AUTOMORPHIC COHOMOLOGY, ARTHUR'S CONJECTURES AND APPLICATIONS TO G_2

SAM MUNDY

Abstract. We develop some techniques for distinguishing between different automorphic representations of a reductive group G and for locating all occurrences of a given irreducible admissible representation of $G(\mathbb{A}_f)$ in the automorphic cohomology of G. We apply these methods to a specific class of automorphic representations of the split exceptional group G_2 and locate every occurrence of their finite parts in Eisenstein cohomology and, assuming Arthur's conjectures, in cuspidal cohomology as well.

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Introduction

This paper has been written primarily to serve two purposes. One is to provide the automorphic background for another paper [Mun] which will study the Bloch–Kato Selmer group attached to the symmetric cube of a modular eigenform, and we will discuss this briefly below. The other is to explain how explicit computations involving the decomposition of Franke–Schwermer [FS98] of the space of automorphic forms on a reductive group G, along with Arthur's conjectures on the endoscopic classification of representations occurring in the discrete spectrum of G, can give very refined information about the automorphic cohomology of G; even more, we will show that, in certain cases, a study of the Franke–Schwermer decomposition can lead to information about the cuspidality of nontempered representations in the discrete spectrum of G, where Arthur's conjectures alone cannot provide any such information.

In more detail, let G be a reductive group over $\mathbb{Q}$. Let A_G be the maximal split torus in the center of G, so that G has a Langlands decomposition $G = G_0 A_G$. Let $\mathfrak{g}$ be the complexified Lie algebra of G, and $\mathfrak{g}_0$ that of G_0, and let K_∞ be a maximal compact subgroup of the real group $G(\mathbb{R})$. If we denote by $\mathcal{A}(G)$ the space of automorphic forms on G which transform trivially under the connected component $A_G(\mathbb{R})^\circ$ of the identity in $A_G(\mathbb{R})$, then $\mathcal{A}(G)$ is a $G(\mathbb{A}_f) \times (\mathfrak{g}_0, K_\infty)$-module; here $\mathbb{A}_f$ denotes the ring of finite adeles in the full ring $\mathbb{A}$ of adeles of $\mathbb{Q}$.

Given a finite dimensional irreducible representation E of $G(\mathbb{C})$, we can consider the sub-$G(\mathbb{A}_f) \times (\mathfrak{g}_0, K_\infty)$-module $\mathcal{A}_E(G)$ of $\mathcal{A}(G)$; this is defined to be the space of automorphic forms which are killed by a power of the annihilator $\mathcal{J}_E$ of the dual $E^\vee$ of E in the center of the universal enveloping algebra of $\mathfrak{g}$, and which transform trivially under the connected component $A_G(\mathbb{R})^\circ$ of $A_G(\mathbb{R})$. Then we can define the *automorphic cohomology* of G with coefficients in E as the $(\mathfrak{g}_0, K_\infty)$-cohomology space

$$H^*(\mathfrak{g}_0, K_\infty; E) = H^*(\mathfrak{g}_0, K_\infty; \mathcal{A}_E(G) \otimes E).$$

(Often in the literature this space is denoted $H^*(G, E)$, but we choose to write it this way in this paper to emphasize that we are computing it using $(\mathfrak{g}_0, K_\infty)$-cohomology.) This is naturally a $G(\mathbb{A}_f)$-module, which is admissible. In fact, it is naturally isomorphic to the inductive limit over all level subgroups of the cohomology of the local system coming from E on the locally symmetric spaces attached to G; this was a conjecture of Borel which was proven by Franke in [Fra98].

We would like to propose the following problem.

Problem. Given an irreducible admissible $G(\mathbb{A}_f)$-module σ, in exactly which ways can σ be realized as a subquotient of $H^*(\mathfrak{g}_0, K_\infty; E)$, and in which degrees?

We believe that this problem is interesting in its own right, but besides this it does have applications to other questions, including some which are arithmetic in nature; see below for a brief discussion of how the solution to this problem can allow one to make p-adic deformations of the $G(\mathbb{A}_f)$-module σ in certain cases.

One approach to this problem, which is the approach developed in this paper, begins with splitting the automorphic cohomology into the direct sum of two $G(\mathbb{A}_f)$-submodules called, respectively, the *Eisenstein cohomology* and *cuspidal cohomology*:

$$H^*(\mathfrak{g}_0, K_\infty; E) = H^*_{\mathrm{Eis}}(\mathfrak{g}_0, K_\infty; E) \oplus H^*_{\mathrm{cusp}}(\mathfrak{g}_0, K_\infty; E).$$

The Eisenstein cohomology is just the $(\mathfrak{g}_0, K_\infty)$-cohomology of the space, tensored with E, of all Eisenstein series, their residues, and the derivatives of such, in $\mathcal{A}_E(G)$. Franke and Schwermer in [FS98] give an extremely refined direct sum decomposition of this subspace of $\mathcal{A}_E(G)$ in terms of certain classes φ of cuspidal representations of Levi factors.

Thus, to try to solve the problem above, one could try to first find all pieces of this Franke–Schwermer decomposition which contain σ as the finite part of a constituent, show that this list is exhaustive, and then compute the $(\mathfrak{g}_0, K_\infty)$-cohomology of all such pieces. To illustrate this technique, in this paper we carry it out for a certain irreducible admissible $G_2(\mathbb{A}_f)$-module, where G_2 denotes the split exceptional group of that type.

More precisely, let F be a cuspidal holomorphic eigenform with even weight $k \geq 4$ and trivial nebentypus, and let π_F be the unitary cuspidal automorphic representation of $GL_2(\mathbb{A})$ attached to it. In the group G_2, there is a maximal parabolic subgroup, denoted P_α in this paper, whose Levi factor M_α contains the unipotent group attached to the long simple root α of G_2. Then $M_\alpha \cong GL_2$, and we can consider the finite part $\mathcal{L}_\alpha(\pi_F, 1/10)_f$ of the unique irreducible quotient $\mathcal{L}_\alpha(\pi_F, 1/10)$ of the unitary induction of $\pi_F \otimes |\det|^{1/2}$ to $G_2(\mathbb{A})$ along $P_\alpha(\mathbb{A})$. (The 1/10 present in this notation is to signify that the modulus character of $P_\alpha(\mathbb{A})$ to the power 1/10 coincides with the twisting factor $|\det|^{1/2}$.) We make an explicit study of the Franke–Schwermer decomposition for G_2 and solve the Eisenstein-component of the problem above for $G = G_2$ and $\sigma = \mathcal{L}_\alpha(\pi_F, 1/10)_f$, at least when $L(1/2, \pi_F, \mathrm{Sym}^3) = 0$; the theorem (see Theorem 2.3.9) is as follows.

Theorem A. *Let λ_0 be the weight*

$$\lambda_0 = \frac{k-4}{2}(2\alpha + 3\beta),$$

where α is the long simple root for G_2 and β is the short one, and let E_{λ_0} be the representation of $G_2(\mathbb{C})$ of highest weight λ_0. Assume

$$L(1/2, \pi_F, \mathrm{Sym}^3) = 0.$$

Then there is a unique summand isomorphic to the unitary induction

$$\iota_{P_\alpha(\mathbb{A}_f)}^{G_2(\mathbb{A}_f)}(\pi_{F,f} \otimes |\det|^{1/2})$$

in the Eisenstein cohomology of G_2 with coefficients in E_{λ_0},

$$H^*_{\mathrm{Eis}}(\mathfrak{g}_2, K_\infty; E_{\lambda_0}),$$

and all irreducible subquotients of this cohomology which are isomorphic to $\mathcal{L}_\alpha(\pi_F, 1/10)_f$, except perhaps at finitely many finite places, appear in this summand. Moreover, this summand appears in middle degree 4.

We note that if $L(1/2, \pi_F, \mathrm{Sym}^3) \neq 0$ then one can still say a lot, but the story as of now is still incomplete; see Remark 3.5.5. We also remark that there are many modular forms F as in the theorem such that $L(1/2, \pi_F, \mathrm{Sym}^3) = 0$, or even with $\epsilon(1/2, \pi_F, \mathrm{Sym}^3) = -1$. For example, a quick exercise with archimedean root numbers shows that any holomorphic modular eigenform of level 1 has $\epsilon(1/2, \pi_F, \mathrm{Sym}^3) = -1$, hence also $L(1/2, \pi_F, \mathrm{Sym}^3) = 0$.

The second part of our technique to try to solve the problem we posed above must then study the occurrence of σ in cuspidal cohomology space $H^*_{\mathrm{cusp}}(\mathfrak{g}_0, K_\infty; E)$; this space is simply the $(\mathfrak{g}_0, K_\infty)$-cohomology of the subspace of cusp forms in $\mathcal{A}_E(G)$ tensored with E. Our approach to this involves the endoscopic classification of representations occurring in the discrete spectrum of G, and thus we now assume Arthur's conjectures.

Arthur's conjectures give a classification of the representations occurring in the discrete spectrum of G in terms of certain parameters ψ, often called discrete Arthur parameters. One then has a strategy for locating σ in cuspidal cohomology, and it proceeds with the following five steps:

(1) Classify all local archimedean Arthur parameters ψ_∞ whose associated packets Π_{ψ_∞} contain a cohomological representation.
(2) Find all discrete global Arthur parameters ψ whose archimedean component ψ_∞ is of the type described in Step (1) and whose associated packets Π_ψ contain a representation π with finite part π_f isomorphic to σ;
(3) Using Arthur's multiplicity formula, for every π as in Step (2), compute the multiplicity $m_{\mathrm{disc}}(\pi)$ of π in the discrete spectrum.
(4) For every π as in Step (3) with $m_{\mathrm{disc}}(\pi) > 0$, compute the multiplicity $m_{\mathrm{res}}(\pi)$ of π in the residual spectrum. Then $m_{\mathrm{cusp}}(\pi) = m_{\mathrm{disc}}(\pi) - m_{\mathrm{res}}(\pi)$ is the multiplicity of π in the cuspidal spectrum.
(5) For every π as in Step (4) with $m_{\mathrm{cusp}}(\pi) > 0$, compute the $(\mathfrak{g}_0, K_\infty)$-cohomology of $\pi_\infty \otimes E$.

In the case of $G = G_2$, $\sigma = \mathcal{L}_\alpha(\pi_F, 1/10)_f$, and $E = E_{\lambda_0}$ with $\lambda_0 = \frac{k-4}{2}(2\alpha + 3\beta)$ as above, we carry out these five steps. Step (1) is accomplished in Proposition 3.4.1. Following Gan and Gurevich [GG09], we define a parameter ψ_F in Section 3.5 whose associated packet contains representations with finite part $\mathcal{L}_\alpha(\pi_F, 1/10)_f$. We then explain why no other cohomological parameter besides ψ_F can contain a representation with finite part $\mathcal{L}_\alpha(\pi_F, 1/10)_f$, thus accomplishing Step (2). Then Step (3) in this case follows from [GG09] and implies there is a unique discrete automorphic representation Π_F with finite part $\Pi_{F,f}$ isomorphic to $\mathcal{L}_\alpha(\pi_F, 1/10)_f$, and that in fact Π_F occurs in the discrete spectrum with multiplicity one.

For Step (4), we use the techniques developed in this paper around Eisenstein series to show that Π_F is cuspidal if and only if $L(1/2, \pi_F, \mathrm{Sym}^3) = 0$. This is done in Proposition 3.5.3, and involves showing that the only piece of the Franke–Schwermer decomposition for G_2 containing a residual representation with finite part $\mathcal{L}_\alpha(\pi_F, 1/10)_f$ is the one studied above if $L(1/2, \pi_F, \mathrm{Sym}^3) \neq 0$.

Finally, Step (5) is accomplished by computing the representation $\Pi_{F,\infty}$ in terms of Adams–Johnson packets [AJ87], and using well known results on the cohomology of representations in such packets. We obtain the following (see Theorem 3.5.4).

Theorem B. *Notation as above, assume*

$$L(1/2, \pi_F, \mathrm{Sym}^3) = 0.$$

Then under Arthur's conjectures, we have

$$H^i_{\text{cusp}}(\mathfrak{g}_2, K_\infty; E)[\Pi_{F,f}] = \begin{cases} \Pi_{F,f} & \text{if } i = 4 \text{ and } \epsilon(1/2, \pi_F, \text{Sym}^3) = -1, \\ & \text{or if } i = 3, 5 \text{ and } \epsilon(1/2, \pi_F, \text{Sym}^3) = +1; \\ 0 & \text{otherwise.} \end{cases}$$

Here, $\Pi_{F,f} = \mathcal{L}_\alpha(\pi_F, 1/10)_f$ *and the brackets in the formula above denote the* $\Pi_{F,f}$*-isotypic component.*

The dependence of this result on the sign $\epsilon(1/2, \pi_F, \text{Sym}^3)$ of the symmetric cube functional equation comes from Arthur's multiplicity formula. In fact, the local archimedean packet for ψ_F contains two representations, and can be described in terms of the theory of Adams–Johnson [AJ87]. Only one will occur as $\Pi_{F,\infty}$, and which one occurs depends on this sign. Moreover, we determine in this paper not just the cohomology of the representations in this packet, but also the representations themselves. The result is the following (see Theorem 3.3.3).

Theorem C. *The local archimedean Arthur packet* $\Pi_{\psi_{F,\infty}}$*, computed as an Adams–Johnson packet [AJ87], consists of the nontempered Langlands quotient* $\mathcal{L}_\alpha(\pi_F, 1/10)_\infty$*, and the quaternionic discrete series representation of weight* $k/2$ *(in the terminology of Gan–Gross–Savin [GGS02]).*

We remark that the above result has already found use in work of R. Dalal on counting quaternionic G_2-automorphic representations; see [Dal23].

We imagine that the methods of this paper may apply to a number of other situations, especially to classical groups. For example, these methods apply to give the existence, multiplicity, and even the cuspidality of the automorphic representations of $PGSp_4(\mathbb{A})$ generated by the Saito–Kurokawa liftings constructed by Piatetski-Shapiro [Pia83], without constructing them directly via theta correspondence; one can even reprove the description of their infinite components using our method, and one can also locate their finite parts in cohomology precisely. One can then use this cohomological information to find them on the eigenvariety of Urban [Urb11] when they are holomorphic at the infinite place, and to prove the main result of Skinner–Urban [SU06a] on the nontriviality of the Bloch–Kato Selmer groups for the p-adic Galois representations attached to certain modular forms, in the case that those forms are not ordinary at p; the argument for this is sketched in [Urb11, §5.5].

As for applications of the results of this paper themselves, in a sequel paper [Mun], we will construct nontrivial elements in the Bloch–Kato Selmer groups attached to (a suitable twist of) the Galois representation $\text{Sym}^3(\rho_F)$, where ρ_F is the p-adic Galois representation attached to F, under the hypothesis that $\epsilon(1/2, \pi_F, \text{Sym}^3) = -1$, at least assuming Arthur's conjectures. This application follows the method of Skinner and Urban [SU06a; SU06b]. As a preview, we describe the construction now.

First we must locate the $G_2(\mathbb{A}_f)$-representation $\mathcal{L}_\alpha(\pi_F, 1/10)_f$ as a point on Urban's eigenvariety, and the first step to this is to find the *classical multiplicity* of (a critical p-stabilization of) $\mathcal{L}_\alpha(\pi_F, 1/10)_f$ as defined in [Urb11]. The results of this paper do just that.

One can then compute from this the *cuspidal overconvergent multiplicity* of this p-stabilization of $\mathcal{L}_\alpha(\pi_F, 1/10)_f$ and show it is nonzero if the sign $\epsilon(1/2, \pi_F, \mathrm{Sym}^3) = -1$, meaning it lies on Urban's eigenvariety. Then passing to the Galois side one gets a p-adic family of G_2-valued Galois representations specializing to the reducible one attached to $\mathcal{L}_\alpha(\pi_F, 1/10)_f$. Then one can construct the desired nontrivial symmetric cube Selmer class as an extension occurring in the specialization at the point corresponding to $\mathcal{L}_\alpha(\pi_F, 1/10)_f$ of a suitably chosen lattice in this family.

We remark that there are a number of other papers on the automorphic cohomology of G_2; see, for example, [LS93; BG22].

This paper is organized as follows. We begin in Section 1 by reviewing the various structural aspects of the group G_2 which will be used in the sections which follow.

Section 2 then studies Eisenstein cohomology. We begin that section with a very general setup; Subsection 2.1 reviews the Franke–Schwermer decomposition for a general group G, and then Subsection 2.2 studies the cohomology of induced representations for G. These two subsections are in no way original, but instead serve as a place to gather useful general information about Eisenstein cohomology. We hope some readers will find this to be a helpful reference. However, we switch back to the setting of G_2 in Subsection 2.3 and apply the material of the preceding two subsections to determine completely the occurrence of $\mathcal{L}_\alpha(\pi_F, 1/10)_f$ in the Eisenstein cohomology of G_2 with coefficients in E_{λ_0}.

Section 3 then determines the location of $\mathcal{L}_\alpha(\pi_F, 1/10)_f$ in the corresponding cuspidal cohomology space under Arthur's conjectures. In particular, we explain how the combination of the methods of Section 2 along with Arthur's conjectures are enough to describe when $\mathcal{L}_\alpha(\pi_F, 1/10)_f$ occurs as the finite part of a *cuspidal* automorphic representation, rather than just a discrete one.

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Notation and conventions

The following conventions will be used throughout this entire paper.

Groups and Lie algebras. In general, our convention is to use uppercase roman letters to denote groups over $\mathbb{Q}$, such as G, and to use the corresponding lowercase fraktur letters to denote complex Lie algebras. So for example, $\mathfrak{g}$ will always denote the complexified Lie algebra of the $\mathbb{Q}$-group G. The only exceptions to this convention occur in Section 3.1; see that section for the notation used there.

When working with a group G, we will often fix a parabolic subgroup P of G along with a Levi decomposition $P = MN$. In this decomposition, M will always denote the Levi factor and N the unipotent radical. If we have another parabolic subgroup with fixed Levi decomposition, then we use subscripts on the notation for its fixed Levi factor and its unipotent radical to distinguish them from those of P; so if Q is another parabolic subgroup, we will write $Q = M_Q N_Q$ for its Levi decomposition.

For any parabolic subgroup P as above, the notation A_P will denote the maximal $\mathbb{Q}$-split torus in the center of the Levi factor M of P. Then there is a Langlands decomposition $P = M_0 A_P N$. This applies in particular to $P = G$, and so $G = G_0 A_G$.

Now we have the complexified Lie algebras $\mathfrak{g}$, $\mathfrak{g}_0$, $\mathfrak{p}$, $\mathfrak{m}$, $\mathfrak{m}_0$, $\mathfrak{n}$, $\mathfrak{a}_P$ of, respectively, G, G_0 P, M, M_0, N, and A_P. We also write $\mathfrak{p}_0 = \mathfrak{p} \cap \mathfrak{g}_0$ and $\mathfrak{a}_{P,0} = \mathfrak{a}_P \cap \mathfrak{g}_0$. Then there are decompositions

$$\mathfrak{p} = \mathfrak{m}_0 \oplus \mathfrak{a}_P \oplus \mathfrak{n}_P,$$

and

$$\mathfrak{p}_0 = \mathfrak{m}_0 \oplus \mathfrak{a}_{P,0} \oplus \mathfrak{n}_P.$$

We will always write ρ_P for the character $\rho_P : \mathfrak{a}_P \to \mathbb{C}$ given by

$$\rho_P(X) = \mathrm{Tr}(\mathrm{ad}(X)|\mathfrak{n}_P), \qquad X \in \mathfrak{a}_P.$$

Most of the time we will be working with the group G_2, which we introduce in Section 1.1. The objects associated with this group have various pieces of notation attached to them as well, and we refer to that section for those notations.

Points of groups. When v is a place of $\mathbb{Q}$, we write $\mathbb{Q}_v$ for the completion of $\mathbb{Q}$ at v. Then $\mathbb{R} = \mathbb{Q}_\infty$. The group of $\mathbb{Q}_v$-points of any affine algebraic group over $\mathbb{Q}$ is always given the usual topology induced from $\mathbb{Q}_v$.

We write $\mathbb{A}$ for the adeles of $\mathbb{Q}$ and $\mathbb{A}_f$ for the finite adeles. If v is a fixed finite place of $\mathbb{Q}$, then $\mathbb{A}_f^v$ will denote the finite adeles away from v. The groups of $\mathbb{A}$-points, $\mathbb{A}_f$-points, or $\mathbb{A}_f^v$-points of any affine algebraic group over $\mathbb{Q}$ are given their standard topologies.

When a parabolic subgroup $P = MN$ of a group G is fixed as above, we will often consider the associated height function H_P. This is a function

$$H_P : G(\mathbb{A}) \to \mathfrak{a}_{P,0}.$$

To define it, we must fix a maximal compact subgroup $K \subset G(\mathbb{A})$. We assume $K = K_{f,\max} K_\infty$ where K_∞ is a fixed maximal compact subgroup of $G(\mathbb{R})$, where $K_{f,\max} = \prod_{v<\infty} K_v$, and where the groups K_v are maximal compact subgroups of $G(\mathbb{Q}_v)$. We moreover assume K to be in good position with respect to a fixed minimal parabolic subgroupsinside P. In particular, the Iwasawa decomposition holds for $P(\mathbb{A})$ and K.

Write $\langle\cdot,\cdot\rangle$ for the natural pairing

$$\langle\cdot,\cdot\rangle : \mathfrak{a}_{P,0} \times \mathfrak{a}_{P,0}^{\vee} \to \mathbb{C}$$

given by evaluation, where $\mathfrak{a}_{P,0}^{\vee} = \mathrm{Hom}_{\mathbb{C}}(\mathfrak{a}_{P,0}, \mathbb{C})$. Write $X^*(M)$ for the group of algebraic characters of M. Then H_P is defined first on the subgroup $M(\mathbb{A})$ by requiring

$$e^{\langle H_P(m), d\Lambda\rangle} = |\Lambda(m)|, \qquad m \in M(\mathbb{A}),\ \Lambda \in X^*(M),$$

where $d\Lambda$ denotes the restriction to $\mathfrak{a}_{P,0}$ of the differential at the identity of the restriction of Λ to $A_P(\mathbb{R})$, and $|\cdot|$ is the usual adelic absolute value. Then H_P is defined in general by declaring it to be left invariant with respect to $N(\mathbb{A})$ and right invariant with respect to K.

If R is one of the rings $\mathbb{Q}_v$, $\mathbb{A}$, or $\mathbb{A}_f$, we use the notation $\delta_{P(R)}$ to denote the modulus character of $P(R)$, and similarly for other parabolic subgroups.

Automorphic representations. When G is a reductive $\mathbb{Q}$-group, we take the point of view that an "automorphic representation" of $G(\mathbb{A})$ is (among other things) an irreducible object in the category of admissible $G(\mathbb{A}_f) \times (\mathfrak{g}, K_\infty)$-modules, where K_∞ is, like above, a maximal compact subgroup in $G(\mathbb{R})$. We often even view automorphic representations as $G(\mathbb{A}_f)\times(\mathfrak{g}_0, K_\infty)$-modules by restriction. We let $\mathcal{A}(G)$ denote the space of all automorphic forms on $G(\mathbb{A})$ which transform trivially under the connected component $A_G(\mathbb{R})^\circ$ of the identity in $A_G(\mathbb{R})$.

If Π is an automorphic representation of $G(\mathbb{A})$ and v is a place of $\mathbb{Q}$, we will denote by Π_v the local component of Π at v. If v is finite, then this is an irreducible admissible representation of $G(\mathbb{Q}_v)$, and if $v = \infty$, then this is an irreducible admissible $(\mathfrak{g}, K_\infty)$-module. We also let Π_f denote the associated representation of $G(\mathbb{A}_f)$, so that $\Pi \cong \Pi_f \otimes \Pi_\infty$.

If $P = MN$ is a parabolic subgroup of G and π an automorphic representation of $M(\mathbb{A})$, then we denote the nonunitary parabolic induction of π to G along P by $\mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})}(\pi)$, and the unitary parabolic induction by $\iota_{P(\mathbb{A})}^{G(\mathbb{A})}(\pi)$. So

$$\iota_{P(\mathbb{A})}^{G(\mathbb{A})}(\pi) = \mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})}(\pi \otimes \delta_{P(\mathbb{A})}^{1/2}).$$

More generally, if $\lambda \in \mathfrak{a}_P^{\vee}$, we write

$$\iota_{P(\mathbb{A})}^{G(\mathbb{A})}(\pi, \lambda) = \iota_{P(\mathbb{A})}^{G(\mathbb{A})}(\pi \otimes e^{\langle H_P(\cdot), \lambda\rangle}) = \mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})}(\pi \otimes e^{\langle H_P(\cdot), \lambda + \rho_P\rangle})$$

We similarly write $\mathrm{Ind}_{P(\mathbb{A}_f)}^{G(\mathbb{A}_f)}$ and $\iota_{P(\mathbb{A}_f)}^{G(\mathbb{A}_f)}$ for the corresponding functors on smooth admissible representations of $M(\mathbb{A}_f)$, and $\mathrm{Ind}_{P(\mathbb{Q}_v)}^{G(\mathbb{Q}_v)}$ and $\iota_{P(\mathbb{Q}_v)}^{G(\mathbb{Q}_v)}$ for their local analogues.

Duals. We use the symbol $(\cdot)^\vee$ in various ways to denote duality. If $\mathfrak{a}$ is an abelian Lie algebra, we write $\mathfrak{a}^\vee = \mathrm{Hom}_{\mathbb{C}}(\mathfrak{a}, \mathbb{C})$. If R is a complex representation of a group, then $R^\vee$ is the usual dual (contragredient) representation over $\mathbb{C}$. If G is our reductive $\mathbb{Q}$-group, then $G^\vee(\mathbb{C})$ will denote the dual group over the algebraically closed field $\mathbb{C}$.

1. The group G_2

We begin by collecting various facts about the group G_2 itself and consolidating them here for the convenience of the reader. We explain various structural aspects of G_2 involving its root system, its parabolic subgroups, and its real points.

1.1. **Structure of the group G_2.** We define G_2 to be the split simple group over $\mathbb{Q}$ with Dynkin diagram as in Figure 1.1. Fixing a maximal $\mathbb{Q}$-split torus T in G_2, we choose a long simple root α and a short simple root β, as notated in the Dynkin diagram. The group G_2 has trivial center.

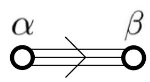

Figure 1.1. The Dynkin diagram of G_2

It is worth noting that G_2 does not have a very nice matricial definition, at least not one that is as nice as for, say, the group Sp_4 of rank 2. Consequently, we will study G_2 from the point of view of its root system, which we discuss now.

The root lattice. The root lattice of G_2 looks as in Figure 1.2. There, the positive Weyl chamber is shaded.

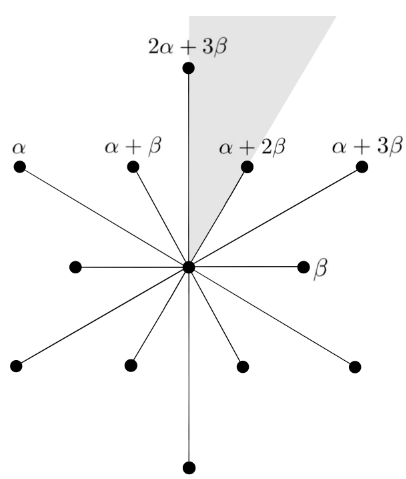

Figure 1.2. The root lattice of G_2

Write Δ for the set of roots of T in G_2, and write Δ^+ for the subset of positive roots. So we have

$$\Delta^+ = \{\alpha, \beta, \alpha+\beta, \alpha+2\beta, \alpha+3\beta, 2\alpha+3\beta\}.$$

One nice feature of G_2 is that the $\mathbb{Z}$-span of the root lattice equals the character group of T:

$$X^*(T) = \mathbb{Z}\alpha \oplus \mathbb{Z}\beta.$$

Since the Cartan matrix of G_2 has determinant 1, an analogous fact holds for the cocharacter group.

Parabolic subgroups. Let B denote the standard Borel subgroup of G_2, defined with respect to Δ^+. We write $B = TU$ for its Levi decomposition. Besides B, there are two other proper standard parabolic subgroups, and they are maximal. Let P_α denote the standard parabolic subgroup whose Levi contains α, and write $P_\alpha = M_\alpha N_\alpha$ for its Levi decomposition. Similarly define $P_\beta = M_\beta N_\beta$.

For $\gamma \in \Delta$ a root, write

$$\mathbf{x}_\gamma : \mathbb{G}_a \to G_2$$

for the corresponding root group homomorphism, where $\mathbb{G}_a$ is the additive group scheme. The Levi subgroups M_α and M_β are both isomorphic to GL_2. We write

$$i_\alpha : GL_2 \to M_\alpha \quad \text{and} \quad i_\beta : GL_2 \to M_\beta$$

for the isomorphisms which send the upper triangular matrix $\left(\begin{smallmatrix} 1 & a \\ 0 & 1 \end{smallmatrix}\right)$ in GL_2 to the element $\mathbf{x}_\alpha(a)$ and $\mathbf{x}_\beta(a)$, respectively. We also often write

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}_\gamma = i_\gamma\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right), \qquad \gamma \in \{\alpha, \beta\}$$

for elements in the image of these maps. We also write

$$\det_\gamma = \det \circ i_\gamma^{-1}, \qquad \gamma \in \{\alpha, \beta\}.$$

The standard representation. The smallest fundamental weight of G_2 is $\alpha + 2\beta$, and the representation attached to it is seven dimensional. We denote it by R_7 and call it the *standard representation* of G_2; it is the representation one naturally gets when defining G_2 through its action on traceless split octonions.

Let V_7 be the space of R_7. This representation contains weight vectors for the seven weights given by the six short roots together with the zero weight; see Figure 1.3.

We then have the following representations of the standard maximal Levi subgroups of G_2:

$$R_7 \circ i_\alpha = \det{}^{-1} \oplus \mathrm{Std}^\vee \oplus 1 \oplus \mathrm{Std} \oplus \det, \tag{1.1.1}$$

where Std is the standard representation of GL_2, and

$$R_7 \circ i_\beta = \mathrm{Std}^\vee \oplus \mathrm{Ad} \oplus \mathrm{Std}, \tag{1.1.2}$$

where $\mathrm{Ad} = \mathrm{Sym}^2(\mathrm{Std}) \otimes \det^{-1}$ is the (three dimensional) adjoint representation of GL_2. These can be seen by looking at strings in the directions of α and β in the weight diagram as in Figure 1.4.

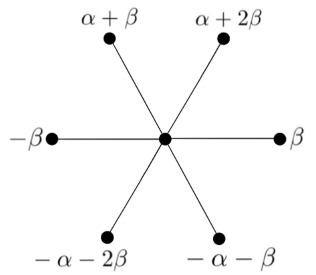

FIGURE 1.3. The weights of R_7

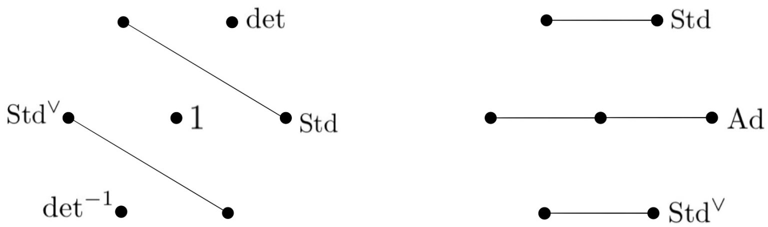

FIGURE 1.4. The standard maximal Levi subgroups of G_2 under R_7

Duality. The group G_2 is self dual, and identifying G_2 with its dual group switches the long and short simple roots. More explicitly, fix identifications $GL_2^\vee \cong GL_2$ and $G_2 \cong G_2^\vee$ so that positive coroots correspond on the dual side to positive roots. Identify M_α and M_β with GL_2 via the maps i_α and i_β introduced above. Then $M_\alpha^\vee$ and $M_\beta^\vee$ are identified with $GL_2^\vee$, and we have commuting diagrams

$$\begin{array}{ccccc} GL_2^\vee & \xrightarrow{\sim} & M_\alpha^\vee & \hookrightarrow & G_2^\vee \\ \downarrow{\scriptstyle\sim} & & \downarrow{\scriptstyle\sim} & & \downarrow{\scriptstyle\sim} \\ GL_2 & \xrightarrow{i_\beta} & M_\beta & \hookrightarrow & G_2, \end{array}$$

and

$$\begin{array}{ccccc} GL_2^\vee & \xrightarrow{\sim} & M_\beta^\vee & \hookrightarrow & G_2^\vee \\ \downarrow{\scriptstyle\sim} & & \downarrow{\scriptstyle\sim} & & \downarrow{\scriptstyle\sim} \\ GL_2 & \xrightarrow{i_\alpha} & M_\alpha & \hookrightarrow & G_2. \end{array}$$

The Weyl group. Let $W = W(T, G_2)$ be the Weyl group of G_2. The group W is isomorphic to the dihedral group D_6 with 12 elements acting naturally on the root lattice.

For $\gamma \in \Delta$, let w_γ be the reflection about the line perpendicular to γ. Then W is generated by the simple reflections w_α and w_β. We use the following notation for amalgamations of such elements: Write $w_{\alpha\beta} = w_\alpha w_\beta$, $w_{\alpha\beta\alpha} = w_\alpha w_\beta w_\alpha$, and so on. Then

$$W = \{1, w_\alpha, w_\beta, w_{\alpha\beta}, w_{\beta\alpha}, w_{\alpha\beta\alpha}, w_{\beta\alpha\beta}, w_{\alpha\beta\alpha\beta}, w_{\beta\alpha\beta\alpha}, w_{\alpha\beta\alpha\beta\alpha}, w_{\beta\alpha\beta\alpha\beta}, w_{-1}\}.$$

The elements above are written minimally in terms of products of the simple reflections w_α and w_β, except for the final element w_{-1}. This is the element that acts by negation on the root lattice, and it is of length 6, equal to both $w_{\alpha\beta\alpha\beta\alpha\beta}$ and $w_{\beta\alpha\beta\alpha\beta\alpha}$.

For $P = MN$ one of the standard parabolic subgroups of G_2, we write as usual

$$W^P = \{w \in W \mid w^{-1}\gamma > 0 \text{ for all positive roots } \gamma \text{ in } M\}$$

for the set of representatives of minimal length for the quotient $W_M\backslash W$, where $W_M = W(T, M)$ is the Weyl group of T in M. Then

$$W^{P_\alpha} = \{1, w_\beta, w_{\beta\alpha}, w_{\beta\alpha\beta}, w_{\beta\alpha\beta\alpha}, w_{\beta\alpha\beta\alpha\beta}\},$$
$$W^{P_\beta} = \{1, w_\alpha, w_{\alpha\beta}, w_{\alpha\beta\alpha}, w_{\alpha\beta\alpha\beta}, w_{\alpha\beta\alpha\beta\alpha}\},$$

and $W^B = W$.

The group $G_2(\mathbb{R})$. The real Lie group $G_2(\mathbb{R})$ is connected and has discrete series. Fix a maximal compact torus T_c in $G_2(\mathbb{R})$. Then T_c is 2-dimensional and lies in a maximal compact subgroup of $G_2(\mathbb{R})$, which we denote by K_∞. Then K_∞ is connected and 6-dimensional. In fact

$$K_\infty \cong (SU(2) \times SU(2))/\mu_2,$$

where $\mu_2 = \{\pm 1\}$ is diagonally embedded in $SU(2) \times SU(2)$.

Let $\mathfrak{t}_c$ be the complexified Lie algebra of T_c, and $\mathfrak{k}$ that of K_∞. We abuse notation and write $\Delta = \Delta(\mathfrak{t}_c, \mathfrak{g}_2)$ for the roots of $\mathfrak{t}_c$ in $\mathfrak{g}_2$. Let $\Delta_c = \Delta(\mathfrak{t}_c, \mathfrak{k})$ denote the set of compact roots. There are four roots in Δ_c consisting of a pair of short roots and a pair of long roots. The short compact roots are orthogonal to the long ones.

Again, abusing notation, choose two simple roots α, β of $\mathfrak{t}_c$ in $\mathfrak{g}_2$ with α long and β short, and choose them so that β is compact. Then

$$\Delta_c = \{\pm\beta, \pm(2\alpha + 3\beta)\}.$$

The compact Weyl group $W_c = W(\mathfrak{t}_c, \mathfrak{k})$ has four elements and is isomorphic to $(\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/2\mathbb{Z})$. In fact, we have

$$W_c = \{1, w_\beta, w_{\alpha\beta\alpha\beta\alpha}, w_{-1}\},$$

and $w_{\alpha\beta\alpha\beta\alpha}$ equals the reflection across the line perpendicular to $2\alpha + 3\beta$. It follows from the theory of Harish-Chandra parameters that the discrete series representations of $G_2(\mathbb{R})$ are parameterized by integral weights in the union of the three chambers between β and $2\alpha + 3\beta$ which are far enough from the walls of those chambers.

2. Eisenstein cohomology

We now introduce some general background on automorphic forms, Eisenstein series and cohomology, which will work for any reductive algebraic group G. We will then specialize to G_2, define the automorphic representation $\mathcal{L}_\alpha(\pi_F, 1/10)$ we will be interested in, and apply the general theory to this representation. We will locate its finite part up to near equivalence in cohomology.

2.1. **Background on the Franke–Schwermer decomposition.** Throughout this subsection we fix a reductive group G over $\mathbb{Q}$. We start with a cuspidal automorphic representation π of $M(\mathbb{A})$, where M is a Levi factor of a parabolic subgroup P in G defined over $\mathbb{Q}$. Let χ be the central character of π, and assume χ is trivial on $A_G(\mathbb{R})^\circ$, where A_G is the maximal $\mathbb{Q}$-split torus in the center of G. So if

$$L^2(M(\mathbb{Q})A_G(\mathbb{R})^\circ\backslash M(\mathbb{A}), \chi)$$

denotes the space of functions on $M(\mathbb{Q})A_G(\mathbb{R})^\circ\backslash M(\mathbb{A})$ which are square integrable modulo center and which transform under the center with respect to χ, then π occurs in the cuspidal spectrum

$$L^2_{\mathrm{cusp}}(M(\mathbb{Q})A_G(\mathbb{R})^\circ\backslash M(\mathbb{A}), \chi) \subset L^2(M(\mathbb{Q})A_G(\mathbb{R})^\circ\backslash M(\mathbb{A}), \chi).$$

Write $d\chi \in \mathfrak{a}^\vee_{P,0}$ for the differential of the restriction of χ to $A_P(\mathbb{R})^\circ/A_G(\mathbb{R})^\circ$. Then we consider the unitary automorphic representation

$$\tilde{\pi} = \pi \otimes e^{-\langle H_P(\cdot), d\chi\rangle}.$$

(See the section on notation and conventions in the introduction.) If π is realized on a space of functions

$$V_\pi \subset L^2_{\mathrm{cusp}}(M(\mathbb{Q})A_G(\mathbb{R})^\circ\backslash M(\mathbb{A}), \chi),$$

then $\tilde{\pi}$ is realized on the space

$$V_{\tilde{\pi}} = \{e^{-\langle H_P(\cdot), d\chi_\pi\rangle} f \mid f \in V_\pi\},$$

which is a subspace of $L^2_{\mathrm{cusp}}(M(\mathbb{Q})A_P(\mathbb{R})^\circ\backslash M(\mathbb{A}))$.

Let $W_{P,\tilde{\pi}}$ be the space of smooth, K-finite, $\mathbb{C}$-valued functions ϕ on

$$M(\mathbb{Q})N(\mathbb{A})A_P(\mathbb{R})^\circ\backslash G(\mathbb{A})$$

such that, for all $g \in G(\mathbb{A})$, the function

$$m \mapsto \phi(mg), \qquad m \in M(\mathbb{A}),$$

lies in the $\tilde{\pi}$-isotypic subspace

$$L^2_{\mathrm{cusp}}(M(\mathbb{Q})A_P(\mathbb{R})^\circ\backslash M(\mathbb{A}))[\tilde{\pi}].$$

The space $W_{P,\tilde{\pi}}$ lets us build Eisenstein series. In fact, let $\phi \in W_{P,\tilde{\pi}}$. We define, for $\lambda \in \mathfrak{a}^\vee_{P,0}$ and $g \in G(\mathbb{A})$, the Eisenstein series $E(\phi, \lambda)$ by

$$E(\phi, \lambda)(g) = \sum_{\gamma \in P(\mathbb{Q})\backslash G(\mathbb{Q})} \phi(\gamma g) e^{\langle H_P(g), \lambda + \rho_P\rangle}.$$

This series only converges for λ sufficiently far inside the positive Weyl chamber, but it defines a holomorphic function there in the variable λ which continues meromorphically to all of $\mathfrak{a}^\vee_{P,0}$; see [Lan76], [MW95], or alternatively [BL20], where a different and much simpler proof is given.

Now let E be a complex, irreducible, finite dimensional representation of $G(\mathbb{C})$. Then the annihilator of the dual $E^\vee$ of E in the center of the universal enveloping algebra of $\mathfrak{g}$ is an ideal, and we denote it by $\mathcal{J}_E$. Denote by $\mathcal{A}_E(G)$ the space of automorphic forms on $G(\mathbb{A})$ which are annihilated by a power of $\mathcal{J}_E$, and which transform trivially under $A_G(\mathbb{R})^\circ$. The forms in $\mathcal{A}_E(G)$

are the ones that can possibly contribute to the cohomology with respect to E, as we will discuss later.

Given two parabolic subgroups of G defined over $\mathbb{Q}$, we say that they are *associate* if their Levi factors are conjugate by an element of $G(\mathbb{Q})$. Let $\mathcal{C}$ be the finite set of equivalence classes for this relation. We write $[P]$ for the equivalence in $\mathcal{C}$ represented by a parabolic subgroup P.

Now given a parabolic subgroup Q of G defined over $\mathbb{Q}$ with Levi factor M_Q, we say a function $f \in \mathcal{A}_E(G)$ is *negligible along* Q if for any $g \in G(\mathbb{A})$, the function given by

$$m \mapsto f(mg), \qquad m \in M_Q(\mathbb{Q})A_G(\mathbb{R})^\circ\backslash M_Q(\mathbb{A}),$$

is orthogonal to the space of cuspidal functions on $M_Q(\mathbb{Q})A_G(\mathbb{R})^\circ\backslash M_Q(\mathbb{A})$. Let $\mathcal{A}_{E,[P]}(G)$ be the subspace of all functions in $\mathcal{A}_E(G)$ which are negligible along any parabolic subgroup $Q \notin [P]$. It is a theorem of Langlands that

$$\mathcal{A}_E(G) = \bigoplus_{C\in\mathcal{C}} \mathcal{A}_{E,C}(G) \tag{2.1.1}$$

as $G(\mathbb{A}_f) \times (\mathfrak{g}_0, K_\infty)$-modules. The summand $\mathcal{A}_{E,[G]}(G)$ is the space of cusp forms in $\mathcal{A}_E(G)$.

The Franke–Schwermer decomposition refines this even further using cuspidal automorphic representations of the Levi factors of the parabolic subgroups in each class $C \in \mathcal{C}$. We briefly recall how.

Let φ be an *associate class of cuspidal automorphic representations of* M. We do not recall here the exact definition of this notion, referring instead to [FS98, §1.2] or [LS04, §1.3]. Each φ is a collection of automorphic representations of the groups $M_{P'}(\mathbb{A})$ for each $P' \in [P]$ with Levi decomposition $P' = M_{P'}N_{P'}$, finitely many for each such P', and each such representation π' must occur in $L^2_{\mathrm{cusp}}(M_{P'}(\mathbb{Q})\backslash M_{P'}(\mathbb{A}), \chi')$, where χ' is the central character of π'. Conversely, any automorphic representation π of $M(\mathbb{A})$ with central character χ occurring in $L^2_{\mathrm{cusp}}(M(\mathbb{Q})\backslash M(\mathbb{A}), \chi)$ determines a unique φ. We let $\Phi_{E,[P]}$ denote the set of all associate classes of cuspidal automorphic representations of M.

Now given a $\varphi \in \Phi_{E,[P]}$, let π' be one of the representations comprising φ; say π' is an automorphic representation of $M_{P'}(\mathbb{A})$, where $M_{P'}$ is a Levi factor of a parabolic subgroup P' associate to P. Form the space $W_{P',\tilde{\pi}'}$ and let $d\chi'$ be the differential of the central character χ' of π' at the archimedean place. Then for any $\phi \in W_{P',\tilde{\pi}'}$ we can form the Eisenstein series $E(\phi, \lambda)$, $\lambda \in \mathfrak{a}^\vee_{P',0}$.

Depending on the choice of ϕ, the Eisenstein series $E(\phi, \lambda)$ may have a pole at $\lambda = d\chi'$. Nevertheless, one can still take residues of $E(\phi, \lambda)$ at $\lambda = d\chi'$ to obtain residual Eisenstein series. We let $\mathcal{A}_{E,[P],\varphi}(G)$ be the collection of all possible Eisenstein series, residual Eisenstein series, and partial derivatives of such with respect to λ, evaluated at $\lambda = d\chi'$, built from any $\phi \in W_{P',\tilde{\pi}'}$ for any $\pi' \in \varphi$ with central character χ' such that $d\chi'$ is in the positive Weyl chamber defined by P'. For a more precise description of this space, see [FS98, §1.3] or [LS04, §1.4]. There is also a more intrinsic definition of this space, defined without reference to Eisenstein series, in [FS98, §1.2] or [LS04, §1.4], which is proved to be equivalent to this description in [FS98].

We can now state the Franke–Schwermer decomposition of $\mathcal{A}_E(G)$.

Theorem 2.1.1 (Franke–Schwermer [FS98])**.** *There is a direct sum decomposition of $G(\mathbb{A}_f) \times (\mathfrak{g}_0, K_\infty)$-modules*

$$\mathcal{A}_E(G) = \bigoplus_{C \in \mathcal{C}} \bigoplus_{\varphi \in \Phi_{E,C}} \mathcal{A}_{E,C,\varphi}(G).$$

We now introduce certain explicit $G(\mathbb{A}_f) \times (\mathfrak{g}_0, K_\infty)$-modules and explain how they can be related to the pieces of the Franke–Schwermer decomposition. Almost everything in the rest of this section is done in Franke's paper [Fra98, pp. 218, 234], but without taking into consideration the associate classes φ.

With π as above, for brevity, let us write $V[\tilde{\pi}]$ for the smooth, K-finite vectors in the $\tilde{\pi}$-isotypic component of $L^2_{\mathrm{cusp}}(M(\mathbb{A})A_P(\mathbb{R})^\circ \backslash M(\mathbb{A}))$. Then $V[\tilde{\pi}]$ is a $M(\mathbb{A}_f) \times (\mathfrak{m}_0, K_\infty \cap P(\mathbb{R}))$-module, and we extend this structure to one of a $P(\mathbb{A}_f) \times (\mathfrak{p}_0, K_\infty \cap P(\mathbb{R}))$-module by letting $\mathfrak{a}_{P,0}$ and $\mathfrak{n}$ act trivially, as well as $A_P(\mathbb{A}_f)$ and $N(\mathbb{A}_f)$. Here N is the unipotent radical of P.

Fix for the rest of this subsection a point $\mu \in \mathfrak{a}^\vee_{P,0}$. Let $\mathrm{Sym}(\mathfrak{a}^\vee_{P,0})_\mu$ be the symmetric algebra on the vector space $\mathfrak{a}^\vee_{P,0}$; we view this space as the space of differential operators on $\mathfrak{a}^\vee_{P,0}$ at the point μ. So if $H(\lambda)$ is a holomorphic function on $\mathfrak{a}^\vee_{P,0}$, then $D \in \mathrm{Sym}(\mathfrak{a}^\vee_{P,0})_\mu$ acts on H by taking a sum of iterated partial derivatives of H and evaluating the result at the point μ. In this way, every $D \in \mathrm{Sym}(\mathfrak{a}^\vee_{P,0})_\mu$ can be viewed as a distribution on holomorphic functions on $\mathfrak{a}^\vee_{P,0}$ supported at the point μ.

With this point of view, these distributions can be multiplied by holomorphic functions on $\mathfrak{a}^\vee_{P,0}$; just multiply the test function by the given holomorphic function before evaluating the distribution. With this in mind, we can define an action of $\mathfrak{a}_{P,0}$ on $\mathrm{Sym}(\mathfrak{a}^\vee_{P,0})_\mu$ by

$$(XD)(f) = D(\langle X, \cdot \rangle f), \qquad X \in \mathfrak{a}_{P,0},\ D \in \mathrm{Sym}(\mathfrak{a}^\vee_{P,0})_\mu.$$

We also let $\mathfrak{m}_0$ and $\mathfrak{n}$ act trivially on $\mathrm{Sym}(\mathfrak{a}^\vee_{P,0})_\mu$, which gives us an action of $\mathfrak{p}_0$ on $\mathrm{Sym}(\mathfrak{a}^\vee_{P,0})_\mu$. In addition, let $K_\infty \cap P(\mathbb{R})$ act trivially on $\mathrm{Sym}(\mathfrak{a}^\vee_{P,0})_\mu$. Since the Lie algebra of $K_\infty \cap P(\mathbb{R})$ lies in $\mathfrak{m}_0$, this is consistent with the $\mathfrak{p}_0$ action just defined and makes $\mathrm{Sym}(\mathfrak{a}^\vee_{P,0})_\mu$ a $(\mathfrak{p}_0, K_\infty \cap P(\mathbb{R}))$-module. Finally, let $P(\mathbb{A}_f)$ act on $\mathrm{Sym}(\mathfrak{a}^\vee_{P,0})_\mu$ by the formula

$$(pD)(f) = D(e^{\langle H_P(p), \cdot \rangle} f), \qquad p \in P(\mathbb{A}_f),\ D \in \mathrm{Sym}(\mathfrak{a}^\vee_{P,0})_\mu.$$

Then with the actions just defined, $\mathrm{Sym}(\mathfrak{a}^\vee_{P,0})_\mu$ gets the structure of a $P(\mathbb{A}_f) \times (\mathfrak{p}_0, K_\infty \cap P(\mathbb{R}))$-module.

Now we form the tensor product $V[\tilde{\pi}] \otimes \mathrm{Sym}(\mathfrak{a}^\vee_{P,0})_\mu$, which carries a natural $P(\mathbb{A}_f) \times (\mathfrak{p}_0, K_\infty \cap P(\mathbb{R}))$-module structure coming from those on the two factors. We will consider in what follows the induced $G(\mathbb{A}_f) \times (\mathfrak{g}_0, K_\infty)$-module

$$\mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})}(V[\tilde{\pi}] \otimes \mathrm{Sym}(\mathfrak{a}^\vee_{P,0})_\mu).$$

This space turns out to be isomorphic to the tensor product

$$W_{P,\tilde{\pi}} \otimes \mathrm{Sym}(\mathfrak{a}_{P,0}^{\vee})_{\mu}.$$

While the first factor in this tensor product is a $G(\mathbb{A}_f) \times (\mathfrak{g}_0, K_\infty)$-module, the second is only a $P(\mathbb{A}_f) \times (\mathfrak{p}_0, K_\infty \cap P(\mathbb{R}))$-module, and so we do not immediately get a $G(\mathbb{A}_f) \times (\mathfrak{g}_0, K_\infty)$-module structure on the tensor product. However, one can endow this space with a $G(\mathbb{A}_f) \times (\mathfrak{g}_0, K_\infty)$-module structure by viewing it as a space of distributions as described in [Fra98, p. 218] and as follows.

Since $W_{P,\tilde{\pi}}$ is a space of functions on $G(\mathbb{A})$, the space $W_{P,\tilde{\pi}} \otimes \mathrm{Sym}(\mathfrak{a}_{P,0}^{\vee})_{\mu}$ may be viewed as a space of distributions on the space $G(\mathbb{A}) \times \mathfrak{a}_{P,0}^{\vee}$. Then for $\phi \in W_{P,\tilde{\pi}}$ and $D \in \mathrm{Sym}(\mathfrak{a}_{P,0}^{\vee})_{\mu}$, we can make $X \in \mathfrak{g}_0$ act on $\phi \otimes D$ by the formula

$$\begin{aligned}(X(\phi \otimes D))(f(g, A)) \\ = ((X\phi) \otimes D)(f(g, A)) + (\phi \otimes D)(\langle (XH)(g), A + \rho_P \rangle f(g, A)),\end{aligned}$$

where f is a test function on $G(\mathbb{A}) \times \mathfrak{a}_{P,0}^{\vee}$ which is smooth and compactly supported in $g \in G(\mathbb{A})$ and holomorphic in $A \in \mathfrak{a}_{P,0}^{\vee}$ near μ. In the expression $(XH)(g)$, we are letting X act through the archimedean component of the variable g. The right hand side of the formula above defines a distribution which one checks is in $W_{P,\tilde{\pi}} \otimes \mathrm{Sym}(\mathfrak{a}_{P,0}^{\vee})_{\mu}$. We also let $k \in K_\infty$ act on such $\phi \otimes D$ by

$$k(\phi \otimes D) = (k\phi) \otimes D.$$

Finally, we let $h \in G(\mathbb{A}_f)$ act on such $\phi \otimes D$ by the formula

$$(h(\phi \otimes D))(f(g, A)) = (\phi \otimes D)(e^{\langle H(gh) - H(g), A + \rho_P \rangle} f(gh, A)).$$

One checks that this makes $W_{P,\tilde{\pi}} \otimes \mathrm{Sym}(\mathfrak{a}_{P,0}^{\vee})_{\mu}$ a $G(\mathbb{A}_f) \times (\mathfrak{g}_0, K_\infty)$-module.

The point is the following proposition, whose proof is an exercise using the definitions, and we omit it for sake of space.

Proposition 2.1.2. *There is an isomorphism of $G(\mathbb{A}_f) \times (\mathfrak{g}_0, K_\infty)$-modules*

$$W_{P,\tilde{\pi}} \otimes \mathrm{Sym}(\mathfrak{a}_{P,0}^{\vee})_{\mu} \cong \mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})}(V[\tilde{\pi}] \otimes \mathrm{Sym}(\mathfrak{a}_{P,0}^{\vee})_{\mu}).$$

More generally, if E is a finite dimensional representation of $G(\mathbb{C})$, then we also have an isomorphism

$$W_{P,\tilde{\pi}} \otimes \mathrm{Sym}(\mathfrak{a}_{P,0}^{\vee})_{\mu} \otimes E \cong \mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})}(V[\tilde{\pi}] \otimes \mathrm{Sym}(\mathfrak{a}_{P,0}^{\vee})_{\mu} \otimes E),$$

where on the left hand side, E is being viewed as a $(\mathfrak{g}_0, K_\infty)$-module, and on the right, it is viewed as a $(\mathfrak{p}_0, K_\infty \cap P(\mathbb{R}))$-module by restriction.

Now we come back to Eisenstein series. Assume π is such that there is an irreducible finite dimensional representation E of $G(\mathbb{C})$ such that the associate class φ containing π is in $\Phi_{E,[P]}$. Assume moreover that π is such that its central character χ has the property that $d\chi$ is in the positive Weyl chamber defined by P. Then we can construct elements of the piece $\mathcal{A}_{E,[P],\varphi}(G)$ of the Franke–Schwermer decomposition from elements of $W_{P,\tilde{\pi}} \otimes \mathrm{Sym}(\mathfrak{a}_{P,0}^{\vee})_{\mu}$ using Eisenstein series as follows.

Write

$$\iota_{P(\mathbb{A})}^{G(\mathbb{A})}(V[\tilde{\pi}],\lambda) = \mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})}(V[\tilde{\pi}]\otimes e^{H_P(\cdot),\lambda+\rho_P}), \qquad \lambda\in\mathfrak{a}_{P,0}^\vee,$$

for the unitary induction of $V[\tilde{\pi}]$, so that we have

$$W_{P,\tilde{\pi}} \cong \iota_{P(\mathbb{A})}^{G(\mathbb{A})}(V[\tilde{\pi}],-\rho_P).$$

Elements $\phi \in \iota_{P(\mathbb{A})}^{G(\mathbb{A})}(V[\tilde{\pi}],-\rho_P)$ fit into flat sections $\phi_\lambda \in \iota_{P(\mathbb{A})}^{G(\mathbb{A})}(V[\tilde{\pi}],\lambda)$ where λ varies in $\mathfrak{a}_{P,0}^\vee$. Then for such ϕ we have $\phi=\phi_{-\rho_P}$. In what follows, we will identify elements of $W_{P,\tilde{\pi}}$ with elements of $\iota_{P(\mathbb{A})}^{G(\mathbb{A})}(V[\tilde{\pi}],-\rho_P)$, and then use this notation to vary them in flat sections.

With $d\chi$ as above, let h_0 be a holomorphic function on $\mathfrak{a}_{P,0}^\vee$ such that, for any $\phi\in W_{P,\tilde{\pi}}$, the product $h_0(\lambda)E(\phi,\lambda)$ is holomorphic near $\lambda=d\chi+\rho_P$. Then we define a map

$$\mathcal{E}_{h_0}: W_{P,\tilde{\pi}}\otimes \mathrm{Sym}(\mathfrak{a}_{P,0}^\vee)_{d\chi+\rho_P}\to \mathcal{A}_{E,[P],\varphi}(G)$$

by

$$\phi\otimes D\mapsto D(h_0(\lambda)E(\phi,\lambda)).$$

The map $\mathcal{E}_{h_0}$ is surjective by our definition of $\mathcal{A}_{E,[P],\varphi}(G)$. If all the Eisenstein series $E(\phi,\lambda)$, for $\phi\in W_{P,\tilde{\pi}}$, are holomorphic at $\lambda=d\chi$, then we write $\mathcal{E}=\mathcal{E}_1$ for the map just defined with $h_0(\lambda)=1$.

Proposition 2.1.3. *The map $\mathcal{E}_{h_0}: W_{P,\tilde{\pi}}\otimes \mathrm{Sym}(\mathfrak{a}_{P,0}^\vee)_{d\chi+\rho_P}\to \mathcal{A}_{E,[P],\varphi}(G)$ defined just above is a surjective map of $G(\mathbb{A}_f)\times(\mathfrak{g}_0,K_\infty)$-modules. Furthermore, if all the Eisenstein series $E(\phi,\lambda)$ arising from $\phi\in W_{P,\tilde{\pi}}$ are holomorphic at $\lambda=d\chi+\rho_P$, then the map $\mathcal{E}$ is an isomorphism.*

Proof. To check that $\mathcal{E}_{h_0}$ is a map of $G(\mathbb{A}_f)\times(\mathfrak{g}_0,K_\infty)$-modules, one just needs to use the formulas defining the $G(\mathbb{A}_f)\times(\mathfrak{g}_0,K_\infty)$-module structure on $W_{P,\tilde{\pi}}\otimes\mathrm{Sym}(\mathfrak{a}_{P,0}^\vee)_\lambda$ and show they are preserved when forming Eisenstein series and taking derivatives; this can be checked when λ is in the region of convergence for the Eisenstein series, and then this extends to all λ by analytic continuation. We omit the precise details of this check.

For the second claim in the proposition, that $\mathcal{E}$ is an isomorphism, this follows from [Fra98, Theorem 14]; this theorem implies that $\mathcal{E}$ injective, since it equals the restriction of Franke's mean value map **MW** to $W_{P,\tilde{\pi}}\otimes \mathrm{Sym}(\mathfrak{a}_{P,0}^\vee)_{d\chi+\rho_P}$. Whence by surjectivity and the first part of the proposition, we are done. ☐

The space $\mathcal{A}_{E,[P],\varphi}(G)$ carries a filtration by $G(\mathbb{A}_f)\times(\mathfrak{g}_0,K_\infty)$-modules which is due to Grbac [Grb12], using an analogous filtration on $\mathcal{A}_E(G)$ Franke in [Fra98]. One might call the resulting filtration the *Franke–Schwermer–Grbac filtration* on the Franke–Schwermer piece $\mathcal{A}_{E,[P],\varphi}(G)$. For our purposes, we will not need the precise definition of this filtration, but just a rough description of its graded pieces. This is described in the following theorem.

Theorem 2.1.4. *Let $C\in\mathcal{C}$. There is a decreasing filtration*

$$\cdots\supset \mathrm{Fil}^i\,\mathcal{A}_{E,C,\varphi}(G)\supset \mathrm{Fil}^{i+1}\,\mathcal{A}_{E,C,\varphi}(G)\supset\cdots$$

of $G(\mathbb{A}_f) \times (\mathfrak{g}_0, K_\infty)$-modules on $\mathcal{A}_{E,C,\varphi}(G)$, for which we have

$$\mathrm{Fil}^0 \mathcal{A}_{E,C,\varphi}(G) = \mathcal{A}_{E,C,\varphi}(G)$$

and

$$\mathrm{Fil}^m \mathcal{A}_{E,C,\varphi}(G) = 0$$

for some $m > 0$ (depending on φ) and whose graded pieces have the property described below.

Fix π in φ, and say π is an automorphic representation of $M(\mathbb{A})$ with M a Levi factor of a parabolic subgroup P in C. Let $d\chi$ be the differential of the archimedean component of the central character χ of π, and assume $d\chi$ is in the positive Weyl chamber defined by P. Let $\mathcal{M}$ be the set of quadruples (Q, ν, Π, μ) where:

- *Q is a parabolic subgroup of G which contains a parabolic subgroup in the associate class $[P]$;*
- *ν is an element of $(\mathfrak{a}_P \cap \mathfrak{m}_{Q,0})^\vee$;*
- *Π is a unitary automorphic representation of $M_Q(\mathbb{A})$ occurring in*
$$L^2_{\mathrm{disc}}(M_Q(\mathbb{Q})A_Q(\mathbb{R})^\circ \backslash M_Q(\mathbb{A}))$$
and which is spanned by residues at the point ν of Eisenstein series parabolically induced from $(P \cap M_Q)(\mathbb{A})$ to $M_Q(\mathbb{A})$ by representations in φ; and
- *μ is an element of $\mathfrak{a}^\vee_{Q,0}$ whose real part in $\mathrm{Lie}(A_G(\mathbb{R})\backslash A_{M_Q}(\mathbb{R}))^\vee$ is in the closure of the positive Weyl chamber, and such that the following relation between μ, ν and π holds: Let $\lambda_{\tilde{\pi}}$ be the infinitesimal character of the archimedean component of $\tilde{\pi}$. Then*
$$\lambda_{\tilde{\pi}} + \nu + \mu$$
may be viewed as a collection of weights of a Cartan subalgebra of $\mathfrak{g}_0$, and the condition we impose is that these weights are in the support of the infinitesimal character of E.

For such a quadruple $(Q, \nu, \Pi, \mu) \in \mathcal{M}$, let $V_d[\Pi]$ denote the Π-isotypic component of the space

$$L^2_{\mathrm{disc}}(M_Q(\mathbb{Q})A_Q(\mathbb{R})^\circ \backslash M_Q(\mathbb{A})) \cap \mathcal{A}_{E,[P\cap M_Q],\varphi|_{M_Q}}(M_P).$$

Then the property of the graded pieces of the filtration above is that, for every i with $0 \le i < m$, there is a subset $\mathcal{M}^i_\varphi \subset \mathcal{M}$ and an isomorphism of $G(\mathbb{A}_f) \times (\mathfrak{g}_0, K_\infty)$-modules

$$\mathrm{Fil}^i \mathcal{A}_{E,C,\varphi}(G)/\mathrm{Fil}^{i+1} \mathcal{A}_{E,C,\varphi}(G) \cong \bigoplus_{(Q,\nu,\Pi,\mu)\in\mathcal{M}^i_\varphi} \mathrm{Ind}^{G(\mathbb{A})}_{Q(\mathbb{A})}(V_d[\Pi] \otimes \mathrm{Sym}(\mathfrak{a}^\vee_{Q,0})_{\mu+\rho_Q}).$$

Proof. While this essentially follows again from the work of Franke [Fra98], in this form, this theorem is a consequence of [Gro13, Theorem 4]; the latter takes into account the presence of the class φ while the former does not. □

Remark 2.1.5. In the context of Proposition 2.1.3 and Theorem 2.1.4, when all the Eisenstein series $E(\phi, \lambda)$ arising from $\phi \in W_{P,\tilde{\pi}}$ are holomorphic at $\lambda = d\chi$, what happens is that the filtration of Theorem 2.1.4 collapses to a single step. The nontrivial piece of this filtration is then given by $\mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})}(V[\tilde{\pi}] \otimes \mathrm{Sym}(\mathfrak{a}_{P,0}^\vee)_{d\chi+\rho_P})$ through the map $\mathcal{E}$ along with the isomorphism of Proposition 2.1.2.

When P is a maximal parabolic subgroup, the filtration of Theorem 2.1.4 becomes particularly simple. To describe it, we set some notation.

Assuming P is maximal, if $\tilde{\pi}$ is a unitary cuspidal automorphic representation of $M(\mathbb{A})$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 0$, let us write

$$\mathcal{L}_{P(\mathbb{A})}^{G(\mathbb{A})}(\tilde{\pi}, s)$$

for the Langlands quotient of

$$\iota_{P(\mathbb{A})}^{G(\mathbb{A})}(\tilde{\pi}, 2s\rho_P).$$

Then we have

Theorem 2.1.6 (Grbac [Grb12])**.** *In the setting above, with P maximal and $\mathrm{Re}(s) > 0$, assume $\tilde{\pi}$ defines an associate class $\varphi \in \Phi_{E,[P]}$. If any of the Eisenstein series $E(\phi, \lambda)$ coming from $\phi \in W_{\tilde{\pi}}$ have a pole at $\lambda = 2s\rho_P$, then there is an exact sequence of $G(\mathbb{A}_f) \times (\mathfrak{g}_0, K_\infty)$-modules as follows:*

$$0 \to \mathcal{L}_{P(\mathbb{A})}^{G(\mathbb{A})}(\tilde{\pi}, s) \to \mathcal{A}_{E,[P],\varphi}(G) \to \mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})}(V[\tilde{\pi}] \otimes \mathrm{Sym}(\mathfrak{a}_{P,0}^\vee)_{(2s+1)\rho_P}) \to 0.$$

Proof. This follows from [Grb12, Theorem 3.1]. □

2.2. **Cohomology of induced representations.** We now calculate the cohomology of representations of G that are parabolically induced from automorphic representations of Levi factors, and hence give a tool for computing the cohomology of the graded pieces of the Franke–Schwermer–Grbac filtration described in Theorem 2.1.4. The computations done in this section were essentially carried out by Franke in [Fra98, §7.4], but not in so much detail. We fill in just a few of the details and give a version of Franke's result which focuses on one representation of a Levi factor at a time. The method is essentially that of the proof of [BW00, Theorem III.3.3]. This method also appears in the computations of Grbac–Grobner [GG13] and Grbac–Schwermer [GS11].

Let the notation be as in the previous section. Then we have our group G and $P \subset G$ a parabolic subgroup defined over $\mathbb{Q}$ with Levi decomposition $P = MN$. Fix a compact subgroup K'_∞ of $G(\mathbb{R})$ such that $K_\infty^\circ \subset K'_\infty \subset K_\infty$, and fix also an irreducible finite dimensional representation E of $G(\mathbb{C})$. We first make the following definition.

Definition 2.2.1. Let the notation be as above. We define the *automorphic cohomology* of G with coefficients in E by

$$H^*(\mathfrak{g}_0, K'_\infty; E) = H^*(\mathfrak{g}_0, K'_\infty; \mathcal{A}_E(G) \otimes E).$$

Then we define the *cuspidal cohomology* of G with coefficients in E by

$$H^*_{\mathrm{cusp}}(\mathfrak{g}_0, K'_\infty; E) = H^*(\mathfrak{g}_0, K'_\infty; \mathcal{A}_{E,[G]}(G) \otimes E),$$

and the *Eisenstein cohomology* of G with coefficients in E by

$$H^*_{\mathrm{Eis}}(\mathfrak{g}_0, K'_\infty; E) = H^*(\mathfrak{g}_0, K'_\infty; \bigoplus_{\substack{[P]\in\mathcal{C} \\ [P]\neq[G]}} \mathcal{A}_{E,[P]}(G) \otimes E),$$

where the notation in the sum is as in (2.1.1).

We note that

$$H^*(\mathfrak{g}_0, K'_\infty; E) = H^*_{\mathrm{cusp}}(\mathfrak{g}_0, K'_\infty; E) \oplus H^*_{\mathrm{Eis}}(\mathfrak{g}_0, K'_\infty; E),$$

and that

$$H^*_{\mathrm{cusp}}(\mathfrak{g}_0, K'_\infty; E) = H^*_{\mathrm{cusp}}(\mathfrak{g}_0, K'_\infty; L^2_{\mathrm{cusp}}(G(\mathbb{Q})A_G(\mathbb{R})^\circ\backslash G(\mathbb{A})) \otimes E).$$

This latter identification is standard and follows from the fact that $(\mathfrak{g}_0, K'_\infty)$-cohomology is invariant under passage to smooth vectors.

Now fix an automorphic representation (not necessarily cuspidal) π of $M(\mathbb{A})$ with central character χ, occurring in the discrete spectrum

$$L^2_{\mathrm{disc}}(M(\mathbb{Q})\backslash M(\mathbb{A}), \chi).$$

Then the unitarization $\tilde{\pi}$ occurs in

$$L^2_{\mathrm{disc}}(M(\mathbb{Q})A_P(\mathbb{R})^\circ\backslash M(\mathbb{A})).$$

Assume χ is trivial on $A_G(\mathbb{R})^\circ$. Let $d\chi$ denote the differential of the archimedean component of χ.

We will compute the $(\mathfrak{g}_0, K'_\infty)$-cohomology space

$$H^i(\mathfrak{g}_0, K'_\infty; \mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})}(\tilde{\pi} \otimes \mathrm{Sym}(\mathfrak{a}_{P,0})_{d\chi+\rho_P}) \otimes E)$$

in terms of $(\mathfrak{m}_0, K'_\infty \cap P(\mathbb{R}))$-cohomology spaces attached to π. We will require the following lemma.

Lemma 2.2.2. *Let $\mu, \mu' \in \mathfrak{a}_{P,0}^\vee$. Let $\mathbb{C}_{\mu'}$ denote the one dimensional $\mathfrak{a}_{P,0}$-module on which $X \in \mathfrak{a}_{P,0}$ acts through multiplication by $\langle X, \mu'\rangle$. Then there is an isomorphism of $P(\mathbb{A}_f)$-modules*

$$H^i(\mathfrak{a}_{P,0}, \mathrm{Sym}(\mathfrak{a}_{P,0}^\vee)_\mu \otimes \mathbb{C}_{\mu'}) \cong \begin{cases} \mathbb{C}(e^{\langle H_P(\cdot),\mu\rangle}) & \text{if } \mu' = -\mu \text{ and } i = 0; \\ 0 & \text{if } \mu' \neq -\mu \text{ or } i > 0. \end{cases}$$

Here, $\mathbb{C}(e^{\langle H_P(\cdot),\mu\rangle})$ is just the one dimensional representation of $P(\mathbb{A}_f)$ on which $p \in P(\mathbb{A}_f)$ acts via $e^{\langle H_P(p),\mu\rangle}$.

Proof. It will be convenient to work in coordinates. So let $\lambda_1, \dots, \lambda_r$ be a basis of $\mathfrak{a}_{P,0}^\vee$ and $X_1, \dots, X_r \in \mathfrak{a}_{P,0}$ the dual basis. Then the elements of $\mathrm{Sym}(\mathfrak{a}_{P,0}^\vee)_\mu$ may be viewed as polynomials in $\lambda_1, \dots, \lambda_r$.

Let $\alpha = (\alpha_1, \dots, \alpha_r)$ be a multi-index. By definition, the monomial $\lambda^\alpha = \lambda_1^{\alpha_1} \cdots \lambda_r^{\alpha_r}$ acts as a distribution on holomorphic functions f on $\mathfrak{a}_{P,0}^\vee$ via the formula

$$\lambda^\alpha f = \frac{\partial^\alpha}{\partial \lambda^\alpha} f(\lambda)|_{\lambda=\mu}.$$

Also by definition, if $X \in \mathfrak{a}_{P,0}$, then $X\lambda^\alpha$ acts as

$$(X\lambda^\alpha)f = \frac{\partial^\alpha}{\partial \lambda^\alpha}(\langle X, \lambda\rangle f(\lambda))|_{\lambda=\mu}.$$

Let $P(\lambda) \in \mathrm{Sym}(\mathfrak{a}_{P,0}^\vee)_\mu$ be a polynomial in λ. Then a quick induction using the above formulas shows that $X \in \mathfrak{a}_{P,0}$ acts on $P(\lambda)$ as

$$X(P(\lambda)) = \langle X, \mu\rangle P(\lambda) + \sum_{i=1}^{r} \frac{\partial}{\partial \lambda_i} P(\lambda).$$

Hence X acts on the element $P(\lambda) \otimes 1$ in $\mathrm{Sym}(\mathfrak{a}_{P,0})_\mu \otimes \mathbb{C}_{\mu'}$ by

$$X(P(\lambda) \otimes 1) = \langle X, \mu + \mu'\rangle (P(\lambda) \otimes 1) + \sum_{i=1}^{r} \left(\frac{\partial}{\partial \lambda_i} P(\lambda) \otimes 1 \right).$$

It follows from this that the decomposition

$$\mathfrak{a}_{P,0} = \mathbb{C}X_1 \oplus \cdots \oplus \mathbb{C}X_r$$

realizes $\mathrm{Sym}(\mathfrak{a}_{P,0})_\mu \otimes \mathbb{C}_{\mu'}$ as an exterior tensor product of analogously defined single-variable symmetric powers:

$$\mathrm{Sym}(\mathfrak{a}_{P,0})_\mu \otimes \mathbb{C}_{\mu'} \cong (\mathrm{Sym}(\mathbb{C}\lambda_1)_{\mu_1} \otimes \mathbb{C}_{\mu'_1}) \otimes \cdots \otimes (\mathrm{Sym}(\mathbb{C}\lambda_r)_{\mu_r} \otimes \mathbb{C}_{\mu'_r}),$$

where $\mu_i, \mu'_i \in \mathbb{C}\lambda$ are the ith components of μ, μ' in the basis $\lambda_1, \ldots, \lambda_r$. By the Künneth formula, if we ignore for now the $P(\mathbb{A}_f)$-action, we then reduce to checking the one-dimensional analog of the lemma, that

$$H^i(\mathbb{C}X_i, \mathrm{Sym}(\mathbb{C}\lambda_i)_{\mu_i} \otimes \mathbb{C}_{\mu'_i}) \cong \begin{cases} \mathbb{C} & \text{if } \mu' = -\mu \text{ and } i = 0; \\ 0 & \text{if } \mu' \neq -\mu \text{ or } i > 0. \end{cases}$$

This can be checked just by writing down the complex that computes this cohomology. Furthermore, $H^0(\mathfrak{a}_{P,0}, \mathrm{Sym}(\mathfrak{a}_{P,0})_\mu \otimes \mathbb{C}_{-\mu})$ can be identified with subspace of $\mathrm{Sym}(\mathfrak{a}_{P,0})_\mu$ consisting of constants. By definition, this space has an action of $P(\mathbb{A}_f)$ given by the character $e^{\langle H_P(\cdot), \mu\rangle}$, which proves our lemma. □

Let $\mathfrak{h} \subset \mathfrak{g}$ be a Cartan subalgebra, and assume $\mathfrak{h} \subset \mathfrak{m}$. Fix an ordering on the roots of $\mathfrak{h}$ in $\mathfrak{g}$ which makes $\mathfrak{p}$ standard. If $W(\mathfrak{h}, \mathfrak{g})$ denotes the Weyl group of $\mathfrak{h}$ in $\mathfrak{g}$, then write

$$W^P = \{w \in W(\mathfrak{h}, \mathfrak{g}) \mid w^{-1}\alpha > 0 \text{ for all positive roots } \alpha \text{ in } \mathfrak{m}\}.$$

Then W^P is the set of representatives of minimal length for $W(\mathfrak{h}, \mathfrak{g})$ modulo the Weyl group $W(\mathfrak{h} \cap \mathfrak{m}_0, \mathfrak{m}_0)$ of $\mathfrak{h} \cap \mathfrak{m}_0$ in $\mathfrak{m}_0$. Write ρ for half the sum of the positive roots of $\mathfrak{h}$ in $\mathfrak{g}$.

If $\Lambda \in \mathfrak{h}^\vee$ is a dominant weight, write E_Λ for the representation of $\mathfrak{g}$ of highest weight Λ. If $\nu \in \mathfrak{h}^\vee$ is a weight which is dominant for $\mathfrak{m}$ we denote by F_ν the representation of $\mathfrak{m}$ of highest weight ν. Then we have the Kostant decomposition [Kos61]:

$$H^i(\mathfrak{n}, E_\Lambda) \cong \bigoplus_{\substack{w \in W^P \\ \ell(w) = i}} F_{w(\Lambda+\rho)-\rho},$$

where $\ell(w)$ denotes the length of the Weyl group element w.

Now we are ready to state the main theorem of this subsection.

Theorem 2.2.3. *Notation as above, let $\Lambda \in \mathfrak{h}^\vee$ be a dominant weight such that $E = E_\Lambda$. Assume that the cohomology space*

$$H^i(\mathfrak{g}_0, K'_\infty; \mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})}(\tilde{\pi} \otimes \mathrm{Sym}(\mathfrak{a}_{P,0}^\vee)_{d\chi+\rho_P}) \otimes E) \tag{2.2.1}$$

is nontrivial for some i. Then there is a unique $w \in W^P$ such that

$$-w(\Lambda + \rho)|_{\mathfrak{a}_{P,0}} = d\chi$$

and such that the infinitesimal character of the archimedean component of $\tilde{\pi}$ contains $-w(\Lambda+\rho)|_{\mathfrak{h}\cap\mathfrak{m}_\circ}$. Furthermore, if $\ell(w)$ is the length of such an element w, then for any i we have

$$\begin{aligned} &H^i(\mathfrak{g}_0, K'_\infty; \mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})}(\tilde{\pi} \otimes \mathrm{Sym}(\mathfrak{a}_{P,0}^\vee)_{d\chi+\rho_P}) \otimes E) \\ &\qquad \cong \iota_{P(\mathbb{A}_f)}^{G(\mathbb{A}_f)}(\pi_f) \otimes H^{i-\ell(w)}(\mathfrak{m}_0, K'_\infty \cap P(\mathbb{R}); \tilde{\pi}_\infty \otimes F_{w(\Lambda+\rho)-\rho,0}), \end{aligned}$$

where ι denotes a normalized parabolic induction functor, and $F_{w(\Lambda+\rho)-\rho,0}$ denotes the restriction to $\mathfrak{m}_0$ of the representation of $\mathfrak{m}$ of highest weight $w(\Lambda+\rho)-\rho$.

Proof. The proof is almost exactly the same as the proof of [BW00, Theorem III.3.3], but we give the details for the convenience of the interested reader.

Let us first prove the uniqueness of the element w in the theorem. Note first that

$$\mathfrak{h} \cap \mathfrak{g}_0 = \mathfrak{a}_{P,0} \oplus (\mathfrak{h} \cap \mathfrak{m}_0).$$

Because Λ is dominant, we know $(\Lambda+\rho)$ is regular, and the conditions in the theorem therefore pin down the element $w(\Lambda+\rho)$ uniquely up to the Weyl group $W(\mathfrak{h}\cap\mathfrak{m}_0, \mathfrak{m}_0)$ of $\mathfrak{h}\cap\mathfrak{m}_0$ in $\mathfrak{m}_0$. But it is well known that W^P is a set of representatives for $W(\mathfrak{h}, \mathfrak{g})$ modulo $W(\mathfrak{h}\cap\mathfrak{m}_0, \mathfrak{m}_0)$. Therefore $w(\Lambda+\rho)$ lies in a unique Weyl chamber, and so w is determined.

Let i be an integer. We now begin to compute the cohomology space

$$H^i(\mathfrak{g}_0, K_\infty; \mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})}(\tilde{\pi} \otimes \mathrm{Sym}(\mathfrak{a}_{P,0}^\vee)_{d\chi+\rho_P}) \otimes E).$$

First, Proposition 2.1.2 allows us to pull the tensor product with E inside the induction, whence by Frobenius reciprocity, we have

$$\begin{aligned} &H^i(\mathfrak{g}_0, K_\infty; \mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})}(\tilde{\pi} \otimes \mathrm{Sym}(\mathfrak{a}_{P,0}^\vee)_{d\chi+\rho_P}) \otimes E) \\ &\qquad \cong \mathrm{Ind}_{P(\mathbb{A}_f)}^{G(\mathbb{A}_f)}(H^i(\mathfrak{p}_0, K_\infty \cap P(\mathbb{R}); \tilde{\pi} \otimes \mathrm{Sym}(\mathfrak{a}_{P,0}^\vee)_{d\chi+\rho_P} \otimes E)). \end{aligned} \tag{2.2.2}$$

It is our goal, therefore, to compute

$$H^i(\mathfrak{p}_0, K_\infty \cap P(\mathbb{R}); \tilde{\pi} \otimes \mathrm{Sym}(\mathfrak{a}_{P,0}^\vee)_{d\chi+\rho_P} \otimes E).$$

Now, as $(\mathfrak{p}_0, K_\infty \cap P(\mathbb{R}))$-modules, the space $\tilde{\pi}$ comes from a $(\mathfrak{m}_0, K_\infty \cap P(\mathbb{R}))$-module and $\mathrm{Sym}(\mathfrak{a}_{P,0}^\vee)_{d\chi+\rho_P}$ comes from an $\mathfrak{a}_{P,0}$-module. Thus, using

$$\mathfrak{p}_0 = (\mathfrak{m}_0 \oplus \mathfrak{a}_{P,0}) \oplus \mathfrak{n},$$

we get a spectral sequence whose E_2 page is

$$E_2^{j,k} = H^j(\mathfrak{m}_0 \oplus \mathfrak{a}_{P,0}, K_\infty \cap P(\mathbb{R}); \tilde{\pi} \otimes \mathrm{Sym}(\mathfrak{a}_{P,0}^\vee)_{d\chi+\rho_P} \otimes H^k(\mathfrak{n}; E))$$

and which degenerates to the cohomology space above with $i = j + k$. We will eventually be able to say that this spectral sequence degenerates on its E_2 page, but this will follow from the vanishing of enough of its terms. So we compute this page now.

By the Kostant decomposition [BW00, Theorem III.3.1], the (j, k)-term on this E_2 page is

$$\bigoplus_{\substack{w' \in W^P \\ \ell(w)=k}} H^j(\mathfrak{m}_0 \oplus \mathfrak{a}_{P,0}, K_\infty \cap P(\mathbb{R}); \tilde{\pi} \otimes \mathrm{Sym}(\mathfrak{a}_{P,0}^\vee)_{d\chi+\rho_P} \otimes F_{w'(\Lambda+\rho)-\rho}).$$

Write $\nu(w') = (w'(\Lambda + \rho) - \rho)|_{\mathfrak{a}_{P,0}}$, As an $(\mathfrak{m}_0 \oplus \mathfrak{a}_{P,0})$-module, the representation $F_{w'(\Lambda+\rho)-\rho}$ decomposes as

$$F_{w'(\Lambda+\rho)-\rho} = F_{w'(\Lambda+\rho)-\rho,0} \otimes \mathbb{C}_{\nu(w')},$$

as an exterior tensor product over the direct sum $\mathfrak{m}_0 \oplus \mathfrak{a}_{P,0}$. Thus by the Künneth formula, we get

$$\begin{aligned} &H^*(\mathfrak{m}_0 \oplus \mathfrak{a}_{P,0}, K_\infty \cap P(\mathbb{R}); \tilde{\pi} \otimes \mathrm{Sym}(\mathfrak{a}_{P,0}^\vee)_{d\chi+\rho_P} \otimes F_{w'(\Lambda+\rho)-\rho}) \cong \\ &H^*(\mathfrak{m}_0, K_\infty \cap P(\mathbb{R}); \tilde{\pi} \otimes F_{w'(\Lambda+\rho)-\rho,0}) \otimes H^*(\mathfrak{a}_{P,0}, \mathrm{Sym}(\mathfrak{a}_{P,0}^\vee)_{d\chi+\rho_P} \otimes \mathbb{C}_{\nu(w')}). \end{aligned}$$

By Lemma 2.2.2, the second factor here is nonvanishing if and only if

$$d\chi_\pi + \rho_P = -\nu(w'),$$

and the first factor is nonvanishing only if the infinitesimal character of $F_{w'(\Lambda+\rho)-\rho,0}$ matches the negative of that of the archimedean component of $\tilde{\pi}$. Since P is standard, we have $\rho_P = \rho|_{\mathfrak{a}_{P,0}}$, which implies

$$\nu(w') = w'(\Lambda + \rho)|_{\mathfrak{a}_{P,0}} - \rho_P$$

and so this first nonvanishing condition is equivalent to

$$= w'(\Lambda + \rho)|_{\mathfrak{a}_{P,0}} = d\chi;$$

the second of these nonvanishing conditions is just that $-w'(\Lambda + \rho)$ occurs in the infinitesimal character of the archimedean component of $\tilde{\pi}$. As shown at the beginning of this proof, there is only one w' satisfying these two conditions, and we will denote it by w.

Thus, by Lemma 2.2.2, we get

$$\begin{aligned} H^*(\mathfrak{m}_0, K_\infty \cap P(\mathbb{R}); \tilde{\pi} \otimes F_{w(\Lambda+\rho)-\rho,0}) \otimes H^*(\mathfrak{a}_{P,0}, \mathrm{Sym}(\mathfrak{a}_{P,0}^\vee)_{d\chi+\rho_P} \otimes \mathbb{C}_{\nu(w)}) \\ \cong H^*(\mathfrak{m}_0, K_\infty \cap P(\mathbb{R}); \tilde{\pi} \otimes F_{w(\Lambda+\rho)-\rho,0}) \otimes \mathbb{C}(e^{\langle H_P(\cdot), d\chi+\rho_P \rangle}), \end{aligned}$$

where the factor $\mathbb{C}(e^{\langle H_P(\cdot), d\chi+\rho_P \rangle})$ is concentrated in degree zero.

Retracing our steps, we have thus computed the E_2 page of our spectral sequence. The term $E_2^{j,k}$ is

$$\begin{cases} H^j(\mathfrak{m}_0, K_\infty \cap P(\mathbb{R}); \tilde{\pi} \otimes F_{w(\Lambda+\rho)-\rho,0}) \otimes \mathbb{C}(e^{\langle H_P(\cdot), d\chi+\rho_P \rangle}) & \text{if } k = \ell(w); \\ 0 & \text{if } k \neq \ell(w). \end{cases}$$

The E_2 page therefore consists only of one row, and thus our spectral sequence degenerates. Hence we have shown

$$H^i(\mathfrak{p}_0, K_\infty \cap P(\mathbb{R}); \tilde{\pi} \otimes \mathrm{Sym}(\mathfrak{a}_{P,0}^\vee)_{d\chi+\rho_P} \otimes E) \cong H^{i-\ell(w)}(\mathfrak{m}_0, K_\infty \cap P(\mathbb{R}); \tilde{\pi} \otimes F_{w(\Lambda+\rho)-\rho,0}) \otimes \mathbb{C}(e^{\langle H_P(\cdot), d\chi+\rho_P\rangle})$$

Now we rewrite

$$\begin{aligned} &H^{i-\ell(w)}(\mathfrak{m}_0, K_\infty \cap P(\mathbb{R}); \tilde{\pi} \otimes F_{w(\Lambda+\rho)-\rho,0}) \otimes \mathbb{C}(e^{\langle H_P(\cdot), d\chi+\rho_P\rangle}) \\ &\cong \tilde{\pi}_f \otimes \mathbb{C}(e^{\langle H_P(\cdot), d\chi+\rho_P\rangle}) \otimes H^{i-\ell(w)}(\mathfrak{m}_0, K_\infty \cap P(\mathbb{R}); \tilde{\pi}_\infty \otimes F_{w(\Lambda+\rho)-\rho,0}) \\ &\cong \pi_f \otimes \mathbb{C}(e^{\langle H_P(\cdot), \rho_P\rangle}) \otimes H^{i-\ell(w)}(\mathfrak{m}_0, K_\infty \cap P(\mathbb{R}); \tilde{\pi}_\infty \otimes F_{w(\Lambda+\rho)-\rho,0}), \end{aligned}$$

so that

$$H^i(\mathfrak{p}_0, K_\infty \cap P(\mathbb{R}); \tilde{\pi} \otimes \mathrm{Sym}(\mathfrak{a}_{P,0}^\vee)_{d\chi+\rho_P} \otimes E) \cong \pi_f \otimes \mathbb{C}(e^{\langle H_P(\cdot), \rho_P\rangle}) \otimes H^{i-\ell(w)}(\mathfrak{m}_0, K_\infty \cap P(\mathbb{R}); \tilde{\pi}_\infty \otimes F_{w(\Lambda+\rho)-\rho,0}).$$

We therefore have, by (2.2.2),

$$\begin{aligned} &H^i(\mathfrak{g}_0, K'_\infty; \mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})}(\tilde{\pi} \otimes \mathrm{Sym}(\mathfrak{a}_{P,0})_{d\chi_\pi+\rho_P}) \otimes E) \\ \cong\ &\mathrm{Ind}_{P(\mathbb{A}_f)}^{G(\mathbb{A}_f)}(\pi_f \otimes \mathbb{C}(e^{\langle H_P(\cdot), \rho_P\rangle})) \otimes H^{i-\ell(w)}(\mathfrak{m}_0, K'_\infty \cap P(\mathbb{R}); \tilde{\pi}_\infty \otimes F_{w(\Lambda+\rho)-\rho,0}), \end{aligned}$$

which is what we wanted to prove. □

2.3. **Location of a Langlands quotient in Eisenstein cohomology.** We now return to the G_2 setting. In this subsection we will apply the theory above to study the occurrence of a particular Langlands quotient in the Eisenstein cohomology of G_2. This will be done in Theorem 2.3.9. As mentioned in the introduction, this Langlands quotient will be the automorphic representation of G_2 which we will p-adically deform in a sequel paper [Mun]. In order to precisely locate it in Eisenstein cohomology, we will need to be able to distinguish between different representations in that cohomology. When these representations are coming from different parabolic subgroups, we can use L-functions to do this, as in the proposition below. To state it, we require a little setup.

Recall that two automorphic representations π and π' of a reductive group G are *nearly equivalent* if for all but finitely many places v, the local components π_v and π'_v are isomorphic.

Identify the long root Levi factor M_α of P_α and the short root Levi factor M_β of P_β with GL_2 via the maps i_α and i_β of Section 1.1. Then the modulus characters are given by

$$\delta_{P_\alpha(\mathbb{A})}(A) = |\det(A)|^5, \qquad \delta_{P_\beta(\mathbb{A})}(A) = |\det(A)|^3,$$

for $A \in GL_2(\mathbb{A})$. We will consider in what follows the unitary parabolic induction functors $\iota_{P(\mathbb{A})}^{G_2(\mathbb{A})}$ for $P \in \{B, P_\alpha, P_\beta\}$; see the section on notation in the introduction.

Proposition 2.3.1. *Let π_α and π_β be unitary, cuspidal automorphic representations of $GL_2(\mathbb{A})$, viewed respectively as representations of $M_\alpha(\mathbb{A})$ and $M_\beta(\mathbb{A})$. Let ψ be a quasicharacter of $T(\mathbb{Q})\backslash T(\mathbb{A})$, and let $s_\alpha, s_\beta \in \mathbb{C}$. Then given any irreducible subquotients*

$$\Pi_\alpha \quad \textit{of} \quad \iota_{P_\alpha(\mathbb{A})}^{G_2(\mathbb{A})}(\pi_\alpha, s_\alpha)$$

and

$$\Pi_\beta \quad \textit{of} \quad \iota_{P_\beta(\mathbb{A})}^{G_2(\mathbb{A})}(\pi_\beta, s_\beta)$$

and

$$\Pi_0 \quad \textit{of} \quad \iota_{B(\mathbb{A})}^{G_2(\mathbb{A})}(\psi),$$

we have that no two of Π_α, Π_β and Π_0 are nearly equivalent.

Before proving this proposition, we first require some setup about L-functions for automorphic representations of $GL_2(\mathbb{A})$. Let π be an automorphic representation of $GL_2(\mathbb{A})$, and let S be a finite set of places of $\mathbb{Q}$ containing the archimedean place and those places at which π is ramified. For a place $v \notin S$, let $t_v = \mathrm{diag}(t_{v,1}, t_{v,2})$ be the usual Satake parameter for π at v, lying in the diagonal torus of $GL_2(\mathbb{C})$. Then given an algebraic, finite dimensional representation r of $GL_2(\mathbb{C})$, we define the local L-factor of π and r at v by

$$L_v(s, \pi, r) = \det(1 - q_v^{-s} r(t_v)),$$

where $s \in \mathbb{C}$ and q_v is the rational prime corresponding to v. Then the global S-partial L-function is defined as

$$L^S(s, \pi, r) = \prod_{v \notin S} L_v(s, \pi, r),$$

for $\mathrm{Re}(s)$ sufficiently large, and by meromorphic continuation otherwise, when possible.

We will be interested in nonvanishing properties of these L-functions for a few specific representations r. Let Std denote the 2-dimensional standard representation of $GL_2(\mathbb{C})$, and det the 1-dimensional determinant representation of $GL_2(\mathbb{C})$. Then we define

$$\mathrm{Ad} = \mathrm{Sym}^2(\mathrm{Std}) \otimes \det{}^{-1}, \qquad \mathrm{Ad}^3 = \mathrm{Sym}^3(\mathrm{Std}) \otimes \det{}^{-1},$$

which are, respectively, 3-dimensional and 4-dimensional representations of $GL_2(\mathbb{C})$. Explicitly, the local L-factors for these representations at $v \notin S$ are given by

$$L_v(s, \pi, \mathrm{Ad}) = (1 - q_v^{-s} t_{1,v} t_{2,v}^{-1})(1 - q_v^{-s})(1 - q_v^{-s} t_{1,v}^{-1} t_{2,v})$$

and

$$L_v(s, \pi, \mathrm{Ad}^3) = (1 - q_v^{-s} t_{1,v}^2 t_{2,v}^{-1})(1 - q_v^{-s} t_{1,v})(1 - q_v^{-s} t_{2,v})(1 - q_v^{-s} t_{1,v}^{-1} t_{2,v}^2).$$

The L-function $L^S(s, \pi, \det)$ is just the usual partial Hecke L-function associated with the central character of π, and the L-function $L(s, \pi, \mathrm{Std})$ is just the usual degree 2 L-function associated with π.

We will also have occasion to consider twisted L-functions. If $\chi : \mathbb{Q}^\times \backslash \mathbb{A}^\times \to \mathbb{C}^\times$ is a Hecke character for $\mathbb{Q}$ and our set S also contains all the places where χ is ramified, then we define

$$L_v(s, r(\pi) \times \chi) = \det(1 - \chi_v(q_v) q_v^{-s} r(t_v)),$$

where χ_v is the local component of χ at v, and then

$$L^S(s, r(\pi) \times \chi) = \prod_{v \notin S} L_v(s, r(\pi) \times \chi).$$

Lemma 2.3.2. *Let π be a unitary, cuspidal automorphic representation of $GL_2(\mathbb{A})$ and let χ be a unitary Hecke character for $\mathbb{Q}$. Let S be any finite set of places of $\mathbb{Q}$ containing the archimedean place as well as those where π is ramified. Then if $r \in \{\det, \mathrm{Std}, \mathrm{Ad}, \mathrm{Ad}^3\}$, then the partial L-function $L^S(s, r(\pi) \times \chi)$ has a meromorphic continuation to $s \in \mathbb{C}$ and is nonvanishing when* $\mathrm{Re}(s) \geq 1$.

Proof. When $r = \det$ this lemma is well known and classical, using that χ and the central character of π are unitary.

Let $\mathrm{Ad}(\pi)$ denote the Gelbart–Jacquet lift of π to GL_3 [GJ78]. Then the S-partial L-function of $\mathrm{Ad}(\pi)$ coincides with the L-function $L^S(s, r(\pi))$ for $r = \mathrm{Ad}$ defined above. Assume first that $\mathrm{Ad}(\pi)$ is not cuspidal. Then (see [GJ78, Remark 9.9]) $\pi \cong \pi \otimes \eta$ where η is a nontrivial quadratic character of $\mathbb{A}^\times$. Let F be the quadratic extension cut out by η. Then there is a unitary Hecke character ω of F such that π is the automorphic induction from F to $\mathbb{Q}$ of ω. Therefore for $r \in \{\det, \mathrm{Std}, \mathrm{Ad}, \mathrm{Ad}^3\}$, the L-function $L^S(s, r(\pi) \times \chi)$ is a product of L-functions defined over $\mathbb{Q}$ or F of unitary characters, so again they are meromorphic and do not vanish for $\mathrm{Re}(s) \geq 1$.

So assume that $\mathrm{Ad}(\pi)$ is cuspidal. We will show that the lemma is a consequence of the Langlands–Shahidi method; there is, in each case, an Eisenstein series for a group larger than GL_2 whose constant term sees the L-function in question. In more detail, let $G \in \{GL_3, GL_4, GL_5\}$ and, correspondingly, let $P_G = M_G N_G \subset G$ be the standard maximal parabolic subgroup with Levi factor $M_G = GL_2 \times GL_1$ if $G = GL_3$, that with $M_G = GL_3 \times GL_1$ if $G = GL_4$, or that with $M_G = GL_3 \times GL_2$ if $G = GL_5$. Then, in these respective situations, let π'_χ be the automorphic representation of $M_G(\mathbb{A})$ given by

$$\pi'_\chi = (\pi \otimes \chi) \boxtimes 1, \quad \text{or} \quad \pi'_\chi = (\mathrm{Ad}(\pi) \otimes \chi) \boxtimes 1, \quad \text{or} \quad \pi'_\chi = (\mathrm{Ad}(\pi) \otimes \chi) \boxtimes \pi^\vee.$$

Now in any of these three cases, fix an element

$$\tilde{\phi} \in \iota_{P_G(\mathbb{A})}^{G(\mathbb{A})}(\pi'_\chi)$$

which is factorizable, so $\tilde{\phi} = \otimes_v \tilde{\phi}_v$, the tensor product being over all places v, and

$$\tilde{\phi}_v \in \iota_{P_G(\mathbb{Q}_v)}^{G(\mathbb{Q}_v)}(\pi'_{\chi,v}).$$

Assume $\tilde{\phi}_v$ is right-$G(\mathbb{Z}_v)$-invariant for all places $v \notin S$. Then, viewing $\pi'_\chi \subset L^2_{\mathrm{cusp}}(M_G(\mathbb{Q}) \backslash M_G(\mathbb{A}))$, we can form the function $\phi : G(\mathbb{A}) \to \mathbb{C}$ given by

$$\phi(g) = \tilde{\phi}(g)(1),$$

where the element 1 is the identity element of $M_G(\mathbb{A})$. Then $\phi \in W_{P_G,\pi'_\chi}$ as in the beginning of Section 2.1. (Strictly speaking, to fit into the setting of Section 2.1, we should take the unitarization $\tilde{\pi}'_\chi$ of π'_χ. This would have the effect of shifting our L-functions below by a purely imaginary complex number, and hence this has no effect on the statement of the lemma. Otherwise, we can just note that everything we say about Eisenstein series in this proof works whether or not we have taken the unitarization of π'_χ, as π'_χ is already unitary.)

Now for $s \in \mathbb{C}$, we consider the Eisenstein series $E(\phi, s)(g) = E(\phi, s\rho_{P_G})(g)$. For any place v, let

$$M_v(s) : \iota_{P_G(\mathbb{Q}_v)}^{G(\mathbb{Q}_v)}(\pi'_{\chi,v}, s) \to \iota_{P_G(\mathbb{Q}_v)}^{G(\mathbb{Q}_v)}(\pi'_{\chi,v}, -s)$$

be the usual intertwining operator. Also, let $\tilde{\phi}_s(g) = \tilde{\phi}(g)e^{\langle H_{P_G}(g), s\rho_{P_G}\rangle}$, and similarly for the local section $\tilde{\phi}_{v,s}(g)$. Then by the Langlands–Shahidi method, we have that the constant term $E(\phi, s)_{N_G}(g)$ of $E(\phi, s)(g)$ along N_G has

$$E(\phi, s)_{N_G}(g) = \tilde{\phi}_s(g)(1) + C^S(s)\left(\bigotimes_{v \notin s} \phi_{v,s} \otimes \bigotimes_{v \in S} M_v(s)(\tilde{\phi}_{v,s})\right)(1),$$

where $C^S(s)$ is the meromorphic function given by

$$C^S(s) = \begin{cases} \frac{L^S(3s, \pi\otimes\chi)}{L^S(1+3s, \pi\otimes\chi)} & \text{if } G = GL_3; \\ \frac{L^S(4s, \mathrm{Ad}(\pi)\times\chi)}{L^S(1+4s, \mathrm{Ad}(\pi)\times\chi)} & \text{if } G = GL_4; \\ \frac{L^S(5s, \mathrm{Ad}^3(\pi)\times\chi)L^S(5s, \pi\otimes\chi)}{L^S(1+5s, \mathrm{Ad}^3(\pi)\times\chi)L^S(1+5s, \pi\otimes\chi)} & \text{if } G = GL_5. \end{cases}$$

See, for example, [Sha10, Chapter 6].

The meromorphy statement of the lemma follows immediately from the meromorphy of $E(\phi, s)$. Moreover, the proof of [Sha81, Theorem 5.1] shows that there is a choice of $\tilde{\phi}$ such that the zeros of the L-functions

$$\begin{cases} L(1 + 3s, \pi \otimes \chi) & \text{if } G = GL_3; \\ L(1 + 4s, \mathrm{Ad}(\pi) \times \chi) & \text{if } G = GL_4; \\ L(1 + 5s, \mathrm{Ad}^3(\pi) \times \chi)L(1 + 5s, \pi \otimes \chi) & \text{if } G = GL_5, \end{cases}$$

are among the poles of $E(\phi, s)$ in these respective cases. Shahidi deduces immediately in *loc. cit.* that these L-functions are nonvanishing for purely imaginary values of s, so we are reduced to checking that the L-functions above are nonvanishing at the poles of $E(\phi, s)$ for $\mathrm{Re}(s) > 0$.

Now in all three cases we are considering, the possible poles of $E(\phi, s)$ for any $\tilde{\phi}$, and any s with $\mathrm{Re}(s) > 0$, have been classified; since P_G is a maximal parabolic subgroup of G, the poles of $E(\phi, s)$ are simple and hence give rise to residual Eisenstein series, and the residual spectra for GL_n have been fully described by Mœglin and Waldspurger in [MW89]. In all cases, we have that $E(\phi, s)$ cannot have a pole for $\mathrm{Re}(s) > 0$ because the blocks of the Levi M_G have different sizes. Thus there is nothing to check, and this proves the lemma fully for $r = \mathrm{Std}$ and $r = \mathrm{Ad}$, as well as the analogue of the lemma for the product of L-functions

$$L^S(s, \mathrm{Ad}^3(\pi) \times \chi) L^S(s, \pi \otimes \chi).$$

But then the nonvanishing statement for $L^S(s, \mathrm{Ad}^3(\pi) \times \chi)$ follows from this and the holomorphy of $L^S(s, \pi \otimes \chi)$. □

Proof (of Proposition 2.3.1). We first note that we may assume $\mathrm{Re}(s_\alpha) \geq 0$ and $\mathrm{Re}(s_\beta) \geq 0$; indeed, if $\gamma \in \{\alpha, \beta\}$, then π_γ is unitary. So if $\mathrm{Re}(s_\gamma) < 0$, and if v is finite place which is unramified for π_γ, then there is a nonzero intertwining operator

$$\iota_{P_\gamma(\mathbb{Q}_v)}^{G_2(\mathbb{Q}_v)}(\pi_{\gamma,v}^\vee, -s_\gamma) \to \iota_{P_\gamma(\mathbb{Q}_v)}^{G_2(\mathbb{Q}_v)}(\pi_{\gamma,v}, s_\gamma),$$

whose image is isomorphic to the unique unramified quotient of the source representation. It follows that any irreducible admissible subquotients of

$$\iota_{P_\gamma(\mathbb{A}_f)}^{G_2(\mathbb{A}_f)}(\pi_\gamma^\vee, -s_\gamma) \qquad \text{and} \qquad \iota_{P_\gamma(\mathbb{A}_f)}^{G_2(\mathbb{A}_f)}(\pi_\gamma, s_\gamma)$$

are nearly equivalent.

Thus assume $\mathrm{Re}(s_\alpha) \geq 0$ and $\mathrm{Re}(s_\beta) \geq 0$. For Π an automorphic representation of $G_2(\mathbb{A})$, χ a unitary character of $\mathbb{Q}^\times \backslash \mathbb{A}^\times$ and S a finite set of places of $\mathbb{Q}$ including the archimedean place and the ramified places for Π and χ, we will consider the partial L-function $L^S(s, R_7(\Pi) \times \chi)$, where R_7 is the standard 7-dimensional representation of G_2 (see Section 1.1). The definition of this L-function is as follows. If $v \notin S$ is a place, $\chi_v : \mathbb{Q}_v^\times \to \mathbb{C}^\times$ is the local component of χ at v, and $t_v \in G_2(\mathbb{C})$ is the Satake parameter of Π_v, then the corresponding local L-factor is defined to be

$$L_v(s, R_7(\Pi) \times \chi) = \det(1 - \chi_v(q_v) q_v^{-s} R_7(t_v)),$$

where q_v is the prime corresponding to v. Then the global L-function is defined as

$$L^S(s, R_7(\Pi) \times \chi) = \prod_{v \notin S} L_v(s, R_7(\Pi) \times \chi),$$

for $\mathrm{Re}(s)$ sufficiently large, and by meromorphic continuation otherwise, when possible; this L-function will have a meromorphic continuation in all cases considered below, because we will obtain factorizations for it and all of the factors will have meromorphic continuations.

Now let Π_α and Π_β be as in the statement of the proposition. Then it follows from (1.1.2) that for S sufficiently large, we have a factorization into partial L-functions

$$L^S(s, R_7(\Pi_\alpha) \times \chi) = L^S(s + s_\alpha, \pi_\alpha \otimes \chi) L^S(s - s_\alpha, \pi_\alpha^\vee \otimes \chi) L^S(s, \mathrm{Ad}(\pi_\alpha) \times \chi),$$

all L-functions on the right hand side being defined as above Lemma 2.3.2. If we write ω_{π_β} for the central character of π_β, then we also have, by (1.1.1), that

$$\begin{aligned} &L^S(s, R_7(\Pi_\beta) \times \chi) \\ &= L^S(s+s_\beta, \pi_\beta \otimes \chi) L^S(s-s_\beta, \pi_\beta^\vee \otimes \chi) L^S(s+s_\beta, \omega_{\pi_\beta}\chi) L^S(s-s_\beta, \omega_{\pi_\beta}^{-1}\chi) \zeta^S(s), \end{aligned}$$

where $\zeta^S(s)$ is the usual Riemann zeta function with the Euler factors at places in S removed.

Now we examine the poles of both expressions above. On the one hand, since π_α is assumed to be cuspidal and unitary, by the work of Gelbart–Jacquet [GJ78, Theorem 9.3.1, Remark 9.9], the L-function $L^S(s, \mathrm{Ad}^2(\pi_\alpha) \times \chi)$ has at worst a simple pole, and this pole can only occur at $s = 1$. Therefore the same is true for $L^S(s, R_7(\Pi_\alpha) \times \chi)$.

On the other hand, by Lemma 2.3.2,

- The L-function $L^S(s + s_\beta, \pi_\beta \otimes \chi)$ does not vanish when $s = 1 + s_\beta$;
- Similarly, the L-function $L^S(s - s_\beta, \pi_\beta^\vee \otimes \chi)$ does not vanish when $s = 1 + s_\beta$;
- The L-function $L^S(s + s_\beta, \omega_{\pi_\beta}\chi)$ and the zeta function $\zeta^S(s)$ do not vanish when $s = 1 + s_\beta$, and $\zeta^S(s)$ has a pole at $s = 1$;
- The L-function $L^S(s - s_\beta, \omega_{\pi_\beta}^{-1}\chi)$ has a pole when $\chi = \omega_{\pi_\beta}$ and $s = 1 + s_\beta$.

Therefore, the L-function $L^S(s, R_7(\Pi_\beta) \times \omega_{\pi_\beta})$ has a pole at $s = 1 + s_\beta$, which is simple if $s_\beta \neq 0$, and is at least double otherwise. Since we already noted that $L^S(s, R_7(\Pi_\alpha) \times \chi)$ does not have this property, we cannot have that Π_α and Π_β are nearly equivalent.

We now distinguish Π_α from Π_0. For Π again an automorphic representation of $G_2(\mathbb{A})$, π an automorphic representation of $GL_2(\mathbb{A})$, and S again a set of places containing the archimedean place and the bad places for Π and π, we consider the degree 14 partial L-function

$$L^S(s, \Pi \times \pi, R_7 \otimes \mathrm{Std}),$$

defined in the obvious way. Then on the one hand, we have,

$$\begin{aligned} &L^S(s, \Pi_\alpha \times \pi_\alpha, R_7 \otimes \mathrm{Std}) \\ &\quad = L^S(s + s_\alpha, \pi_\alpha \times \pi_\alpha) L^S(s - s_\alpha, \pi_\alpha^\vee \times \pi_\alpha) L^S(s, \mathrm{Ad}^3(\pi_\alpha)) L^S(s, \pi_\alpha), \end{aligned}$$

where the first two factors are Rankin–Selberg L-functions; we have further decompositions

$$L^S(s + s_\alpha, \pi_\alpha \times \pi_\alpha) = L^S(s + s_\alpha, \mathrm{Ad}(\pi_\alpha) \times \omega_{\pi_\alpha}) L^S(s + s_\alpha, \omega\pi_\alpha),$$

where ω_{π_α} is the central character of π_α, and

$$L^S(s - s_\alpha, \pi_\alpha^\vee \times \pi_\alpha) = L^S(s - s_\alpha, \mathrm{Ad}(\pi_\alpha)) \zeta^S(s - s_\alpha).$$

Now by Lemma 2.3.2, all of the L-functions in the expression for $L^S(s, \Pi_\alpha \times \pi_\alpha, R_7 \otimes \mathrm{Std})$ above are meromorphic and nonvanishing when $\mathrm{Re}(s) \geq 1 + \mathrm{Re}(s_\alpha)$. Moreover, the second one, $L^S(s, \pi_\alpha \times \pi_\alpha^\vee)$, has a pole at $s = 1 + s_\alpha$. Thus $L^S(s, \Pi_\alpha \times \pi_\alpha, R_7 \otimes \mathrm{Std})$ has a pole at $s = 1 + s_\alpha$. On the other hand, $L^S(s, \Pi_0 \times \pi_\alpha, R_7 \otimes \mathrm{Std})$ is a product of seven L-functions of various character twists of π_α. Since π_α is cuspidal, these L-functions are entire, whence Π_0 cannot be nearly equivalent to Π_α.

A completely analogous argument to this, using twists by π_β instead of π_α, distinguishes Π_β from Π_0 as well; we omit the details. □

Remark 2.3.3. An earlier version of this paper proved only a weaker version of the above result, and used Galois representations to do it. We are grateful to Sug Woo Shin for suggesting there should be a purely automorphic proof of this result along these lines.

We will also need to distinguish between representations occurring in the Eisenstein cohomology of G_2 which come from the same maximal parabolic subgroup of G_2. To do this, we will appeal to strong multiplicity one for the Levi factor, as in the following proposition.

Proposition 2.3.4. *Let π, π' be unitary, cuspidal automorphic representations of $GL_2(\mathbb{A})$, viewed as representations of $M_\alpha(\mathbb{A})$, and assume π and π' are tempered at all but finitely many finite places. Let $s, s' > 0$. If there are irreducible subquotients*

$$\Pi \textit{ of } \iota_{P_\alpha(\mathbb{A})}^{G_2(\mathbb{A})}(\pi, s)$$

and

$$\Pi' \textit{ of } \iota_{P_\alpha(\mathbb{A})}^{G_2(\mathbb{A})}(\pi', s')$$

such that Π and Π' are nearly equivalent, then $s = s'$ and $\pi \cong \pi'$.

Proof. Let v be a finite place which is unramified for both Π and Π', where π_v and π'_v are tempered and where $\Pi_v \cong \Pi'_v$. Then π_v and π'_v are unramified. Since π_v and π'_v are tempered and unitary, there are unramified unitary characters χ_v, χ'_v of $T(\mathbb{Q}_v)$ such that π_v is the unique unramified subquotient of

$$\iota_{(B\cap M_\alpha)(\mathbb{Q}_v)}^{M_\alpha(\mathbb{Q}_v)}(\chi)$$

and π'_v is the unique unramified subquotient of

$$\iota_{(B\cap M_\alpha)(\mathbb{Q}_v)}^{M_\alpha(\mathbb{Q}_v)}(\chi')$$

By induction in stages and the fact that $\Pi_v \cong \Pi'_v$, we have that the unique unramified subquotients of

$$\iota_{B(\mathbb{Q}_v)}^{G_2(\mathbb{Q}_v)}(\chi\delta_{M_\alpha(\mathbb{Q}_v)}^{s}) \quad \text{and} \quad \iota_{B(\mathbb{Q}_v)}^{G_2(\mathbb{Q}_v)}(\chi'\delta_{M_\alpha(\mathbb{Q}_v)}^{s'})$$

coincide. By the theory of Satake parameters, this implies that there is an element w in the Weyl group $W(G_2, T)$ such that

$$w(\chi\delta_{M_\alpha(\mathbb{Q}_v)}^{s}) = \chi'\delta_{M_\alpha(\mathbb{Q}_v)}^{s'}.$$

Now since χ and χ' are unitary, taking absolute values gives

$$(w\delta_{M_\alpha(\mathbb{A})})^{s}(t) = \delta_{M_\alpha(\mathbb{A})}^{s'}(t)$$

for any $t \in T(\mathbb{Q}_p)$. Let γ be a root and consider the equation above with $t = \gamma^\vee(q_v^{-1})$ where q_v is the prime corresponding to the place v; since

$$\delta_{M_\alpha(\mathbb{Q}_v)}(\gamma^\vee(q_v^{-1})) = q_v^{5\langle \alpha+2\beta, \gamma^\vee\rangle},$$

this gives

$$q_v^{5s\langle w(\alpha+2\beta), \gamma^\vee\rangle} = q_v^{5s'\langle \alpha+2\beta, \gamma^\vee\rangle}.$$

Since γ was arbitrary, since $s, s' > 0$, and since $W(G_2, T)$ acts on the short roots of G_2 with the stabilizer of $\alpha + 2\beta$ being $\{1, w_\alpha\}$, this forces $s = s'$ and also $w = 1$ or $w = w_\alpha$. But the induced representations

$$\iota_{(B\cap M_\alpha)(\mathbb{Q}_v)}^{M_\alpha(\mathbb{Q}_v)}(\chi\delta^s_{M_\alpha(\mathbb{Q}_v)}) \quad \text{and} \quad \iota_{(B\cap M_\alpha)(\mathbb{Q}_v)}^{M_\alpha(\mathbb{Q}_v)}(w_\alpha(\chi\delta^s_{M_\alpha(\mathbb{Q}_v)}))$$

have the same unramified subquotients. Thus $\pi_v \cong \pi'_v$. Since this holds for almost all v, $\pi \cong \pi'$ by strong multiplicity one for GL_2. □

Remark 2.3.5. An analogous result as the above proposition holds with P_β or B in place of P_α, and the proof is also completely analogous in either case. We only need this proposition for P_α in this paper, however.

We now begin to examine cohomology spaces for G_2, starting with the following proposition.

Proposition 2.3.6. *Let E be an irreducible, finite dimensional representation of $G_2(\mathbb{C})$, and say that E has highest weight Λ. Write*

$$\Lambda = c_1(2\alpha + 3\beta) + c_2(\alpha + 2\beta)$$

with $c_1, c_2 \in \mathbb{Z}_{\geq 0}$. Let F be a cuspidal eigenform of weight k and trivial nebentypus and π_F its associated automorphic representation, and let $s \in \mathbb{C}$ with $\mathrm{Re}(s) \geq 0$. Assume

$$H^i(\mathfrak{g}_2, K_\infty; \mathrm{Ind}_{P_\alpha(\mathbb{A})}^{G_2(\mathbb{A})}(\pi_F \otimes \mathrm{Sym}(\mathfrak{a}^\vee_{P_\alpha,0})_{(2s+1)\rho_{P_\alpha}}) \otimes E) \neq 0,$$

where, as described in Section 2.1, the subscript on $\mathrm{Sym}(\mathfrak{a}^\vee_{P_\alpha,0})_{(2s+1)\rho_{P_\alpha}}$ indicates a twisted $P_\alpha(\mathbb{A}_f) \times (\mathfrak{p}_{\alpha,0}, K_\infty \cap P_\alpha(\mathbb{R}))$-module structure determined by the weight $(2s+1)\rho_{P_\alpha}$. Then either:

(i) We have

$$i = 4, \qquad k = 2c_1 + c_2 + 4, \qquad s = \frac{c_2 + 1}{10},$$

and

$$H^4(\mathfrak{g}_2, K_\infty; \mathrm{Ind}_{P_\alpha(\mathbb{A})}^{G_2(\mathbb{A})}(\pi_F \otimes \mathrm{Sym}(\mathfrak{a}^\vee_{P_\alpha,0})_{(2s+1)\rho_{P_\alpha}}) \otimes E) \cong \iota_{P_\alpha(\mathbb{A}_f)}^{G_2(\mathbb{A}_f)}(\pi_{F,f}, \tfrac{c_2+1}{10}),$$

(ii) We have

$$i = 5, \qquad k = c_1 + c_2 + 3, \qquad s = \frac{3c_1 + c_2 + 4}{10},$$

and

$$H^5(\mathfrak{g}_2, K_\infty; \mathrm{Ind}_{P_\alpha(\mathbb{A})}^{G_2(\mathbb{A})}(\pi_F \otimes \mathrm{Sym}(\mathfrak{a}^\vee_{P_\alpha,0})_{(2s+1)\rho_{P_\alpha}}) \otimes E) \cong \iota_{P_\alpha(\mathbb{A}_f)}^{G_2(\mathbb{A}_f)}(\pi_{F,f}, \tfrac{3c_1+c_2+4}{10}),$$

(iii) We have

$$i = 6, \qquad k = c_1 + 2, \qquad s = \frac{3c_1 + 2c_2 + 5}{10},$$

and

$$H^6(\mathfrak{g}_2, K_\infty; \mathrm{Ind}_{P_\alpha(\mathbb{A})}^{G_2(\mathbb{A})}(\pi_F \otimes \mathrm{Sym}(\mathfrak{a}^\vee_{P_\alpha,0})_{(2s+1)\rho_{P_\alpha}}) \otimes E) \cong \iota_{P_\alpha(\mathbb{A}_f)}^{G_2(\mathbb{A}_f)}(\pi_{F,f}, \tfrac{3c_1+2c_2+5}{10}).$$

Proof. Note that we have a decomposition

$$\mathfrak{t} = (\mathfrak{m}_{\alpha,0} \cap \mathfrak{t}) \oplus \mathfrak{a}_{P_\alpha,0},$$

and $(\alpha + 2\beta)$ acts by zero on the first component while α acts by zero on the second. By Theorem 2.2.3, in order for our cohomology space to be nontrivial, there needs to be a $w \in W^{P_\alpha}$ with

$$-w(\Lambda + \rho)|_{\mathfrak{a}_{P_\alpha,0}} = 2s\rho_{P_\alpha} = 10s\frac{\alpha + 2\beta}{2},$$

and

$$-w(\Lambda + \rho)|_{\mathfrak{m}_{\alpha,0}} = \pm(k-1)\frac{\alpha}{2}.$$

One computes that, since $\mathrm{Re}(s) \geq 0$, the first of these conditions is possible only when $w = w_{\beta\alpha\beta}, w_{\beta\alpha\beta\alpha}, w_{\beta\alpha\beta\alpha\beta}$. In case $w = w_{\beta\alpha\beta}$, we have

$$-w_{\beta\alpha\beta}(\Lambda + \rho) = -(2c_1 + c_2 + 3)\frac{\alpha}{2} + (c_2 + 1)\frac{\alpha + 2\beta}{2}.$$

Then Theorem 2.2.3 implies (i). The other two cases are similar. □

The following lemma is key; it is the place where we use the vanishing hypothesis for the symmetric cube L-function in the course of proving Theorem 2.3.9.

Lemma 2.3.7. *Let F be a cuspidal eigenform of weight $k \geq 2$ and trivial nebentypus, and π_F its associated unitary automorphic representation. If*

$$L(1/2, \pi_F, \mathrm{Sym}^3) = 0,$$

then for any flat section $\phi_s \in \iota_{P_\alpha(\mathbb{A})}^{G_2(\mathbb{A})}(\pi_F, s)$, the Eisenstein series $E(\phi, 2s\rho_{P_\alpha})$ is holomorphic at $s = 1/10$.

Proof. It follows from the Langlands–Shahidi method and the vanishing hypothesis on the L-function that the constant term of such an Eisenstein series as in the proposition is holomorphic at $s = 1/10$; see, for example, [Sha89], [Kim96] or [Žam97]. □

Let F be a cuspidal eigenform of weight $k \geq 4$ and trivial nebentypus, and let π_F be the automorphic representation attached to F. For $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 0$, let us write

$$\mathcal{L}_\alpha(\pi_F, s) = \text{Unique irreducible quotient of } \iota_{P_\alpha(\mathbb{A})}^{G_2(\mathbb{A})}(\pi_F, s),$$

where we view π_F as an automorphic representation of $M_\alpha(\mathbb{A})$, as usual. Then $\mathcal{L}_\alpha(\pi, s)$ is isomorphic to the restricted tensor product over all places v of the Langlands quotients of the induced representations $\iota_{P_\alpha(\mathbb{Q}_v)}^{G_2(\mathbb{Q}_v)}(\pi_v, s)$. For this reason, we will abusively call the global object $\mathcal{L}_\alpha(\pi, s)$ the *Langlands quotient* of $\iota_{P_\alpha(\mathbb{A})}^{G_2(\mathbb{A})}(\pi, s)$.

We will now study the appearance of $\mathcal{L}_\alpha(\pi_F, 1/10)$ in the Eisenstein cohomology of G_2 (Definition 2.2.1). In doing so, we will require the following lemma.

Lemma 2.3.8. *Let $\gamma \in \{\alpha, \beta\}$, and let π be a unitary, cuspidal automorphic representation of $M_\gamma(\mathbb{A})$. Let $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 0$. Let Π be an irreducible subquotient of $\iota_{P_\alpha(\mathbb{A})}^{G_2(\mathbb{A})}(\pi, s)$. Assume that for some finite dimensional representation E of $G_2(\mathbb{C})$, we have that*

$$H^*(\mathfrak{g}_2, K_\infty; \Pi \otimes E) \tag{2.3.1}$$

is nontrivial and nearly equivalent to $\mathcal{L}_\alpha(\pi_F, 1/10)_f$. Then $\gamma = \alpha$, $s = 1/10$, and $\pi = \pi_F$.

Proof. A priori, we could have that π_∞ is in the complementary series. The first thing to do is show that this is not the case, and the rest of the proof will not be long from there.

By Proposition 2.3.1, we already have $\gamma = \alpha$, and by the nonvanishing hypothesis on (2.3.1), the infinitesimal character of $\mathcal{L}_\gamma(\pi, s)$ is integral and regular, and $s > 0$. Thus the infinitesimal character of π_∞ is also integral and regular. Therefore π is cohomological, and since it is cuspidal, it is generic, and hence tempered at infinity. But then since π_∞ is cohomological and tempered, it is discrete series, and thus π is tempered at all places (by Deligne [Del71; Del74]). We may then apply Proposition 2.3.4 to conclude that $s = 1/10$ and $\pi = \pi_F$, as desired. □

Theorem 2.3.9. *Let F be a cuspidal eigenform of weight $k \geq 4$ and trivial nebentypus, and let π_F be the automorphic representation of $GL_2(\mathbb{A})$ attached to it. Let*

$$\lambda_0 = \frac{k-4}{2}(2\alpha + 3\beta)$$

and let E_{λ_0} be the representation of $G_2(\mathbb{C})$ of highest weight λ_0. Assume

$$L(1/2, \pi_F, \mathrm{Sym}^3) = 0.$$

Then there is a unique summand isomorphic to

$$\iota_{P_\alpha(\mathbb{A}_f)}^{G_2(\mathbb{A}_f)}(\pi_{F,f}, 1/10)$$

in the Eisenstein cohomology

$$H^*_{\mathrm{Eis}}(\mathfrak{g}_2, K_\infty; E_{\lambda_0}),$$

and all irreducible subquotients of this cohomology which are nearly equivalent to $\mathcal{L}_\alpha(\pi_F, 1/10)_f$ appear in this summand. Moreover, this summand appears in middle degree 4.

Proof. Let φ_F be the associate class of automorphic representations of $M_\alpha(\mathbb{A})$ containing $\pi_F \otimes \delta_{P_\alpha(\mathbb{A})}^{1/10}$. Then by Proposition 2.1.3 and Lemma 2.3.7, we have

$$\mathcal{A}_{E_{\lambda_0},[P_\alpha],\varphi_F} \cong \iota_{P_\alpha(\mathbb{A})}^{G_2(\mathbb{A})}(\pi_F \otimes \mathrm{Sym}(\mathfrak{a}^\vee_{P_\alpha,0})_{(6/5)\rho_{P_\alpha}}).$$

By Proposition 2.3.6, we therefore have

$$H^4(\mathfrak{g}_2, K_\infty; \mathcal{A}_{E_{\lambda_0},[P_\alpha],\varphi_F}(G_2) \otimes E_{\lambda_0}) \cong \iota_{P_\alpha(\mathbb{A}_f)}^{G_2(\mathbb{A}_f)}(\pi_{F,f}, 1/10),$$

and that this cohomology vanishes in all other degrees.

By the Franke–Schwermer decomposition, Theorem 2.1.1, in order to prove our theorem, it now suffices to show that

$$H^*(\mathfrak{g}_2, K_\infty; \mathcal{A}_{E_{\lambda_0},[P],\varphi}(G_2) \otimes E_{\lambda_0})$$

contains no irreducible subquotients nearly equivalent to $\mathcal{L}_\alpha(\pi_F, 1/10)_f$ for any proper parabolic subgroup $P \subset G_2$ and any associate class φ, except for $P = P_\alpha$ and $\varphi = \varphi_F$. We do this for the maximal parabolic subgroups P_α and P_β and for B separately; thus the following two lemmas complete the proof of the theorem. □

Lemma 2.3.10. *Let $\gamma \in \{\alpha, \beta\}$, and let φ be an associate class for P_γ. Assume that for some i,*

$$H^i(\mathfrak{g}_2, K_\infty; \mathcal{A}_{E_{\lambda_0},[P_\gamma],\varphi}(G_2) \otimes E_{\lambda_0})$$

contains a subquotient nearly equivalent to $\mathcal{L}_\alpha(\pi_F, 1/10)_f$. Then $\gamma = \alpha$ and $\varphi = \varphi_F$.

Proof. The class φ contains a cuspidal automorphic representation of $M_\gamma(\mathbb{A}) \cong GL_2(\mathbb{A})$, and which therefore must be of the form

$$\pi \otimes \delta^s_{P_\gamma(\mathbb{A})},$$

where π is a unitary cuspidal automorphic representation of $GL_2(\mathbb{A})$ and $s \in \mathbb{C}$. After possibly conjugating by the longest element in the set W^{P_γ}, we may even assume $\mathrm{Re}(s) \geq 0$.

Next we note that the infinitesimal character of $\mathcal{A}_{E_{\lambda_0},[P_\gamma],\varphi}(G_2)$ as a $(\mathfrak{g}_2, K_\infty)$-module must match that of E_{λ_0}, i.e.,

$$\lambda_\pi + 2s\rho_{P_\gamma} = \lambda_0 + \rho,$$

where λ_π is the infinitesimal character of π. But $\lambda_0 + \rho$ is regular and real, and so since λ_π is a multiple of the root γ and ρ_{P_γ} is a multiple of the positive root orthogonal to γ, it follows that λ_π and s are real and nonzero. In particular, $s > 0$ since we assumed $\mathrm{Re}(s) \geq 0$.

Now we apply Theorem 2.1.6 and Proposition 2.1.3 to find that the cohomology space

$$H^*(\mathfrak{g}_2, K_\infty; \mathcal{A}_{E_{\lambda_0},[P_\gamma],\varphi}(G_2) \otimes E_{\lambda_0}),$$

if nontrivial, is made up of subquotients of the cohomology spaces

$$H^*(\mathfrak{g}_2, K_\infty; \mathcal{L}_\gamma(\pi, s) \otimes E_{\lambda_0}) \tag{2.3.2}$$

and

$$H^*(\mathfrak{g}_2, K_\infty; \mathrm{Ind}_{P_\gamma(\mathbb{A})}^{G_2(\mathbb{A})}(\pi \otimes \mathrm{Sym}(\mathfrak{a}^\vee_{P_\gamma,0})_{(2s+1)\rho_{P_\gamma}}) \otimes E_{\lambda_0}). \tag{2.3.3}$$

If (2.3.2) is nonzero, then we conclude by Lemma 2.3.8. Otherwise, if (2.3.3) is nonzero, then π is cohomological; indeed, the cohomology in (2.3.3) is computed in terms of that of π by Theorem 2.2.3. Thus π is tempered by Deligne [Del71; Del74], and so by Proposition 2.3.1, $\gamma = \alpha$, and then by Proposition 2.3.4, $\pi = \pi_F$ and $s = 1/10$. Whence also $\varphi = \varphi'$, as desired. □

Lemma 2.3.11. *Let φ be an associate class for B. Then the cohomology*

$$H^*(\mathfrak{g}_2, K_\infty; \mathcal{A}_{E_{\lambda_0},[B],\varphi}(G_2) \otimes E_{\lambda_0})$$

does not contain any subquotient nearly equivalent to $\mathcal{L}_\alpha(\pi_F, 1/10)_f$.

Proof. Note that the class φ must contain a character of $T(\mathbb{Q})\backslash T(\mathbb{A})$ of the form

$$\psi e^{\langle H_B(\cdot), s_1\alpha + s_2\beta\rangle},$$

where ψ is of finite order and $s_1, s_2 \in \mathbb{C}$. We will study the piece $\mathcal{A}_{E_{\lambda_0},[B],\varphi}(G_2)$ of the Franke–Schwermer decomposition using the Franke–Schwermer–Grbac filtration of Theorem 2.1.4. By that theorem, there is a filtration on the space $\mathcal{A}_{E_{\lambda_0},[B],\varphi}(G_2)$ whose graded pieces are parametrized by certain quadruples (Q, ν, Π, μ). For the convenience of the reader, we recall what these quadruples consist of now:

- Q is a standard parabolic subgroup of G_2;
- ν is an element of $(\mathfrak{t} \cap \mathfrak{m}_{Q,0})^\vee$;
- Π is an automorphic representation of $M_Q(\mathbb{A})$ occurring in

$$L^2_{\mathrm{disc}}(M_Q(\mathbb{Q})A_Q(\mathbb{R})^\circ \backslash M_Q(\mathbb{A}))$$

  and which is spanned by values at, or residues at, the point ν of Eisenstein series parabolically induced from $(B \cap M_Q)(\mathbb{A})$ to $M_Q(\mathbb{A})$ by representations in φ; and
- μ is an element of $\mathfrak{a}^\vee_{Q,0}$ whose real part in $\mathrm{Lie}(A_{M_Q}(\mathbb{R}))$ is in the closure of the positive Weyl chamber, and such that $\nu + \mu$ lies in the Weyl orbit of $\lambda_0 + \rho$.

Then the graded pieces of $\mathcal{A}_{E,[B],\varphi}(G_2)$ are isomorphic to direct sums of $G_2(\mathbb{A}_f) \times (\mathfrak{g}_2, K_\infty)$-modules of the form

$$\mathrm{Ind}_{Q(\mathbb{A})}^{G_2(\mathbb{A})}(\Pi \otimes \mathrm{Sym}(\mathfrak{a}^\vee_{Q,0})_{\mu+\rho_Q})$$

for certain quadruples (Q, ν, Π, μ) of the form just described.

For each of the four possible parabolic subgroups Q and any corresponding quadruple (Q, ν, Π, μ) as above, we will show using Proposition 2.3.1 that the cohomology

$$H^*(\mathfrak{g}_2, K_\infty; \mathrm{Ind}_{Q(\mathbb{A})}^{G_2(\mathbb{A})}(\Pi \otimes \mathrm{Sym}(\mathfrak{a}^\vee_{Q,0})_{\mu+\rho_Q})) \tag{2.3.4}$$

cannot have $\mathcal{L}_\alpha(\pi_{F,f}, 1/10)$ as a subquotient, which will finish the proof.

So first assume we have a quadruple (Q, ν, Π, μ) as above where $Q = B$. Then $\mathfrak{m}_{Q,0} = 0$, forcing $\nu = 0$. The entry Π is the unitarization of a representation in φ, and thus must be a character ψ' of $T(\mathbb{A})$ conjugate by $G_2(\mathbb{A})$ to ψ. Finally, we have μ is Weyl conjugate to $\lambda_0 + \rho$. Therefore the cohomology (2.3.4) is isomorphic, by Theorem 2.2.3, to a finite sum of copies of

$$\iota_{B(\mathbb{A}_f)}^{G_2(\mathbb{A}_f)}(\psi'_f, \mu).$$

By Proposition 2.3.1, $\mathcal{L}_\alpha(\pi_F, 1/10)_f$ cannot be nearly equivalent to a subquotient of this space, and we conclude in the case when $Q = B$.

If now we have a quadruple (Q, ν, Π, μ) where $Q = P_\alpha$, then we find that Π is a representation generated by residual Eisenstein series at the point ν and is therefore a subquotient of the normalized induction

$$\iota_{(B\cap M_\alpha)(\mathbb{A})}^{M_\alpha(\mathbb{A})}(\psi', \nu),$$

where ψ' is as above. Then by Theorem 2.2.3 and induction in stages, (2.3.4) is isomorphic to a subquotient of a finite sum of copies of

$$\iota_{B(\mathbb{A}_f)}^{G_2(\mathbb{A}_f)}(\psi'_f, \nu + \mu).$$

We then conclude in this case as well using Proposition 2.3.1.

The case when $Q = P_\beta$ is completely similar, and we omit the details. When $Q = G_2$, it is once again similar, but we do not need to use induction in stages. So we are done. □

3. Consequences of Arthur's conjectures and cuspidal cohomology

This section is devoted to stating a conjecture, namely Conjecture 3.5.1, and explaining how it follows from Arthur's conjectures. Section 3.1 briefly reviews the results of Adams–Johnson [AJ87] on certain Arthur packets for real groups. In Section 3.2 we make a rough classification of Arthur parameters for $G_2(\mathbb{R})$, and then we explicitly describe the Arthur packets for a particular family of archimedean parameters of Adams–Johnson type in Section 3.3; this latter result is important for describing the archimedean component of the automorphic representations appearing in Conjecture 3.5.1. Then in Section 3.4 we classify the cohomological Arthur parameters of $G_2(\mathbb{R})$. Finally, in Section 3.5, we state Conjecture 3.5.1, explain how it follows from Arthur's conjectures, and prove consequences of it for the cuspidal cohomology of G_2.

For precise statements of Arthur's conjectures, we refer the reader to [Art84] and [Art90, §4]; Section 8 of the latter contains a precise statement of Arthur's conjectural multiplicity formula. The appendix to the book [BC09] contains a nice account of Arthur's conjectures as well.

3.1. **The construction of Adams and Johnson.**

We consider in this section the real case of Arthur's conjectures, and we warn the reader that in this section some of the notation differs in meaning from its use in other sections throughout the paper. So fix G a reductive algebraic group defined over $\mathbb{R}$ with complex Lie algebra $\mathfrak{g}$. We identify G with its $\mathbb{R}$-points. Fix a Cartan involution θ for G and let K be the maximal compact subgroup of G defined by θ. We will assume that G has discrete series, so that there is a maximal torus T for G contained in K.

We will consider the L-group ${}^LG = G^\vee(\mathbb{C}) \rtimes W_\mathbb{R}$. Write j for the element of the Weil group $W_\mathbb{R}$ of $\mathbb{R}$ such that $j^2 = -1$ and $jzj^{-1} = \bar{z}$ for $z \in \mathbb{C}^\times \subset W_\mathbb{R}$.

Recall that an Arthur parameter ψ for G is a homomorphism

$$\psi : W_\mathbb{R} \times SL_2(\mathbb{C}) \to {}^LG.$$

whose restriction to the $W_{\mathbb{R}}$ factor defines a tempered L-parameter for G. Given an Arthur parameter ψ, we define its associated L-parameter ϕ_ψ (which is, in general, different from the restriction of ψ to $W_{\mathbb{R}}$) by

$$\phi_{psi}(w) = \psi\left(w \times \begin{pmatrix} |z|^{1/2} & 0 \\ 0 & |z|^{-1/2} \end{pmatrix}\right), \qquad w \in W_{\mathbb{R}},$$

where $|\cdot| : W_{\mathbb{R}} \to \mathbb{R}_{>0}$ is as usual the homomorphism that restricts to the usual absolute value on $\mathbb{C}^\times \subset W_{\mathbb{R}}$, and such that $|j| = 1$.

Let $\mathfrak{q} \subset \mathfrak{g}$ be a θ-stable parabolic subalgebra containing the complexified Lie algebra $\mathfrak{t}$ of T. The corresponding θ-stable Levi subgroup of $G(\mathbb{R})$ is defined to be the stabilizer of $\mathfrak{q}$ under the adjoint action of $G(\mathbb{R})$. Given such a θ-stable Levi subgroup $L \subset G(\mathbb{R})$ corresponding to a θ-stable parabolic subalgebra $\mathfrak{q}$ containing $\mathfrak{t}$, there is the following natural way to embed ${}^L L$ into ${}^L G$.

Fix an ordering on the roots of the complex Lie algebra $\mathfrak{t}$ of T in $\mathfrak{g}$ which makes $\mathfrak{q}$ standard. Then naturally $T^\vee(\mathbb{C}) \subset L^\vee(\mathbb{C}) \subset G^\vee(\mathbb{C})$. Let n_L be any element of the derived group of $L^\vee(\mathbb{C})$ which sends positive roots in $L^\vee(\mathbb{C})$ for $T^\vee(\mathbb{C})$ to negative ones, and similarly define n_G. Let ρ_L be the half-sum of the positive roots of $T(\mathbb{C})$ in $L(\mathbb{C})$, and similarly for ρ_G. Then we define the embedding $\xi_L : {}^L L \hookrightarrow {}^L G$ first on $\mathbb{C}^\times \subset W_{\mathbb{R}}$ as follows. For $z \in \mathbb{C}^\times$, let $\xi_L(z) = \chi(z) \rtimes z$ where $\chi : \mathbb{C}^\times \to T^\vee(\mathbb{C})$ is the unique homomorphism such that

$$\lambda^\vee(\chi(z)) = z^{\langle \rho_G - \rho_L, \lambda^\vee \rangle} \bar{z}^{-\langle \rho_G - \rho_L, \lambda^\vee \rangle}, \tag{3.1.1}$$

for any cocharacter $\lambda^\vee : \mathbb{C}^\times \to T^\vee(\mathbb{C})$. Then we define

$$\xi_L(j) = n_G n_L^{-1} \rtimes j. \tag{3.1.2}$$

We consider this embedding in the following definition.

Definition 3.1.1. An Arthur parameter $\psi : W_{\mathbb{R}} \times SL_2(\mathbb{C}) \to {}^L G$ is said to be *of Adams–Johnson type* if there is a θ-stable Levi subgroup L of G containing the compact maximal torus T such that the three points below are satisfied.

- The restriction $\psi|_{SL_2(\mathbb{C})}$ sends $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ to a principal unipotent element of $L^\vee(\mathbb{C})$.
- The image of $\psi|_{\mathbb{C}^\times}$ lies in $Z(L^\vee(\mathbb{C})) \rtimes W_{\mathbb{R}}$, where Z denotes the center, considered as a subgroup of ${}^L G$ via the embedding ξ_L defined above.

These two points imply that ϕ_ψ has image contained in ${}^L L$ and that it defines a one dimensional representation π_L of L. Let λ be the restriction of this representation to T. Then we also require:

- L is the stabilizer of a θ-stable parabolic subalgebra $\mathfrak{q}$ with radical $\mathfrak{u}$ such that, for all roots γ in $\mathfrak{u}$, we have $\mathrm{Re}\langle \lambda + \rho_G, \gamma^\vee \rangle \geq 0$.

Adams and Johnson in [AJ87] have constructed packets for any parameter ψ of Adams–Johnson type. Fix such a parameter ψ. Let $\mathfrak{q}$ be the θ-stable parabolic subalgebra of $\mathfrak{g}$ such that $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$ where

$$\mathfrak{l} = \mathfrak{t} \oplus \bigoplus_{\langle\lambda,\gamma^\vee\rangle=0} \mathfrak{g}^\gamma, \qquad \mathfrak{u} = \bigoplus_{\langle\lambda,\gamma^\vee\rangle>0} \mathfrak{g}^\gamma,$$

where the sums are over roots γ of $\mathfrak{t}$ in $\mathfrak{g}$ and $\mathfrak{g}^\gamma$ denotes the one dimensional subspace of $\mathfrak{g}$ corresponding to the root γ.

Adams and Johnson then construct the packet attached to ψ by defining certain Weyl twists of the one-dimensional representation π_L from the second point above, perhaps defined on other Levi subgroups which are inner forms of L, and cohomologically inducing them to G. We now review this construction.

Let W be the Weyl group of $\mathfrak{t}$ in $\mathfrak{g}$. For $w \in W$, we can consider the θ-stable parabolic subalgebra $\mathfrak{q}_w$ of $\mathfrak{g}$ given as follows. Let $\Delta(\mathfrak{q}, \mathfrak{t})$ be the set of roots of $\mathfrak{t}$ in $\mathfrak{q}$, so that

$$\mathfrak{q} = \bigoplus_{\gamma\in\Delta(\mathfrak{q},\mathfrak{t})} \mathfrak{g}^\gamma \oplus \mathfrak{t}.$$

Then we define $\mathfrak{q}_w$ by

$$\mathfrak{q}_w = \bigoplus_{\gamma\in\Delta(\mathfrak{q},\mathfrak{t})} \mathfrak{g}^{w\gamma} \oplus \mathfrak{t}.$$

Let L_w be the θ-stable Levi subgroup corresponding to $\mathfrak{q}_w$.

Adams and Johnson show that there is a one dimensional representation of L_w whose restriction to T is $w\lambda$. Let π_{L_w} be any such representation.

For $i \geq 0$, we consider the cohomological induction functors $\mathcal{R}^i_{\mathfrak{q}}$ from $(\mathfrak{l}, L \cap K)$-modules to $(\mathfrak{g}, K)$-modules as defined in [KV95, Section V.1]. These are normalized so that if Z is an $(\mathfrak{l}, L\cap K)$-module with infinitesimal character given by $\Lambda \in \mathfrak{t}^\vee$, then $\mathcal{R}^i_{\mathfrak{q}}(Z)$ has infinitesimal character given by $\Lambda + \rho(\mathfrak{u})$, where $\rho(\mathfrak{u})$ is half the sum of the roots of $\mathfrak{t}$ in $\mathfrak{u}$ (see [KV95, Corollary 5.25]).

For $w \in W$, let

$$S_w = \frac{1}{2}(\dim(K) - \dim(L_w \cap K)).$$

Let

$$A_{\mathfrak{q}_w}(w\lambda) = \mathcal{R}^{S_w}_{\mathfrak{q}_w}(\pi_{L_w}).$$

Let W_L be the Weyl group of $\mathfrak{t}$ in $\mathfrak{l}$, and W_c the Weyl group of $\mathfrak{t}$ in the complex Lie algebra of K. Adams and Johnson show that if $w, w' \in W$ define the same double coset in

$$W_c\backslash W/W_L,$$

then $A_{\mathfrak{q}_w}(w\lambda) \cong A_{\mathfrak{q}_{w'}}(w'\lambda)$.

Definition 3.1.2. With ψ a parameter of Adams–Johnson type as above, we define the corresponding Adams–Johnson packet to be

$$\Pi^{\mathrm{AJ}}_\psi = \{A_{\mathfrak{q}_w}(w\lambda) \mid w \in W_c\backslash W/W_L\}.$$

As mentioned before, these packets satisfy the conclusion of Arthur's conjecture in the archimedean case. In the next two sections, we will study the Arthur parameters for $G_2(\mathbb{R})$ whose restriction to $SL_2(\mathbb{C})$ is nontrivial. Many of these will turn out to be of Adams–Johnson type, and we will be able to compute the corresponding Adams–Johnson packets.

3.2. **Arthur parameters for $G_2(\mathbb{R})$.** We now study the Arthur parameters for $G_2(\mathbb{R})$. Let $\psi : W_{\mathbb{R}} \times SL_2(\mathbb{C}) \to {}^L(G_2(\mathbb{R}))$ be an Arthur parameter which is nontrivial on the $SL_2(\mathbb{C})$. By the Jacobson–Morozov theorem, the conjugacy class of the homomorphism $\psi|_{SL_2(\mathbb{C})}$ is determined by the conjugacy class of the unipotent element $\psi(\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right))$. Such a class is called a unipotent orbit, and there are five such orbits for G_2, described as follows. For a root γ of the split maximal torus T in G_2, let X_γ be a root vector in the Lie algebra $\mathfrak{g}_2$ corresponding to γ. Then:

- There is the orbit of the identity element, for which we write $\mathcal{O}_0$.
- There is the long root orbit, which is that of $\exp(X_\alpha)$. We write $\mathcal{O}_l$ for this orbit.
- There is the short root orbit, which is that of $\exp(X_\beta)$. We write $\mathcal{O}_s$ for this orbit.
- There is the subregular orbit, which is that of $\exp(X_\alpha + X_{\alpha+3\beta})$. We write $\mathcal{O}_{sr}$ for this orbit.
- There is the regular orbit, which is that of $\exp(X_\alpha + X_\beta)$. We write $\mathcal{O}_r$ for this orbit.

The respective dimensions of these orbits are 0, 6, 8, 10, and 12. The closure of each contains the previous one.

Let us write $\Psi_?(G_2(\mathbb{R}))$, for $? \in \{0, l, s, sr, r\}$, for the set of Arthur parameters ψ such that $\psi|_{SL_2(\mathbb{C})}$ corresponds to the orbit $\mathcal{O}_?$. We aim to classify the Arthur parameters ψ for which $\psi|_{SL_2(\mathbb{C})}$ is nontrivial; that is, we will describe $\Psi_?(G_2(\mathbb{R}))$ for $? \neq 0$.

For γ a root of T, let us write $SL_{2,\gamma}(\mathbb{C})$ for the SL_2-subgroup of $G_2(\mathbb{C})$ corresponding to γ. Then if γ and γ' are orthogonal roots, then $SL_{2,\gamma}(\mathbb{C})$ and $SL_{2,\gamma'}(\mathbb{C})$ are mutual centralizers and their inclusions into $G_2(\mathbb{C})$ induce a map,

$$\iota_{\gamma,\gamma'} : SL_{2,\gamma}(\mathbb{C}) \times SL_{2,\gamma'}(\mathbb{C}) \to G_2(\mathbb{C})$$

with kernel $\{\pm 1\}$ embedded diagonally.

Let us fix a compact maximal torus $T_c \subset G_2(\mathbb{R})$ contained in a maximal compact subgroup K of $G_2(\mathbb{R})$, and let θ be a Cartan involution giving K. Let $\mathfrak{t}_c$ be the complex Lie algebra of T_c. We identify the root system of $\mathfrak{t}_c$ in $\mathfrak{g}_2$ with that of T in G_2 in such a way that β and $2\alpha + 3\beta$ are the positive compact roots.

Let $\mathfrak{q}$ be the θ-stable parabolic subalgebra of $\mathfrak{g}_2$ whose Levi factor $\mathfrak{l}$ contains the roots $\pm\beta$, and whose radical $\mathfrak{u}$ contains the positive roots different from β. Let L be the θ-stable Levi subgroup corresponding to $\mathfrak{q}$. It is isomorphic to the unitary group $U(2)$ since the roots $\pm\beta$ are compact. On the dual side, we identify $L^\vee(\mathbb{C})$ with $GL_2(\mathbb{C})$.

Given an even integer $k \geq 2$, we let $\psi_{L,k} : W_\mathbb{R} \times SL_2(\mathbb{C}) \to {}^L L$ be the homomorphism given by

$$\psi_{L,k}(z) = \begin{pmatrix} (z/|z|)^{k-4} & 0 \\ 0 & (z/|z|)^{k-4} \end{pmatrix} \times z \in L^\vee(\mathbb{C}) \rtimes W_\mathbb{R}, \qquad z \in \mathbb{C}^\times,$$
$$\psi_{L,k}(j) = 1 \times j \in L^\vee(\mathbb{C}) \rtimes W_\mathbb{R},$$
$$\psi_{L,k}(A) = A \times 1 \in SL_2(\mathbb{C}) \rtimes W_\mathbb{R} \subset L^\vee(\mathbb{C}) \rtimes W_\mathbb{R}, \qquad A \in SL_2(\mathbb{C}).$$

Note that this is indeed a homomorphism because k is even, so that $\psi_{L,k}(j)^2 = \psi_{L,k}(-1)$. Let $\xi_L : {}^L L \to {}^L(G_2(\mathbb{R}))$ be the embedding considered in (3.1.1) and (3.1.2), and define

$$\psi_k = \xi_L \circ \psi_{L,k}.$$

Then ψ_k is an Arthur parameter of Adams–Johnson type. As in Definition 3.1.1, the Langlands parameter $\phi_{\psi_{L,k}}$, when viewed as having target ${}^L L$, defines a one dimensional representation of L and hence a character λ_k of T_c. One checks that $\lambda_k = \frac{k-4}{2}(2\alpha + 3\beta)$.

Given an Arthur parameter ψ for $G_2(\mathbb{R})$, we let C_ψ be the centralizer in $G_2(\mathbb{C})$ of the image of ψ in ${}^L G_2(\mathbb{C})$, and $\mathcal{C}_\psi$ the group of connected components of C_ψ. (We do not need to quotient by the center because the center of G_2 is trivial.) The group $\mathcal{C}_\psi$ is called the *component group* of ψ.

Proposition 3.2.1. *Let $\psi \in \Psi_l(G_2(\mathbb{R}))$. Then ψ factors as*

$$\psi : W_\mathbb{R} \times SL_2(\mathbb{C}) \xrightarrow{\phi \times \mathrm{id}_{SL_2(\mathbb{C})}} W_\mathbb{R} \times SL_{2,\beta}(\mathbb{C}) \times SL_{2,2\alpha+3\beta}(\mathbb{C})$$
$$\xrightarrow{\mathrm{id}_{W_\mathbb{R}} \times \iota_{\beta,2\alpha+3\beta}} W_\mathbb{R} \times G_2(\mathbb{C}) = {}^L(G_2(\mathbb{R})),$$

where $\phi : W_\mathbb{R} \to W_\mathbb{R} \times SL_2(\mathbb{C})$ is a tempered Langlands parameter for $PGL_2(\mathbb{R})$. If ψ is unipotent, then $\phi|_{\mathbb{C}^\times}$ is trivial and $\phi(j) \in \{\pm 1, \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)\}$. Otherwise, $\psi = \psi_k$ for some $k \geq 2$ even; in this case, ϕ is the Langlands parameter of the discrete series of weight k for $PGL_2(\mathbb{R})$.

Finally, the component groups are given as follows. If ψ is unipotent, then $\mathcal{C}_\psi$ is trivial. Otherwise $\mathcal{C}_\psi$ has two elements.

Proof. Since $\psi \in \Psi_l(G_2(\mathbb{R}))$, by conjugating we may assume ψ identifies $SL_2(\mathbb{C})$ with $SL_{2,2\alpha+3\beta}(\mathbb{C})$. Then since $SL_{2,\beta}(\mathbb{C})$ is the centralizer of $SL_{2,2\alpha+3\beta}(\mathbb{C})$ in $G_2(\mathbb{C})$, $\psi|_{W_\mathbb{R}}$ must factor through $SL_{2,\beta}(\mathbb{C}) \times W_\mathbb{R}$. Since $\psi|_{W_\mathbb{R}}$ has bounded image, it therefore defines a tempered Langlands parameter, which we take to be our ϕ, of $PGL_2(\mathbb{R})$.

If ψ is unipotent, then $\phi(j)$ is of order 2, and must be either ± 1 or a conjugate of $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. Otherwise ϕ is a discrete series parameter of positive even weight k. Assuming this, we will now show $\psi = \psi_k$.

Recall that the discrete series of weight k for $PGL_2(\mathbb{R})$ has Langlands parameter given by

$$z \mapsto z \times \begin{pmatrix} (z/|z|)^{k-1} & 0 \\ 0 & (z/|z|)^{1-k} \end{pmatrix} \in W_\mathbb{R} \times SL_2(\mathbb{C}), \qquad z \in \mathbb{C}^\times,$$

and

$$j \mapsto j \times \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in W_{\mathbb{R}} \times SL_2(\mathbb{C}).$$

The two equations above therefore define $\phi = \psi|_{W_{\mathbb{R}}}$ when identifying both groups $SL_2(\mathbb{C})$ with $SL_{2,\beta}(\mathbb{C})$. Recall also that the Levi subgroup L used to define ψ_k is a short root Levi subgroup, and therefore we may identify $L^\vee(\mathbb{C})$ with the long root Levi subgroup in $G_2(\mathbb{C})$ given by

$$T_\beta \times SL_{2,2\alpha+3\beta}(\mathbb{C})/\{\pm 1\},$$

where T_β is the maximal torus in $SL_{2,\beta}(\mathbb{C})$. That $\psi = \psi_k$ then follows from a straightforward computation using the definitions; one must use (3.1.1) and (3.1.2), noting that $\rho_G - \rho_L = \frac{3}{2}(2\alpha + 3\beta)$ and that we may take

$$n_G n_L^{-1} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \times 1 \in SL_{2,\beta}(\mathbb{C}) \times SL_{2,2\alpha+3\beta}(\mathbb{C}).$$

Finally, for the component groups, we note that since $\psi|_{SL_2(\mathbb{C})}$ has image $SL_{2,2\alpha+3\beta}(\mathbb{C})$, the centralizer C_ψ is the subgroup of $SL_{2,\beta}(\mathbb{C})$ which centralizes the image of $\psi|_{W_{\mathbb{R}}}$. If ψ is unipotent and $\psi(j)$ is central, then $C_\psi = SL_{2,\beta}(\mathbb{C})$; if ψ is unipotent and $\psi(j) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, then $C_\psi = T_\beta$; if $\psi = \psi_k$, then $C_\psi = \{\pm 1\}$. The result follows. □

We omit the details, but a completely analogous analysis can be made of $\Psi_s(G_2(\mathbb{R}))$ by switching the roles of long and short roots.

Now we look at the subregular parameters.

Proposition 3.2.2. *There are two parameters in $\Psi_{sr}(G_2(\mathbb{R}))$ and they are both unipotent. One has component group S_3, the symmetric group on three elements, and the other has component group $\mathbb{Z}/2\mathbb{Z}$.*

Proof. Since $\mathcal{O}_{sr}$ has dimension 10, the centralizer of any of its representatives is 4 dimensional. On the level of Lie algebras, $X_\alpha + X_{\alpha+3\beta}$ is centralized by the four independent nilpotent elements

$$X_{\alpha+\beta}, \quad X_{\alpha+2\beta}, \quad X_{2\alpha+3\beta}, \quad X_\alpha + X_{\alpha+3\beta},$$

which therefore span the centralizer of $X_\alpha + X_{\alpha+3\beta}$. The other nilpotent element in the $\mathfrak{sl}_2$-triple containing $X_\alpha + X_{\alpha+3\beta}$ is (up to scalar) $X_{-\alpha} + X_{-\alpha-3\beta}$, which is centralized by

$$X_{-\alpha-\beta}, \quad X_{-\alpha-2\beta}, \quad X_{-2\alpha-3\beta}, \quad X_{-\alpha} + X_{-\alpha-3\beta}.$$

Thus the centralizer of the entire $\mathfrak{sl}_2$-triple in $\mathfrak{g}_2$ is trivial.

It follows that the centralizer of the image of the homomorphism $SL_2(\mathbb{C}) \to G_2(\mathbb{C})$ corresponding to $\mathcal{O}_{sr}$ is discrete. Now it is a fact that the component group of the centralizer of $X_\alpha + X_{\alpha+3\beta}$ is S_3. It is easy to check that the Weyl group element w_β, along with $\beta^\vee(\zeta_3)$ with ζ_3 a third root of unity, centralize this $SL_2(\mathbb{C})$ and generate a group isomorphic to S_3 which is therefore the centralizer of this $SL_2(\mathbb{C})$ in $G_2(\mathbb{C})$.

Now let $\psi \in \Psi_{sr}(G_2(\mathbb{C}))$. Then $\psi|_{W_{\mathbb{R}}}$ must have image contained in this subgroup isomorphic to S_3, which implies ψ is unipotent and $\psi(j) = 1$ or $\psi(j)$ is an order two element of this S_3 (all of which are conjugate, even in S_3 itself). In the former case, $C_\psi = S_3$, and in the latter case, $C_\psi = \{1, \psi(j)\}$. □

Finally, we examine $\Psi_r(G_2(\mathbb{R}))$. The analysis is similar to the above proposition.

Proposition 3.2.3. *There is only one parameter in $\Psi_r(G_2(\mathbb{R}))$, and it is trivial on $W_\mathbb{R}$. Its component group is trivial.*

Proof. The component group of the centralizer of $X_\alpha + X_\beta$ in $G_2(\mathbb{C})$ is known to be trivial. The centralizer of this element in $\mathfrak{g}_2$ is 2 dimensional (because $\mathcal{O}_r$ has dimension 12) and contains (and is thus spanned by) $X_{\alpha+2\beta}$ and $X_{2\alpha+3\beta}$. Since $X_{-\alpha} + X_{-\beta}$ is centralized by $X_{-\alpha-2\beta}$ and $X_{-2\alpha-3\beta}$, this implies that the centralizer of the regular $SL_2(\mathbb{C})$ in $G_2(\mathbb{C})$ is discrete, hence trivial.

Thus if $\psi \in \Psi_r(G_2(\mathbb{R}))$, then $\psi|_{W_\mathbb{R}}$ is trivial, and so is the centralizer of the image of ψ. The proposition follows. □

3.3. **Determination of the packet $\Pi^{\mathrm{AJ}}_{\psi_k}$.** For an even integer $k \geq 2$, let $\psi_k \in \Psi_l(G_2(\mathbb{R}))$ be the parameter of Adams–Johnson type defined in the previous section. It gives rise to the character $\lambda_k = \frac{k-4}{2}(2\alpha + 3\beta)$, where $2\alpha + 3\beta$ is viewed as a long compact root for a compact torus T_c. We also let $\mathfrak{q}$ and L be as in the construction of ψ_k. Then $L \cong U(2)$.

For the remainder of this section, we let $w \in W$ be the rotation counterclockwise by $\pi/3$ in the root system, so $w(2\alpha + 3\beta) = \alpha$ and $w\beta = \alpha + 2\beta$. It represents the nontrivial double coset in

$$W_c \backslash W(G_2, T_c)/W_L.$$

By definition, the Adams–Johnson packet for ψ_k is

$$\Pi^{\mathrm{AJ}}_{\psi_k} = \{A_\mathfrak{q}(\lambda_k), A_{\mathfrak{q}_w}(w\lambda_k)\}.$$

We determine these representations when $k \geq 4$. Write $\rho = \rho_{G_2} = 3\alpha + 5\beta$.

Proposition 3.3.1. *Let $k \geq 4$. Then the representation $A_\mathfrak{q}(\lambda_k)$ is the discrete series representation with Harish-Chandra parameter $\frac{k-4}{2}(2\alpha+3\beta)+\rho$. In the terminology of Gan–Gross–Savin [GGS02], this is the quaternionic discrete series of weight $k/2$.*

Proof. We will use the spectral sequence of [KV95, Theorem 11.77]. Let $\mathfrak{b}$ be the standard Borel subalgebra of $\mathfrak{g}_2$ containing the complex Lie algebra $\mathfrak{t}_c$ of T_c, and $\mathfrak{n}$ its radical. Let $\mathfrak{u}$ be the radical of $\mathfrak{q}$ and $\mathfrak{l}$ its Levi factor. Then in our case, this spectral sequence reads

$$\mathcal{R}^i(\mathcal{R}^j(Z \otimes \mathbb{C}_{-2\rho(\mathfrak{n}\cap\mathfrak{l})}) \otimes \mathbb{C}_{-2\rho(\mathfrak{u})}) \Rightarrow \mathcal{R}^{i+j}(Z \otimes \mathbb{C}_{-2\rho(\mathfrak{n})}),$$

for $(\mathfrak{t}_c, T_c)$-modules Z, where the $\mathcal{R}$'s denote cohomological inductions, the ρ's denote the obvious half sums of roots, and the modules $\mathbb{C}_\mu$ for weights μ are the obvious 1 dimensional modules. (See also [KV95, (11.73)] for the discrepancy that gives rise to these half sums.)

Let λ'_k be the character of T_c given by

$$\lambda'_k = \frac{k-4}{2}(2\alpha + 3\beta) + (6\alpha + 10\beta).$$

We apply the spectral sequence above with $Z = \lambda'_k$. We note that

$$2\rho(\mathfrak{n} \cap \mathfrak{l}) = \beta, \qquad 2\rho(\mathfrak{u}) = 6\alpha + 9\beta, \qquad 2\rho(\mathfrak{n}) = 6\alpha + 10\beta.$$

Now on the one hand, by the classification of discrete series via cohomological induction [KV95, Theorem 11.178(a)],

$$\mathcal{R}^2(\lambda'_k \otimes \mathbb{C}_{-2\rho(\mathfrak{n})}) = \mathcal{R}^2(\tfrac{k-4}{2}(2\alpha + 3\beta))$$

is the discrete series of $G_2(\mathbb{R})$ sought, with Harish-Chandra parameter $\frac{k-4}{2}(2\alpha+3\beta) + \rho$. On the other hand, by the same theorem,

$$\mathcal{R}^1(\lambda'_k \otimes \mathbb{C}_{-2\rho(\mathfrak{n}\cap\mathfrak{l})}) = \mathcal{R}^1(\lambda'_k \otimes \mathbb{C}_{-\beta})$$

is the discrete series representation of L with Harish-Chandra parameter $\lambda'_k - \frac{1}{2}\beta$; i.e., it is the character of $L \cong U(2)$ whose restriction to T_c is $\frac{k-4}{2} + (6\alpha + 9\beta)$. Thus

$$\mathcal{R}^1(\lambda'_k \otimes \mathbb{C}_{-2\rho(\mathfrak{n}\cap\mathfrak{l})}) \otimes \mathbb{C}_{-2\rho(\mathfrak{u})}$$

is the character of L given by $\lambda_k = \frac{k-4}{2}(2\alpha+3\beta)$. Cohomologically inducing again gives

$$\mathcal{R}^1(\mathcal{R}^1(\lambda'_k \otimes \mathbb{C}_{-2\rho(\mathfrak{n}\cap\mathfrak{l})}) \otimes \mathbb{C}_{-2\rho(\mathfrak{u})}) = A_{\mathfrak{q}}(\lambda_k)$$

by definition.

Now since all representations considered here have infinitesimal character in the good range (this is where we use $k \geq 4$) these cohomological inductions are concentrated in one degree (see [KV95, Theorem 0.50]) and the spectral sequence collapses. Thus,

$$\mathcal{R}^1(\mathcal{R}^1(\lambda'_k \otimes \mathbb{C}_{-2\rho(\mathfrak{n}\cap\mathfrak{l})}) \otimes \mathbb{C}_{-2\rho(\mathfrak{u})}) = \mathcal{R}^2(\lambda'_k \otimes \mathbb{C}_{-2\rho(\mathfrak{n})}),$$

which, by the above computations, proves the proposition. □

Now we consider the other representation $A_{\mathfrak{q}_w}(w\lambda_k)$ in the packet. We note that the θ-stable Levi subgroup L_w associated with $\mathfrak{q}_w$ is a $U(1,1)$ because its complex Lie algebra $\mathfrak{l}_w$ contains the root $w\beta = \alpha + 2\beta$, which is noncompact. Also, we have $w\lambda_k = \frac{k-4}{2}\alpha$.

Proposition 3.3.2. *With the notation as above, we have*

$$A_{\mathfrak{q}_w}(w\lambda_k) \cong \mathcal{L}_\alpha(\pi_k, 1/10)$$

if $k \geq 4$, where $\mathcal{L}_\alpha(\pi_k, 1/10)$ is the Langlands quotient of the induction of the discrete series π_k of weight k from the long root parabolic subgroup $P_\alpha(\mathbb{R})$.

Proof. It is easy to see by our description of the parameter ψ_k in Proposition 3.2.1 that the Langlands parameter ϕ_{ψ_k} is that of $\mathcal{L}_\alpha(\pi_k, 1/10)$. Since the Adams–Johnson packet of ψ_k contains the L-packet of ϕ_{ψ_k}, we have that this Langlands quotient is the remaining member of our packet, as desired.

One can also compute the representation $A_{\mathfrak{q}_w}(w\lambda_k)$ directly using [Vog81, Theorem 6.6.15]; we omit the details, but explain the result of this computation.

Let T_0 be the center of L_w, and let A be the θ-stable maximal split torus in the derived group of L_w. Then $H = T_0A$ is a maximal torus in L_w. Let $B_w \subset L_w$ be the standard Borel subgroup containing H. Write

$$\mu = w\lambda_k|_{T_0} = \frac{k-4}{2}\alpha|_{T_0},$$

and

$$\nu = \delta_{B_w}^{-1/2}|_A.$$

Finally, let

$$\mu' = w\lambda_k|_{T_0} + \alpha|_{T_0} = \frac{k-2}{2}\alpha|_{T_0}.$$

Then [Vog81, Theorem 6.6.15] implies the isomorphism of standard modules

$$\mathcal{R}^2(\mathrm{Ind}_{B_w}^{L_w}((\mu \otimes \nu) \otimes \delta_{B_w}^{1/2})) \cong \mathrm{Ind}_{P(\mathbb{R})}^{G_2(\mathbb{R})}(\mathcal{R}^1(\mu' \otimes \nu) \otimes \delta_{P(\mathbb{R})}^{1/2}).$$

The unique irreducible subrepresentation of the left hand side is $A_{\mathfrak{q}_w}(w\lambda_k)$, and that of the right hand side is $\mathcal{L}_\alpha(\pi_k, 1/10)$. ☐

We summarize the above results as a theorem.

Theorem 3.3.3. *The Adams–Johnson packet $\Pi_{\psi_k}^{\mathrm{AJ}}$ consists of the quaternionic discrete series of weight $k/2$, of Harish-Chandra parameter $\frac{k-4}{2}(2\alpha + 3\beta) + \rho$, and the Langlands quotient $\mathcal{L}_\alpha(\pi_k, 1/10)$ of the discrete series of weight k from the long root parabolic subgroup $P_\alpha(\mathbb{R})$ of $G_2(\mathbb{R})$.*

3.4. **Cohomological parameters.** We would like to describe all Arthur parameters for $G_2(\mathbb{R})$ whose associated Arthur packets contain a representation with cohomology. This will not be so difficult from what we have set up. However, we need to specify what we mean by "Arthur packets" for parameters which are not of Adams–Johnson type.

In [ABV92], Adams, Barbasch and Vogan define Arthur packets very generally for parameters for real groups, and prove that their packets satisfy the conclusion of Arthur's conjecture [Art84, Conjecture 1.3.3]. These packets are hard to compute in general, but in the case of unipotent parameters, they can be shown to give the unipotent representations constructed by the methods of Barbasch–Vogan [BV85]; see [ABV92, Corollary 27.13]. They are also known to coincide with the packets constructed by Adams–Johnson for parameters of Adams–Johnson type; see [Ara19]. We will write $\Pi_\psi = \Pi_\psi^{\mathrm{ABV}}$ for the Arthur packet for a real Arthur parameter ψ as constructed by Adams–Barbasch–Vogan.

Let ψ be an Arthur parameter for $G_2(\mathbb{R})$ which is nontrivial on $SL_2(\mathbb{C})$. By the results of Section 3.2, any such parameter is either unipotent or of Adams–Johnson type, and therefore we can compute the representations in the packets Π_ψ via the methods of Adams–Johnson or those of Barbasch–Vogan. If, on the other hand, ψ is trivial on $SL_2(\mathbb{C})$, then $\psi|_{W_\mathbb{R}} = \phi_\psi$. Therefore Π_ψ is just the L-packet attached to the tempered Langlands parameter $\psi|_{W_\mathbb{R}}$.

Proposition 3.4.1. *Let ψ be an Arthur parameter for $G_2(\mathbb{R})$. Assume Π_ψ contains a representation with cohomology. Then exactly one of the following holds.*

- *We have $\psi \in \Psi_0(G_2(\mathbb{R}))$. In this case $\psi|_{W_\mathbb{R}}$ is the Langlands parameter for a discrete series representation, and Π_ψ is the corresponding discrete series L-packet. Its members are thus cohomological for middle degree* 4.
- *We have $\psi \in \Psi_l(G_2(\mathbb{R}))$. In this case $\psi = \psi_k$ for some even $k \geq 4$ in the notation of Proposition 3.2.1, and the representations in Π_ψ are both cohomological for the irreducible representation of G_2 of highest weight $\frac{k-4}{2}(2\alpha+3\beta)$; moreover, the L-packet element has cohomology exactly in degrees* 3 *and* 5*, and the other has cohomology exactly in degree* 4.
- *We have $\psi \in \Psi_s(G_2(\mathbb{R}))$ and ψ is obtained in the same way as ψ_k with $k \geq 4$ as in Proposition 3.2.1, except that $SL_{2,\beta}(\mathbb{C})$ and $SL_{2,2\alpha+3\beta}(\mathbb{C})$ are switched in the construction. There are two representations in Π_ψ and they are both cohomological for the irreducible representation of G_2 of highest weight $\frac{k-4}{2}(\alpha + 2\beta)$. Moreover, the L-packet element has cohomology exactly in degrees* 3 *and* 5*, and the other has cohomology exactly in degree* 4.
- *We have $\psi \in \Psi_r(G_2(\mathbb{R}))$. Then Π_ψ contains only the trivial representation of $G_2(\mathbb{R})$. It is cohomological for degrees* $0, 2, 4, 6, 8$.

Proof. As in Section 3.2, we classify the Arthur parameters according to their restriction to $SL_2(\mathbb{C})$. If $\Psi|_{SL_2(\mathbb{C})}$ is trivial, then $\phi_\psi = \psi|_{W_\mathbb{R}}$, which is tempered by definition, and moreover $\Pi_\psi = \Pi_{\phi_\psi}$. Being cohomological and tempered, ϕ_ψ must correspond to a discrete series L-packet, which proves the proposition in the case that $\psi \in \Psi_0(G_2(\mathbb{R}))$.

Next, let $\psi \in \Psi_l(G_2(\mathbb{R}))$ be cohomological. We note that ψ cannot be unipotent because the representations corresponding to unipotent parameters were classified by Vogan in [Vog94, Theorem 18.3], and they all have irregular infinitesimal character. (Note that Vogan's list contains one extra unipotent representation because he is working with the double cover of $G_2(\mathbb{R})$ and one of the representations he obtains does not factor through the projection to $G_2(\mathbb{R})$.)

Thus $\psi = \psi_k$ for some $k \geq 2$ even, as in Proposition 3.2.1. But we cannot have $k = 2$ because in this case the Adams–Johnson packets for ψ_2 contain only representations with irregular infinitesimal character; they are $A_\mathfrak{q}(\lambda)$'s with $\lambda = -(2\alpha + 3\beta)$ and therefore have infinitesimal character $\rho - (2\alpha + 3\beta) = \alpha + 2\beta$.

Thus $\psi = \psi_k$ for $k \geq 4$, and it follows from Theorem 3.3.3 that the L-packet element of Π_{ψ_k} is cohomological in degrees 3 and 5. See [BW00, Theorem VI.1.7] for the the justification of this claim in the case of trivial coefficients; the proof for twisted coefficients follows the same lines. The other element of Π_{ψ_k} is cohomological in degree 4 because it is discrete series. Alternatively, one can see both of these claims directly from the presentation of these representations as $A_\mathfrak{q}(\lambda)$'s using [VZ84, Theorem 5.5]. This proves the proposition in case $\psi \in \psi_l(G_2(\mathbb{R}))$.

The case of $\psi \in \Psi_s(G_2(\mathbb{R}))$ is completely analogous; one uses instead [Vog94, Theorem 18.4] to handle the unipotent representations.

Next we note that ψ cannot be in $\Psi_{sr}(G_2(\mathbb{R}))$, for then by Proposition 3.2.2, ψ is unipotent and the unipotent representations for these parameters were classified in [Vog94, Theorem 18.5]; they all have irregular infinitesimal character.

Finally, by Proposition 3.2.3, there is only one parameter ψ in $\Psi_r(G_2(\mathbb{R}))$ and $\psi|_{W_\mathbb{R}}$ is trivial. Moreover, $\Pi_\psi = \Pi_{\phi_\psi}$ because the component groups are trivial. We have that the restriction of ψ to the maximal torus of $SL_2(\mathbb{C})$ is $(\alpha+2\beta)^\vee + (2\alpha+3\beta)^\vee = 3\beta^\vee + 5\alpha^\vee$, and therefore ϕ_ψ is the parameter corresponding to the Langlands quotient of the representation induced from the Borel subgroup from the character ρ; that is, it is the trivial representation. The claim about the cohomological degrees of the trivial representation is a straightforward computation which we omit. □

3.5. **Occurrence of $\mathcal{L}_\alpha(\pi_F, 1/10)$ in the cuspidal spectrum.** We begin by making the following conjecture.

Conjecture 3.5.1. *Let F be a cuspidal holomorphic eigenform of weight $k \geq 4$ and trivial nebentypus, π_F its associated automorphic representation, and $\Pi_{F,f} = \mathcal{L}_\alpha(\pi_F, 1/10)_f$ the finite part Langlands quotient of $\pi_F \otimes |\det|^{1/2}$ from the long root parabolic subgroup of G_2. Moreover, let v be a finite place at which π_F is unramified, and let $\Pi_{F,f}^v$ be the component of $\Pi_{F,f}$ away from v. Then the $\Pi_{F,f}^v$-isotypic component of the discrete spectrum is given by*

$$L^2_{\mathrm{disc}}(G_2(\mathbb{Q})\backslash G_2(\mathbb{A}))[\Pi_{F,f}^v] = \Pi_{F,f} \otimes \Pi_\infty,$$

where:

(a) In the case that $\epsilon(1/2, \pi_F, \mathrm{Sym}^3) = 1$, the representation Π_∞ is the archimedean component $\mathcal{L}_\alpha(\pi_F, 1/10)_\infty$ of the same Langlands quotient, or,

(b) In the case that $\epsilon(1/2, \pi_F, \mathrm{Sym}^3) = -1$, the representation Π_∞ is the (quaternionic) discrete series representation of $G_2(\mathbb{R})$ with Harish-Chandra parameter $\frac{k-4}{2}(2\alpha + 3\beta) + \rho$.

Remark 3.5.2. The finite place v in this conjecture is inserted for technical reasons having to do with the p-adic deformation of $\Pi_{F,f}$, which will be the subject of the paper [Mun] which follows this one. In that paper, the place v will be the one corresponding to a prime p at which π_F is unramified, and because of a small technical issue involving p-stabilizations, we will need to be precise about which discrete automorphic representations can have finite part equivalent to $\Pi_{F,f}$ away from p. The conjecture says then that there are no others but the ones which are equivalent to $\Pi_{F,f}$ at all finite places. But for the purposes of this paper alone, the reader may ignore any occurrences of this auxiliary place v.

We will show how the conjecture above follows from Arthur's multiplicity formula, along with the assertion that Adams–Johnson packets should coincide with the packets at infinity for cohomological parameters for G_2. This will use the material from the previous four subsections. We will then show how to use the results of Section 2 to upgrade the statement of this conjecture, which is about the discrete spectrum, to one about the cuspidal spectrum.

Now there are two main assertions from which this conjecture would follow:

(1) There is the assertion that the representation $\Pi_{F,f} \otimes \Pi_\infty$ occurs in the discrete spectrum, with Π_∞ depending on the symmetric cube root number as described, with multiplicity one.
(2) There is the assertion that $\Pi^v_{F,f}$ cannot occur in the discrete spectrum as the finite part away from v of any other representation beside the one described above.

To justify assertion (1) using Arthur's multiplicity formula, we first note that π_F factors through $PGL_2(\mathbb{A})$ and so should define a tempered Langlands parameter $\phi_F : L_{\mathbb{Q}} \to SL_2(\mathbb{C})$, where $L_{\mathbb{Q}}$ is the (conjectural) Langlands group of $\mathbb{Q}$. Following Gan and Gurevich [GG09], we define a global Arthur parameter $\psi_F : L_{\mathbb{Q}} \times SL_2(\mathbb{C}) \to {}^LG_2$ as the composition

$$\begin{aligned}\psi_F : L_{\mathbb{Q}} \times SL_2(\mathbb{C}) &\xrightarrow{\phi_F \times \mathrm{id}_{SL_2(\mathbb{C})}} L_{\mathbb{Q}} \times SL_{2,\beta}(\mathbb{C}) \times SL_{2,2\alpha+3\beta}(\mathbb{C})\\ &\xrightarrow{\mathrm{id}_{W_{\mathbb{R}}} \times \iota_{\beta,2\alpha+3\beta}} L_{\mathbb{Q}} \times G_2(\mathbb{C}) = {}^LG_2,\end{aligned}$$

where the notation is as in Section 3.2. Note that the local component of ψ_F at ∞ is the parameter ψ_k from Section 3.2 by Proposition 3.2.1. The implications of Arthur's multiplicity formula for this parameter ψ_F were explained in [GG09, Section 13.4]. We briefly recall this now. The reader may refer to [Art90, §4] for the definitions of the global component group $\mathcal{C}_\psi$ (called $\mathcal{S}^+_\psi$ in *loc. cit.*), and the character ε_ψ which we use below, as well as a formulation of the multiplicity formula in terms of this data. The local component groups $\mathcal{C}_{\psi_w}$ are defined just as in the archimedean case; see the remark above Proposition 3.2.1.

First, at places w where $\pi_{F,w}$ is discrete series, the component group $\mathcal{C}^+_{\psi_{F,w}}$ has two elements; otherwise it is trivial. Thus we expect the Arthur packets $\Pi_{\psi_{F,w}}$ attached to $\psi_{F,w}$ at places w where $\pi_{F,w}$ is discrete series to have two elements, and we write $\Pi_{\psi_{F,w}} = \{\pi^+_w, \pi^-_w\}$, where π^+_w is the L-packet element. Otherwise we have $\Pi_{\psi_{F,w}} = \{\pi^+_w\}$, the singleton containing the L-packet element. The L-packet elements are visibly the Langlands quotients $\mathcal{L}_\alpha(\pi_{F,w}, 1/10)$ from the long root parabolic subgroup. Note in particular that for $w = v$, the packet $\Pi_{\psi_{F,v}}$ is the singleton containing $\mathcal{L}_\alpha(\pi_{F,v}, 1/10)$.

The global component group $\mathcal{C}_{\psi_F}$ has two elements and the character ε_ψ can be computed to be the character which sends the nontrivial element in $\mathcal{C}_\psi$ to the root number $\epsilon(1/2, \pi_F, \mathrm{Sym}^3)$. Thus, if π is of the form

$$\pi = \bigotimes_w{}' \pi_w, \qquad \pi_w \in \Pi_{\psi_w},$$

and if we let $\epsilon_{\pi_w} = 1$ or -1 according to whether π_w is π^+_w or π^-_w, respectively, then π occurs with multiplicity 1 in $L^2_{\mathrm{disc}}(G_2(\mathbb{Q})\backslash G_2(\mathbb{A}))$ if $\prod_w \epsilon_{\pi_w} = \epsilon(1/2, \pi_F, \mathrm{Sym}^3)$; otherwise π occurs with multiplicity 0. Since the Arthur packet at ∞ coincides with the possibilities for Π_∞ described in our conjecture by Theorem 3.3.3, we have justified the assertion (1) above.

To understand why (2) should hold, we have to examine other global Arthur parameters ψ. More precisely, let Π'_∞ be a representation of $G_2(\mathbb{R})$ and Π'_v a representation of $G_2(\mathbb{Q}_v)$ such that $\Pi^v_{F,f} \otimes \Pi'_v \otimes \Pi'_\infty$ occurs in $L^2_{\text{cusp}}(G_2(\mathbb{Q})\backslash G_2(\mathbb{A}))$, and let ψ be the global Arthur parameter corresponding to this representation. We must show that $\psi = \psi_F$.

To see this, we first break the set of global Arthur parameters for G_2 into five subsets, called $\Psi_?(G_{2/\mathbb{Q}})$, $? \in \{0, l, s, sr, r\}$, based on their restrictions to $SL_2(\mathbb{C})$, just as in Section 3.2; the definitions of these subsets are just as in that section. We note that the restriction of ψ to $SL_2(\mathbb{C})$ is the same as the restriction of any of the local components ψ_w to $SL_2(\mathbb{C})$.

Now if $w \neq v$ is a finite place which is unramified for $\Pi_{F,f}$ and such that the local packet Π_{ψ_w} contains only one element, then the L-parameter ϕ_{ψ_w} is nontempered because it must coincide with the L-parameter $\phi_{\psi_{F,w}}$; indeed, if q_w is the prime corresponding to w and t_w is the Satake parameter of π_F at w, then

$$\phi_{\psi_{F,w}}(\mathrm{Frob}_w^{-1}) = t_w \times \begin{pmatrix} q_w^{1/2} & 0 \\ 0 & q_w^{-1/2} \end{pmatrix} \in SL_{2,\beta}(\mathbb{C}) \times SL_{2,2\alpha+3\beta}(\mathbb{C}),$$

whose powers are unbounded. Thus we cannot have $\psi|_{W'_w} = \phi_{\psi_{F,w}}$, which implies $\psi \notin \Psi_0(G_{2/\mathbb{Q}})$.

Now assume for sake of contradiction that $\psi \in \Psi_r(G_{2/\mathbb{Q}})$. Then $\psi|_{L_\mathbb{Q}}$ is trivial, because the centralizer of the image of the corresponding homomorphism from $SL_2(\mathbb{C})$ is trivial, as we saw in Proposition 3.2.3. Moreover, at a finite place w at which $\Pi_{F,w}$ is unramified and Π_{ψ_w} contains only one element, we have

$$\psi_{\psi_w}(\mathrm{Frob}_w^{-1}) = (3\beta^\vee + 5\alpha^\vee)(q_w^{1/2}),$$

which does not coincide with our expression for $\phi_{\psi_{F,w}}(\mathrm{Frob}_w^{-1})$ above. Thus $\psi \notin \Psi_r(G_{2/\mathbb{Q}})$. A similar argument, applied with Frob_w^{-6}, shows also that $\psi \notin \Psi_{sr}(G_{2/\mathbb{Q}})$, since the centralizer of the image of the subregular homomorphism $SL_2(\mathbb{C}) \to G_2(\mathbb{C})$ is a group of order 6.

Now if $\psi \in \Psi_s(G_{2/\mathbb{Q}})$, then $\psi|_{L_\mathbb{Q}}$ must factor through a long root $SL_2(\mathbb{C})$. Therefore $\Pi_{F,f}$ would be nearly equivalent to a Langlands quotient of a parabolic induction of a tempered representation from both the long root and short root parabolic subgroups. By Proposition 2.3.1, this is impossible.

Thus $\psi \in \Psi_l(G_{2/\mathbb{Q}})$, and $\psi|_{L_\mathbb{Q}}$ factors through a short root $SL_2(\mathbb{C})$. Moreover, by strong multiplicity one for GL_2, the parameter $\psi|_{L_\mathbb{Q}}$ corresponds to π_F, and thus $\psi = \psi_F$, as desired.

This completes our justification of Conjecture 3.5.1 using Arthur's multiplicity formula. We would like to remark, however, that using theta correspondence for $PU(3) \times G_2$, the work [BHLS24] completes the program proposed in [GG09] and proves the multiplicity formula for long root CAP representations of G_2 that are induced from dihedral cuspidal representations of PGL_2. Moreover, by [HPS96, Theorem 5.2], this theta correspondence is functorial at the archimedean place and gives the quaternionic discrete series representation that we obtained in Proposition 3.3.1. Unfortunately, the paper [Mun] which is sequel to this one needs this conjecture in the non-dihedral case. There is still hope that the relative trace formula approach to the endoscopy classification for classical groups can shed light on the conjecture in this case.

We conclude by showing that Conjecture 3.5.1 implies a multiplicity formula for $\Pi_{F,f}$ in the cuspidal spectrum, in light of the material developed in Section 2.

Proposition 3.5.3. *In the notation of Conjecture 3.5.1, assuming that conjecture, we have that $\Pi_{F,f}^v$ is the finite component away from v of a unique discrete automorphic representation, and this representation is cuspidal if and only if $L(1/2, \pi_F, \mathrm{Sym}^3) = 0$.*

Proof. If $L(1/2, \pi_F, \mathrm{Sym}^3) \neq 0$, then any section of $\iota_{P_\alpha(\mathbb{A})}^{G_2(\mathbb{A})}(\pi_F, s)$ mapping nontrivially to the Langlands quotient of this induction at $s = 1/10$ gives rise to an Eisenstein series with a simple pole at $s = 1/10$, and the corresponding residual Eisenstein series generate a copy of $\mathcal{L}_\alpha(\pi_F, 1/10)$ in $L^2_{\mathrm{res}}(G_2(\mathbb{Q})\backslash G_2(\mathbb{A}))$; see [Kim96] or [Žam97]. Conjecture 3.5.1 says that this should be the only discrete automorphic representation with finite part away from v given by $\Pi_{F,f}^v$. Otherwise, if $L(1/2, \pi_F, \mathrm{Sym}^3) = 0$, no such section can give rise to an Eisenstein series with a pole at $s = 1/10$ by Lemma 2.3.7. Thus to prove the proposition, it suffices to show that if $\mathcal{A}_{E,[P],\varphi}(G_2)$ is any Franke–Schwermer piece containing a residual Eisenstein representation Π' which is nearly equivalent to $\mathcal{L}_\alpha(\pi_F, 1/10)$, then $P = P_\alpha$ and φ contains $\pi_F \otimes \delta_{P_\alpha(\mathbb{A})}^{1/10}$. Actually, in this case, Proposition 2.3.1 already implies $P = P_\alpha$, so it suffices to show that φ contains $\pi_F \otimes \delta_{P_\alpha(\mathbb{A})}^{1/10}$.

Let $\pi' \in \varphi$ be a cuspidal representation of $M_\alpha(\mathbb{A})$ with central character given by a the product of finite order character of $\mathbb{A}^\times$ with $\delta_{P_\alpha(\mathbb{A})}^{s}$ with $\mathrm{Re}(s) \geq 0$. By assumption, Conjecture 3.5.1 implies $\Pi'_\infty = \Pi'$, and therefore, by looking at infinitesimal characters, $s > 0$. Moreover, we then have that $\Pi' = \mathcal{L}_\alpha(\pi', s)$, and that this is cohomological (since Π_∞ is). Then Lemma 2.3.8 implies that $\pi' = \pi$ and $s = 1/10$, as desired. ∎

This proposition gives us information about the cuspidal cohomology of G_2 (see Definition 2.2.1).

Theorem 3.5.4. *Let the notation be as in Conjecture 3.5.1, and assume that conjecture. Let*

$$\lambda_0 = \frac{k-4}{2}(2\alpha + 3\beta),$$

and let E_{λ_0} be the representation of $G_2(\mathbb{C})$ of highest weight λ_0. Assume

$$L(1/2, \pi_F, \mathrm{Sym}^3) = 0.$$

Then

$$H^i_{\mathrm{cusp}}(\mathfrak{g}_2, K_\infty; E)[\Pi_{F,f}^v] = \begin{cases} \Pi_{F,f} & \textit{if } i = 4 \textit{ and } \epsilon(1/2, \pi_F, \mathrm{Sym}^3) = -1, \\ & \textit{or if } i = 3,5 \textit{ and } \epsilon(1/2, \pi_F, \mathrm{Sym}^3) = +1; \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. This follows immediately from Conjecture 3.5.1 and Proposition 3.5.3. □

Remark 3.5.5. Theorem 3.5.4 together with Theorem 2.3.9 give a complete description of where $\Pi_{F,f}$ appears in the automorphic cohomology of G_2 with coefficients in E_{λ_0} when $L(1/2, \pi_F, \mathrm{Sym}^3) = 0$. When $L(1/2, \pi_F, \mathrm{Sym}^3) \neq 0$, $\Pi_{F,f}$ can only appear in Eisenstein cohomology by Proposition 3.5.3, and moreover the proof of that proposition shows that the $\Pi_{F,f}$-isotypic component is the space

$$H^*(\mathfrak{g}_2, K_\infty)(\mathcal{A}_{E_{\lambda_0},[P_\alpha],\varphi_F}(G_2) \otimes E_{\lambda_0})$$

where φ_F is the associate class containing $\pi_F \otimes \delta_{P_\alpha(\mathbb{A})}^{1/10}$.

By Grbac's theorem ([Grb12], stated above as Theorem 2.1.6) there is an exact sequence

$$0 \to \mathcal{L}_{P_\alpha(\mathbb{A})}^{G_2(\mathbb{A})}(\pi_F, 1/10) \to \mathcal{A}_{E_{\lambda_0},[P_\alpha],\varphi_F}(G_2) \\ \to \mathrm{Ind}_{P_\alpha(\mathbb{A})}^{G_2(\mathbb{A})}(\pi_F \otimes \mathrm{Sym}(\mathfrak{a}_{P_\alpha,0}^\vee)_{(6/5)\rho_{P_\alpha}}) \to 0.$$

The $(\mathfrak{g}_2, K_\infty)$-cohomology of the first term tensored with E_{λ_0} is just isomorphic to $\Pi_{F,f}$ in degrees 3 and 5; that of the last term is $\iota_{P_\alpha(\mathbb{A}_f)}^{G_2(\mathbb{A}_f)}(\pi_{F,f}, 1/10)$ by Proposition 2.3.6. Thus the long exact sequence in $(\mathfrak{g}_2, K_\infty)$-cohomology obtained from this short exact sequence after tensoring it with E_{λ_0} shows that

$$H^3(\mathfrak{g}_2, K_\infty; \mathcal{A}_{E_{\lambda_0},[P_\alpha],\varphi_F}(G_2) \otimes E_{\lambda_0}) \cong \Pi_{F,f},$$

and that, moreover, there is a boundary map from degrees 4 to 5. This boundary map is a map

$$\iota_{P_\alpha(\mathbb{A}_f)}^{G_2(\mathbb{A}_f)}(\pi_{F,f}, 1/10) \to \Pi_{F,f}.$$

It would be very interesting to know whether this map is nontrivial, and hence a cohomological realization of the intertwining operators at finite places. If this is the case, then we would have

$$H^5(\mathfrak{g}_2, K_\infty; \mathcal{A}_{E_{\lambda_0},[P_\alpha],\varphi_F}(G_2) \otimes E_{\lambda_0}) = 0,$$

and that in middle degree 4 the cohomology would be given by the kernel of the intertwining operator.

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