MAT 449 : Problem Set 10

Due Sunday, December 9

1. Fix a positive integer n. For every $m \in \mathbb{Z}_{\geq 0}$, we denote by $V_m(\mathbb{R}^n)$ the vector space of complex-valued polynomial functions on $\mathbb{R}^n$ that are homogenous of degree m. We define an action of $\mathbf{O}(n)$ on $V_m(\mathbb{R}^n)$ by $(x \cdot f)(v) = f(x^{-1}v)$ if $x \in \mathbf{O}(n)$, $f \in V_m(\mathbb{R}^n)$ and $v \in \mathbb{R}^n$ (in other words, $x \cdot f = L_x f$).

   For $i \in \{1, \dots, n\}$, we denote by ∂_{x_i} the endomorphism $f \mapsto \frac{\partial}{\partial x_i} f$ of $C^\infty(\mathbb{R}^n)$ (the space of smooth functions from $\mathbb{R}^n$ to $\mathbb{C}$), and we set $\Delta = \sum_{i=1}^n (\partial_{x_i})^2$ (this is called the Laplacian operator).

   The space of harmonic polynomials of degree m on $\mathbb{R}^n$ is the space

$$\mathcal{H}_m(\mathbb{R}^n) = \{f \in V_m(\mathbb{R}^n) | \Delta(f) = 0\}.$$

   a) (2 points) Calculate $\dim(V_m(\mathbb{R}^n))$.
   b) (1 point) Show that the action of $\mathbf{O}(n)$ on $V_m(\mathbb{R}^n)$ is a continuous representation.
   c) (2 points) Show that, for every $x \in \mathbf{O}(n)$ and every $f \in C^\infty(\mathbb{R}^n)$, we have $\Delta(L_x f) = L_x(\Delta(f))$.
   d) (1 point) Show that the subspace $\mathcal{H}_m(\mathbb{R}^n)$ of $V_m(\mathbb{R}^n)$ is $\mathbf{O}(n)$-invariant.

Solution.

   a) For every $i \in \{1, \dots, n\}$, denote by $x_i \in V_1(\mathbb{R}^n)$ the function $(z_1, \dots, z_n) \mapsto z_i$. Then $\{x_1^{i_1} \dots x_n^{i_n}, i_1, \dots, i_n \in \mathbb{Z}_{\geq 0}, i_1 + \dots + i_n = m\}$ is a basis of $V_m(\mathbb{R}^n)$. So

$$\dim(V_m(\mathbb{R}^n)) = |\{(i_1, \dots, i_n) \in (\mathbb{Z}_{\geq 0})^n | i_1 + \dots + i_n = m\}|.$$

   This is also equal to

$$|\{(j_1, \dots, j_n) \in (\mathbb{Z}_{\geq 1})^n | j_1 + \dots + j_n = m + n\}|$$

   (take $j_r = i_r + 1$). Choosing $(j_1, \dots, j_n)$ in the set above is equivalent to choosing the numbers $j_1, j_1 + j_2, \dots, j_1 + \dots + j_{n-1}$, which form a subset of $\{1, \dots, n+m-1\}$ of cardinality $n - 1$. So we get

$$\dim(V_m(\mathbb{R}^n)) = \binom{n+m-1}{n-1} = \binom{n+m-1}{m}.$$

   b) If we use the basis of $V_m(\mathbb{R}^n)$ from (a), the action of $x \in \mathbf{O}(n)$ is given by a matrix with coefficients polynomial functions in the entries of x. So, for every $f \in V_m(\mathbb{R}^n)$, the map $\mathbf{O}(n) \to V_m(\mathbb{R}^n)$, $x \cdot f$ is continuous. As $V_m(\mathbb{R}^n)$ is finite-dimensional, this implies that the action is continuous.

c) I don't want to do the calculation, so let's use 2(c). (Sorry.) Note that $\Delta = \partial_{x_1^2+\ldots+x_n^2}$. So, by 2(c), for every $f \in C^\infty(\mathbb{R}^n)$ and every $x \in G$, we have

$$\Delta(x \cdot f) = x \cdot (\partial_g f),$$

where $g = L_{x^T}(x_1^2 + \ldots + x_n)$. So we just need to show that $x_1^2 + \ldots + x_n^2 \in V_2(\mathbb{R}^n)$ is invariant by all the elements of $\mathbf{O}(n)$, which follows directly from the definition of $\mathbf{O}(n)$.

d) Question (c) implies that $\Delta : V_m(\mathbb{R}^n) \to V_{m-2}(\mathbb{R}^n)$ is $\mathbf{O}(n)$-equivariant, and $\mathcal{H}_m(\mathbb{R}^n)$ is its kernel.

$\square$

2. We keep the notation of problem 1. For $i \in \{1, \ldots, n\}$, we denote by x_i the oth coordinate function on $\mathbb{R}^n$.

a) (1 point) Show that the map $x_i \to \partial_{x_i}$ extends to a unique morphism of $\mathbb{C}$-algebras from $\bigoplus_{m\geq 0} V_m(\mathbb{R}^n)$ (the algebra of complex-valued polynomial functions on $\mathbb{R}^n$) to $\mathrm{End}(\mathcal{C}^\infty(\mathbb{R}^n))$. We will denote this morphism by $f \mapsto \partial_f$.

If $f, g \in V_m(\mathbb{R}^n)$, we set $\langle f, g\rangle = \partial_{\overline{g}}(f)$. (Note that $\overline{g}$ is still a polynomial function on $\mathbb{R}^n$.)

b) (3 points) Show that $\langle ., .\rangle$ is an inner form on $V_m(\mathbb{R}^n)$. (Hint : Can you find an orthogonal basis ?)

c) (2 points) Show that, for every $f \in V_m(\mathbb{R}^n)$ and every $y \in \mathbf{O}(n)$, we have $\partial_f \circ L_y = L_y \circ \partial_{L_{y^T} f}$ in $\mathrm{End}(C^\infty(\mathbb{R}^n))$.

d) (1 point) Show that the continuous representation of $\mathbf{O}(n)$ on $V_m(\mathbb{R}^n)$ defined in problem 1 is unitary for the inner product $\langle ., .\rangle$.

e) (1 point) If $m \leq 1$, show that $V_m(\mathbb{R}^n) = \mathcal{H}_m(\mathbb{R}^n)$.

f) (2 points) If $m \geq 2$, show that $\mathcal{H}_m(\mathbb{R}^n)^\perp = |x|^2 V_{m-2}(\mathbb{R}^n)$, where $|x|^2$ is the function $\sum_{i=1}^n x_i^2 \in V_2(\mathbb{R}^n)$.

g) (2 points) Show that

$$V_m(\mathbb{R}^n) = \bigoplus_{k=0}^{\lfloor m/2\rfloor} |x|^{2k}\mathcal{H}_{m-2k}(\mathbb{R}^n),$$

and that this induces a $\mathbf{O}(n)$-equivariant isomorphism

$$V_m(\mathbb{R}^n) = \bigoplus_{k=0}^{\lfloor m/2\rfloor} \mathcal{H}_{m-2k}(\mathbb{R}^n).$$

h) (2 points) If $S \subset \mathbb{R}^n$ is the unit sphere, show that the map $\bigoplus_{m\geq 0} \mathcal{H}_m(\mathbb{R}^n) \to \mathcal{C}(S)$, $f \mapsto f_{|S}$ is injective.

i) (1 point) Show that, for every $f \in V_m(\mathbb{R}^n)$, there is a unique $g \in \bigoplus_{k=0}^{\lfloor m/2\rfloor} \mathcal{H}_{m-2k}(\mathbb{R}^n)$ such that $f_{|S} = g_{|S}$.

Solution.

a) Note that $\bigoplus_{m\geq 0} V_m(\mathbb{R}^n)$ is isomorphic to the polynomial algebra $\mathbb{C}[x_1, \ldots, x_n]$. So we just need to check that ∂_{x_i} and ∂_{x_j} commute for all $i, j \in \{1, \ldots, n\}$. But this is a well-known property of partial derivatives of C^2 functions.

b) First, it is clear from the definition that $\langle .,. \rangle$ is linear in the first variable and antilinear in the second variable. We calculate the matrix of this form in the basis of 1(a).

Let $f = x_1^{i_1} \dots x_n^{i_n}$, with $i_1, \dots, i_n \in \mathbb{Z}_{\geq 0}$ and $i_1 + \dots + i_n = m$. If $r \in \{1, \dots, n\}$ and $a \in \mathbb{Z}_{\geq 0}$, we have

$$\partial_{x_r^a} f = \begin{cases} 0 & \text{if } a > i_r \\ i_r(i_r - 1) \dots (i_r - a + 1) x_r^{i_r - a} \prod_{s \neq r} x_s^{i_s} & \text{if } a \leq i_r. \end{cases}$$

Let $g = x_1^{j_1} \dots x_n^{j_n}$, with $j_1, \dots, j_n \in \mathbb{Z}_{\geq 0}$ and $j_1 + \dots + j_n = m$. As $i_1 + \dots + i_n = j_1 + \dots + j_n$, either there exists $r \in \{1, \dots, n\}$ such that $j_r > i_r$, or $i_r = j_r$ for every $r \in \{1, \dots, n\}$. In the first case, we have $\langle f, g \rangle = \partial_{\overline{g}} f = 0$. In the second case, we have

$$\langle f, g \rangle = \partial_{\overline{g}} f = i_1! i_2! \dots i_n!.$$

So the matrix of $\langle .,. \rangle$ in the basis $\{x_1^{i_1} \dots x_n^{i_n}, i_1, \dots, i_n \in \mathbb{Z}_{\geq 0}, i_1 + \dots + i_n = m\}$ of $V_m(\mathbb{R}^n)$ is diagonal with real positive entries, and in particular Hermitian definite positive. This implies that $\langle .,. \rangle$ is an inner product.

c) The statement is actually true for every $y \in \mathbf{GL}_n(\mathbb{R})$, and we will prove this.

First note that the identity of the statement makes sense for f in the algebra $V(\mathbb{R}^n) := \bigoplus V_m(\mathbb{R}^n)$, and it is linear in f. Also, if it is true for $f, g \in V(\mathbb{R}^n)$, then we have, for $y \in \mathbf{GL}_n(\mathbb{R})$,

$$\begin{aligned} \partial_{fg} \circ L_y &= \partial_f \circ \partial_g \circ L_y \\ &= \partial_f \circ L_y \circ \partial_{L_{y^T} g} \\ &= L_y \circ \partial_{L_{y^T} f} \circ \partial_{L_{y^T} g} \\ &= L_y \circ \partial_{L_{y^T} f L_{y^T} g} \\ &= L_y \circ \partial_{L_{y^T}(fg)}, \end{aligned}$$

that is, the identity also holds for fg. In conclusion, we only need to prove it for the functions $x_1, \dots, x_n$.

Let $i \in \{1, \dots, n\}$, and, let $y \in \mathbf{GL}_n(\mathbb{R})$, and write $y^{-1} = a = (a_{ij}) \in \mathbf{GL}_n(\mathbb{R})$. Then for every $(z_1, \dots, z_n) \in \mathbb{R}^n$, we have

$$(L_{y^T} x_i)(z_1, \dots, z_n) = x_i(a^T(z_1, \dots, z_n)) = \sum_{j=1}^{n} a_{ji} z_j.$$

In other words, we have

$$L_{y^T} x_i = \sum_{j=1}^{n} a_{ji} x_j,$$

so

$$\partial_{L_{y^T} x_i} = \sum_{j=1}^{n} a_{ji} \partial_{x_j}.$$

Let $\varphi \in C^\infty(\mathbb{R}^n)$ and $z = (z_1, \dots, z_n) \in \mathbb{R}^n$. We have

$$\partial_{L_{y^T} x_i} \varphi = \sum_{j=1}^{n} a_{ji} \frac{\partial \varphi}{\partial x_j},$$

so

$$L_y \partial_{L_{y^T} x_i} \varphi(z) = \sum_{j=1}^n a_{ji} \frac{\partial \varphi}{\partial x_j}(az).$$

On the other hand, $L_y \varphi(z) = \varphi(az)$, with

$$az = (\sum_{j=1}^n a_{rj} z_r)_{1 \leq r \leq n},$$

so

$$(\partial_{x_i} L_y \varphi)(z) = \sum_{r=1}^n a_{ri} \partial_{x_r} \varphi(az).$$

We see that we do get the same result for $L_y \partial_{L_{y^T} x_i} \varphi(z)$ and $(\partial_{x_i} L_y \varphi)(z)$.

d) Let $f, g \in V_m(\mathbb{R}^n)$ and $y \in \mathbf{O}(n)$. By (c), we have

$$\langle L_y f, L_y g \rangle = \partial_{\overline{L_y g}} L_y f = L_y \partial_{L_{y^T} L_y \overline{g}} f = L_y \partial_{\overline{g}} f.$$

As $\partial_{\overline{g}} f$ is a constant function, we have

$$L_y \partial_{\overline{g}} f = \partial_{\overline{g}} f = \langle f, g \rangle,$$

which is what we wanted.

e) If $m \geq 1$, then $\Delta = 0$ on $V_m(\mathbb{R}^n)$, so $\mathcal{H}_m(\mathbb{R}^n) = \ker(\Delta) = V_m(\mathbb{R}^n)$.

f) Note that $\Delta = \partial_{|x|^2}$. So, if $f \in V_m(\mathbb{R}^n)$ and $g \in V_{m-2}(\mathbb{R}^n)$, we have

$$\langle f, |x|^2 g \rangle = \partial_{|x|^2 \overline{g}} f = \partial_{\overline{g}}(\partial_{|x|^2} f) = \langle \Delta f, g \rangle.$$

In other words, the map $V_{m-2}(\mathbb{R}^n) \to V_m(\mathbb{R}^n)$, $g \mapsto |x|^2 g$ is the adjoint of $\Delta : V_m(\mathbb{R}^n) \to V_{m-2}(\mathbb{R}^n)$, and so its image is the orthogonal of $\operatorname{Ker} \Delta = \mathcal{H}_m(\mathbb{R}^n)$.

g) The first formula just follows from (f) by an easy induction. For the second formula, we note that, for every $k \in \{0, \dots, \lfloor \frac{m}{2} \rfloor\}$, the injective linear transformation $V_{m-2}(\mathbb{R}^n) \to V_m(\mathbb{R}^n)$, $g \mapsto |x|^{2k} g$ is $\mathbf{O}(n)$-equivariant, because the function $|x|^{2k}$ is invariant by $\mathbf{O}(n)$.

h) We will use polar coordinates on $\mathbb{R}^n$: a point z of $\mathbb{R}^n$ can be written as $z = rs$, with $r \in \mathbb{R}_{\geq 0}$ and $s \in S$, and r and s are uniquely determined if $z \neq 0$.

Let $f \in \mathcal{H}_m(\mathbb{R}^n)$ and $g \in \mathcal{H}_p(\mathbb{R}^n)$. Then, if $r \geq \mathbb{R}_{\geq 0}$ and $s \in S$, we have $f(rs) = r^m f(s)$ and $g(rs) = r^p g(s)$. By Green's second formula, we have

$$\int_B (f \Delta g - g \Delta f) d\lambda = c \int_S (f \frac{\partial g}{\partial r} - g \frac{\partial f}{\partial r}) d\mu,$$

where B is the closed unit ball, λ is Lebesgue measure on $\mathbb{R}^n$, c is a positive constant and and μ is the measure on S defined in 4(g). As f and g are in the kernel of Δ, this gives

$$0 = c(p - m) \int_S f(s) g(s) d\mu(s).$$

If $m \neq p$, we get $\int_S f(s) g(s) d\mu(s) = 0$. So, for $m \neq p$, the subspaces $\mathcal{H}_m(\mathbb{R}^n)_{|S}$ and $\mathcal{H}_p(\mathbb{R}^n)_{|S}$ are orthogonal for the inner product of $L^2(S, \mu)$. In particular, if $f \in \bigoplus_{m \geq 0} \mathcal{H}_m(\mathbb{R}^n)$ is such that $f_{|S} = 0$, then, writing $f = \sum_{m \geq 0} f_m$ with $f_m \in \mathcal{H}_m(\mathbb{R}^n)$, we must have $f_{m|S} = 0$ for every $m \geq 0$. But f_m is homogeneous of degree m, so $f_m(rs) = r^m f_m(s)$ for every $r \in \mathbb{R}_{\geq 0}$ and $s \in S$, so $f_{m|S} = 0$ implies $f_m = 0$.

i) The existence follows from the first identity of (g) (because $|x|^2 = 1$ on S), and the uniqueness from (h).

$\square$

3. Let $n \geq 2$, and embed $\mathbf{O}(n-1)$ into $\mathbf{O}(n)$ by using the map $x \mapsto \begin{pmatrix} 1 & 0 \\ 0 & x \end{pmatrix}$. Let $G = \mathbf{SO}(n)$, and let K be the image of $\mathbf{SO}(n-1)$ in G by the embedding we just defined.

   a) (2 points) Let A be the subset of $\mathbf{SO}(n)$ consisting of matrices of the form
$$\begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & I_{n-2} & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix},$$
   with $\theta \in \mathbb{R}$.

   Show that A is a subgroup of G and that we have $G = KAK$.

   b) (2 points) Show that (G, K) is a Gelfand pair. (You might want to use the involution θ of G defined by $\theta(x) = JxJ$, where J is the diagonal matrix with diagonal coefficients $-1, 1, \ldots, 1$.)

Solution.

a) For every $\theta \in \mathbb{R}$, we write $A_\theta = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & I_{n-2} & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}$. We have $A = \{A_\theta,\ \theta \in \mathbb{R}\}$. We check easily that $A_\theta A_{\theta'} = A_{\theta+\theta'}$, so A is a subgroup of G.

Let $v_0 = (1, 0, \ldots, 0) \in S$. Then the action of $\mathbf{O}(n)$ on $\mathbb{R}^n$ preserves S (and $\mathbf{O}(n)$ acts transitively on S), and K is the stabilizer of v_0 in $\mathbf{O}(n)$. Let $x \in \mathbf{O}(n)$, and write $z = x \cdot v_0 = (z_1, \ldots, z_n)$. We can find $y \in K$ such that $y \cdot z = (z_1, 0, \ldots, 0, c)$, with $c^2 = z_2^2 + \ldots + z_n^2$, and then we can find $a \in A$ such that $a \cdot (y \cdot z) = (1, 0, \ldots, 0) = v_0$. Then we have $(ayx) \cdot v_0 = v_0$, so $ayx \in K$, and $x \in Ka^{-1}y^{-1} \subset KAK$.

b) We want to apply proposition V.2.5 of the notes to θ, where θ sends $x \in G$ to JxJ, with $J = \begin{pmatrix} -1 & 0 \\ 0 & I_{n-1} \end{pmatrix}$. Note that $J^2 = I_n$, so $J = J^{-1}$, so θ is a morphism of groups and an involution. It is also clear that $\theta(K) = K$. We need to check that $\theta(x) \in Kx^{-1}K$ for every $x \in G$. Let $x \in G$, and write $x = kak'$, with $k, k' \in K$ and $a \in A$. Then $\theta(x) = \theta(k)\theta(a)\theta(k')$ and $\theta(k), \theta(k') \in K$, and, if $a = A_\theta$, we have $\theta(a) = A_{-\theta} = a^{-1}$. So $\theta(x) = \theta(k)k'x^{-1}k\theta(k') \in Kx^{-1}K$.

$\square$

4. We use the notation of problems 1 and 2, and the embedding $\mathbf{O}(n-1) \subset \mathbf{O}(n)$ defined in problem 3.

   a) (1 point) Show that we have a $\mathbf{O}(n-1)$-equivariant isomorphism
$$V_m(\mathbb{R}^n) \simeq \bigoplus_{k=0}^{m} V_{m-k}(\mathbb{R}^{n-1}).$$

   b) (3 points) Show that we have a $\mathbf{O}(n-1)$-equivariant isomorphism
$$\mathcal{H}_m(\mathbb{R}^n) \simeq \bigoplus_{k=0}^{m} \mathcal{H}_{m-k}(\mathbb{R}^{n-1}).$$

c) (3 points) If $m \geq 2$, show that $\mathcal{H}_m(\mathbb{R}^2)$ is an irreducible representation of $\mathbf{O}(2)$, but that it is not irreducible as a representation of $\mathbf{SO}(2)$.

From now on, we assume that $n \geq 3$.

d) (2 points) If $m \geq 1$, show that $\mathcal{H}_m(\mathbb{R}^n)^{\mathbf{SO}(n)} = \{0\}$.

e) (1 point) Show that, for every $m \geq 0$, the space $\mathcal{H}_m(\mathbb{R}^n)^{\mathbf{SO}(n-1)}$ is 1-dimensional.

f) (2 points) Let $S \subset \mathbb{R}^n$ be the unit sphere, and let $v_0 = (1, 0, \dots, 0) \in S$. Show that the map $\mathbf{SO}(n) \to S$, $x \mapsto x \cdot v_0$ induces a homeomorphism $\mathbf{SO}(n)/\mathbf{SO}(n-1) \xrightarrow{\sim} S$.

g) (2 points) Show that the measure μ on S defined in 1(f) of problem set 2 (using the normalized Haar measures on $\mathbf{SO}(n)$ and $\mathbf{SO}(n-1)$) is given by $\mu(E) = c\lambda(\{tx,\ t \in [0,1],\ x \in E\})$ for every Borel subset E of S, where λ is Lebesgue measure on $\mathbb{R}^n$ and c^{-1} is the volume of the unit ball (for λ).

h) (2 points) By the previous question, we have the quasi-regular representation of $\mathbf{SO}(n)$ on $L^2(S)$, and it preserves the subspace of continuous functions. If $V \subset \mathcal{C}(S)$ is a nonzero finite-dimensional $\mathbf{SO}(n)$-stable subspace, show that $V^{\mathbf{SO}(n-1)} \neq \{0\}$. (Hint : Start with a function $f \in V$ such that $f(v_0) \neq 0$.)

i) (1 point) Show that the representation $\mathcal{H}_m(\mathbb{R}^n)$ of $\mathbf{SO}(n)$ is irreducible.

j) (2 points) Show that the representations $\mathcal{H}_m(\mathbb{R}^n)$ and $\mathcal{H}_{m'}(\mathbb{R}^n)$ of $\mathbf{SO}(n)$ are not equivalent if $m \neq m'$. (Hint : Compare the dimensions.)

k) (3 points, extra credit) If $m \geq 2$, show that $\mathcal{H}_m(\mathbb{R}^n)$ is spanned by the functions $(z_1, \dots, z_n) \mapsto (a_1 z_1 + \dots a_n z_n)^m$, with $a_1, \dots, a_n \in \mathbb{C}$ such that $a_1^2 + \dots + a_n^2 = 0$.

Solution.

a) If $f \in V_m(\mathbb{R}^n)$, then we can write $f = \sum_{k=0}^m x_1^k f_k$, for uniquely determined $f_k \in V_{m-k}(\mathbb{R}^{n-1})$. As $\mathbf{O}(n-1) \subset \mathbf{O}(n)$ acts trivially on x_1, this gives an $\mathbf{O}(n-1)$-equivariant isomorphism $V_m(\mathbb{R}^n) \simeq \bigoplus_{k=0}^m V_{m-k}(\mathbb{R}^{n-1})$.

b) In this proof, we will use the convention that $V_m(\mathbb{R}^n) = 0$ if $m < 0$. Fix n and m. By 2(f) and 2(g), we have an $\mathbf{O}(n)$-equivariant isomorphism $V_m(\mathbb{R}^n) \simeq \mathcal{H}_m(\mathbb{R}^n) \oplus V_{m-2}(\mathbb{R}^n)$. Using (a), we deduce from this an $\mathbf{O}(m-1)$-equivariant isomorphism

$$V_m(\mathbb{R}^n) \simeq \mathcal{H}_m(\mathbb{R}^n) \oplus \bigoplus_{k=0}^{m-2} V_{m-2-k}(\mathbb{R}^{n-1}).$$

On the other hand, applying (a) to $V_m(\mathbb{R}^n)$ gives an $\mathbf{O}(n-1)$-equivariant isomorphism $V_m(\mathbb{R}^n) \simeq \bigoplus_{k=0}^m V_{m-k}(\mathbb{R}^{n-1})$. Using 2(f) or 2(g) on each summand, we get an $\mathbf{O}(n-1)$-equivariant isomorphism

$$\begin{aligned} V_m(\mathbb{R}^n) &\simeq \bigoplus_{k=0}^{m} (\mathcal{H}_{m-k}(\mathbb{R}^{n-1}) \oplus V_{m-2-k}(\mathbb{R}^{n-1})) \\ &= \left(\bigoplus_{k=0}^{m} \mathcal{H}_{m-k}(\mathbb{R}^{n-1}) \right) \oplus \left(\bigoplus_{k=0}^{m-2} V_{m-2-k}(\mathbb{R}^{n-1}) \right). \end{aligned}$$

Define representations V_1, V_2 and V_3 of $\mathbf{O}(n-1)$ by $V_1 = \mathcal{H}_m(\mathbb{R}^n)$, $V_2 = \bigoplus_{k=0}^m \mathcal{H}_{m-k}(\mathbb{R}^{n-1})$ and $V_3 = \bigoplus_{k=0}^{m-2} V_{m-2-k}(\mathbb{R}^{n-1})$. We have just seen that $V_1 \oplus V_3 \simeq V_2 \oplus V_3$, so $\chi_{V_1} + \chi_{V_3} = \chi_{V_2} + \chi_{V_3}$, so $\chi_{V_1} = \chi_{V_2}$. By corollary IV.5.10 of the notes, this implies that V_1 and V_2 are equivalent.

c) Note that $\mathbf{SO}(2)$ is a commutative group (it is isomorphic to S^1), so its irreducible representations are all 1-dimensional. On the other hand, by 2(f) and 1(a), we have

$$\dim \mathcal{H}_m(\mathbb{R}^2) = \dim V_m(\mathbb{R}^2) - \dim V_{m-2}(\mathbb{R}^2) = \binom{m+1}{m} - \binom{m-1}{m-2} = 2 > 1$$

if $m \geq 2$, so $\mathcal{H}_m(\mathbb{R}^n)$ cannot be an irreducible representation of $\mathbf{SO}(2)$.

If $m = 2$, then a basis of $\mathcal{H}_m(\mathbb{R}^2)$ is given by the functions $x_1^2 - x_2^2$ and $x_1 x_2$, and they both span lines that are stable by the action of $\mathbf{O}(2)$, so $\mathcal{H}_2(\mathbb{R}^2)$ is not an irreducible representation of $\mathbf{O}(2)$.

Suppose that $m \geq 3$. As $\dim \mathcal{H}_m(\mathbb{R}^2) = 2$, if $\mathcal{H}_m(\mathbb{R}^2)$ is not an irreducible representation of $\mathbf{O}(2)$, we must have a nonzero $f \in \mathcal{H}_m(\mathbb{R}^2)$ such that $L_x f \in \mathbb{C} f$ for every $x \in \mathbf{O}(2)$. We identify $\mathbb{R}^2$ with the complex plane $\mathbb{C}$ in the usual way. Then $f(z)$, for $z \in \mathbb{C}$, can be written as $f(z) = \sum_{r=0}^m a_r z^r \overline{z}^{m-r}$, with $a_0, \dots, a_m \in \mathbb{C}$. The action of $\mathbf{SO}(2)$ becomes the action of S^1 on $\mathbb{C}$ by multiplication, and the action of $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ corresponds to complex conjugation. By the assumption on f, for every" $u \in S^1$, the function $f(uz) = \sum_{r=0}^m a_r u^{2r-m} z^r \overline{z}^{m-r}$ is a multiple of f. This is only possible if there exists $r \in \{0, \dots, m\}$ such that $a_s = 0$ for $s \neq r$. So we may assume that $f(z) = z^r \overline{z}^{m-r}$. The function $f(\overline{z}) = z^{m-r} \overline{z}^r$ is also a multiple of f, so we must have $m = 2r$ and $f = |x|^m$. Then $\Delta f = m(m-1)|x|^{m-2}$, which contradicts the fact that $\Delta f = 0$.

d) Let $f \in V_m(\mathbb{R}^n)^{\mathbf{SO}(n)}$. As f is invariant by $\mathbf{SO}(n)$, it is constant on the sphere with center 0, so $f(z) = f(\|z\| v_0)$ for every $z \in \mathbb{R}^n$. As f is homogeneous of degree m, we get that $f(z) = \|z\|^m f(v_0)$, for every $z \in \mathbb{R}^n$. So f is a polynomial if and only if m is even. Also, we check easily that $\Delta f = m(m-1)|x|^{-2} f$, so $f \in \mathcal{H}_m(\mathbb{R}^n)$ if and only if $f = 0$ or $m = 0$.

e) By (b), we have a $\mathbf{SO}(n-1)$-equivariant isomorphism $\mathcal{H}_m(\mathbb{R}^n) \simeq \bigoplus_{k=0}^m Hf_{m-k}(\mathbb{R}^{n-1})$. So, by (d), we get

$$\mathcal{H}_m(\mathbb{R}^n)^{\mathbf{SO}(n-1)} \simeq \mathcal{H}_0(\mathbb{R}^{n-1})^{\mathbf{SO}(n-1)} \simeq \mathbb{C}.$$

f) Let us denote the map $\mathbf{SO}(n) \to S$, $x \mapsto x \cdot v_0$ by φ. First, this map is clearly continuous, and it is surjective because $\mathbf{SO}(n)$ acts transitively on S. (If we have $v_1, v_1' \in S$, we want to find $x \in \mathbf{SO}(n)$ such that $x \cdot v_1 = v_1'$. We can complete v_1 and v_1' to two orthonormal bases $(v_1, \dots, v_n)$ and $(v_1', \dots, v_n')$ of $\mathbb{R}^n$. The change of basis matrix between these two bases is in $\mathbf{O}(n)$. If it is in $\mathbf{SO}(n)$, we are done. Otherwise, the change of basis matrix between $(v_1, \dots, v_n)$ and $(v_1', \dots, v_{n-1}', -v_n')$ will be in $\mathbf{SO}(n)$.)

Also, the stabilizer of v_0 in $\mathbf{SO}(n)$ is the subgroup of $\mathbf{SO}(n)$ whose elements have $\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ as their first column, and we see easily that this is $\mathbf{SO}(n-1)$. So φ induces a continuous bijective map $\mathbf{SO}(n)/\mathbf{SO}(n-1) \overset{\sim}{\to} S$. As $\mathbf{SO}(n)/\mathbf{SO}(n-1)$ is compact, this map is a homeomorphism.

g) Define a regular Borel measure ν on S by

$$\nu(E) = c\lambda(\{tx,\ t \in [0,1],\ x \in E\}).$$

Define a linear functional $I : \mathcal{C}(\mathbf{SO}(n)) \to \mathbb{C}$ by

$$I(f) = \int_S f^{\mathbf{SO}(n-1)}(s) d\nu(s),$$

where

$$f^{\mathbf{SO}(n-1)}(x) = \int_{\mathbf{SO}(n-1)} f(xy)dy$$

(we are using the normalized Haar measure on $\mathbf{SO}(n-1)$) for every $s \in \mathbf{SO}(n)$; the function $f^{\mathbf{SO}(n-1)}$ is right invariant by $\mathbf{SO}(n-1)$, hence can be identified to a function on S by (f).

This is a positive functional on $\mathcal{C}(\mathbf{SO}(n))$, so it comes from a regular Borel measure on $\mathbf{SO}(n)$, say ρ. We want to show that ρ is the normalized Haar measure on $\mathbf{SO}(n)$.

Note that, if f is the constant function 1, then $I(f) = 1$. So, to show that ρ is the normalized Haar measure on $\mathbf{SO}(n)$, it suffices to show that it is left invariant. Let $f \in \mathcal{C}(\mathbf{SO}(n))$ and $y \in \mathbf{SO}(n)$. Then it follows immediately from the definition that $(L_y f)^{\mathbf{SO}(n-1)} = L_y(f^{\mathbf{SO}(n-1)})$, so we only need to show that the measure ν on S is left invariant by the action of $\mathbf{SO}(n)$. But this follows immediately from the fact that Lebesgue measure λ is left invariant by the action of $\mathbf{SO}(n)$ (which we can see using the change of variables formula).

h) Let V be as in the question. As $\mathbf{SO}(n)$ acts transitively on S and V is stable by $\mathbf{SO}(n)$, we can find $f \in V$ such that $f(v_0) \neq 0$. Let $(f_1, \dots, f_r)$ be a basis of V. As V is stable by $\mathbf{SO}(n)$ and as the action of $\mathbf{SO}(n)$ on V is continuous, we can find continuous functions $c_1, \dots, c_r : \mathbf{SO}(n) \to \mathbb{C}$ such that, for every $x \in \mathbf{SO}(n)$ and every $s \in S$, we have $f(x \cdot s) = \sum_{i=1}^r c_i(x) f_i(s)$. Define $\widetilde{f} : S \to \mathbb{C}$ by

$$\widetilde{f}(s) = \int_{\mathbf{SO}(n-1)} f(x \cdot s)dx.$$

Then $\widetilde{f} = \sum_{i=1}^r \left(\int_{\mathbf{SO}(n-1)} c_i(x)dx \right) f_i$, so $\widetilde{f} \in V$. Also, $\widetilde{f}$ is $\mathbf{SO}(n-1)$-invariant by construction. Finally, as $x \cdot v_0 = v_0$ for every $x \in \mathbf{SO}(n-1)$, we have $\widetilde{f}(v_0) = f(v_0) \neq 0$, so $\widetilde{f} \neq 0$.

i) By 2(h), restriction from $\mathbb{R}^n$ to S is injective on $\mathcal{H}_m(\mathbb{R}^n)$, so $\mathcal{H}_m(\mathbb{R}^n)$ is irreducible as a representation of $\mathbf{SO}(n)$ if and only $H_m = \mathcal{H}_m(\mathbb{R}^n)_{|S} \subset \mathcal{C}(S)$ is irreducible as a representation of $\mathbf{SO}(n)$. As $\mathbf{SO}(n)$ is compact, if H_m is not irreducible, then we can write $H_m = V \oplus V'$ with V and V' nonzero $\mathbf{SO}(n)$-invariant subspaces of H_m. By (h), this implies that $\dim(H_m^{\mathbf{SO}(n)}) \geq 2$ and contradicts (d). So H_m is irreducible.

j) Let $d_m = \dim \mathcal{H}_m(\mathbb{R}^n)$. We will show that $d_{m+1} > d_m$ for every $m \in \mathbb{Z}_{\geq 0}$, which implies that $d_m \neq d_{m'}$ if $m \neq m'$, hence that $\mathcal{H}_m(\mathbb{R}^n)$ and $\mathcal{H}_{m'}(\mathbb{R}^n)$ are not equivalent.

If $m \leq 1$, then, by 1(a) and 2(e), we have $d_m = \dim V_m(\mathbb{R}^n) = \binom{m+n-1}{m}$. If $m \geq 2$, then, by 1(a) and 2(f), we have

$$\begin{aligned} d_m &= \dim V_m(\mathbb{R}^n) - \dim V_{m-2}(\mathbb{R}^n) \\ &= \binom{m+n-1}{m} - \binom{m+n-3}{m-2} \\ &= \frac{(m+n-3)!}{(m-2)!(n-1)!} \left(\frac{(m+n-1)(m+n-2)}{m(m-1)} - 1 \right) \\ &= \frac{(2m+n-2)(m+n-3)!}{m!(n-2)!}. \end{aligned}$$

In particular, $d_0 = 1$, $d_1 = n$ and $d_2 = 2n-1$, so $d_2 > d_1 > d_0$. Let $m \geq 2$. Then

$$\begin{aligned}
d_{m+1} - d_m &= \frac{(2(m+1)+n-2)(m+1+n-3)!}{(m+1)!(n-2)!} - \frac{(2m+n-2)(m+n-3)!}{m!(n-2)!} \\
&= \frac{(m+n-3)!}{m!(n-2)!}\left(\frac{(2m+n)(m+n-2)}{m+1} - (2m+n-2)\right) \\
&= \frac{(m+n-3)!}{m!(n-2)!}\frac{(2m+n)(m+n-2)-(2m+n-2)(m+1)}{m+1} \\
&> 0
\end{aligned}$$

(because $m+n-2 \geq m+1$ and $2m+n > 2m+n-2$).

k) Let $a_1, \ldots, a_n \in \mathbb{C}$, and consider $f = (a_1x_1 + \ldots + a_nx_n)^m \in V_m(\mathbb{R}^n)$. Then $\Delta f = m(m-1)(a_1^2 + \ldots + a_n^2)(a_1x_1 + \ldots + a_nx_n)^{m-2}$, so $f \in \mathcal{H}_m(\mathbb{R}^n)$ if and only if $a_1^2 + \ldots + a_n^2 = 0$. Let

$$W = \operatorname{Span}\{(a_1x_1 + \ldots + a_nx_n)^m,\ a_1, \ldots, a_n \in \mathbb{C}, a_1^2 + \ldots + a_n^2 = 0\} \subset \mathcal{H}_m(\mathbb{R}^n).$$

As $W \neq 0$ and $\mathcal{H}_m(\mathbb{R}^n)$ is irreducible as a representation of $\mathbf{SO}(n)$, to show that $W = \mathcal{H}_m(\mathbb{R}^n)$, it suffices to show that W is invariant by $\mathbf{SO}(m)$. Let $x \in \mathbf{SO}(n)$ and $a_1, \ldots, a_n \in \mathbb{C}$, and let $f = (a_1x_1 + \ldots + a_nx_n)^m$. Write $(b_1, \ldots, b_n) = (a_1, \ldots, a_n)x^T$ (we see $(a_1, \ldots, a_n)$ as a row vector). Then $b_1^2 + \ldots + b_n^2 = 0$ because $x \in \mathbf{O}(n)$, and $L_x f = (b_1x_1 + \ldots + b_nx_n)^m$, so $L_x f \in W$.

□

5. We keep the notation of problems 1,2,3,4, and we assume that $n \geq 3$.

a) (2 points) Show that the space $\sum_{m \geq 0} \mathcal{H}_m(\mathbb{R}^n)_{|S}$ is dense in $L^2(S)$ and that the sum is direct.

b) (1 point) Show that the subspaces $\mathcal{H}_m(\mathbb{R}^n)_{|S}$ and $\mathcal{H}_{m'}(\mathbb{R}^n)_{|S}$ of $L^2(S)$ are orthogonal (for the L^2 inner form) if $m \neq m'$.

c) (1 point) Show that every irreducible unitary representation of $\mathbf{SO}(n)$ having a nonzero $\mathbf{SO}(n-1)$-invariant vector is isomorphic to one of the $\mathcal{H}_m(\mathbb{R}^n)$.

Solution.

a) By 2(i), we have

$$\sum_{m \geq 0} \mathcal{H}_m(\mathbb{R}^n)_{|S} = \sum_{m \geq 0} V_m(\mathbb{R}^n)_{|S},$$

an the right hand side is dense in $\mathcal{C}(S)$ (hence in $L^2(S)$) by the Stone-Weierstrass theorem. Also, we have seen in the proof of 2(h) that the spaces $\mathcal{H}_m(\mathbb{R}^n)_{|S}$ are pairwise orthogonal in $L^2(S)$, so they are in direct sum.

b) See (a).

c) Let V be an irreducible unitary representation of $\mathbf{SO}(n)$ such that $V^{\mathbf{SO}(n-1)} \neq 0$. By theorem V.3.2.4 of the notes, V is a subrepresentation of $L^2(S)$. But we have seen that

$$L^2(S) = \overline{\bigoplus_{m \geq 0} \mathcal{H}_m(\mathbb{R}^n)_{|S}}$$

and that all these summands are irreducible, so V is isomorphic to one of them.

□

6. We keep the notation of problems 1,2,3,4,5. We say that a function $\varphi \in \mathcal{C}(S)$ is *zonal* if it is left invariant by $\mathbf{SO}(n-1)$. (As $S = \mathbf{SO}(n)/\mathbf{SO}(n-1)$, we can also see the function φ as a bi-invariant function on $\mathbf{SO}(n)$.) Suppose that $n \geq 3$.

   a) (2 point) Show that $\varphi \in \mathcal{C}(S)$ is zonal if and only if there exists a continuous function $f : [-1,1] \to \mathbb{C}$ such that, for every $z = (z_1, \dots, z_n) \in S$, we have $\varphi(z) = f(z_1)$.

   b) (3 points, extra credit) Show that there exists $c \in \mathbb{R}_{>0}$ such that, for every zonal $\varphi \in \mathcal{C}(S)$, if we define $f : [-1,1] \to \mathbb{C}$ as in (a), then

   $$\int_S \varphi(z) d\mu(z) = c \int_{-1}^{1} f(t)(1-t^2)^{(n-3)/2} dt.$$

   (Hint : You can try using spherical coordinates, as in `https://en.wikipedia.org/wiki/N-sphere#Spherical_coordinates`.)

   c) (1 point) Let $m \geq 0$. If $t \in S$, let f_t be the unique element of $\mathcal{H}_m(\mathbb{R}^n)$ such that, for every $g \in \mathcal{H}_m(\mathbb{R}^n)$, we have $\langle g, f_t \rangle = g(t)$. (Note that we are using the inner form of problem 2.)

   Show that the function $Z_m = f_{v_0 | S}$ (where $v_0 = (1, 0, \dots, 0)$) is a zonal function.

   d) (2 points) Let $f_m : [-1,1] \to \mathbb{C}$ be the continuous function corresponding to Z_m as in question (a). Show that f_m is a polynomial function of degree $\leq m$.

   e) (1 point) If $m \neq m'$, show that $\int_{-1}^{1} f_m(t) \overline{f_{m'}(t)} (1-t^2)^{(n-3)/2} dt = 0$.

   f) (2 points) Show that the degree of f_m is m.

   g) (2 points) Show that $x \mapsto \frac{1}{Z_m(v_0)} Z_m(x \cdot v_0)$ is a spherical function on $\mathbf{SO}(n)$, and that every spherical function is of this form.

   The polynomials $\frac{1}{f_m(1)} f_m$ are called Gegenbauer polynomials (and also Legendre polynomials if $n = 3$).

   We will now give a different formula for the spherical functions.

   h) (1 point, extra credit) Consider the function $f_m \in V_m(\mathbb{R}^n)$ defined by $f_m(z_1, \dots, z_n) = (z_1 + i z_2)^m$. Show that $f_m \in \mathcal{H}_m(\mathbb{R}^n)$.

   i) (3 points, extra credit) Define a function $\psi_m : S \to \mathbb{C}$ by $\psi_m(z) = \int_{\mathbf{SO}(n-1)} f_m(k \cdot z) dk$. Show that ψ_m is left invariant by $\mathbf{SO}(n-1)$, that $\psi_m \in \mathcal{H}_m(\mathbb{R}^n)_{|S}$ and that $\psi_m(v_0) = 1$.

   j) (1 point, extra credit) Show that every spherical function on $\mathbf{SO}(n)$ is of the form $x \mapsto \psi_m(x \cdot v_0)$, for a unique $m \geq 0$.

   We can calculate the integral defining ψ_m, and we get

   $$\psi_m(\cos\varphi, z_2, \dots, z_n) = \frac{\Gamma(\frac{n-1}{2})}{\sqrt{\pi}\Gamma(\frac{n-2}{2})} \int_0^{\pi} (\cos\varphi + i \sin\varphi \cos\theta)^m \sin^{n-3}\theta d\theta.$$

   Solution.

   a) If there exists a continuous function $f : [-1,1] \to \mathbb{C}$ such that $\varphi(z_1, \dots, z_n) = f(z_1)$, then φ is clearly zonal.

   Conversely, suppose that φ is zonal, and define $f : [-1,1] \to \mathbb{C}$ by

   $$(z_1) = \varphi(z_1, 0, \dots, 0, \sqrt{1 - z_1^2}).$$

   Then f is clearly continuous. Let $s = (z_1, \dots, z_n) \in S$. Then there exists $x \in \mathbf{SO}(n-1)$ such that $x \cdot s = (z_1, 0, \dots, 0, \sqrt{1 - z_1^2})$ (we are using the fact that $\mathbf{SO}(n-1)$ acts transitively on any sphere in $\mathbb{R}^{n-1}$). As φ is zonal, we have $\varphi(s) = f(z_1)$.

b) We use spherical coordinates on $\mathbb{R}^n$. That is, given $(z_1,\dots,z_n)\in\mathbb{R}^n$, we write

$$\begin{aligned}
z_1 &= r\cos\phi_1\\
z_2 &= r\sin\phi_1\cos\phi_2\\
&\dots\\
z_{n-1} &= r\sin\phi_1\sin\phi_2\dots\sin\phi_{n-2}\cos\phi_{n-1}\\
z_n &= r\sin\phi_1\sin\phi_2\dots\sin\phi_{n-2}\sin\phi_{n-1},
\end{aligned}$$

with $r=\sqrt{z_1^2+\dots+z_n^2}\in\mathbb{R}_{\geq 0}$, $\phi_1,\dots,\phi_{n-2}\in[0,\pi]$ and $\phi_{n-1}\in[0,2\pi)$ ($\phi_1,\dots,\phi_{n-1}$ are not uniquely determined in general, but they are if for example $z_1,\dots,z_n$ are all nonzero). If dz is Lebesgue measure on $\mathbb{R}^n$, then we have

$$dz = r^{n-1}\sin^{n-2}\phi_1\sin^{n-3}\phi_2\dots\sin\phi_{n-2}drd\phi_1\dots d\phi_{n-1}.$$

Let $\varphi\in\mathcal{C}(S)$ be zonal. Up to a positive real constant, $\int_S\varphi(s)d\mu(s)$ is equal to $\int_{B-\{0\}}\psi(z)dz$, where B is the closed unit ball and $\psi(z)=\varphi(\|z\|^{-1}z)$ for $z\neq 0$. This is equal to teh product of $\int_0^1 r^{n-1}dr$ (another positive real constant) and of

$$\int_0^\pi\dots\int_0^\pi\int_0^{2\pi}\varphi(\cos\phi_1,\sin\phi_1\cos\phi_2,\dots,\sin\phi_1\dots\sin\phi_{n-2}\sin\phi_{n-1})$$
$$\sin^{n-2}\phi_1\sin^{n-3}\phi_2\dots\sin\phi_{n-2}d\phi_1\dots d\phi_{n-1}.$$

As φ is zonal, the big integral above is equal to

$$\int_0^\pi\dots\int_0^\pi\int_0^{2\pi} f(\cos\phi_1)\sin^{n-2}\phi_1\sin^{n-3}\phi_2\dots\sin\phi_{n-2}d\phi_1\dots d\phi_{n-1}.$$

Up to the constant

$$\int_0^\pi\dots\int_0^\pi\int_0^{2\pi}\sin^{n-3}\phi_2\dots\sin\phi_{n-2}d\phi_2\dots d\phi_{n-1}$$

(which has to be positive because it calculates the integral of the constant function 1 up to a positive constant), this is equal to

$$\int_0^\pi f(\cos\phi_1)\sin^{n-2}\phi_1 d\phi_1.$$

Finally, we use the change of variable $t=\cos\phi_1$. We have $dr=-\sin\phi_1 d\phi_1$, so

$$\begin{aligned}
\int_0^\pi f(\cos\phi_1)\sin^{n-2}\phi_1 d\phi_1 &= \int_1^{-1} f(t)(-\sqrt{1-t^2}^{n-3})dt\\
&= \int_{-1}^1 f(1)(1-t^2)^{(n-3)/2}dt.
\end{aligned}$$

c) Let $y\in\mathbf{SO}(n-1)$. By definition of f_{v_0}, we have, for every $g\in\mathcal{H}_m(\mathbb{R}^n)$,

$$\langle g, L_y f_{v_0}\rangle = \langle L_{y^{-1}}g, f_{v_0}\rangle = L_y g(v_0) = g(y^{-1}v_0) = g(v_0) = \langle g, f_{v_0}\rangle$$

(because $\mathbf{SO}(n-1)$ is the stabilizer of v_0 in $\mathbf{SO}(n)$). So $f_{v_0}-L_y f_{v_0}$ is orthogonal to every element of $\mathcal{H}_m(\mathbb{R}^n)$, which implies that $f_{v_0}-L_y f_{v_0}=0$, i.e. that $f_{v_0}=L_y f_{v_0}$. So Z_m is zonal.

d) The function Z_m is the restriction of f_{v_0}, which is an element of $\mathcal{H}_m(\mathbb{R}^n)$, and in particular a polynomial function of degree m. As f_m is defined by $f_m(t) = Z_m(t, 0, \dots, 0, \sqrt{1-t^2})$, we can write

$$f_m(t) = \sum_{k=0}^{m} c_k t^k (1-t^2)^{(m-k)/2},$$

with $c_0, \dots, c_m \in \mathbb{C}$. But we also have $f_m(t) = Z_m(t, 0, \dots, 0, -\sqrt{1-t^2})$, so

$$\sum_{k=0}^{m} c_k t^k (1-t^2)^{(m-k)/2} = \sum_{k=0}^{m} (-1)^{m-k} c_k t^k (1-t^2)^{(m-k)/2}.$$

This forces c_k to be 0 unless $m-k$ is even, and so $f(t)$ is indeed polynomial of degree $\leq m$ in t.

e) The function $s \mapsto Z_m(s)\overline{Z_{m'}(s)}$ is zonal, so we have

$$\int_S Z_m(s)\overline{Z_{m'}(s)} d\mu(s) = c \int_{-1}^{1} f_m(t)\overline{f_{m'}(t)}(1-t^2)^{(n-3)/2} dt$$

by (b). By 5(b), the left hand side is 0.

f) For every $m \geq 0$, the function Z_m is nonzero by definition (and because there exist functions $f \in \mathcal{H}_m(\mathbb{R}^n)$ such that $g(v_0) \neq 0$, see the proof of 4(h)), so $f_m \neq 0$ by 2(h). By (e), the functions $(f_m)_{m \geq 0}$ form an orthogonal family in $L^2([-1,1], (1-t^2)^{(n-3)/2} dt)$, so they also form a linearly independent family. Fix $m \geq 0$. The space P_m of polynomials of degree $\leq m$ is of dimension $m+1$ and contains the linearly independent family $(f_0, \dots, f_m)$, so this family is a basis of P_m. But $f_0, \dots, f_{m-1}$ are of degree $\leq m-1$, so f_m has to be of degree m.

g) By corollary V.7.2 of the notes, if Z is the set of spherical functions on $\mathbf{SO}(n)$, then we have

$$L^2(S) = \widehat{\bigoplus_{\varphi \in Z} V_\varphi}$$

and φ generates $V_\varphi^{\mathbf{SO}(n-1)}$. So, using problem 5, we see that the spherical functions are exactly the generators of the spaces $(\mathcal{H}_m(\mathbb{R}^n)_{|S})^{\mathbf{SO}(n-1)}$ that send v_0 to 1.

For every $m \geq 0$, the function $Z_m \in \mathcal{H}_m(\mathbb{R}^n)_{|S}$ is invariant by $\mathbf{SO}(n-3)$, so it generates the space of $\mathbf{SO}(n-1)$-invariant vectors in $\mathcal{H}_m(\mathbb{R}^n)_{|S}$ and has a multiple which is a spherical functions. Because a spherical function must send v_0 to 1, this multiple is $x \mapsto \frac{1}{Z_m(v_0)} Z_m(x \cdot v_0)$.

h) (Unintentional reuse of notation. Sorry.) This follows from 4(k). It is also easy to prove it directly.

i) This is exactly the same construction as in the proof of 4(h) (with $V = \mathcal{H}_m(\mathbb{R}^n)_{|S}$). The same proof shows that $\psi_m \in \mathcal{H}_m(\mathbb{R}^n)_{|S}$, that ψ_m is left invariant by $\mathbf{SO}(n-1)$ and that $\psi_m(v_0) = f_m(v_0) = 1$.

j) Same idea as in the proof of (g) : we have one spherical function in each $\mathcal{H}_m(\mathbb{R}^n)_{|S}$, and it is the unique $\mathbf{SO}(n-1)$-invariant element of this space that sends v_0 to 1. By (i), the function ψ_m satisfies all the required properties.

□