,m-1}(A)$ in φ_h is $\frac{1}{|S|} \sum_{B \in S} \varphi_h(B)$. On the other hand, $\sum_{B \in S} \varphi_h(B)$ is equal to $\Delta\varphi_h(A)$ by definition of Δ, and this is equal to $(m(n-m) - h(n-h+1))\varphi_h(A)$. Moreover, as φ_h is spherical, we must have $\varphi_h(A) = 1$. Finally, the coefficient of $\sigma_{1,m-1}(A)$ in φ_h is $\frac{1}{|S|}(m(n-m) - h(n-h-1))$. To finish the calculation, we just need to show that $|S| = m(n-m)$. This just follows from the fact that we get every element B of S by removing one element of A (m choices) and adding an element of $\{1, \dots, n\} - A$ ($n-m$ choices).

$\square$