MAT 449 : Problem Set 1

Due Thursday, September 20

In this problem set, $\mathbb{N}$ is the set of nonnegative integers.

Examples of topological groups

1. Let V be a Banach space over $\mathbb{C}$. (That is, V is a normed $\mathbb{C}$-vector space which is complete for the metric given by its norm.) We denote by $\mathcal{L}(V)$ the space of bounded linear operators from V to itself, equipped with the operator norm. Remember that, if $\|.\|$ is the norm on V, then the operator norm $\|.\|_{op}$ is defined by : for every $f \in \mathcal{L}(V)$,

$$\|f\|_{op} = \inf\{c \in \mathbb{R}_{\geq 0} | \forall v \in V,\ \|f(v)\| \leq c\|v\|\} = \sup_{v \in V,\ \|v\|=1} \|f(v)\|$$

   Let $\mathbf{GL}(V)$ be the group of invertible elements in $\mathcal{L}(V)$, with the topology induced by that of $\mathcal{L}(V)$.

   You can do this problem assuming that V is finite-dimensional. You'll get one point of extra credit for every question where you treat the general case (i.e. without any assumption on the dimension of V).

   a) (1) Show that $\mathbf{GL}(V)$ is an open subset of $\mathcal{L}(V)$.
   b) (2) Show that $\mathbf{GL}(V)$ is a topological group.
   c) (1) Show that $\mathbf{GL}(V)$ is locally compact if and only if V is finite-dimensional.

2. Let $(G_i)_{i \in I}$ be a family of topological groups.

   a) (2) Show that $\prod_{i \in I} G_i$ is a topological group (for the product topology).
   b) (2, extra credit) If all the G_i are locally compact, is $\prod_{i \in I} G_i$ always locally compact ? (Give a proof or a counterexample.)

3. Let $(I, \leq)$ be an ordered set. Consider a family $(X_i)_{i \in I}$ of sets and a family $(u_{ij} : X_i \to X_j)_{i \geq j}$ of maps such that :

   - For every $i \in I$, we have $u_{ii} = \mathrm{id}_{X_i}$;
   - For all $i \geq j \geq k$, we have $u_{ik} = u_{ij} \circ u_{jk}$.

   This is called a *projctive system of sets indexed by the ordered set I*. The *projective limit* of this projective system is the subset $\varprojlim_{i \in I} X_i$ of $\prod_{i \in I} X_i$ defined by :

$$\varprojlim_{i \in I} X_i = \{(x_i)_{i \in I} \in \prod_{i \in I} X_i | \forall i, j \in I \text{ such that } i \geq j,\ u_{ij}(x_i) = x_j\}.$$

   a) (1) If all the X_i are Hausdorff topological spaces and all the u_{ij} are continuous maps, show that $\varprojlim_{i \in I} X_i$ is a closed subset of $\prod_{i \in I} X_i$. From now on, we will always put the induced topology on $\varprojlim_{i \in I} X_i$.

b) (1) If all the X_i are compact Hausdorff topological spaces and all the u_{ij} are continuous maps, show that $\varprojlim_{i\in I} X_i$ is also compact Hausdorff. (Hint : Tychonoff's theorem.)

c) (2) If all the X_i are groups (resp. rings) and all the u_{ij} are morphisms of groups (resp. of rings), show that $\varprojlim_{i\in I} X_i$ is a subgroup (resp. a subgroup) of $\prod_{i\in I} X_i$.

d) (2) If all the X_i are topological groups and all the u_{ij} are continuous group morphisms, show that $\varprojlim_{i\in I} X_i$ is a topological group.

e) (2) Let p be a prime number. Take $I = \mathbb{N}$, with the usual order, $X_n = \mathbb{Z}/p^n\mathbb{Z}$ and $u_{nm} : \mathbb{Z}/p^n\mathbb{Z} \to \mathbb{Z}/p^m\mathbb{Z}$ be the reduction modulo p^m map. Show that $\mathbb{Z}_p := \varprojlim_{i\in I} X_i$ is a ring, and a compact topological group for the addition.

4. Let p be a prime number. We define the *p-adic norm* $|.|_p$ on $\mathbb{Q}$ in the following way :
   - $|0|_p = 0$;
   - if x is a nonzero rational number, we write $x = p^n y$ with y a rational number whose numerator and denominator are prime to p, and we set $|x|_p = p^{-n}$.

   a) (2) Show that we have, for every $x, y \in \mathbb{Q}$:
      - $|x + y|_p \leq \max(|x|_p, |y|_p)$, with equality if $|x|_p \neq |y|_p$;
      - $|xy|_p = |x|_p|y|_p$.

   In particular, the p-adic distance function $d(x, y) = |x - y|_p$ is a metric on $\mathbb{Q}$. We denote by $\mathbb{Q}_p$ the completion of $\mathbb{Q}$ for this metric.

   b) (4) Show that the p-adic norm $|.|_p$, the addition and the multiplication of $\mathbb{Q}$ extend to $\mathbb{Q}_p$ by continuity, that $\mathbb{Q}_p$ is a field (called the *field of p-adic numbers*), and that the statements of (a) extend to $\mathbb{Q}_p$.

   c) (1) Show that the additive group of $\mathbb{Q}_p$ is a topological group.

   d) (1) Calculate the subset $|\mathbb{Q}_p|_p$ of $\mathbb{R}$.

   e) (1) Show that every open ball in $\mathbb{Q}_p$ is also a closed ball, and that every closed ball of positive radius in $\mathbb{Q}_p$ is also an open ball.

   f) (1) Show that $\mathbb{Q}_p$ is totally disconnected (i.e. its only nonempty connected subsets are the singletons) but not discrete.

   g) (1) Show that a series $\sum_{n\geq 0} x_n$ is convergent if and only if $\lim_{n\to+\infty} |x_n|_p = 0$.

   h) (1) If $m \in \mathbb{Z}$ and $(c_n)_{n\geq m}$ is a family of integers, show that the series $\sum_{n\geq m} c_n p^n$ converges in $\mathbb{Q}_p$, and that its p-adic absolute value is $\leq p^{-m}$, with equality if c_m is prime to p.

   i) (2) Let $x \in \mathbb{Q}_p - \{0\}$. Show that there exists a unique $m \in \mathbb{Z}$ and a unique family $(c_n)_{n\geq m}$ of elements of $\{0, 1, \ldots, p-1\}$ such that $x_m \neq 0$ and $x = \sum_{n\geq m} c_n p^n$, and that $|x|_p = p^{-m}$.

   j) (2) Let $B = \{x \in \mathbb{Q}_p | \|x\|_p \leq 1\}$. Show that this is a subring of $\mathbb{Q}_p$, and the closure of $\mathbb{Z}$ in $\mathbb{Q}_p$.

   k) (2) We define a map u from B to $\prod_{n\geq 0} \mathbb{Z}/p^n\mathbb{Z}$ in the following way : If $x \in B$, then, by question (e), we can find a Cauchy sequence $(x_n)_{n\geq 0}$ of elements of $\mathbb{Z}$ converging to x. After replacing it by a subsequence, we may assume that $|x - x_n|_p \leq p^{-n}$ for every n. We set $u(x) = (x_n \mod p^n\mathbb{Z})_{n\geq 0}$.

      Show that u is well-defined, a homeomorphism from B to $\mathbb{Z}_p$, and that it is also a morphism of rings. We will use this to identify B and $\mathbb{Z}_p$.

l) (2) We identify $M_n(\mathbb{Q}_p)$ with $\mathbb{Q}_p^{n^2}$, we put the product topology on it, and we use the induced topology on $\mathbf{GL}_n(\mathbb{Q}_p)$. Show that $\mathbf{GL}_n(\mathbb{Q}_p)$ is a locally compact topological group.

m) (2) Show that $\mathbf{GL}_n(\mathbb{Z}_p)$ is an open compact subgroup of $\mathbf{GL}_n(\mathbb{Q}_p)$. (Hint : Show that $\mathbb{Z}_p^\times$ is closed in $\mathbb{Z}_p$.)

Examples of Haar measures

5. (6) Let G be a topological group. Suppose that we have a homeomorphism of G with an open subset of some $\mathbb{R}^N$ (not necessarily compatible with any groups structures), such that left translations on G are given by affine maps. That is, if we identify G with its image in $\mathbb{R}^N$ (as a topological space only !), then, for every $x \in G$, there is a $N \times N$ matrix $A(x) \in M_N(\mathbb{R})$ and an element $b(x) \in \mathbb{R}^N$ such that, for every $y \in G$, we have $xy = A(x)y + b(x)$.

   Show that $|\det A(x)|^{-1}dx$ is a left Haar measure on G, where dx denotes the Lebesgue measure on $\mathbb{R}^N$. (Hint : The change-of-variable formula. Also, start by proving that x uniquely determines $A(x)$ and $b(x)$, and that $x \mapsto A(x)$ is a morphism of groups from G to $\mathbf{GL}_N(\mathbb{R})$.)

6. In this problem, dx will always be the Lebesgue measure on $\mathbb{R}$.

   a) (1) Show that $\frac{dx}{|x|}$ is a Haar measure on the multiplicative group $\mathbb{R}^\times$.

   b) (1) Show that $\frac{dxdy}{x^2+y^2}$ is a Haar measure on the multiplicative group $\mathbb{C}^\times$, with coordinates $z = x + iy$.

   c) (3) Let dT be the Lebesgue measure on $M_n(\mathbb{R})$. Show that $|\det T|^{-n}dT$ is a left and right Haar measure on $\mathbf{GL}_n(\mathbb{R})$.

   d) (2) Let $G = \{\begin{pmatrix} x & y \\ 0 & z \end{pmatrix} | x, z \in \mathbb{R}^\times, y \in \mathbb{R}\}$. Show that $\frac{dxdydz}{x^2|z|}$ is a left Haar measure on G. Is it a right Haar measure ?

7. (extra credit) Consider the group $G = (\mathbb{Z}/2\mathbb{Z})^{\mathbb{N}}$.

   a) (1) Show that there exists a Haar measure μ on G such that $\mu(G) = 1$.

   b) (2) Show that every open subset of G is a countable union of set of the form $U = V \times (\mathbb{Z}/2\mathbb{Z})^{\mathbb{N}_{\geq n+1}}$, with $n \in \mathbb{N}$ and $V \subset (\mathbb{Z}/2\mathbb{Z})^{\{0,\dots,n\}}$, and that we have $\mu(U) = \frac{|V|}{2^{n+1}}$.

   c) (4) Consider the map $u : G \to [0,1]$ sending $(x_n)_{n\in\mathbb{N}} \in G$ to $\sum_{n\geq 0} x_n 2^{-n-1}$. (We identify $\mathbb{Z}/2\mathbb{Z}$ with $\{0,1\}$ in the defintion of u.) Show that u is measurable and maps μ to Lebesgue measure λ on $[0,1]$. That is, show that, if $B \subset [0,1]$ is a Borel set, then $u^{-1}(B)$ is a Borel set and $\lambda(B) = \mu(u^{-1}(B))$. (Hint : Show that the half-open intervals of the form $[j2^{-k}, (j+1)2^{-k}]$ generate the Borel σ-algebra on $[0,1]$, and calculate their inverse images by u.)

8. For $x \in \mathbb{Q}_p$ and $r \in \mathbb{R}$, write $B(x,r) = \{y \in \mathbb{Q}_p | |x-y|_p \leq r\}$ (the closed ball of center x and radius r). Let λ be the Haar measure on $\mathbb{Q}_p$ such that $\lambda(\mathbb{Z}_p) = 1$.

   a) (1) If $x \in \mathbb{Q}_p$ and $m \in \mathbb{Z}$, show that $\lambda(B(x,p^m)) = p^m$.

   b) (2) For every Borel set $X \subset \mathbb{Q}_p$, show that

$$\lambda(X) = \inf\{\sum_{i\geq 0} p^{m_i} | \exists x_0, x_1, \ldots \in \mathbb{Q}_p \text{ with } X \subset \bigcup_{i\geq 0} B(x_i, p^{m_i})\}.$$