MAT 449 : Problem Set 11

Due Sunday, December 16

The goal of this problem set is to study the Gelfand pair $(\mathfrak{S}_n, \mathfrak{S}_r \times \mathfrak{S}_{n-r})$. We will embed $\mathfrak{S}_r \times \mathfrak{S}_{n-r}$ in $\mathfrak{S}_n$ in the following way : If $\sigma \in \mathfrak{S}_r$ and $\tau \mathfrak{S}_{n-r}$, then $\sigma \times \tau \in \mathfrak{S}_n$ is given by

$$(\sigma \times \tau)(i) = \begin{cases} \sigma(i) & \text{if } 1 \leq i \leq r \\ \tau(i-r)+r & \text{if } r+1 \leq i \leq n. \end{cases}$$

If E is a finite set, we will denote by $L(E)$ the space of functions $f : E \to \mathbb{C}$, with the L^2 inner product given by $\langle f, f' \rangle = \sum_{x \in E} f(x)\overline{f'(x)}$.

1. In this problem, we fix a finite group G acting transitively (on the left) on a set E. Let $x_0 \in E$, and let $K \subset G$ be the stabilizer of x_0.
   a) (2) Show that the following conditions are equivalent :
      (i) For all $x, y \in E$, there exists $g \in G$ such that $g \cdot (x, y) = (y, x)$.
      (ii) For every $g \in G$, we have $g^{-1} \in KgK$.
   b) (1) If the conditions of (a) are satisfied, show that (G, K) is a Gelfand pair.

   We now assume that there is a metric $d : E \times E \to \mathbb{R}_{\geq 0}$, and that the group G acts by isometries. Suppose that the action of G on E is *distance-transitive*, that is : for all $(x, y), (x', y') \in E \times E$ such that $d(x, y) = d(x', y')$, there exists $g \in G$ such that $g \cdot (x, y) = (x', y')$.

   c) (1) Show that (G, K) is a Gelfand pair.
   d) (1) Show that the orbits of K on E are the spheres $\{x \in E | d(x, x_0) = j\}$, for $j \in \mathbb{R}_{\geq 0}$.
   e) (1) Let Ω_r be the set of cardinality r subsets of $\{1, \dots, n\}$. Show that the formula $d(A, B) = r - |A \cap B|$ defines a metric on Ω_r.
   f) (2) Show that $(\mathfrak{S}_n, \mathfrak{S}_r \times \mathfrak{S}_{n-r})$ is a Gelfand pair.

2. Let (Ω_r, d) be the finite metric space of problem 1(e). Let N be the *diameter* of Ω_r, that is,

   $$N = \max\{d(A, B),\ A, B \in \Omega_r\}.$$

   For $i \in \{0, \dots, N\}$, we define a linear operator $\Delta_i : L(\Omega_r) \to L(\Omega_r)$ by

   $$\Delta_i f(A) = \sum_{B \in \Omega_r,\ d(A,B)=i} f(B),$$

   for every $f \in L(\Omega_r)$. We also denote by $\mathcal{A}$ the subalgebra of $\mathrm{End}(L(\Omega_r))$ generated by $\Delta_0, \dots, \Delta_N$.

   a) (2) Show that $N = \min(r, n-r)$.

b) (2) Show that there exist integers $b_0, \dots, b_N, c_0, c_1, \dots, c_N$ such that, for every i and all $A, B \in \Omega_r$ such that $d(A,B) = i$, we have

$$|\{C \in \Omega_r | d(A,C) = 1 \text{ and } d(B,C) = i+1\}| = b_i$$

and

$$|\{C \in \Omega_r | d(A,C) = 1 \text{ and } d(B,C) = i-1\}| = c_i.$$

(Of course, $c_0 = 0$ and $c_1 = 1$.)

c) (1) Show that $c_2, \dots, c_N > 0$.

d) (3) If $i \in \{1, \dots, N\}$, show that

$$\Delta_i \Delta_1 = b_{i-1}\Delta_{i-1} + (b_0 - b_i - c_i)\Delta_i + c_{i+1}\Delta_{i+1},$$

with the convention that $\Delta_{N+1} = 0$.

e) (1) Show that there exist polynomials $p_0, \dots, p_N \in \mathbb{R}[t]$ such that $\deg(p_i) = i$ and $\Delta_i = p(\Delta_1)$.

f) (1) Show that $\mathcal{A}$ is the subalgebra of $\mathrm{End}(L(\Omega_r))$ generated by Δ_1.

g) (1) Show that $\mathcal{A}$ is spanned as a $\mathbb{C}$-vector space by $\Delta_0, \dots, \Delta_N$.

h) (2) Show that $\dim_{\mathbb{C}} \mathcal{A} = N + 1$.

i) (1) Show that the endormorphism Δ_1 of $L(\Omega_r)$ is self-adjoint.

j) (2) Show that we have a decomposition into pairwise orthogonal subspaces $L(\Omega_r) = \bigoplus_{i=0}^N V_i$, where $V_0, \dots, V_N$ are the eigenspaces of Δ_1. (Hint : problem 2 of problem set 5.)

3. We use the notation of problem 2. Note that we have an action of $G := \mathfrak{S}_n$ on Ω_r, and that the stabilizer of $\{1, \dots, r\}$ is $K := \mathfrak{S}_r \times \mathfrak{S}_{n-r}$. Let $M^{n-r,r} = L(\Omega_r)$, seen as a representation of $\mathfrak{S}_n$ via the quasi-regular representation (that is, if $g \in G$, $f \in M^{n-r,n}$ and $A \in \Omega_r$, we have $(g \cdot f)(A) = f(g^{-1}A)$).

   We define $d : M^{n-r,r} \to M^{n-r+1,r-1}$ and $d^* : M^{n-r+1,r-1} \to M^{n-r,r}$ by

$$(df)(A) = \sum_{B \in \Omega_r | A \subset B} f(B)$$

   and

$$(d^* f)(B) = \sum_{A \in \Omega_r | A \subset B} f(A).$$

   (If $r = 0$, we take $d = 0$ and $d^* = 0$.)

   We also denote by Δ the operator Δ_1 of problem 2; that is, for every $f \in M^{n-r,r}$, the function $\Delta f \in M^{n-r,r}$ is defined by

$$(\Delta f)(A) = \sum_{B \in \Omega_r | d(A,B) = 1} f(B).$$

   Note that the functions d, d^* and Δ are defined for every r; we will not indicate r in the notation, it should be clear from the context.

   Finally, if $a \in \mathbb{C}$ and $i \in \mathbb{Z}_{\geq 0}$, we write

$$(a)_i = a(a+1)\dots(a+i-1).$$

   For example, we have $(1)_n = n!$ and $\binom{n}{k} = \frac{(n-k+1)_k}{k!}$.

a) (1) Show that $A \mapsto \{1, \dots, n\} - A$ induces a G-equivariant isomorphism $M^{r,n-r} \xrightarrow{\sim} M^{n-r,r}$.

b) (2) Show that d and Δ are G-equivariant.

c) (1) Show that d^* is the adjoint of d.

d) (2) If $f \in M^{n-r,r}$, show that

$$dd^* f = \Delta f + (n-r)f \quad \text{and} \quad d^* df = \Delta f + rf.$$

e) (2) Let $f \in M^{n-r,r}$ and $1 \leq p \leq q \leq n-r$. Show that

$$d(d^*)^q f = (d^*)^q df + q(n - 2r - q + 1)(d^*)^{q-1} f.$$

If moreover $df = 0$, show that

$$d^p (d^*)^q f = (q - p + 1)_p (n - 2r - q + 1)_p (d^*)^{q-p} f.$$

Suppose that $0 \leq r \leq n/2$. If $r > 0$, set $S^{n-r,r} = \mathrm{Ker}(d : M^{n-r,r} \to M^{n-r+1,r-1})$; if $r = 0$, set $S^{n-r,r} = M^{n-r,r}$. This is a G-stable subspace of $M^{n-r,r}$.

f) (2) If $0 \leq m \leq n$ and $0 \leq r \leq \min(m, n-m)$, show that $(d^*)^{m-r} : S^{n-r,r} \to M^{n-m,m}$ is injective. (Hint : calculate $\|(d^*)^{m-r} f\|_2^2$).

g) (1) Under the hypothesis of (f), show that $(d^*)^{m-r}(S^{n-r,r})$ is contained in the eigenspace of Δ for the eigenvalue $m(n-m) - r(n-r+1)$.

h) (1) Show that the orthogonal of $S^{n-m,m}$ in $M^{n-m,m}$ is $d^*(M^{n-m+1,m-1})$, if $1 \leq m \leq n/2$.

i) (1) Show that $S^{n-r,r} \neq 0$ for every r such that $0 \leq r \leq n/2$.

j) (3) If $0 \leq m \leq n$, show that we have

$$M^{n-m,m} = \bigoplus_{r=0}^{\min(m,n-m)} (d^*)^{m-r}(S^{n-r,r}),$$

where the summands are pairwise orthogonal and are exactly the eigenspaces of Δ.

k) (1) Show that $\dim_{\mathbb{C}}(S^{n-r,n}) = \binom{n}{r} - \binom{n}{r-1}$ if $r > 0$.

l) (3) Show that the representations $S^{n-r,r}$, $0 \leq r \leq n/2$, are irreducible and pairwise inequivalent. (Hint : how many irreducible constituents does $M^{m,n-m}$ have ?)

4. We keep the notation of problem 3. If $m, h \in \{0, \dots, n\}$, $A \in \Omega_m$, and $\max(0, h-m) \leq \ell \leq \min(n-m, h)$, we denote by $\sigma_{\ell,h-\ell}(A) \in L(\Omega_h)$ the characteristic function of the set $\{C \in \Omega_h | |A \cap C| = h - \ell\}$. We also write $\sigma_{-1,h+1}(A) = 0$. We fix $A \in \Omega_m$.

a) (1) If $h = m$, show that, for every $\ell \in \{0, \dots, \min(m, n-m)\}$, the function $\sigma_{\ell,m-\ell}(A)$ is the characteristic function of the sphere $\{C \in \Omega_m | d(A, C) = \ell\}$.

b) (2) Show that

$$d(\sigma_{\ell,h-\ell}(A)) = (n - m - \ell + 1)\sigma_{\ell-1,h-\ell}(A) + (m - h + \ell + 1)\sigma_{\ell,h-\ell-1}(A).$$

c) (3, extra credit) If $k \leq h$ and $\max(0, k-m) \leq i \leq \min(k, n-m)$, show that

$$\frac{1}{(h-k)!}(d^*)^{h-k}\sigma_{i,k-i}(A) = \sum_{\ell=\max(i,h-m)}^{\min(h-k+i,n-m)} \binom{\ell}{i}\binom{h-\ell}{k-i}\sigma_{\ell,h-\ell}(A).$$

From now on, we take $A = \{1, \ldots, m\}$.

d) (2) If $0 \leq h \leq \min(m, n-m)$, show that the space of $\mathfrak{S}_m \times \mathfrak{S}_{n-m}$-invariant vectors in $M^{n-h,h}$ is spanned by the functions $\sigma_{\ell,h-\ell}(A)$, for $0 \leq \ell \leq h$.

e) (2, extra credit) If $0 \leq h \leq \min(m, n-m)$, show that the space of $\mathfrak{S}_m \times \mathfrak{S}_{n-m}$-invariant vectors in $S^{n-h.h}$ is spanned by the function

$$\sum_{\ell=0}^{h} \frac{(n-m-h+1)_{h-\ell}}{(-m)_{h-\ell}} \sigma_{\ell,h-\ell}(A).$$

f) (3, extra credit) For $0 \leq h \leq \min(m, n-m)$, let $\varphi_h \in M^{n-m,m}$ be the unique spherical function contained in the summand $(d^*)^{m-h}(S^{n-h,h})$. Show that

$$\varphi_h = \sum_{\ell=0}^{\min(m,n-m)} \varphi(n, m, h; \ell) \sigma_{\ell,m-\ell}(A),$$

where

$$\varphi(n, m, h; \ell) = (-1)^h \frac{1}{\binom{n-m}{h}} \sum_{i=\max(0,\ell-m+h)}^{\min(\ell,h)} \binom{m-\ell}{h-i} \binom{\ell}{i} \frac{(n-m-h+1)_{h-i}}{(-m)_{h-i}}.$$

g) (3) Fix h such that $0 \leq h \leq \min(m, n-m)$. Show that the coefficient of $\sigma_{1,m-1}(A)$ in φ_h is $1 - \dfrac{h(n-h+1)}{m(n-m)}$. (Remark : there is a way to solve this question with minimal calculations and without using any of the extra credit questions.)