MAT 449 : Problem Set 10

Due Sunday, December 9

1. Fix a positive integer n. For every $m \in \mathbb{Z}_{\geq 0}$, we denote by $V_m(\mathbb{R}^n)$ the vector space of complex-valued polynomial functions on $\mathbb{R}^n$ that are homogenous of degree m. We define an action of $\mathbf{O}(n)$ on $V_m(\mathbb{R}^n)$ by $(x \cdot f)(v) = f(x^{-1}v)$ if $x \in \mathbf{O}(n)$, $f \in V_m(\mathbb{R}^n)$ and $v \in \mathbb{R}^n$ (in other words, $x \cdot f = L_x f$).

   For $i \in \{1, \dots, n\}$, we denote by ∂_{x_i} the endomorphism $f \mapsto \frac{\partial}{\partial x_i} f$ of $C^\infty(\mathbb{R}^n)$ (the space of smooth functions from $\mathbb{R}^n$ to $\mathbb{C}$), and we set $\Delta = \sum_{i=1}^n (\partial_{x_i})^2$ (this is called the Laplacian operator).

   The space of harmonic polynomials of degree m on $\mathbb{R}^n$ is the space

$$\mathcal{H}_m(\mathbb{R}^n) = \{f \in V_m(\mathbb{R}^n) | \Delta(f) = 0\}.$$

   a) (2 points) Calculate $\dim(V_m(\mathbb{R}^n))$.
   b) (1 point) Show that the action of $\mathbf{O}(n)$ on $V_m(\mathbb{R}^n)$ is a continuous representation.
   c) (2 points) Show that, for every $x \in \mathbf{O}(n)$ and every $f \in C^\infty(\mathbb{R}^n)$, we have $\Delta(L_x f) = L_x(\Delta(f))$.
   d) (1 point) Show that the subspace $\mathcal{H}_m(\mathbb{R}^n)$ of $V_m(\mathbb{R}^n)$ is $\mathbf{O}(n)$-invariant.

2. We keep the notation of problem 1. For $i \in \{1, \dots, n\}$, we denote by x_i the oth coordinate function on $\mathbb{R}^n$.

   a) (1 point) Show that the map $x_i \to \partial_{x_i}$ extends to a unique morphism of $\mathbb{C}$-algebras from $\bigoplus_{m \geq 0} V_m(\mathbb{R}^n)$ (the algebra of complex-valued polynomial functions on $\mathbb{R}^n$) to $\mathrm{End}(\mathcal{C}^\infty(\mathbb{R}^n))$. We will denote this morphism by $f \mapsto \partial_f$.

   If $f, g \in V_m(\mathbb{R}^n)$, we set $\langle f, g \rangle = \partial_{\overline{g}}(f)$. (Note that $\overline{g}$ is still a polynomial function on $\mathbb{R}^n$.)

   b) (3 points) Show that $\langle .,. \rangle$ is an inner form on $V_m(\mathbb{R}^n)$. (Hint : Can you find an orthogonal basis ?)
   c) (2 points) Show that, for every $f \in V_m(\mathbb{R}^n)$ and every $y \in \mathbf{O}(n)$, we have $\partial_f \circ L_y = L_y \circ \partial_{L_{y^T} f}$ in $\mathrm{End}(C^\infty(\mathbb{R}^n))$.
   d) (1 point) Show that the continuous representation of $\mathbf{O}(n)$ on $V_m(\mathbb{R}^n)$ defined in problem 1 is unitary for the inner product $\langle .,. \rangle$.
   e) (1 point) If $m \leq 1$, show that $V_m(\mathbb{R}^n) = \mathcal{H}_m(\mathbb{R}^n)$.
   f) (2 points) If $m \geq 2$, show that $\mathcal{H}_m(\mathbb{R}^n)^\perp = |x|^2 V_{m-2}(\mathbb{R}^n)$, where $|x|^2$ is the function $\sum_{i=1}^n x_i^2 \in V_2(\mathbb{R}^n)$.
   g) (2 points) Show that

$$V_m(\mathbb{R}^n) = \bigoplus_{k=0}^{\lfloor m/2 \rfloor} |x|^{2k} \mathcal{H}_{m-2k}(\mathbb{R}^n),$$

and that this induces a $\mathbf{O}(n)$-equivariant isomorphism

$$V_m(\mathbb{R}^n) = \bigoplus_{k=0}^{\lfloor m/2 \rfloor} \mathcal{H}_{m-2k}(\mathbb{R}^n).$$

h) (2 points) If $S \subset \mathbb{R}^n$ is the unit sphere, show that the map $\bigoplus_{m \geq 0} \mathcal{H}_m(\mathbb{R}^n) \to \mathcal{C}(S)$, $f \mapsto f_{|S}$ is injective.

i) (1 point) Show that, for every $f \in V_m(\mathbb{R}^n)$, there is a unique $g \in \bigoplus_{k=0}^{\lfloor m/2 \rfloor} \mathcal{H}_{m-2k}(\mathbb{R}^n)$ such that $f_{|S} = g_{|S}$.

3. Let $n \geq 2$, and embed $\mathbf{O}(n-1)$ into $\mathbf{O}(n)$ by using the map $x \mapsto \begin{pmatrix} 1 & 0 \\ 0 & x \end{pmatrix}$. Let $G = \mathbf{SO}(n)$, and let K be the image of $\mathbf{SO}(n-1)$ in G by the embedding we just defined.

   a) (2 points) Let A be the subset of $\mathbf{SO}(n)$ consisting of matrices of the form

$$\begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & I_{n-2} & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix},$$

   with $\theta \in \mathbb{R}$.

   Show that A is a subgroup of G and that we have $G = KAK$.

   b) (2 points) Show that (G, K) is a Gelfand pair. (You might want to use the involution θ of G defined by $\theta(x) = JxJ$, where J is the diagonal matrix with diagonal coefficients $-1, 1, \ldots, 1$.)

4. We use the notation of problems 1 and 2, and the embedding $\mathbf{O}(n-1) \subset \mathbf{O}(n)$ defined in problem 3.

   a) (1 point) Show that we have a $\mathbf{O}(n-1)$-equivariant isomorphism

$$V_m(\mathbb{R}^n) \simeq \bigoplus_{k=0}^{m} V_{m-k}(\mathbb{R}^{n-1}).$$

   b) (3 points) Show that we have a $\mathbf{O}(n-1)$-equivariant isomorphism

$$\mathcal{H}_m(\mathbb{R}^n) \simeq \bigoplus_{k=0}^{m} \mathcal{H}_{m-k}(\mathbb{R}^{n-1}).$$

   c) (3 points) If $m \geq 2$, show that $\mathcal{H}_m(\mathbb{R}^2)$ is an irreducible representation of $\mathbf{O}(2)$, but that it is not irreducible as a representation of $\mathbf{SO}(2)$.

   From now on, we assume that $n \geq 3$.

   d) (2 points) If $m \geq 1$, show that $\mathcal{H}_m(\mathbb{R}^n)^{\mathbf{SO}(n)} = \{0\}$.

   e) (1 point) Show that, for every $m \geq 0$, the space $\mathcal{H}_m(\mathbb{R}^n)^{\mathbf{SO}(n-1)}$ is 1-dimensional.

   f) (2 points) Let $S \subset \mathbb{R}^n$ be the unit sphere, and let $v_0 = (1, 0, \ldots, 0) \in S$. Show that the map $\mathbf{SO}(n) \to S$, $x \mapsto x \cdot v_0$ induces a homeomorphism $\mathbf{SO}(n)/\mathbf{SO}(n-1) \overset{\sim}{\to} S$.

   g) (2 points) Show that the measure μ on S defined in 1(f) of problem set 2 (using the normalized Haar measures on $\mathbf{SO}(n)$ and $\mathbf{SO}(n-1)$) is given by $\mu(E) = c\lambda(\{tx,\ t \in [0,1],\ x \in E\})$ for every Borel subset E of S, where λ is Lebesgue measure on $\mathbb{R}^n$ and c^{-1} is the volume of the unit ball (for λ).

h) (2 points) By the previous question, we have the quasi-regular representation of $\mathbf{SO}(n)$ on $L^2(S)$, and it preserves the subspace of continuous functions. If $V \subset \mathcal{C}(S)$ is a nonzero finite-dimensional $\mathbf{SO}(n)$-stable subspace, show that $V^{\mathbf{SO}(n-1)} \neq \{0\}$. (Hint : Start with a function $f \in V$ such that $f(v_0) \neq 0$.)

i) (1 point) Show that the representation $\mathcal{H}_m(\mathbb{R}^n)$ of $\mathbf{SO}(n)$ is irreducible.

j) (2 points) Show that the representations $\mathcal{H}_m(\mathbb{R}^n)$ and $\mathcal{H}_{m'}(\mathbb{R}^n)$ of $\mathbf{SO}(n)$ are not equivalent if $m \neq m'$. (Hint : Compare the dimensions.)

k) (3 points, extra credit) If $m \geq 2$, show that $\mathcal{H}_m(\mathbb{R}^n)$ is spanned by the functions $(z_1, \dots, z_n) \mapsto (a_1 z_1 + \dots a_n z_n)^m$, with $a_1, \dots, a_n \in \mathbb{C}$ such that $a_1^2 + \dots + a_n^2 = 0$.

5. We keep the notation of problems 1,2,3,4, and we assume that $n \geq 3$.

a) (2 points) Show that the space $\sum_{m \geq 0} \mathcal{H}_m(\mathbb{R}^n)_{|S}$ is dense in $L^2(S)$ and that the sum is direct.

b) (1 point) Show that the subspaces $\mathcal{H}_m(\mathbb{R}^n)_{|S}$ and $\mathcal{H}_{m'}(\mathbb{R}^n)_{|S}$ of $L^2(S)$ are orthogonal (for the L^2 inner form) if $m \neq m'$.

c) (1 point) Show that every irreducible unitary representation of $\mathbf{SO}(n)$ having a nonzero $\mathbf{SO}(n-1)$-invariant vector is isomorphic to one of the $\mathcal{H}_m(\mathbb{R}^n)$.

6. We keep the notation of problems 1,2,3,4,5. We say that a function $\varphi \in \mathcal{C}(S)$ is *zonal* if it is left invariant by $\mathbf{SO}(n-1)$. (As $S = \mathbf{SO}(n)/\mathbf{SO}(n-1)$, we can also see the function φ as a bi-invariant function on $\mathbf{SO}(n)$.) Suppose that $n \geq 3$.

a) (2 point) Show that $\varphi \in \mathcal{C}(S)$ is zonal if and only if there exists a continuous function $f : [-1,1] \to \mathbb{C}$ such that, for every $z = (z_1, \dots, z_n) \in S$, we have $\varphi(z) = f(z_1)$.

b) (3 points, extra credit) Show that there exists $c \in \mathbb{R}_{>0}$ such that, for every zonal $\varphi \in \mathcal{C}(S)$, if we define $f : [-1,1] \to \mathbb{C}$ as in (a), then

$$\int_S \varphi(z) d\mu(z) = c \int_{-1}^{1} f(t)(1-t^2)^{(n-3)/2} dt.$$

(Hint : You can try using spherical coordinates, as in `https://en.wikipedia.org/wiki/N-sphere#Spherical_coordinates`.)

c) (1 point) Let $m \geq 0$. If $t \in S$, let f_t be the unique element of $\mathcal{H}_m(\mathbb{R}^n)$ such that, for every $g \in \mathcal{H}_m(\mathbb{R}^n)$, we have $\langle g, f_t \rangle = g(t)$. (Note that we are using the inner form of problem 2.)

Show that the function $Z_m = f_{v_0|S}$ (where $v_0 = (1, 0, \dots, 0)$) is a zonal function.

d) (2 points) Let $f_m : [-1,1] \to \mathbb{C}$ be the continuous function corresponding to Z_m as in question (a). Show that f_m is a polynomial function of degree $\leq m$.

e) (1 point) If $m \neq m'$, show that $\int_{-1}^{1} f_m(t)\overline{f_{m'}(t)}(1-t^2)^{(n-3)/2} dt = 0$.

f) (2 points) Show that the degree of f_m is m.

g) (2 points) Show that $x \mapsto \frac{1}{Z_m(v_0)} Z_m(x \cdot v_0)$ is a spherical function on $\mathbf{SO}(n)$, and that every spherical function is of this form.

The polynomials $\frac{1}{f_m(1)} f_m$ are called Gegenbauer polynomials (and also Legendre polynomials if $n = 3$).

We will now give a different formula for the spherical functions.

h) (1 point, extra credit) Consider the function $f_m \in V_m(\mathbb{R}^n)$ defined by $f_m(z_1, \dots, z_n) = (z_1 + iz_2)^m$. Show that $f_m \in \mathcal{H}_m(\mathbb{R}^n)$.

i) (3 points, extra credit) Define a function $\psi_m : S \to \mathbb{C}$ by $\psi_m(z) = \int_{\mathbf{SO}(n-1)} f_m(k \cdot z)dk$. Show that ψ_m is left invariant by $\mathbf{SO}(n-1)$, that $\psi_m \in \mathcal{H}_m(\mathbb{R}^n)_{|S}$ and that $\psi_m(v_0) = 1$.

j) (1 point, extra credit) Show that every spherical function on $\mathbf{SO}(n)$ is of the form $x \mapsto \psi_m(x \cdot v_0)$, for a unique $m \geq 0$.

We can calculate the integral defining ψ_m, and we get

$$\psi_m(\cos\varphi, z_2, \ldots, z_n) = \frac{\Gamma(\frac{n-1}{2})}{\sqrt{\pi}\Gamma(\frac{n-2}{2})} \int_0^\pi (\cos\varphi + i\sin\varphi\cos\theta)^m \sin^{n-3}\theta d\theta.$$