

UNIFORM BOUNDS IN EXCELLENT RINGS AND APPLICATIONS TO SEMICONTINUITY

SHIJI LYU

ABSTRACT. This is a draft of a paper in preparation on certain uniform behaviours on the spectrum of an excellent \mathbf{F}_p -algebra.

1. SUMMARY OF RESULTS

In this section, let R be an excellent (Noetherian) \mathbf{F}_p -algebra.

Theorem 1.0.1 (a uniform version of the Cohen-Gabber theorem; see Theorem 3.4.3). *Assume that R is (R_0) . Then there exist constants $\delta, \mu, \Delta \in \mathbf{Z}_{\geq 0}$ depending only on R , and a quasi-finite, syntomic ring map $R \rightarrow S$, such that for all $\mathfrak{p} \in \operatorname{Spec}(R)$, there exist a $\mathfrak{q} \in \operatorname{Spec}(S)$ above \mathfrak{p} and a ring map $P \rightarrow S_{\mathfrak{q}}^{\wedge}$ that satisfy the followings.*

- (i) (P, \mathfrak{m}_P) is a formal power series ring over a field.
- (ii) $P/\mathfrak{m}_P = \kappa(\mathfrak{q})$.
- (iii) $P \rightarrow S_{\mathfrak{q}}^{\wedge}$ is finite and generically étale of generic degree $\leq \delta$.
- (iv) $\mathfrak{q}S_{\mathfrak{q}}^{\wedge}/\mathfrak{m}_PS_{\mathfrak{q}}^{\wedge}$ is generated by at most μ elements.
- (v) There exist $e_1, \dots, e_n \in S_{\mathfrak{q}}^{\wedge}$ that map to a basis of $S_{\mathfrak{q}}^{\wedge} \otimes_P \operatorname{Frac}(P)$ (as an $\operatorname{Frac}(P)$ -vector space), such that $\operatorname{Disc}_{S_{\mathfrak{q}}^{\wedge}/P}(e_1, \dots, e_n) \notin \mathfrak{m}_P^{\Delta+1}$.

The next few results are established in [Smi16] and [Pol18] for F -finite rings or rings essentially of finite type over an excellent local ring.

Theorem 1.0.2 (see Corollary 5.2.5). *For every finite R -module M , there exists a constant $C(M)$ with the following property. For all $\mathfrak{p} \in \operatorname{Spec}(R)$, all ideals $I \subseteq J$ of $R_{\mathfrak{p}}$ with $l_{R_{\mathfrak{p}}}(J/I) < \infty$, and all $e \leq e' \in \mathbf{Z}_{\geq 1}$, the following holds.*

$$\left| \frac{1}{p^{e \dim M_{\mathfrak{p}}}} l_{R_{\mathfrak{p}}} \left(\frac{J^{[p^e]} M_{\mathfrak{p}}}{I^{[p^e]} M_{\mathfrak{p}}} \right) - \frac{1}{p^{e' \dim M_{\mathfrak{p}}}} l_{R_{\mathfrak{p}}} \left(\frac{J^{[p^{e'}]} M_{\mathfrak{p}}}{I^{[p^{e'}]} M_{\mathfrak{p}}} \right) \right| \leq C(M) p^{-e} l_{R_{\mathfrak{p}}}(J/I).$$

Here by convention the left hand side is zero if $M_{\mathfrak{p}} = 0$.

Theorem 1.0.3 (see Theorem 6.1.3). *Assume that R is locally equidimensional. Then the function $\mathfrak{p} \mapsto e_{\operatorname{HK}}(R_{\mathfrak{p}})$ is upper semi-continuous on R .*

Theorem 1.0.4 (see Theorem 6.2.6 and Corollary 6.2.7; restriction comes from [EY11]). *Assume that R is either a quotient of a regular ring, or Gorenstein. Then the function $\mathfrak{p} \mapsto s(R_{\mathfrak{p}})$ is lower semi-continuous on R , and the strongly F -regular locus of R is open.*

2. PRELIMINARIES

2.1. Local Bertini. We recall the following classical theorem. See also [Tri94] and [OS15].

Theorem 2.1.1 ([Fle77, Satz 2.1]). *Let (A, \mathfrak{m}) be a Noetherian local ring containing a field.¹ Let I be a proper ideal of A . Let $D(I)$ be the open subset $\text{Spec}(A) \setminus V(I)$ of $\text{Spec}(A)$. Let Σ be a finite subset of $D(I)$.*

Then there exists an element $a \in I$ that is not contained in any prime in Σ , and is not contained in $\mathfrak{p}^{(2)}$ for any $\mathfrak{p} \in D(I)$.

We shall use the following consequence.

Lemma 2.1.2. *Let (A, \mathfrak{m}) be a Noetherian J-2 local ring that is (R_0) . Assume $d := \dim A \geq 1$. Then there exist elements $a_1, \dots, a_{d-1} \in \mathfrak{m}$ such that*

- (i) a_{j+1} is not in any minimal prime of (a_1, \dots, a_j) ; and that
- (ii) $A/(a_1, \dots, a_{d-1})$ is (R_0) .

Proof. This follows from the argument in [Fle77, §3]. We reproduce the proof for the reader's convenience.

We can assume $d > 1$. By induction, it suffices to find an element $a_1 = a \in \mathfrak{m}$ not in any minimal prime of A such that A/aA is (R_0) .

Since A is J-2, the singular locus $\text{Sing}(A)$ is closed in $\text{Spec}(A)$. Since A is (R_0) , $\Sigma_1 = \{\mathfrak{p} \in \text{Sing}(A) \mid \text{ht } \mathfrak{p} \leq 1\}$ is finite. Let Σ_2 be the set of minimal primes of A . Then $\mathfrak{m} \notin \Sigma_1 \cup \Sigma_2$ since $d > 1$.

By Theorem 2.1.1, we can find $a \in \mathfrak{m}$ such that a is not in any prime in $\Sigma_1 \cup \Sigma_2$ and that $a \notin \mathfrak{p}^{(2)}$ for all $\mathfrak{p} \in \text{Spec}(A) \setminus \{\mathfrak{m}\}$. It is then straightforward to verify that A/aA is (R_0) . \square

2.2. Discriminant. We review some basic facts about the discriminant we will use, which is related to the Dedekind different, [Stacks, Tag 0BW0], cf. [Lan86, Chapter III].

We let A be a normal domain, K its fraction field, B a finite A -algebra, and we assume $B \otimes_A K$ finite étale over K of degree n .

Definition 2.2.1. Let $e_1, \dots, e_n \in B$ be elements that map to a basis of $B \otimes_A K$. The *discriminant* of e_1, \dots, e_n is

$$\text{Disc}_{B/A}(e_1, \dots, e_n) = \det(\text{Tr}_{B/A}(e_i e_j)_{i,j}).$$

where $\text{Tr}_{B/A}$ denotes the Galois-theoretic trace map $B \otimes_A K \rightarrow K$.

Since B is integral over A and A is normal, we have $\text{Tr}_{B/A}(B) \subseteq A$, and thus $\text{Disc}_{B/A}(e_1, \dots, e_n) \in A$. Moreover, it is clear that the discriminant is unchanged along a flat base change $A \rightarrow A'$ of normal domains.

This notion is useful to us later because of the following result.

¹Flenner stated the theorem for mixed characteristic rings as well, but his proof in that case was wrong. This is explained and fixed in [Tri94].

Lemma 2.2.2. *Let A, B, e_1, \dots, e_n be as above. If B is a torsion-free A -module and A contains \mathbf{F}_p , then for any $m \in \mathbf{Z}_{\geq 1}$, as subsets of $(B \otimes_A K)^{1/p^m}$*

$$\text{Disc}_{B/A}(e_1, \dots, e_n) \cdot B^{1/p^m} \subseteq A^{1/p^m}[B].$$

Proof. This is [HH90, Lemma 6.5] for $(-)^{1/p^\infty}$, but the same proof works in the case of $(-)^{1/p^m}$. \square

We need a compatibility result.

Lemma 2.2.3. *Assume that (A, \mathfrak{m}) is local and that $A \rightarrow B$ is finite étale. Let $e_1, \dots, e_n \in B$ be a basis of B as an A -module. Then*

$$\overline{\text{Disc}_{B/A}(e_1, \dots, e_n)} = \text{Disc}_{(B/\mathfrak{m}B)/(A/\mathfrak{m})}(\overline{e_1}, \dots, \overline{e_n})$$

where $\overline{(-)}$ means mod \mathfrak{m} or mod $\mathfrak{m}B$.

Proof. Let z_{ijkl} be elements of B such that $e_i e_j e_k = \sum_l z_{ijkl} e_l$. Then $\text{Tr}_{B/A}(e_i e_j) = \sum_k z_{ijkk}$, so $\text{Disc}_{B/A}(e_1, \dots, e_n) = \det((\sum_k z_{ijkk})_{i,j})$. The same formulas compute the right hand side, showing the desired identity. \square

We need an explicit computation.

Lemma 2.2.4. *Let $A \rightarrow B$ be a finite map of DVRs. Let $K = \text{Frac}(A)$, $L = \text{Frac}(B)$, and $s = [L : K]$. Assume that $s \in A^\times$, and that the residue fields of A and B are the same (i.e. L/K is totally tamely ramified). Let v_A and v_B be the discrete valuation of A and B respectively.*

Assume that there exists $y \in B$ such that $v_B(y)$ and s are relatively prime, and that $x := y^s \in A$. Then $v_A(\text{Disc}_{B'/A}(y, \dots, y^{s-1}, y^s)) = (s+1)v_A(x)$ for any sub- A -algebra B' of B that contains y .

Proof. By assumptions L/K is separable and $v_B|_A = s v_A$. Since $v_B(y)$ and s are relatively prime, it is clear that y, \dots, y^{s-1}, y^s is a basis of L/K , thus for any sub- A -algebra B' that contains y , $L = \text{Frac}(B')$, so we may assume $B' = B$.

Since $x = y^s \in A$, we can easily write down the matrix of a power of y as a linear transformation with respect to the basis y, \dots, y^{s-1}, y^s , and it follows that $\text{Tr}(y^{bs}) = s x^b$ and $\text{Tr}(y^a) = 0$ if s does not divide a . Thus the matrix $\text{Tr}(y^i y^j)$ has exactly one nonzero entry in each row, which is $s x$ in the first $s-1$ rows and $s x^2$ in the last one. Since $s \in A^\times$, $v_A(\text{Disc}_{B'/A}(y, \dots, y^{s-1}, y^s)) = (s+1)v_A(x)$ as desired. \square

2.3. A non-completed version of Cohen-Gabber. We will need the following version of the Cohen-Gabber structure theorem [GO08, Théorème 7.1].

Theorem 2.3.1. *Let $(A^{nc}, \mathfrak{m}^{nc}, k)$ be a Noetherian local \mathbf{F}_p -algebra and let (A, \mathfrak{m}, k) be the reduction of the completion of A^{nc} . Assume that A is equidimensional, and assume that for each minimal prime \mathfrak{p} of A^{nc} , there is exactly one minimal prime of A above \mathfrak{p} .*

Let $d = \dim A$. Then there exists a set $\Lambda \subseteq A^{nc}$ and a system of parameters $t_1, \dots, t_d \in \mathfrak{m}^{nc}$ with the following properties.

- (i) Λ maps to a p -basis of k .
- (ii) For the unique coefficient field κ of A containing Λ (see [Bou83, chapitre IX, §2, Théorème 1]), A is finite and generically étale over the subring $\kappa[[t_1, \dots, t_d]]$.

Proof. We run the argument in [GO08, §7] for the ring A , while making sure that the elements of concern belong in the ring A^{nc} . We start with the constructions in [GO08, (7.2)]. Let $\mathfrak{p}_1^{nc}, \dots, \mathfrak{p}_c^{nc}$ be the minimal primes of A^{nc} , so by our assumption, A has exactly c minimal primes $\mathfrak{p}_1, \dots, \mathfrak{p}_c$ with $\mathfrak{p}_i \cap A^{nc} = \mathfrak{p}_i^{nc}$. Fix a set $\Lambda \subseteq A^{nc}$ that maps to a p -basis of k and let κ be the unique coefficient field of A containing Λ . For a finite set $e \subseteq \Lambda$, let $\kappa_e = \kappa^p(\Lambda \setminus e)$.

By [GO08, (7.3)], we can find an e such that for each ring $B = A/\mathfrak{p}_i$, we have

$$\text{rank } \hat{\Omega}_{B/\kappa_e}^1 = d + |e|.$$

Now, we claim that for every ideal I of A^{nc} , the sets $\{d(i) \mid i \in I\}$ and $\{d(i) \mid i \in IB\}$ generates the same submodule of $\hat{\Omega}_{B/\kappa_e}^1$. This is because $\hat{\Omega}_{B/\kappa_e}^1$ is a finite module over B , so all its submodules are closed, and because $\{d(i) \mid i \in I\}$ is dense in $\{d(i) \mid i \in IB\}$, since $d(\mathfrak{m}^N B) \subseteq \mathfrak{m}^{N-1} \hat{\Omega}_{B/\kappa_e}^1$. Applying this to $I = \mathfrak{p}_1^{nc} \cap \dots \cap \mathfrak{p}_j^{nc}$, noting that $IA \not\subseteq \mathfrak{p}_{j+1}$ since $\mathfrak{p}_{j+1} \cap A^{nc} = \mathfrak{p}_{j+1}^{nc}$, we see that the elements m_i, m'_i in [GO08, (7.4)] can be chosen to be in A^{nc} . Finally, applying the claim to $I = A^{nc}$, we see that the elements f_i in [GO08, (7.5)] can be chosen in A^{nc} . This concludes the proof. \square

3. TAME RAMIFICATION

3.1. A condition of one-dimensional local rings. We consider the following condition of a Noetherian local ring A of dimension 1.

Condition 3.1.1.

- (i) A^\wedge is (R_0) .
- (ii) $(A/\mathfrak{p})^\nu$ is local for all minimal primes \mathfrak{p} of A .²
- (iii) The map $A \rightarrow (A/\mathfrak{p})^\nu$ induces an isomorphism of residue fields for all minimal primes \mathfrak{p} of A .³

Note that if A is complete, or more generally Henselian, then (ii) is automatic; see [Stacks, Tag 0BQ0].

Lemma 3.1.2. *Let A be a Noetherian local ring of dimension 1. The followings are equivalent:*

- (i) the completion A^\wedge of A is (R_0) ; and
- (ii) A is (R_0) , and the normalization of A is finite.

²In other words, A/\mathfrak{p} is unibranch [Stacks, Tag 0BPZ].

³In particular, A/\mathfrak{p} is geometrically unibranch [Stacks, Tag 0BPZ].

If this holds, then $(A^\wedge)_{\text{red}} = (A_{\text{red}})^\wedge$ and $A^{\wedge\nu} = A^\nu \otimes_A A^\wedge$.

Proof. Assume first that A^\wedge is (R_0) , so A is also (R_0) . Let \mathfrak{N} be the nil-radical of A . Then $\mathfrak{N}(A^\wedge)_P = 0$ for all minimal primes P of A^\wedge , thus $A^\wedge/\mathfrak{N}A^\wedge = (A_{\text{red}})^\wedge$ is (R_0) . Since A is one-dimensional, A_{red} is Cohen-Macaulay, thus $(A_{\text{red}})^\wedge$ is Cohen-Macaulay, thus reduced since it is (R_0) . Finiteness of normalization is then classical, see for example [Stacks, Tag 032Y].

Now assume (ii). We need to show (i) and $A^{\wedge\nu} = A^\nu \otimes_A A^\wedge$. Since A^ν is finite over A , we see $A^\nu \otimes_A A^\wedge$ is the completion of A^ν as a semi-local ring. Since A^ν is normal of dimension 1, it is regular, hence so is $A^\nu \otimes_A A^\wedge$. Since A is (R_0) , for any $f \in \mathfrak{m}$, $A_f = (A^\nu)_f$, thus $(A^\wedge)_f = (A^\nu \otimes_A A^\wedge)_f$, so A^\wedge is (R_0) and $A^\nu \otimes_A A^\wedge = A^{\wedge\nu}$. \square

Condition 3.1.1 implies desired tame behavior, Proposition 3.1.4 below. Before that, some notations.

Notation 3.1.3. Let A be a Noetherian local ring of dimension 1 that satisfies Condition 3.1.1.

Let \mathfrak{p} be a minimal prime of A . $(A/\mathfrak{p})^\nu$ is finite over A by Lemma 3.1.2, thus a DVR.⁴ Denote by $v_{\mathfrak{p}} : A \rightarrow \mathbf{Z}_{\geq 0} \cup \{\infty\}$ the corresponding valuation composed with the map $A \rightarrow (A/\mathfrak{p})^\nu$; by Condition 3.1.1(iii), we see $v_{\mathfrak{p}}(a) = l_A((A/\mathfrak{p})^\nu/a(A/\mathfrak{p})^\nu)$. Let $\beta(\mathfrak{p}) \in \mathbf{Z}_{\geq 0}$ be the minimal β such that there exists an element $s \in A$ in all minimal primes of A other than \mathfrak{p} and that $v_{\mathfrak{p}}(s) = \beta$.

Denote by $\mathfrak{c}_{\mathfrak{p}}$ the conductor of the extension $A/\mathfrak{p} \rightarrow (A/\mathfrak{p})^\nu$, i.e., $\mathfrak{c}_{\mathfrak{p}} = \{a \in (A/\mathfrak{p})^\nu \mid a(A/\mathfrak{p})^\nu \subseteq A/\mathfrak{p}\}$. Note that $A/\mathfrak{p} \rightarrow (A/\mathfrak{p})^\nu$ is finite by Lemma 3.1.2, so $\mathfrak{c}_{\mathfrak{p}}$ is nonzero. Denote by $\gamma_0(\mathfrak{p})$ the number $l_A((A/\mathfrak{p})^\nu/\mathfrak{c}_{\mathfrak{p}})$.

Assume now that A contains \mathbf{F}_p . We denote by $\gamma(\mathfrak{p})$ the minimal integer γ such that $\gamma \geq \gamma_0(\mathfrak{p}) + \beta(\mathfrak{p})$ and that γ is not divisible by p .

Finally, let $\delta(A) = \sum_{\mathfrak{p}} \gamma(\mathfrak{p})$ and $\Delta(A) = \sum_{\mathfrak{p}} (\gamma(\mathfrak{p}) + 1)^2$, where the sum is over all minimal primes.

We now present the main result of this subsection. Our idea has some overlap with [Ska16].

Proposition 3.1.4. *Let (A, \mathfrak{m}) be a Noetherian local ring of dimension 1 that satisfies Condition 3.1.1. Then the followings hold.*

- (i) *Let \mathfrak{p} be a minimal prime of A , and let $n_{\mathfrak{p}} \in \mathbf{Z}$, $n_{\mathfrak{p}} \geq \gamma_0(\mathfrak{p})$. Then there exists an element $t_{\mathfrak{p}} \in A$ lying in all minimal primes other than \mathfrak{p} , such that $v_{\mathfrak{p}}(t_{\mathfrak{p}}) = n_{\mathfrak{p}} + \beta(\mathfrak{p})$.*
- (ii) *Assume that A contains \mathbf{F}_p . Then there exists $t \in \mathfrak{m}$ such that for all minimal primes \mathfrak{p} of A , $v_{\mathfrak{p}}(t) = \gamma(\mathfrak{p})$.*
- (iii) *Assume that A is complete and contains \mathbf{F}_p . For any t as in (ii), and any choice of a coefficient field $k \subseteq A$, the map $k[[T]] \rightarrow A$ mapping T to t is finite and generically étale of generic degree $n = \delta(A)$.*

⁴In fact, the normalization of a one-dimensional Noetherian domain is always Noetherian by the theorem of Krull-Akizuki [Stacks, Tag 00PG].

(iv) For any $k[[T]] \rightarrow A$ as in (iii), there exist elements e_1, \dots, e_n of A mapping to a basis of $A[\frac{1}{T}]$ over $k((T))$ such that the T -adic valuation of the discriminant $\text{Disc}_{A/k[[T]]}(e_1, \dots, e_n)$ is $\Delta(A)$.

Proof. By the definition of the conductor, we see that there exists $r_{\mathfrak{p}} \in A$ such that $v_{\mathfrak{p}}(r_{\mathfrak{p}}) = n_{\mathfrak{p}}$. Let $s_{\mathfrak{p}}$ be an element of A contained in all other minimal primes of A and satisfies $v_{\mathfrak{p}}(s_{\mathfrak{p}}) = \beta(\mathfrak{p})$. Then $t_{\mathfrak{p}} = s_{\mathfrak{p}}r_{\mathfrak{p}}$ satisfies $v_{\mathfrak{p}}(t_{\mathfrak{p}}) = n_{\mathfrak{p}} + \beta(\mathfrak{p})$, showing (i).

For (ii), let $n_{\mathfrak{p}} = \gamma(\mathfrak{p}) - \beta(\mathfrak{p})$ for each \mathfrak{p} , and let $t_{\mathfrak{p}}$ be as in (i). Then $t = \sum_{\mathfrak{p}} t_{\mathfrak{p}}$ works. Note that t must be in \mathfrak{m} since $\gamma(\mathfrak{p}) > 0$.

Now we prove (iii). Let $t \in \mathfrak{m}$ be such that for all minimal primes \mathfrak{p} of A , $v_{\mathfrak{p}}(t) = \gamma(\mathfrak{p})$. In particular, t is a parameter of A . Let $k \subseteq A$ be an arbitrary coefficient field, so the map $k[[T]] \rightarrow A$ mapping T to t is finite. Since $v_{\mathfrak{p}}(t) = \gamma(\mathfrak{p})$ is not divisible by p and since the residue field of $(A/\mathfrak{p})^\nu$ is k (Condition 3.1.1(iii)), we see that $k[[T]] \rightarrow (A/\mathfrak{p})^\nu$ is totally tamely ramified of index $\gamma(\mathfrak{p})$. In particular, $k[[T]] \rightarrow A/\mathfrak{p}$ is generically étale, thus so is $k[[T]] \rightarrow A$ since A is (R_0) . That $k[[T]] \rightarrow A$ has generic degree $\delta(A)$ is clear.

It remains to show (iv). Let \mathfrak{p} be a minimal prime of A . Let s be an element of A contained in all other minimal primes of A and satisfies $v_{\mathfrak{p}}(s) = \beta(\mathfrak{p})$. In $(A/\mathfrak{p})^\nu$ we can write $s^{\gamma(\mathfrak{p})} = t^{\beta(\mathfrak{p})}u$, where $u \in (A/\mathfrak{p})^{\nu \times}$. Then we can write $u = vw_1^{-1}$, with $v \in k^\times$ and w_1 has residue class 1 in the residue field of $(A/\mathfrak{p})^\nu$, since the residue field of $(A/\mathfrak{p})^\nu$ is k (Condition 3.1.1(iii)). Since p does not divide $\gamma(\mathfrak{p})$, by Hensel's Lemma $w_1 = w^{\gamma(\mathfrak{p})}$ for some $w \in (A/\mathfrak{p})^\nu$ with residue class 1. Then $(ws)^{\gamma(\mathfrak{p})} = t^{\beta(\mathfrak{p})}v$.

Let $y \in A$ be such that the image of y in A/\mathfrak{p} is in $\mathfrak{c}_{\mathfrak{p}}$ and that $v_{\mathfrak{p}}(y) = n_{\mathfrak{p}} := \gamma(\mathfrak{p}) - \beta(\mathfrak{p}) + 1$. This is possible because $n_{\mathfrak{p}} \geq \gamma_0(\mathfrak{p})$. Then similarly we can write $(w'y)^{\gamma(\mathfrak{p})} = t^{n_{\mathfrak{p}}}v' \in (A/\mathfrak{p})^\nu$, where $v' \in k^\times$ and w' has residue class 1. Now, since the image of y in A/\mathfrak{p} is in $\mathfrak{c}_{\mathfrak{p}}$, there exists $z \in A$ such that the $z = yww' \in A/\mathfrak{p}$. Finally, let $x_{\mathfrak{p}}$ be the element sz . Then $x_{\mathfrak{p}}$ is in all minimal ideals other than \mathfrak{p} , and $x_{\mathfrak{p}}^{\gamma(\mathfrak{p})} = t^{\gamma(\mathfrak{p})+1}vv' \in (A/\mathfrak{p})^\nu$, where $v, v' \in k^\times$.

We have that $v_{\mathfrak{p}}(x_{\mathfrak{p}}) = \gamma(\mathfrak{p}) + 1$ and $\gamma(\mathfrak{p})$ are relatively prime, so we see that $x_{\mathfrak{p}}, \dots, x_{\mathfrak{p}}^{\gamma(\mathfrak{p})}$ is a basis of $\text{Frac}(A/\mathfrak{p})$ over $k((T))$. Since $x_{\mathfrak{p}}$ is in all minimal primes other than \mathfrak{p} , we see that $\cup_{\mathfrak{p}} \{x_{\mathfrak{p}}, \dots, x_{\mathfrak{p}}^{\gamma(\mathfrak{p})}\}$ is a basis of $A[\frac{1}{T}]$ over $k((T))$. It suffices to show the discriminant of this basis has T -adic valuation $\Delta(A)$; thus it suffices to show the discriminant of the basis $x_{\mathfrak{p}}, \dots, x_{\mathfrak{p}}^{\gamma(\mathfrak{p})}$ of $\text{Frac}(A/\mathfrak{p})$ over $k((T))$ has T -adic valuation $(\gamma(\mathfrak{p})+1)^2$. Since $x_{\mathfrak{p}}^{\gamma(\mathfrak{p})}$ is the image of $T^{\gamma(\mathfrak{p})+1}vv' \in k[[T]]$, this follows from Lemma 2.2.4. \square

We will need to move between a local ring and its completion.

Lemma 3.1.5. *Let A be a Noetherian local ring of dimension 1. Assume that A satisfies Condition 3.1.1(i)(ii). Then the followings hold.*

- (i) The map $\mathfrak{p} \mapsto \mathfrak{p}A^\wedge$ is a bijection between the minimal primes of A and those of A^\wedge .
- (ii) A satisfies Condition 3.1.1(iii) if and only if A^\wedge does.
- (iii) If (ii) is the case, then the map in (i) identifies β, γ_0 and γ . In particular, $\delta(A) = \delta(A^\wedge)$ and $\Delta(A) = \Delta(A^\wedge)$.

Proof. Let \mathfrak{p} be a minimal prime of A . By Lemma 3.1.2, $A/\mathfrak{p} \rightarrow (A/\mathfrak{p})^\nu$ is finite, so by Lemma 3.1.2 again we see that $(A/\mathfrak{p})^{\nu^\wedge} = (A/\mathfrak{p})^{\wedge\nu}$ is normal. Condition 3.1.1(ii) says that $(A/\mathfrak{p})^\nu$ is local, thus $(A/\mathfrak{p})^{\nu^\wedge}$ is local, hence a DVR, and its subring $(A/\mathfrak{p})^\wedge$ is then an integral domain. So $\mathfrak{p}A^\wedge$ is a minimal prime of A^\wedge , showing (i), and $(A/\mathfrak{p})^{\nu^\wedge} = (A^\wedge/\mathfrak{p}A^\wedge)^\nu$, showing (ii).

For (iii), by previous discussions $v_{\mathfrak{p}A^\wedge}|_A = v_{\mathfrak{p}}$. Since taking conductor and finite intersection commute with flat base change, it is clear that $\gamma_0(\mathfrak{p}A^\wedge) = \gamma_0(\mathfrak{p})$ and $\beta(\mathfrak{p}A^\wedge) = \beta(\mathfrak{p})$. Therefore $\gamma(\mathfrak{p}A^\wedge) = \gamma(\mathfrak{p})$. \square

3.2. Tame curves.

Definition 3.2.1. Let A be a Noetherian local ring, $d = \dim A$. We say a proper ideal \mathfrak{a} of A defines a tame curve if

- (i) all minimal primes of \mathfrak{a} have height $d - 1$; and
- (ii) A/\mathfrak{a} satisfies Condition 3.1.1.

Lemma 3.2.2. Let A be a Noetherian local ring, \mathfrak{a} a proper ideal of A . If \mathfrak{a} defines a tame curve, so does $\mathfrak{a}A^\wedge \subseteq A^\wedge$.

Proof. Since $A \rightarrow A^\wedge$ is flat, every minimal prime of $\mathfrak{a}A^\wedge$ has the same height as some minimal prime of \mathfrak{a} . This takes care of (i) in Definition 3.2.1. For (ii), Condition 3.1.1(i) for A/\mathfrak{a} and $A^\wedge/\mathfrak{a}A^\wedge$ are the same, (ii) is automatic for the complete local ring $A^\wedge/\mathfrak{a}A^\wedge$, and $A^\wedge/\mathfrak{a}A^\wedge$ satisfies (iii) by Lemma 3.1.5. \square

Theorem 3.2.3. Let (A, \mathfrak{m}) be a Noetherian local \mathbf{F}_p -algebra, $d = \dim A$. Assume that there exist elements a_1, \dots, a_{d-1} such that $\mathfrak{a} = (a_1, \dots, a_{d-1})$ defines a tame curve.

Then there exists $t \in \mathfrak{m}$ such that for any coefficient field k of A^\wedge , the map $P := k[[X_1, \dots, X_{d-1}, T]] \rightarrow A^\wedge$ mapping X_i to a_i and T to t is finite of generic degree $n = \delta(A/\mathfrak{a})$, and is étale at the prime $\mathfrak{P} = (X_1, \dots, X_{d-1})$, and there exists a basis e_1, \dots, e_n of $A^\wedge \otimes_P \text{Frac}(P)$ over $\text{Frac}(P)$ such that

$$\text{Disc}_{A^\wedge/P}(e_1, \dots, e_n) \notin \mathfrak{P} + T^{\Delta(A/\mathfrak{a})+1}P$$

See Notation 3.1.3 for $\delta(-)$ and $\Delta(-)$ in the statement.

Proof. The completion of A satisfies the same assumptions by Lemmas 3.2.2 and 3.1.5. We will show that if A is complete, and $t \in A$ is such that the image of t in A/\mathfrak{a} is as in Proposition 3.1.4(ii), then t works. This proves the theorem, since the set of t indicated in Proposition 3.1.4(ii) is open in the adic topology.

Assume A and t are as above. Let k be an arbitrary coefficient field and let $P \rightarrow A$ and \mathfrak{P} be as in the statement of our theorem. Note that $P \rightarrow A$

is finite and $\mathfrak{a} = \mathfrak{P}A$. Since every minimal prime of \mathfrak{a} has height $d - 1$, every maximal ideal of $A_{\mathfrak{P}}$ has height $d - 1$. Since $P_{\mathfrak{P}}/\mathfrak{P}P_{\mathfrak{P}} \rightarrow A_{\mathfrak{P}}/\mathfrak{P}A_{\mathfrak{P}}$ is finite étale of degree $n = \delta(A/\mathfrak{a})$ (Proposition 3.1.4(iii)), and since $P_{\mathfrak{P}}$ is normal of dimension $d - 1$, $P_{\mathfrak{P}} \rightarrow A_{\mathfrak{P}}$ is finite étale of degree n , see for example [Stacks, Tag 0GSC].

Find n elements of $A/\mathfrak{a} = A/\mathfrak{P}A$ as in Proposition 3.1.4(iv) and lift them to elements $e_1, \dots, e_n \in A$. Then e_1, \dots, e_n is a basis of $A_{\mathfrak{P}}$ over $P_{\mathfrak{P}}$, and Lemma 2.2.3 gives $\text{Disc}_{A/P}(e_1, \dots, e_n) \notin \mathfrak{P} + T^{\Delta(A/\mathfrak{a})+1}P$, as desired. \square

3.3. Finding tame curves. The goal of this subsection is to show that tame curves in the spectrum of a local ring can be found after a reasonable extension (Proposition 3.3.2).

Lemma 3.3.1. *Let R be a Noetherian local \mathbf{F}_p -algebra of dimension 1. Assume that R^\wedge is (R_0) , and assume that R/\mathfrak{p} is geometrically unibranch for all minimal primes \mathfrak{p} of R . Then the followings hold.*

- (i) *There exist a subset Λ of R and a parameter $t \in R$ that satisfies the conclusions of Theorem 2.3.1.*
- (ii) *For Λ, t as in (i), put $\kappa_0 = \mathbf{F}_p(\Lambda)$. Then there exists a finite purely inseparable extension κ_1/κ_0 such that $R \otimes_{\kappa_0} \kappa_1$ satisfies Condition 3.1.1.*

Proof. For (i), we need to verify the conditions of Theorem 2.3.1. Since R is one-dimensional, $A = (R^\wedge)_{\text{red}}$ is equidimensional. Since R/\mathfrak{p} is (geometrically) unibranch for each minimal prime \mathfrak{p} of R and since R^\wedge is (R_0) , R satisfies Condition 3.1.1(i)(ii), so by Lemma 3.1.5, $\mathfrak{p} \mapsto \mathfrak{p}A$ is a bijection between the minimal primes of R and A . Thus the conditions of Theorem 2.3.1 are satisfied.

Now fix Λ and t as in (i) and let $\kappa_0 = \mathbf{F}_p(\Lambda)$. Let κ be the unique coefficient field of R^\wedge containing κ_0 , so A is finite and generically étale over $\kappa[[t]]$, see Theorem 2.3.1. Since R^\wedge is (R_0) , we see R^\wedge is finite and generically étale over $\kappa[[t]]$ as well.

We fix a perfect closure κ_0^{perf} and denote by $\kappa_1, \kappa_2, \dots$ the finite purely inseparable extensions of κ_0 inside κ_0^{perf} . For a κ_1 , denote by R_1 the ring $R \otimes_{\kappa_0} \kappa_1$, so R_1 is a Noetherian local ring with residue field $k \otimes_{\kappa_0} \kappa_1$ where k is the residue field of R . Let \tilde{R} be the local ring $R \otimes_{\kappa_0} \kappa_0^{\text{perf}}$, and let R^* be the ring $R^\wedge \otimes_{\kappa[[t]]} \kappa^{\text{perf}}[[t]]$. Since Λ is a p -basis of κ , we have $\kappa^{\text{perf}} = \kappa \otimes_{\kappa_0} \kappa_0^{\text{perf}}$, so we have canonical maps $R_1 \rightarrow \tilde{R} \rightarrow R^*$. Note that R^* is finite and generically étale over $\kappa^{\text{perf}}[[t]]$, hence is complete, Noetherian, and (R_0) . The map $R_1 \rightarrow R^*$ is faithfully flat, and \tilde{R} is the union of all such rings R_1 , so $\mathfrak{a} = \mathfrak{a}R^* \cap \tilde{R}$ for every ideal \mathfrak{a} of \tilde{R} . Thus \tilde{R} is Noetherian, and it is clear that $\tilde{R}^\wedge = R^*$. By Lemma 3.1.2, the normalization of \tilde{R} is finite, so we can find a κ_1 such that $R_1^\nu \otimes_{\kappa_1} \kappa_0^{\text{perf}} = \tilde{R}^\nu$, hence for all κ_2/κ_1 , we have $R_1^\nu \otimes_{\kappa_1} \kappa_2 = R_2^\nu$.

For a κ_2/κ_1 , consider the quantity $\lambda(\kappa_2) = l_{R_2'}(R_2'/tR_2')$. Then $\lambda(\kappa_2) \leq l_{R_2}(R_2'/tR_2')$, and since $R_1' \otimes_{\kappa_1} \kappa_2 = R_2'$, this latter quantity is equal to $l_{R_1}(R_1'/tR_1')$. Thus the quantities $\lambda(\kappa_2)$ is upper bounded, so we may take a κ_2/κ_1 that achieves the maximal $\lambda(\kappa_2)$. We claim that κ_2/κ_0 is what we wanted. Condition 3.1.1(i) is clear since $R_2 \rightarrow R^*$ is faithfully flat; we need the rest two items.

Note that $R \rightarrow R_2$ is finite and radicial, so for each minimal prime \mathfrak{p}_2 of R_2 , $\mathfrak{p}_2 \cap R$ is a minimal prime R and we have $(R/\mathfrak{p}_2 \cap R)^{sh} \otimes_{R/\mathfrak{p}_2 \cap R} R_2/\mathfrak{p}_2 = (R_2/\mathfrak{p}_2)^{sh}$. Since $R/\mathfrak{p}_2 \cap R$ is geometrically unibranch, R_2/\mathfrak{p}_2 is geometrically unibranch, see [Stacks, Tag 06DM]. In particular, Condition 3.1.1(ii) holds for R_2 .

Now we show Condition 3.1.1(iii) holds for R_2 . Let $\mathfrak{m}_1, \dots, \mathfrak{m}_c$ be the maximal ideals of R_2' . For every κ_3/κ_2 , since $R_2' \otimes_{\kappa_2} \kappa_3 = R_3'$, $\mathfrak{n}_i = \sqrt{\mathfrak{m}_i R_3'}$ are exactly the maximal ideals of R_3' , and $(R_2')_{\mathfrak{m}_i} \otimes_{\kappa_2} \kappa_3 = (R_3')_{\mathfrak{n}_i}$. Thus we see

$$\begin{aligned} l_{R_3'}(R_3'/tR_3') &= \sum_i l_{R_3'}((R_3')_{\mathfrak{n}_i}/t(R_3')_{\mathfrak{n}_i}) \\ &= \sum_i \frac{1}{[\kappa(\mathfrak{n}_i) : \kappa(\mathfrak{m}_i)]} l_{R_2'}((R_3')_{\mathfrak{n}_i}/t(R_3')_{\mathfrak{n}_i}) \\ &= \sum_i \frac{[\kappa_3 : \kappa_2]}{[\kappa(\mathfrak{n}_i) : \kappa(\mathfrak{m}_i)]} l_{R_2'}((R_2')_{\mathfrak{m}_i}/t(R_2')_{\mathfrak{m}_i}) \end{aligned}$$

and we always have

$$l_{R_2'}(R_2'/tR_2') = \sum_i l_{R_2'}((R_2')_{\mathfrak{m}_i}/t(R_2')_{\mathfrak{m}_i}).$$

For each i we have $[\kappa_3 : \kappa_2] \geq [\kappa(\mathfrak{n}_i) : \kappa(\mathfrak{m}_i)]$ since $\kappa(\mathfrak{n}_i)$ is a quotient of $\kappa(\mathfrak{m}_i) \otimes_{\kappa_2} \kappa_3$. Thus the maximality of $\lambda(\kappa_2)$ gives $[\kappa_3 : \kappa_2] = [\kappa(\mathfrak{n}_i) : \kappa(\mathfrak{m}_i)]$ and thus $\kappa(\mathfrak{n}_i) = \kappa(\mathfrak{m}_i) \otimes_{\kappa_2} \kappa_3$. Since κ_3/κ_2 was arbitrary, we must have $\kappa(\mathfrak{m}_i)$ separable over κ_2 for all i . Since R_2/\mathfrak{p}_2 is geometrically unibranch for every minimal prime \mathfrak{p}_2 of R_1 , we see $\kappa(\mathfrak{m}_i)$ is purely inseparable over κ_2 , so $\kappa(\mathfrak{m}_i) = \kappa_2$, which is Condition 3.1.1(iii), as desired. \square

Proposition 3.3.2. *Let (A, \mathfrak{m}, k) be a Noetherian local \mathbf{F}_p -algebra of dimension d . Let \mathfrak{a} be a proper ideal of A . Assume that all minimal primes of \mathfrak{a} has height $d - 1$ and that $(A/\mathfrak{a})^\wedge$ is (R_0) .*

Then there exist a syntomic-local ring map $A \rightarrow B^5$ such that $B/\mathfrak{m}B$ is finite over k and that $\mathfrak{a}B$ defines a tame curve.

Proof. Any étale-local ring map $A/\mathfrak{a} \rightarrow E$ is syntomic-local by [Stacks, Tag 00UE], and E^\wedge is (R_0) since it is étale over $(A/\mathfrak{a})^\wedge$. Take an E such that E/\mathfrak{p} is geometrically unibranch for all minimal primes \mathfrak{p} of E , cf. [Stacks, Tag 0CB4]. By Lemma 3.3.1, there exists a finite syntomic E -algebra C that

⁵This, by convention, means that $A \rightarrow B$ is a local map of local rings such that B is the localization of a syntomic A -algebra at a prime ideal.

is local and satisfies Condition 3.1.1. Note that $A/\mathfrak{a} \rightarrow C$ is also syntomic-local, and $C/\mathfrak{m}C$ is finite over k .

By [Stacks, Tag 07M8], we can lift C to a syntomic-local A -algebra B . By our choice, $B/\mathfrak{a}B = C$ satisfies Definition 3.2.1(ii), and $B/\mathfrak{m}B = C/\mathfrak{m}C$ is finite over k . By flatness, $\dim B = d$ and all minimal primes of $\mathfrak{a}B$ have height $d - 1$, giving Definition 3.2.1(i). \square

3.4. Local uniformity. The goal of this subsection is to prove the following statement.

Theorem 3.4.1. *Let R be a Noetherian ring, $\mathfrak{p} \in \operatorname{Spec}(R)$, $d = \operatorname{ht} \mathfrak{p}$.*

Let $\mathfrak{a} \subseteq \mathfrak{p}$ be an ideal of R such that $\mathfrak{a}R_{\mathfrak{p}}$ defines a tame curve (Definition 3.2.1). Then, upon replacing R by R_g for some $g \notin \mathfrak{p}$, the followings hold.

- (i) *For all $\mathfrak{P} \in V(\mathfrak{p})$ such that $R_{\mathfrak{P}}/\mathfrak{p}R_{\mathfrak{P}}$ is Cohen-Macaulay, $\operatorname{ht}(\mathfrak{P}) = d + \operatorname{ht}(\mathfrak{P}/\mathfrak{p})$.⁶*
- (ii) *For all $\mathfrak{P} \in V(\mathfrak{p})$ such that $R_{\mathfrak{P}}/\mathfrak{p}R_{\mathfrak{P}}$ is regular, and all sequence of elements $\pi_1, \dots, \pi_\delta \in R_{\mathfrak{P}}$ that maps to a regular system of parameters of $R_{\mathfrak{P}}/\mathfrak{p}R_{\mathfrak{P}}$, $\mathfrak{A} := \mathfrak{a}R_{\mathfrak{P}} + (\pi)$ defines a tame curve.*
- (iii) *Notations as in (ii) and Notation 3.1.3. If R/\mathfrak{a} contains \mathbf{F}_p , then for \mathfrak{P}, π as in (ii), we have $\delta(R_{\mathfrak{P}}/\mathfrak{A}) = \delta(R_{\mathfrak{p}}/\mathfrak{a})$ and $\Delta(R_{\mathfrak{P}}/\mathfrak{A}) = \Delta(R_{\mathfrak{p}}/\mathfrak{a})$.*

We first prove (i). After localizing R , we may assume that $\mathfrak{p}^n/\mathfrak{p}^{n+1}$ is a free (R/\mathfrak{p}) -module for all $n \geq 0$ [EGA IV₂, Proposition 6.10.2]. For $\mathfrak{P} \in V(\mathfrak{p})$ such that $R_{\mathfrak{P}}/\mathfrak{p}R_{\mathfrak{P}}$ is Cohen-Macaulay, let $h = \operatorname{ht}(\mathfrak{P}/\mathfrak{p})$, and let π_1, \dots, π_h be elements of $R_{\mathfrak{P}}$ that map to a system of parameters of $R_{\mathfrak{P}}/\mathfrak{p}R_{\mathfrak{P}}$. Let $a \in \mathbf{Z}_{\geq 1}$ be such that $\mathfrak{P}^a \subseteq \mathfrak{p}R_{\mathfrak{P}} + (\pi)$. Then for all $n \in \mathbf{Z}_{\geq 1}$

$$\mathfrak{P}^{2ahn}R_{\mathfrak{P}} \subseteq (\mathfrak{p}R_{\mathfrak{P}} + (\pi))^{2hn} \subseteq \mathfrak{p}^{hn}R_{\mathfrak{P}} + (\pi)^{hn} \subseteq \mathfrak{p}^{hn}R_{\mathfrak{P}} + (\pi^n) \subseteq \mathfrak{P}^nR_{\mathfrak{P}}$$

and thus

$$l_{R_{\mathfrak{P}}}(R_{\mathfrak{P}}/\mathfrak{P}^{2ahn}R_{\mathfrak{P}}) \geq l_{R_{\mathfrak{P}}}(R_{\mathfrak{P}}/(\mathfrak{p}^{hn}R_{\mathfrak{P}} + (\pi^n))) \geq l_{R_{\mathfrak{P}}}(R_{\mathfrak{P}}/\mathfrak{P}^nR_{\mathfrak{P}}).$$

We know that R/\mathfrak{p}^{hn} is a successive extension of isomorphic copies of R/\mathfrak{p} , and the number of copies is $l_{R_{\mathfrak{p}}}(R_{\mathfrak{p}}/\mathfrak{p}^{hn}R_{\mathfrak{p}})$. Since π^n is a regular sequence in $R_{\mathfrak{P}}/\mathfrak{p}R_{\mathfrak{P}}$, the middle term equals $l_{R_{\mathfrak{p}}}(R_{\mathfrak{p}}/\mathfrak{p}^{hn}R_{\mathfrak{p}})n^hl(R_{\mathfrak{P}}/(\mathfrak{p}R_{\mathfrak{P}} + (\pi)))$, asymptotically cn^{d+h} . Thus $\operatorname{ht}(\mathfrak{P}) = d + h$.

Before going into the proof of (ii) and (iii), we note the following, which has some overlap with the argument above.

Discussion 3.4.2 (cf. [EY11, Lemma 3.3]). Let R be a Noetherian ring, $\mathfrak{p} \in \operatorname{Spec}(R)$. Let M be a finite R -module. Then, upon replacing R by R_g for some $g \notin \mathfrak{p}$, there is a filtration $M = M_n \supsetneq M_{n-1} \supsetneq \dots \supsetneq M_1 \supsetneq M_0 = 0$ such that $M_j/M_{j-1} \cong R/\mathfrak{p}_j$ where $\mathfrak{p}_j \subseteq \mathfrak{p}$. In particular, if $M_{\mathfrak{p}}$ is of finite

⁶If R is catenary, we can just localize R so that all minimal primes of R are contained in \mathfrak{p} , then this identity is automatic and does not require Cohen-Macaulayness.

length, then M is a successive extension of R/\mathfrak{p} . Thus if $\mathfrak{P} \in V(\mathfrak{p})$, π_1, \dots, π_h elements of $R_{\mathfrak{P}}$ that are a regular sequence in $R_{\mathfrak{P}}/\mathfrak{p}R_{\mathfrak{P}}$, then π_1, \dots, π_h is a regular sequence in $M_{\mathfrak{P}}$. Consequently, if

$$0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow M \longrightarrow 0$$

is a short exact sequence of R -modules, then

$$0 \longrightarrow (M_1)_{\mathfrak{P}}/(\underline{\pi}) \longrightarrow (M_2)_{\mathfrak{P}}/(\underline{\pi}) \longrightarrow M_{\mathfrak{P}}/(\underline{\pi}) \longrightarrow 0$$

is exact. Moreover, if $h = \dim R_{\mathfrak{P}}/\mathfrak{p}R_{\mathfrak{P}}$ (so in particular $R_{\mathfrak{P}}/\mathfrak{p}R_{\mathfrak{P}}$ is Cohen-Macaulay), then looking at the prime filtration we see $l(M_{\mathfrak{P}}/(\underline{\pi})) = l(M_{\mathfrak{P}})l(R_{\mathfrak{P}}/(\mathfrak{p}R_{\mathfrak{P}} + (\underline{\pi})))$.

Now we continue the proof of Theorem 3.4.1.

Step 1. Let $\mathfrak{q}_1, \dots, \mathfrak{q}_m$ be the minimal primes of \mathfrak{a} . Localize R , we may assume $\mathfrak{q}_i \subseteq \mathfrak{p}$ for all i .

Step 2. For each i , the normalization of $R_{\mathfrak{p}}/\mathfrak{q}_i R_{\mathfrak{p}}$ is finite (Lemma 3.1.2). Thus there exists a finite extension R'_i of R/\mathfrak{q}_i in its fraction field such that $(R'_i)_{\mathfrak{p}} = (R_{\mathfrak{p}}/\mathfrak{q}_i R_{\mathfrak{p}})^\nu$.

Step 3. By Condition 3.1.1(ii), $(R'_i)_{\mathfrak{p}}$ is local, so R'_i has exactly one prime \mathfrak{p}'_i above \mathfrak{p} . Localizing R we may assume $\mathfrak{p}'_i = \sqrt{\mathfrak{p}R'_i}$. By Condition 3.1.1(iii), $(R'_i)_{\mathfrak{p}}/\mathfrak{p}'_i(R'_i)_{\mathfrak{p}} = \kappa(\mathfrak{p})$, so after localizing R we may assume $R/\mathfrak{p} = R'_i/\mathfrak{p}'_i$. In particular, for each $\mathfrak{P} \in V(\mathfrak{p})$, there is a unique prime \mathfrak{P}'_i of R'_i above \mathfrak{P} , and $R_{\mathfrak{P}}/\mathfrak{p}R_{\mathfrak{P}} = (R'_i)_{\mathfrak{P}'_i}/\mathfrak{p}'_i(R'_i)_{\mathfrak{P}'_i}$, in particular $\kappa(\mathfrak{P}'_i) = \kappa(\mathfrak{P})$.

Step 4. Since $(R'_i)_{\mathfrak{p}}$ is a DVR, after localizing R we may assume that \mathfrak{p}'_i is a principal ideal. Let τ_i be a generator, so $R/\mathfrak{p} = R'_i/\tau_i R'_i$. For $\mathfrak{P}, \underline{\pi}$ as in (ii), $\tau_i, \underline{\pi}$ is then a regular sequence in $(R'_i)_{\mathfrak{P}'_i} = (R'_i)_{\mathfrak{P}}$ that generates the maximal ideal. Thus $(R'_i)_{\mathfrak{P}}$ is regular and $\tau_i, \underline{\pi}$ is a regular system of parameters. In particular, $\underline{\pi}$ is a regular sequence in $(R'_i)_{\mathfrak{P}}$ and $(R'_i)_{\mathfrak{P}}/(\underline{\pi})$ is a DVR.

Step 5. Apply Discussion 3.4.2 to $M = \frac{R'_i}{R/\mathfrak{q}_i}$, we see that after localizing R , we may assume that for all $\mathfrak{P}, \underline{\pi}$, $R_{\mathfrak{P}}/(\mathfrak{q}_i R_{\mathfrak{P}} + (\underline{\pi})) \rightarrow (R'_i)_{\mathfrak{P}}/(\underline{\pi})$ is injective with finite length cokernel. Since $(R'_i)_{\mathfrak{P}}/(\underline{\pi})$ is a DVR (Step 4), it is the normalization of the integral domain $R_{\mathfrak{P}}/(\mathfrak{q}_i R_{\mathfrak{P}} + (\underline{\pi}))$. In particular, $(\mathfrak{q}_i R_{\mathfrak{P}} + (\underline{\pi}))$ is a prime ideal, and $\dim R_{\mathfrak{P}}/(\mathfrak{q}_i R_{\mathfrak{P}} + (\underline{\pi})) = 1$.

Step 6. Apply Discussion 3.4.2 to $M = R/(\mathfrak{q}_i + \mathfrak{q}_j)$ ($i \neq j$), we see that after localizing R , we may assume that for all $\mathfrak{P}, \underline{\pi}$, $R_{\mathfrak{P}}/(\mathfrak{q}_i R_{\mathfrak{P}} + \mathfrak{q}_j R_{\mathfrak{P}} + (\underline{\pi}))$ has finite length. Thus $\mathfrak{q}_i R_{\mathfrak{P}} + (\underline{\pi}) \neq \mathfrak{q}_j R_{\mathfrak{P}} + (\underline{\pi})$.

Step 7. Apply Discussion 3.4.2 to $M = \frac{\oplus_i R/\mathfrak{q}_i}{R/\sqrt{\mathfrak{a}}}$, we see that after localizing R , we may assume that for all $\mathfrak{P}, \underline{\pi}$, $\cap_i (\mathfrak{q}_i R_{\mathfrak{P}} + (\underline{\pi})) = \sqrt{\mathfrak{a}} R_{\mathfrak{P}} + (\underline{\pi})$. Thus $\mathfrak{q}_i R_{\mathfrak{P}} + (\underline{\pi})$ are precisely all the minimal primes of $\mathfrak{a} R_{\mathfrak{P}} + (\underline{\pi})$.

Step 8. Apply Discussion 3.4.2 to $M = \sqrt{\mathfrak{a}}/\mathfrak{a}$, we see that after localizing R , we may assume that for all $\mathfrak{P}, \underline{\pi}$, $\frac{\sqrt{\mathfrak{a}} R_{\mathfrak{P}} + (\underline{\pi})}{\mathfrak{a} R_{\mathfrak{P}} + (\underline{\pi})}$ has finite length. Thus $R_{\mathfrak{P}}/(\mathfrak{a} R_{\mathfrak{P}} + (\underline{\pi}))$ is (R_0) .

At this point, with the characterization of minimal primes and normalizations in the previous steps, and with Lemma 3.1.2, we conclude that for all \mathfrak{P}, π , the ring $R_{\mathfrak{P}}/(\mathfrak{a}R_{\mathfrak{P}} + (\pi))$ has dimension 1 and satisfies Condition 3.1.1. To see $\mathfrak{a}R_{\mathfrak{P}} + (\pi)$ defines a tame curve, we must show $\text{ht}(\mathfrak{q}_i R_{\mathfrak{P}} + (\pi)) = \text{ht}(\mathfrak{P}) - 1$ for all i .

By what we have done in Steps 4 and 5, π is a regular sequence in both $(R'_i)_{\mathfrak{P}}$ and $\left(\frac{R'_i}{R/\mathfrak{q}_i}\right)_{\mathfrak{P}}$. Thus π is a regular sequence in $R_{\mathfrak{P}}/\mathfrak{q}_i R_{\mathfrak{P}}$. Therefore $\text{ht}(\mathfrak{q}_i R_{\mathfrak{P}} + (\pi)) \geq \text{ht}(\mathfrak{q}_i R_{\mathfrak{P}}) + h$, where $h = \text{ht}(\mathfrak{P}/\mathfrak{p})$. Since $\text{ht}(\mathfrak{q}_i R_{\mathfrak{P}}) = \text{ht}(\mathfrak{q}_i R_{\mathfrak{p}}) = d - 1$ (Definition 3.2.1(i)), we see $\text{ht}(\mathfrak{q}_i R_{\mathfrak{P}} + (\pi)) \geq d + h - 1$. Since $\text{ht}(\mathfrak{P}) = d + h$ (by (i)), we see that $\text{ht}(\mathfrak{q}_i R_{\mathfrak{P}} + (\pi)) = d + h - 1$, thus $\mathfrak{a}R_{\mathfrak{P}} + (\pi)$ defines a tame curve.

It remains to show the agreement of δ and Δ , assuming R/\mathfrak{a} contains \mathbf{F}_p . By definition (Notation 3.1.3), it suffices to show, for each i , that $\beta(\bar{\mathfrak{q}}_i) = \beta(\mathfrak{q}_i R_{\mathfrak{p}}/\mathfrak{a}R_{\mathfrak{p}})$ and same for γ_0 . Here $\bar{\mathfrak{q}}_i$ denotes $\frac{\mathfrak{q}_i R_{\mathfrak{P}} + (\pi)}{\mathfrak{a}R_{\mathfrak{P}} + (\pi)}$.

Step 9. Fix an index i . Let $\mathfrak{r}_i = \cap_{j \neq i} \mathfrak{q}_j$, so $\beta(\mathfrak{q}_i R_{\mathfrak{p}}/\mathfrak{a}R_{\mathfrak{p}}) = l_{R_{\mathfrak{p}}}(M_{\mathfrak{p}})$, where M is the finite R -module $R'_i/\mathfrak{r}_i R'_i$. ($R_{\mathfrak{p}}/\mathfrak{a}R_{\mathfrak{p}}$ satisfies Condition 3.1.1(iii) so we can calculate the length over $R_{\mathfrak{p}}$.) As in Step 7, after localizing R we may assume for all \mathfrak{P}, π , $\mathfrak{r}_i R_{\mathfrak{P}} + (\pi) = \cap_{j \neq i} (\mathfrak{q}_j R_{\mathfrak{P}} + (\pi))$. Thus $\beta(\bar{\mathfrak{q}}_i) = l(M_{\mathfrak{P}}/(\pi)M_{\mathfrak{P}})$. Apply Discussion 3.4.2, we see $\beta(\bar{\mathfrak{q}}_i) = \beta(\mathfrak{q}_i R_{\mathfrak{p}}/\mathfrak{a}R_{\mathfrak{p}})$ after localizing R again.

Step 10. Again, fix an index i . Let $\mathfrak{c}_i = \{a \in R \mid \mathfrak{a}R'_i \subseteq R/\mathfrak{q}_i\}$, the conductor of R'_i over R/\mathfrak{q}_i computed in R , and let $M = \frac{R'_i}{R/\mathfrak{q}_i}$. If x_1, \dots, x_l generate M as an R -module, then we have an injection

$$\begin{aligned} R/\mathfrak{c}_i &\rightarrow M^{\oplus l} \\ a &\mapsto (ax_1, \dots, ax_l) \end{aligned}$$

The cokernel of this map has finite length at \mathfrak{p} since M does. Apply Discussion 3.4.2, we see that after localizing R , $\mathfrak{c}_i R_{\mathfrak{P}} + (\pi)$ is the conductor of the normalization over $R_{\mathfrak{P}}/(\mathfrak{q}_i R_{\mathfrak{P}} + (\pi))$ computed in $R_{\mathfrak{P}}$. We have $\gamma(\mathfrak{q}_i R_{\mathfrak{p}}/\mathfrak{a}R_{\mathfrak{p}}) = l_{R_{\mathfrak{p}}}((R'_i/\mathfrak{c}_i R'_i)_{\mathfrak{p}})$ and similar for $\gamma(\bar{\mathfrak{q}}_i)$. Thus applying Discussion 3.4.2 again, we see that after localizing R again, $\gamma(\mathfrak{q}_i R_{\mathfrak{p}}/\mathfrak{a}R_{\mathfrak{p}}) = \gamma(\bar{\mathfrak{q}}_i)$.

The proof of Theorem 3.4.1 is now finished.

We arrive at the main theorem of the section.

Theorem 3.4.3. *Let R be a Noetherian \mathbf{F}_p -algebra. Assume the followings hold.*

- (1) R is J -2.
- (2) For all primes $\mathfrak{p}' \subset \mathfrak{p}$ of R with $\text{ht}(\mathfrak{p}/\mathfrak{p}') = 1$, $R_{\mathfrak{p}}^{\wedge}/\mathfrak{p}'R_{\mathfrak{p}}^{\wedge}$ is (R_0) .
- (3) R is (R_0) .

Then there exist constants $\delta, \mu, \Delta \in \mathbf{Z}_{\geq 0}$ depending only on R , and a quasi-finite syntomic ring map $R \rightarrow S$, such that for all $\mathfrak{p} \in \text{Spec}(R)$, there exist a $\mathfrak{q} \in \text{Spec}(S)$ above \mathfrak{p} and a ring map $P \rightarrow S_{\mathfrak{q}}^{\wedge}$ that satisfy the followings.

- (i) (P, \mathfrak{m}_P) is a formal power series ring over a field.
- (ii) $P/\mathfrak{m}_P = \kappa(\mathfrak{q})$.
- (iii) $P \rightarrow S_{\mathfrak{q}}^{\wedge}$ is finite and generically étale of generic degree $\leq \delta$.
- (iv) $\mathfrak{q}S_{\mathfrak{q}}^{\wedge}/\mathfrak{m}_P S_{\mathfrak{q}}^{\wedge}$ is generated by at most μ elements.
- (v) There exist $e_1, \dots, e_n \in S_{\mathfrak{q}}^{\wedge}$ that map to a basis of $S_{\mathfrak{q}}^{\wedge} \otimes_P \text{Frac}(P)$, such that $\text{Disc}_{S_{\mathfrak{q}}^{\wedge}/P}(e_1, \dots, e_n) \notin \mathfrak{m}_P^{\Delta+1}$.

Proof. Let $\mathfrak{p} \in \text{Spec}(R)$, $d = \text{ht}(\mathfrak{p})$. We shall find constants $\delta_{\mathfrak{p}}, \mu_{\mathfrak{p}}, \Delta_{\mathfrak{p}}$, a syntomic ring map $R \rightarrow S(\mathfrak{p})$, and a constructible subset $\mathcal{C}(\mathfrak{p}) \subseteq \text{Spec}(R)$ containing \mathfrak{p} such that for all $\mathfrak{P} \in \mathcal{C}(\mathfrak{p})$, there exists $\mathfrak{Q} \in \text{Spec}(S(\mathfrak{p}))$ above \mathfrak{P} and a ring map $P \rightarrow S(\mathfrak{p})_{\mathfrak{Q}}^{\wedge}$ that satisfy (i)–(v). If this is possible, then we win since the constructible topology is compact [Stacks, Tag 0901] and we can take a finite product of $S(\mathfrak{p})$'s with the corresponding $\mathcal{C}(\mathfrak{p})$'s covering $\text{Spec}(R)$.

By assumptions, $R_{\mathfrak{p}}$ is J-2 and (R_0) . If $d = 0$, then there is some $f \notin \mathfrak{p}$ such that R_f is regular, so we can just take $\mathcal{C}(\mathfrak{p}) = D(f)$, $S(\mathfrak{p}) = R_f$, $\delta_{\mathfrak{p}} = 1$, $\mu_{\mathfrak{p}} = \Delta_{\mathfrak{p}} = 0$.

Assume $d \geq 1$. By Lemma 2.1.2, there exist elements $a_1, \dots, a_{d-1} \in \mathfrak{p}R_{\mathfrak{p}}$ such that all minimal primes of (\underline{a}) are of height $d-1$ and that $R_{\mathfrak{p}}/(\underline{a})$ is (R_0) . Note that then $(R_{\mathfrak{p}}/(\underline{a}))^{\wedge}$ is (R_0) by assumption (2).

Let $R_{\mathfrak{p}} \rightarrow B$ be as in Proposition 3.3.2 for the ideal $\mathfrak{a} = (\underline{a})$. B is a localization of a syntomic $R_{\mathfrak{p}}$ -algebra, and $B/\mathfrak{p}B$ is finite over $\kappa(\mathfrak{p})$, thus $B = S_{\mathfrak{q}}$ for some quasi-finite syntomic R -algebra S and some $\mathfrak{q} \in \text{Spec}(S)$. We also have $\mathfrak{a}B = \mathfrak{b}_0B$ for some ideal \mathfrak{b}_0 of S generated by $d-1$ elements.

Since R is J-2, we can localize S near \mathfrak{q} to assume S/\mathfrak{q} regular. Let $\delta_{\mathfrak{p}} = \delta(B/\mathfrak{b}_0B)$, $\Delta_{\mathfrak{p}} = \Delta(B/\mathfrak{b}_0B)$ (Notation 3.1.3), and let $\mu_{\mathfrak{p}}$ be the number of generators of $\mathfrak{q}/\mathfrak{b}_0$. Find $g \notin \mathfrak{q}$ as in Theorem 3.4.1 (for $R = S$, $\mathfrak{a} = \mathfrak{b}_0$, and $\mathfrak{p} = \mathfrak{q}$), and let $S(\mathfrak{p}) = S_g$. Then for all $\mathfrak{Q} \in V(\mathfrak{q}S(\mathfrak{p}))$, there exists an ideal \mathfrak{B} of $S(\mathfrak{p})_{\mathfrak{Q}}$ generated by $\text{ht}(\mathfrak{Q}) - 1$ elements defining a tame curve (Theorem 3.4.1(ii)) and satisfying $\delta(S(\mathfrak{p})_{\mathfrak{Q}}/\mathfrak{B}) = \delta_{\mathfrak{p}}$ and $\Delta(S(\mathfrak{p})_{\mathfrak{Q}}/\mathfrak{B}) = \Delta_{\mathfrak{p}}$ (Theorem 3.4.1(iii)). The form of \mathfrak{B} as in Theorem 3.4.1(ii) tells us that $\mathfrak{Q}/\mathfrak{B}$ is generated by at most $\mu_{\mathfrak{p}}$ elements.

Let $P \rightarrow S(\mathfrak{p})_{\mathfrak{Q}}^{\wedge}$ be a map as in Theorem 3.2.3, so (ii) is true by construction and (i)(iii)(v) follow from the theorem. By construction, $\mathfrak{B}S(\mathfrak{p})_{\mathfrak{Q}}^{\wedge} \subseteq \mathfrak{m}_P S(\mathfrak{p})_{\mathfrak{Q}}^{\wedge}$, so we get (iv). Finally, we let $\mathcal{C}(\mathfrak{p})$ be the image of $V(\mathfrak{q}S(\mathfrak{p}))$ in $\text{Spec}(R)$, which is constructible since $R \rightarrow S(\mathfrak{p})$ is of finite type. This finishes the proof. \square

4. MORE PRELIMINARIES

4.1. Local equidimensionality.

Lemma 4.1.1. *Let R be a Noetherian ring that is locally equidimensional and universally catenary. Let $R \rightarrow S$ be a flat ring map of finite type. If all nonempty generic fibers of $R \rightarrow S$ are equidimensional and have the same dimension, then S is locally equidimensional.*

Proof. Let d be the generic fiber dimension. Let $\mathfrak{q} \in \operatorname{Spec}(S)$ be above some $\mathfrak{p} \in \operatorname{Spec}(R)$. Let \mathfrak{q}_0 be an arbitrary minimal prime of S contained in \mathfrak{q} , lying over $\mathfrak{p}_0 \in \operatorname{Spec}(R)$. Then \mathfrak{p}_0 is a minimal prime of R by flatness.

By [Stacks, Tag 021J], $\operatorname{ht}(\mathfrak{q}/\mathfrak{q}_0) = \operatorname{ht}(\mathfrak{p}/\mathfrak{p}_0) + \operatorname{trdeg}_{\kappa(\mathfrak{p}_0)} \kappa(\mathfrak{q}_0) - \operatorname{trdeg}_{\kappa(\mathfrak{p})} \kappa(\mathfrak{q})$. By our assumptions, $\operatorname{trdeg}_{\kappa(\mathfrak{p}_0)} \kappa(\mathfrak{q}_0) = d$ is independent of \mathfrak{q}_0 chosen. Also $\operatorname{ht}(\mathfrak{p}/\mathfrak{p}_0)$ does not depend on the choice of \mathfrak{q}_0 since R is locally equidimensional. Thus $\operatorname{ht}(\mathfrak{q}/\mathfrak{q}_0)$ does not depend on the choice of \mathfrak{q}_0 , so S is locally equidimensional. \square

4.2. Formally (S_1) rings. The purpose of this subsection is to relax the excellence hypothesis in our main theorems. The “excellent” reader can skip this subsection.

Definition 4.2.1. Let R be a Noetherian ring. We say R is *formally (S_1)* if $R_{\mathfrak{p}}^{\wedge}$ is (S_1) for all $\mathfrak{p} \in \operatorname{Spec}(R)$.

Lemma 4.2.2. *Let (R, \mathfrak{m}) be a Noetherian local ring. Then R is formally (S_1) if and only if R^{\wedge} is (S_1) .*

Proof. If R^{\wedge} is (S_1) , then R^{\wedge} is formally (S_1) since a complete local ring is a G-ring [Stacks, Tag 07PS] and the property (S_1) ascends [Stacks, Tag 0339]. Since $R \rightarrow R^{\wedge}$ is faithfully flat it is clear that R is formally (S_1) . \square

Lemma 4.2.3. *Let R be a Noetherian ring, $R \rightarrow S$ a ring map of finite type. Assume that $R \rightarrow S$ is flat with (S_1) fibers. Then if R is formally (S_1) , so is S .*

Proof. Let $\mathfrak{q} \in \operatorname{Spec}(S)$, we want to show $S_{\mathfrak{q}}^{\wedge}$ is (S_1) . We may assume (R, \mathfrak{m}) local and $\mathfrak{q} \cap R = \mathfrak{m}$. Let $\mathfrak{Q} \in \operatorname{Spec}(S \otimes_R R^{\wedge})$ be above \mathfrak{q} , so we have

$$\begin{array}{ccccccc} R & \longrightarrow & S & \longrightarrow & S_{\mathfrak{q}} & \longrightarrow & S_{\mathfrak{q}}^{\wedge} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ R^{\wedge} & \longrightarrow & S \otimes_R R^{\wedge} & \longrightarrow & (S \otimes_R R^{\wedge})_{\mathfrak{Q}} & \longrightarrow & (S \otimes_R R^{\wedge})_{\mathfrak{Q}}^{\wedge} \end{array}$$

where the vertical maps are faithfully flat. It suffices to show $(S \otimes_R R^{\wedge})_{\mathfrak{Q}}^{\wedge}$ is (S_1) . Since $S \otimes_R R^{\wedge}$ is of finite type over R^{\wedge} , it is a G-ring [Stacks, Tag 07PX], so by [Stacks, Tag 0339] it suffices to show $S \otimes_R R^{\wedge}$ is (S_1) . By [Stacks, Tag 0339] again it suffices to show the fibers of $R^{\wedge} \rightarrow S \otimes_R R^{\wedge}$ are (S_1) .

Since the fibers of $R \rightarrow S$ are (S_1) , it suffices to show if k is a field, K/k is a field extension, A is a finite type k -algebra that is (S_1) , then $A \otimes_k K$ is (S_1) . By [Stacks, Tag 0339], applied to the map $A \rightarrow A \otimes_k K$, it suffices to show $k' \otimes_k K$ is (S_1) for all finitely generated extensions k'/k . This ring is actually Cohen-Macaulay, see [Stacks, Tag 045M]. \square

Lemma 4.2.4. *Let R be a Noetherian local ring. If R^{\wedge} is (S_1) , then $R^{\wedge}/\mathfrak{p}R^{\wedge}$ is (S_1) for all minimal primes $\mathfrak{p} \in \operatorname{Spec}(R)$.*

Proof. Since $\mathfrak{p} \in \operatorname{Ass}_R(R)$, $\operatorname{Ass}_{R^{\wedge}}(R^{\wedge}/\mathfrak{p}R^{\wedge}) \subseteq \operatorname{Ass}_{R^{\wedge}}(R^{\wedge})$, cf. [Stacks, Tag 0312]. Thus $R^{\wedge}/\mathfrak{p}R^{\wedge}$ has no embedded primes, as desired. \square

4.3. Number of generators.

Lemma 4.3.1. *Let P be a normal domain, A a finite torsion-free P -algebra of generic degree $\delta \geq 1$. Let $\mu \in \mathbf{Z}_{\geq 0}$ be such that A is generated by μ elements as a P -algebra.*

Assume that $A \otimes_P \text{Frac}(P)$ is a product of fields. Then A is generated by at most δ^μ elements as a P -module.

Proof. Let $a \in A$. In each factor of $A \otimes_P \text{Frac}(P)$, a has a monic minimal polynomial whose coefficients are in P since P is normal. Since $A \rightarrow A \otimes_P \text{Frac}(P)$ is injective, the product of these minimal polynomials is a monic polynomial of degree $\leq \delta$ with coefficients in P that has a as a root. The rest is clear. \square

4.4. An easy estimate.

Lemma 4.4.1. *Let (P, \mathfrak{m}, k) be a regular local ring containing \mathbf{F}_p . Let $d = \dim P$. Let $n \in \mathbf{Z}_{\geq 0}$, $F \in P$, $F \notin \mathfrak{m}^{n+1}$. Then for all $e \in \mathbf{Z}_{\geq 1}$, $l(P/((F) + \mathfrak{m}^{[p^e]})) \leq np^{e(d-1)}$.*

Proof. We may assume $F \in \mathfrak{m}^n$ and k infinite. Arguing as in [Nag62, (40.2)], we can find a regular system of parameters x_1, \dots, x_d of P such that $l(P/(F, x_2, \dots, x_d)) = n$ and that F, x_2, \dots, x_d is a regular sequence in P . Then $l(P/((F) + \mathfrak{m}^{[p^e]})) \leq l(P/((F) + (x_2, \dots, x_d)^{[p^e]})) = np^{e(d-1)}$. \square

5. UNIFORM BOUND

5.1. Bound from a single Cohen-Gabber type normalization.

Lemma 5.1.1 (cf. [Pol18, proof of Corollary 3.4]). *Let A be an \mathbf{F}_p -algebra, \mathfrak{m} a maximal ideal of A , I an ideal of A , u an element of A such that $(I : u) = \mathfrak{m}$, e a positive integer. Let M be an A -module.*

Write $J = I + (u)$. Then the followings hold.

- (i) $M/(I^{[p^e]}M :_M u^{p^e}) \cong J^{[p^e]}M/I^{[p^e]}M$.
- (ii) *If A/\mathfrak{m} is perfect and M is finitely generated, then for all $t \in \mathbf{Z}_{\geq 0}$,*

$$l_A \left(\frac{F_*^t M}{(I^{[p^e]} F_*^t M :_{F_*^t M} u^{p^e})} \right) = l_A \left(\frac{J^{[p^{e+t}]} M}{I^{[p^{e+t}]} M} \right) < \infty.$$

Proof. There is a canonical surjection $M \rightarrow J^{[p^e]}M/I^{[p^e]}M$ sending m to $u^{p^e}m$, showing (i). For (ii), finiteness follows from the fact that $\frac{J^{[p^{e+t}]}M}{I^{[p^{e+t}]}M}$ is a finitely generated $(A/\mathfrak{m}^{[p^{e+t}]})$ -module. To see the identity, notice that $\frac{F_*^t M}{(I^{[p^e]} F_*^t M :_{F_*^t M} u^{p^e})} = F_*^t \left(\frac{M}{(I^{[p^{e+t}]} M :_M u^{p^{e+t}})} \right)$, and that calculating the length of an $(F_*^t \mathfrak{m})$ -primary $(F_*^t A)$ -module over $F_*^t A$ and A are the same since A/\mathfrak{m} is perfect. \square

Proposition 5.1.2. *Let (P, \mathfrak{m}_P, k) be a regular local ring of dimension d containing \mathbf{F}_p , $K = \text{Frac}(P)$. Let A be a finite, generically étale, and torsion-free P -algebra generated by $m \in \mathbf{Z}_{>0}$ elements as a P -module.*

Let $\Delta \in \mathbf{Z}_{\geq 0}$. Assume that there exist $e_1, \dots, e_n \in A$ that map to a basis of $A \otimes_P K$ such that $D := \text{Disc}_{A/P}(e_1, \dots, e_n) \notin \mathfrak{m}_P^{\Delta+1}$.

Then for all $e \leq e' \in \mathbf{Z}_{\geq 1}$, and all ideals $I \subseteq J$ of A with $l_A(J/I) < \infty$, we have

$$\left| \frac{1}{p^{ed}} l_A \left(\frac{J^{[p^e]}}{I^{[p^e]}} \right) - \frac{1}{p^{e'd}} l_A \left(\frac{J^{[p^{e'}]}}{I^{[p^{e'}]}} \right) \right| \leq m \Delta p^{-e} l_A(J/I).$$

Proof. If $I \subseteq J_1 \subseteq J$ and the statement is true for both inclusions, then it is true for $I \subseteq J$ by additivity. Thus we may assume $l_A(J/I) = 1$. In particular $\mathfrak{m}J \subseteq I$ for a unique maximal ideal \mathfrak{m} of A . For a finite \mathfrak{m} -primary A -module X , we have $l_P(X) = [\kappa(\mathfrak{m}) : k] l_A(X)$, so it suffices to show

$$(1) \quad \left| \frac{1}{p^{ed}} l_P \left(\frac{J^{[p^e]}}{I^{[p^e]}} \right) - \frac{1}{p^{e'd}} l_P \left(\frac{J^{[p^{e'}]}}{I^{[p^{e'}]}} \right) \right| \leq m \Delta p^{-e} l_P(J/I)$$

when $l_A(J/I) = 1$.

Let $(P, \mathfrak{m}_P, k) \rightarrow (P', \mathfrak{m}_{P'}, k')$ be a flat map of regular local rings with $\mathfrak{m}_P P' = \mathfrak{m}_{P'}$, and let $A' = A \otimes_P P'$. Then it is clear that A' is a finite, generically étale, and torsion-free P' -algebra generated by $m \in \mathbf{Z}_{>0}$ elements as a P' -module. The discriminant does not change, and $\mathfrak{m}_{P'}^{\Delta+1} \cap P = \mathfrak{m}_P^{\Delta+1}$ by flatness, so all assumptions hold for $P' \rightarrow A'$. For any finite length P -module X , $l_P(X) = l_{P'}(X \otimes_P P')$. Thus to show (1) when $l_A(J/I) = 1$, it suffices to show (1) for $A = A'$ with $l_A(J/I)$ arbitrary, and thus it suffices to show (1) for $A = A'$ with $l_A(J/I) = 1$. Thus we may assume P complete and k algebraically closed. In particular, for any finite P -algebra Q and any finite length Q -module Y , we have $l_P(Y) = l_Q(Y)$.

Write $t = e' - e$. Then P^{1/p^t} is a free P -module of rank p^{td} . Write $H = P^{1/p^t} \otimes_P A$. We have an exact sequence

$$H \longrightarrow A^{1/p^t} \longrightarrow L \longrightarrow 0$$

of H -modules, where L is generated by m elements as a P^{1/p^t} -module (since A^{1/p^t} is) and is annihilated by D (Lemma 2.2.2; here we use A torsion-free).

Write $J = I + (u)$, so $\mathfrak{m}_P u \subseteq I$, and we get an exact sequence

$$\frac{H}{(I^{[p^e]} H :_H u^{p^e})} \longrightarrow \frac{A^{1/p^t}}{(I^{[p^e]} A^{1/p^t} :_{A^{1/p^t}} u^{p^e})} \longrightarrow L' \longrightarrow 0$$

of H -modules with L' a quotient of $L/\mathfrak{m}_P^{[p^e]} L$, see [Pol18, proof of Corollary 3.4] for more details. Note that H is a free A -module of rank p^{td} . Lemma 5.1.1 gives the first inequality in the following chain, and the other two follows

from constructions:

$$\begin{aligned} -p^{td}l_P\left(J^{[p^e]}/I^{[p^e]}\right) + l_P\left(J^{[p^{e'}]}/I^{[p^{e'}]}\right) &\leq l_P(L') \\ &\leq l_P\left(L/\mathfrak{m}_P^{[p^e]}L\right) \\ &\leq ml_P\left(\frac{P^{1/p^t}}{\mathfrak{m}_P^{[p^e]}P^{1/p^t} + D.P^{1/p^t}}\right). \end{aligned}$$

Note that $\mathfrak{m}_P^{[p^e]}P^{1/p^t} = (\mathfrak{m}_P^{1/p^t})^{[p^e]}$, and $D \notin (\mathfrak{m}_P^{1/p^t})^{p^t\Delta+1}$ since $D \notin \mathfrak{m}_P^{\Delta+1}$. By Lemma 4.4.1, the last quantity is at most $mp^t\Delta p^{e'(d-1)}$. Therefore (recall $t = e' - e$)

$$(2) \quad -\frac{1}{p^{ed}}l_P\left(J^{[p^e]}/I^{[p^e]}\right) + \frac{1}{p^{e'd}}l_P\left(J^{[p^{e'}]}/I^{[p^{e'}]}\right) \leq m\Delta p^{-e}.$$

Note that $H \rightarrow A^{1/p^t}$ is injective since A is generically étale and torsion-free over P . By Lemma 2.2.2 again, we have an exact sequence

$$D.A^{1/p^t} \longrightarrow H \longrightarrow L_1 \longrightarrow 0$$

where, again, L_1 is generated by m elements over P^{1/p^t} since H is, and is annihilated by D by construction. Since A is torsion-free over P , D^{p^t} is a nonzerodivisor on A , thus $D.A^{1/p^t} \cong A^{1/p^t}$. By the same argument as above, we get (2) with the signs on the left hand side reversed. This shows (1) and thus the proposition. \square

Proposition 5.1.3. *Let (P, \mathfrak{m}_P, k) be a regular local ring of dimension d containing \mathbf{F}_p , $K = \text{Frac}(P)$. Let A be a P -algebra, \mathfrak{N} an ideal of A , and M a finite A -module. Let $\Delta \in \mathbf{Z}_{\geq 0}, m, e_0, b \in \mathbf{Z}_{\geq 1}$.*

Write $\bar{A} = A/\mathfrak{N}$. Assume the followings hold.

- (i) *\bar{A} is a finite, generically étale, and torsion-free P -algebra generated by m elements as a P -module.*
- (ii) *There exist $e_1, \dots, e_n \in \bar{A}$ that map to a basis of $\bar{A} \otimes_P K$ such that $D := \text{Disc}_{\bar{A}/P}(e_1, \dots, e_n) \notin \mathfrak{m}_P^{\Delta+1}$.*
- (iii) *$\mathfrak{N}^{[p^{e_0}]} = 0$.*
- (iv) *M has a filtration $M = M_b \supsetneq M_{b-1} \supsetneq \dots \supsetneq M_0 = 0$ such that $M_j/M_{j-1} \cong \bar{A}$ as A -modules.*

Then for all $e \in \mathbf{Z}, e > e_0$, and all ideals $I \subseteq J$ of A with $l_A(J/I) < \infty$, we have

$$\left| \frac{b}{p^{(e-e_0)d}}l_A\left(\frac{J^{[p^{e-e_0}]\bar{A}}}{I^{[p^{e-e_0}]\bar{A}}}\right) - \frac{1}{p^{ed}}l_A\left(\frac{J^{[p^e]M}}{I^{[p^e]M}}\right) \right| \leq p^{e_0}b^2m\Delta p^{-e}l_A(J/I).$$

Proof. As before, we may assume $l_A(J/I) = 1$, $J = I + (u)$; and we may assume P complete and k algebraically closed. Calculation of lengths therefore does not depend on the base ring chosen.

Write $H = F_*^{e_0} P \otimes_P A$ and $\overline{H} = F_*^{e_0} P \otimes_P \overline{A}$. As seen in the proof of Proposition 5.1.2, there exists an exact sequence

$$0 \longrightarrow \overline{H} \longrightarrow F_*^{e_0} \overline{A} \longrightarrow L \longrightarrow 0$$

where L is annihilated by D and is generated by m elements as a $F_*^{e_0} P$ -module. By (iv), as an $F_*^{e_0} A$ -module, $F_*^{e_0} M$ is a successive extension of b isomorphic copies of $F_*^{e_0} \overline{A}$, thus the same is true for $F_*^{e_0} M$ as an H -module. By (iii), $F_*^{e_0} M$ is an \overline{H} -module. Thus the exact sequence above implies the existence of an exact sequence of \overline{H} -modules

$$0 \longrightarrow \overline{H}^{\oplus b} \longrightarrow F_*^{e_0} M \longrightarrow L' \longrightarrow 0$$

where L' is a successive extension of b isomorphic copies of L . In particular, L' is annihilated by $D^b \notin \mathfrak{m}_P^{b\Delta+1}$ and is generated by bm elements as an $F_*^{e_0} P$ -module.

We now proceed as in the proof of Proposition 5.1.2. Taking colon with respect to $I^{[p^{e-e_0}]}$ and $u^{p^{e-e_0}}$, Lemma 5.1.1 gives

$$-bp^{e_0 d} l_P \left(\frac{J^{[p^{e-e_0}]} \overline{A}}{I^{[p^{e-e_0}]} \overline{A}} \right) + l_P \left(\frac{J^{[p^e]} M}{I^{[p^e]} M} \right) \leq l_P \left(\frac{L'}{\mathfrak{m}_P^{[p^{e-e_0}]} L'} \right).$$

By Lemma 4.4.1, $l_P(L'/\mathfrak{m}_P^{[p^{e-e_0}]} L') \leq b m p^{e_0} b \Delta p^{e(d-1)}$. Thus

$$(3) \quad -\frac{b}{p^{(e-e_0)d}} l_A \left(\frac{J^{[p^{e-e_0}]} \overline{A}}{I^{[p^{e-e_0}]} \overline{A}} \right) + \frac{1}{p^{ed}} l_A \left(\frac{J^{[p^e]} M}{I^{[p^e]} M} \right) \leq p^{e_0} b^2 m \Delta p^{-e}.$$

The exact sequence above gives

$$D^b \cdot F_*^{e_0} M \longrightarrow \overline{H}^{\oplus b} \longrightarrow L'' \longrightarrow 0$$

where L'' is annihilated by D^b by construction, and is generated by bm elements as a $F_*^{e_0} P$ -module since \overline{A} is generated by m elements as a P -module. By (iv), M is a torsion-free P -module. Thus D^b is a nonzerodivisor on $F_*^{e_0} M$ and $D^b \cdot F_*^{e_0} M \cong F_*^{e_0} M$. This gives the inequality (3) with signs on the left hand side reversed, showing the proposition. \square

Corollary 5.1.4. *Notations and assumptions as in Proposition 5.1.3. Then for all $e \leq e' \in \mathbf{Z}$, $e > e_0$, and all ideals $I \subseteq J$ of A with $l_A(J/I) < \infty$, we have*

$$\left| \frac{1}{p^{ed}} l_A \left(\frac{J^{[p^e]} M}{I^{[p^e]} M} \right) - \frac{1}{p^{e'd}} l_A \left(\frac{J^{[p^{e'}]} M}{I^{[p^{e'}]} M} \right) \right| \leq (1 + (1 + p^{e-e'}) p^{e_0} b) b m \Delta p^{-e} l_A(J/I).$$

Note that $p^{e-e'} \leq 1$.

Proof. Immediate from Propositions 5.1.2 and 5.1.3. \square

5.2. Uniform bound in excellent and less-excellent rings. We shall use the following fact.

Theorem 5.2.1. *Let R be a Noetherian \mathbf{F}_p -algebra. Assume that R/\mathfrak{p} is J-0 for all $\mathfrak{p} \in \text{Spec}(R)$.*

Then for every R -module M , there exists a constant $C = C(M)$ such that for all $\mathfrak{p} \in \text{Spec}(R)$ and all $e \in \mathbf{Z}_{\geq 1}$, $l(M_{\mathfrak{p}}/\mathfrak{p}^{[p^e]}M_{\mathfrak{p}}) \leq Cp^{e \dim M_{\mathfrak{p}}}$.

Proof. This is [PTY, Theorem 2.11], and also follows from [Smi16, Lemma 15], where R is assumed to be excellent. However, both proofs work under the assumption R/\mathfrak{p} is J-0 for all $\mathfrak{p} \in \text{Spec}(R)$. \square

Consider the following condition on a Noetherian ring R .

Condition 5.2.2.

- (i) R is J-2.
- (ii) For all primes $\mathfrak{p}' \subset \mathfrak{p}$ of R with $\text{ht}(\mathfrak{p}/\mathfrak{p}') = 1$, $(R_{\mathfrak{p}}/\mathfrak{p}'R_{\mathfrak{p}})^{\wedge}$ is (R_0) .
- (iii) R is universally catenary.

Remark 5.2.3. An excellent R , or more generally a J-2, Nagata, and universally catenary R , satisfies Condition 5.2.2; and R_{red} is formally (S_1) (Definition 4.2.1) for such R . See [Stacks, Tag 0BJ0].

Theorem 5.2.4 (cf. [Pol18, Theorem 4.4]). *Let R be a Noetherian \mathbf{F}_p -algebra that satisfies Condition 5.2.2.*

Then for every finite R -module M , there exists a constant $C(M)$ with the following property. For all $\mathfrak{p} \in \text{Spec}(R)$ such that $(R/\sqrt{\text{Ann}_R(M)})_{\mathfrak{p}}^{\wedge}$ is (S_1) , all ideals $I \subseteq J$ of $R_{\mathfrak{p}}$ with $l_{R_{\mathfrak{p}}}(J/I) < \infty$, and all $e \leq e' \in \mathbf{Z}_{\geq 1}$, the following holds.

$$\left| \frac{1}{p^{e \dim M_{\mathfrak{p}}}} l_{R_{\mathfrak{p}}} \left(\frac{J^{[p^e]}M_{\mathfrak{p}}}{I^{[p^e]}M_{\mathfrak{p}}} \right) - \frac{1}{p^{e' \dim M_{\mathfrak{p}}}} l_{R_{\mathfrak{p}}} \left(\frac{J^{[p^{e'}]}M_{\mathfrak{p}}}{I^{[p^{e'}]}M_{\mathfrak{p}}} \right) \right| \leq C(M)p^{-e} l_{R_{\mathfrak{p}}}(J/I).$$

Here by convention the left hand side is zero if $M_{\mathfrak{p}} = 0$.

Proof. We may replace R by $R/\text{Ann}_R(M)$, so $\dim M_{\mathfrak{p}} = \text{ht } \mathfrak{p}$ for all \mathfrak{p} . Let \mathfrak{p}_0 be a minimal prime of R . Then there exists a submodule $N = N(\mathfrak{p}_0)$ of M that is a successive extension of isomorphic copies of R/\mathfrak{p}_0 , such that $N_{\mathfrak{p}_0} = M_{\mathfrak{p}_0}$, by the theory of associated primes.

Let $N' = \oplus_{\mathfrak{p}_0} N(\mathfrak{p}_0)$ (not necessarily a submodule of M), so M and N' are isomorphic at all minimal primes of R , in particular $\text{Ann}_R(N')$ is nilpotent. Apply the argument in [Pol18, proof of Corollary 3.4], using Theorem 5.2.1 instead of [Pol18, Proposition 3.3], we see that it suffices to prove the result for N' .

In fact, it suffices to prove the result for each $N(\mathfrak{p}_0)$. Indeed, assume the result is true for each $N(\mathfrak{p}_0)$ and let $C(\mathfrak{p}_0)$ be the corresponding constant. Let $C'(\mathfrak{p}_0)$ be the constant as in Theorem 5.2.1 for $N(\mathfrak{p}_0)$ and $C''(\mathfrak{p}_0) = \max\{2C'(\mathfrak{p}_0), C(\mathfrak{p}_0)\}$. We claim that $\sum_{\mathfrak{p}_0} C''(\mathfrak{p}_0)$ works for N' . To see this,

let $\mathfrak{p} \in \operatorname{Spec}(R)$ be such that $(R_{\text{red}})_{\mathfrak{p}}^{\wedge}$ is (S_1) . Let \mathfrak{p}_0 be a minimal prime of R contained in \mathfrak{p} . Since N' is the direct sum of all $N(\mathfrak{p}_0)$, it suffices to show

$$\left| \frac{1}{p^{e \operatorname{ht} \mathfrak{p}}} l_{R_{\mathfrak{p}}} \left(\frac{J^{[p^e]} N(\mathfrak{p}_0)_{\mathfrak{p}}}{I^{[p^e]} N(\mathfrak{p}_0)_{\mathfrak{p}}} \right) - \frac{1}{p^{e' \operatorname{ht} \mathfrak{p}}} l_{R_{\mathfrak{p}}} \left(\frac{J^{[p^{e'}]} N(\mathfrak{p}_0)_{\mathfrak{p}}}{I^{[p^{e'}]} N(\mathfrak{p}_0)_{\mathfrak{p}}} \right) \right| \leq C''(\mathfrak{p}_0) p^{-e} l_{R_{\mathfrak{p}}}(J/I).$$

If $\operatorname{ht}(\mathfrak{p}/\mathfrak{p}_0) < \operatorname{ht}(\mathfrak{p})$, then this follows from [Pol18, Lemma 3.2] and the choice of $C'(\mathfrak{p}_0)$. Otherwise, $\operatorname{ht}(\mathfrak{p}/\mathfrak{p}_0) = \operatorname{ht}(\mathfrak{p})$. Since $(R/\mathfrak{p}_0)_{\mathfrak{p}}^{\wedge}$ is (S_1) , see Lemma 4.2.4, the desired inequality follows from the choice of $C(\mathfrak{p}_0)$.

Thus we may assume M is a successive extension of isomorphic copies of R/\mathfrak{p}_0 where \mathfrak{p}_0 is a fixed minimal prime of R . Replace R by $R/\operatorname{Ann}_R(M)$ once again, we may assume \mathfrak{p}_0 is the nilradical of R . Write $\bar{R} = R/\mathfrak{p}_0$. Let $b = l_{R_{\mathfrak{p}_0}}(M_{\mathfrak{p}_0})$ and let $e_0 \in \mathbf{Z}_{\geq 1}$ be such that $(\mathfrak{p}_0)^{[p^{e_0}]} = 0$. By Theorem 5.2.1 and [Pol18, Lemma 3.2], it suffices to find a constant $C = C(M)$ such that the desired inequality holds for all $e' \geq e > e_0$.

Note that \bar{R} is (R_0) since it is an integral domain. Let $\bar{R} \rightarrow \bar{S}$ and δ, μ and Δ be as in Theorem 3.4.3. We shall show that $C = (1 + 2p^{e_0}b)b\delta^{\mu}\Delta$ works. Let $R \rightarrow S$ be a syntomic ring map that lifts $\bar{R} \rightarrow \bar{S}$, see [Stacks, Tag 07M8]. Then $\mathfrak{p}_0 S$ is a nilpotent ideal of S , so we can identify $\operatorname{Spec}(S)$ and $\operatorname{Spec}(\bar{S})$. Fix $\mathfrak{p} \in \operatorname{Spec}(R)$ with $\bar{R}_{\mathfrak{p}}^{\wedge} (S_1)$, and let $\mathfrak{q} \in \operatorname{Spec}(\bar{S})$, $P \rightarrow \bar{S}_{\mathfrak{q}}^{\wedge}$ be as in the statement of Theorem 3.4.3. Lift the map $P \rightarrow \bar{S}_{\mathfrak{q}}^{\wedge}$ to a ring map $P \rightarrow S_{\mathfrak{q}}^{\wedge}$, possible as P is formally smooth over \mathbf{F}_p [Stacks, Tag 07NL]. Note that $R \rightarrow S$ is flat quasi-finite, so $R_{\mathfrak{p}} \rightarrow S_{\mathfrak{q}}^{\wedge}$ is flat local with zero-dimensional closed fiber. Thus for all finite length $R_{\mathfrak{p}}$ -modules X , $l_{R_{\mathfrak{p}}}(X) l_{S_{\mathfrak{q}}^{\wedge}}(S_{\mathfrak{q}}^{\wedge}/\mathfrak{p}S_{\mathfrak{q}}^{\wedge}) = l_{S_{\mathfrak{q}}^{\wedge}}(X \otimes_{R_{\mathfrak{p}}} S_{\mathfrak{q}}^{\wedge})$. Thus it suffices to prove an estimate as in the statement of Corollary 5.1.4 for the $S_{\mathfrak{q}}^{\wedge}$ -module $M \otimes_R S_{\mathfrak{q}}^{\wedge}$ with the correct constants $b, m = \delta^{\mu}$, and Δ .

It thus suffices to verify the assumptions of Corollary 5.1.4 for $P \rightarrow S_{\mathfrak{q}}^{\wedge}$. Recall that \bar{R} is an integral domain, and is universally catenary by assumption. Thus $\bar{S}_{\mathfrak{q}}$ is equidimensional (Lemma 4.1.1 since $\bar{R} \rightarrow \bar{S}$ is flat quasi-finite) and universally catenary, hence $\bar{S}_{\mathfrak{q}}^{\wedge}$ is equidimensional (Ratliff's result, [Stacks, Tag 0AW3]). Thus all minimal primes of $\bar{S}_{\mathfrak{q}}^{\wedge}$ are above $(0) \subseteq P$. By Lemmas 4.2.2 and 4.2.3, $\bar{S}_{\mathfrak{q}}^{\wedge}$ is (S_1) , thus we see $\bar{S}_{\mathfrak{q}}^{\wedge}$ is a torsion-free P -module. Note that $\bar{S}_{\mathfrak{q}}^{\wedge}$ is a finite and generically étale P -algebra (Theorem 3.4.3(iii)).

By Theorem 3.4.3(iv), we can find $y_1, \dots, y_{\mu} \in \mathfrak{q}$ such that $\mathfrak{q}\bar{S}_{\mathfrak{q}}^{\wedge} = \mathfrak{m}_P \bar{S}_{\mathfrak{q}}^{\wedge} + \langle y \rangle$. Let $S' = P[y_1, \dots, y_{\mu}] \subseteq \bar{S}_{\mathfrak{q}}^{\wedge}$, and $\mathfrak{m}' = \mathfrak{m}_P S' + \langle y \rangle$. Then we see that (S', \mathfrak{m}') is a local ring and that $\bar{S}_{\mathfrak{q}}^{\wedge} = S' + \mathfrak{m}' \bar{S}_{\mathfrak{q}}^{\wedge}$ by Theorem 3.4.3(ii). Therefore $S' = \bar{S}_{\mathfrak{q}}^{\wedge}$. By Lemma 4.3.1 (and Theorem 3.4.3(iii)), we see that $\bar{S}_{\mathfrak{q}}^{\wedge}$ is generated by at most δ^{μ} elements as a P -module.

Let $e_1, \dots, e_n \in \overline{S}_q^\wedge$ be as in Theorem 3.4.3(v), and let $D = \text{Disc}_{\overline{S}_q^\wedge/P}(e_1, \dots, e_n)$. Then $D \notin \mathfrak{m}_P^{\Delta+1}$.

We have $(\mathfrak{p}_0 S_q^\wedge)^{[p^{e_0}]} = 0$ since $\mathfrak{p}_0^{[p^{e_0}]} = 0$. Since M is a successive extension of b isomorphic copies of \overline{R} , $M \otimes_R S_q^\wedge$ is a successive extension of b isomorphic copies of \overline{S}_q^\wedge . We have verified all assumptions of and checked all constants in Corollary 5.1.4, showing what we want. \square

In view of Remark 5.2.3, the following is a special case of the theorem.

Corollary 5.2.5. *Let R be a Noetherian \mathbf{F}_p -algebra. Assume that R is excellent, or more generally J -2, Nagata, and universally catenary.*

Then for every finite R -module M , there exists a constant $C(M)$ with the following property. For all $\mathfrak{p} \in \text{Spec}(R)$, all ideals $I \subseteq J$ of $R_{\mathfrak{p}}$ with $l_{R_{\mathfrak{p}}}(J/I) < \infty$, and all $e \leq e' \in \mathbf{Z}_{\geq 1}$, the following holds.

$$\left| \frac{1}{p^{e \dim M_{\mathfrak{p}}}} l_{R_{\mathfrak{p}}} \left(\frac{J^{[p^e]} M_{\mathfrak{p}}}{I^{[p^e]} M_{\mathfrak{p}}} \right) - \frac{1}{p^{e' \dim M_{\mathfrak{p}}}} l_{R_{\mathfrak{p}}} \left(\frac{J^{[p^{e'}]} M_{\mathfrak{p}}}{I^{[p^{e'}]} M_{\mathfrak{p}}} \right) \right| \leq C(M) p^{-e} l_{R_{\mathfrak{p}}}(J/I).$$

6. APPLICATIONS: SEMI-CONTINUITY

6.1. Hilbert-Kunz multiplicity. For a Noetherian local \mathbf{F}_p -algebra (R, \mathfrak{m}) , denote by $\lambda_e(R)$ the number $\frac{l(R/\mathfrak{m}^{[p^e]})}{p^{e \dim R}}$. We have, by definition, $e_{\text{HK}}(R) = \lim_e \lambda_e(R)$, and the limit exists [Mon83].

The following slightly strengthens [SB79].

Lemma 6.1.1. *Let R be a Noetherian \mathbf{F}_p -algebra, $\mathfrak{p} \in \text{Spec}(R)$. Assume that R/\mathfrak{p} is J -0.*

Let e be a positive integer. Then for some $g \notin \mathfrak{p}$ and all $\mathfrak{P} \in D(g) \cap V(\mathfrak{p})$, $\lambda_e(R_{\mathfrak{p}}) = \lambda_e(R_{\mathfrak{P}})$.

Proof. We may assume R/\mathfrak{p} regular. By Theorem 3.4.1(i), we may assume for all $\mathfrak{P} \in V(\mathfrak{p})$, $\text{ht}(\mathfrak{P}) = \text{ht}(\mathfrak{p}) + \text{ht}(\mathfrak{P}/\mathfrak{p})$. It remains to apply Discussion 3.4.2 to the module $M = R/\mathfrak{p}^{[p^e]}$ and the regular sequence $\pi_1 = t_1^{p^e}, \dots, \pi_h = t_h^{p^e}$, where $t_1, \dots, t_h \in R_{\mathfrak{p}}$ map to a regular sequence of parameters of $R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$. \square

Corollary 6.1.2. *Let R be a Noetherian \mathbf{F}_p -algebra. Assume that R/\mathfrak{p} is J -0 for all $\mathfrak{p} \in \text{Spec}(R)$, and that R is catenary and locally equidimensional.*

Let e be a positive integer. Then the function $\mathfrak{p} \mapsto \lambda_e(R_{\mathfrak{p}})$ is constructible and upper semi-continuous.

Proof. By Lemma 6.1.1 our function is constructible. We have $\text{ht}(\mathfrak{P}) = \text{ht}(\mathfrak{p}) + \text{ht}(\mathfrak{P}/\mathfrak{p})$ for all $\mathfrak{p} \subseteq \mathfrak{P} \in \text{Spec}(R)$, since R is catenary and locally equidimensional. By [Kun76, Corollary 3.8], our function is non-decreasing along specialization. Thus our function is upper semi-continuous by general topology [Stacks, Tag 0542]. \square

Theorem 6.1.3 (cf. [Smi16, Theorem 23]). *Let R be a Noetherian \mathbf{F}_p -algebra. Assume that R satisfies Condition 5.2.2, and that R_{red} is formally (S_1) (Definition 4.2.1). (For example, if R is excellent, or if R is J -2, Nagata, and universally catenary, see Remark 5.2.3.)*

If R is locally equidimensional, then the function $\mathfrak{p} \mapsto e_{\text{HK}}(R_{\mathfrak{p}})$ is upper semi-continuous.

Proof. Apply Theorem 5.2.4 to $M = R, I = \mathfrak{p}R_{\mathfrak{p}}, J = R_{\mathfrak{p}}$, we see that our function is the uniform limit of the functions $\mathfrak{p} \mapsto \lambda_e(R_{\mathfrak{p}})$. These functions are upper semi-continuous by Corollary 6.1.2. Thus our function is upper semi-continuous as well. \square

6.2. F -signature. For a Noetherian local \mathbf{F}_p -algebra (R, \mathfrak{m}) , denote by $s_e(R)$ the e th normalized F -splitting number as in [EY11, Definition 1.1]. The limit $s(R) = \lim s_e(R)$ is called the F -signature of R . The limit was first shown to exist in [Tuc12]. (We also recover the existence in Proposition 6.2.4 below.)

We use the following facts.

Fact 6.2.1. Let $(R, \mathfrak{m}) \rightarrow (R', \mathfrak{m}')$ be a flat map of Noetherian local \mathbf{F}_p -algebras with $\mathfrak{m}R' = \mathfrak{m}'$. Then $s_e(R) = s_e(R')$ for all e , see [Yao06, Remark 2.3(3)].

Fact 6.2.2. For a Noetherian local \mathbf{F}_p -algebra (R, \mathfrak{m}) , $s_e(R) > 0$ for some e if and only if $s_e(R) > 0$ for all e , if and only if R is F -pure. Indeed, using the notations preceding [EY11, Definition 1.1], $s_e(R) > 0$ if and only if $R^{(e)} \otimes_R k \rightarrow R^{(e)} \otimes_R E$ is nonzero, if and only if k is not killed in $R^{(e)} \otimes_R E$, if and only if $R \rightarrow R^{(e)}$ is pure, see [Fed83, Proposition 1.3(5)].

Fact 6.2.3. Let (R, \mathfrak{m}) be a Noetherian local \mathbf{F}_p -algebra. For two positive integers e, e' , there exists an \mathfrak{m} -primary ideal I and an element $u \in (I : \mathfrak{m})$ such that $s_e(R) = p^{-e \dim R} l((I, u)^{[p^e]} / I^{[p^e]})$ and $s_{e'}(R) = p^{-e' \dim R} l((I, u)^{[p^{e'}]} / I^{[p^{e'}]})$. Indeed, by Fact 6.2.1 we may assume R complete, and by Fact 6.2.2 we may assume R F -pure (otherwise take $I = \mathfrak{m}$ and $u = 0$), in particular reduced, so [Pol18, Lemma 5.4] applies.

Proposition 6.2.4. *Let R be a Noetherian ring that satisfies Condition 5.2.2. Let $C = C(R)$ be as in Theorem 5.2.4.*

Then for all $\mathfrak{p} \in \text{Spec}(R)$ and all $e \leq e' \in \mathbf{Z}_{\geq 1}$, $|s_e(R_{\mathfrak{p}}) - s_{e'}(R_{\mathfrak{p}})| \leq Cp^{-e}$.

Proof. Let $\mathfrak{p} \in \text{Spec}(R)$. If $R_{\mathfrak{p}}^{\wedge}$ is not reduced, then $s_e(R_{\mathfrak{p}}) = s_{e'}(R_{\mathfrak{p}}) = 0$, see Facts 6.2.2 and 6.2.1. So we only need to show the inequality for those \mathfrak{p} with $R_{\mathfrak{p}}^{\wedge}$ reduced. By Fact 6.2.3, we need to show

$$\left| \frac{1}{p^{e \text{ht } \mathfrak{p}}} l_{R_{\mathfrak{p}}} \left(J^{[p^e]} / I^{[p^e]} \right) - \frac{1}{p^{e' \text{ht } \mathfrak{p}}} l_{R_{\mathfrak{p}}} \left(J^{[p^{e'}]} / I^{[p^{e'}]} \right) \right| \leq Cp^{-e}.$$

where I is a $\mathfrak{p}R_{\mathfrak{p}}$ -primary ideal of $R_{\mathfrak{p}}$, and $J = (I, u)$ for some $u \in (I : \mathfrak{p}R_{\mathfrak{p}})$. In particular $l_{R_{\mathfrak{p}}}(J/I) \leq 1$. The inequality now follows from Theorem 5.2.4. \square

Lemma 6.2.5. *Let R be a Noetherian ring such that R/\mathfrak{p} is J-0 for all $\mathfrak{p} \in \operatorname{Spec}(R)$.*

Assume that either R is Gorenstein, or that R is a locally equidimensional quotient of a regular Noetherian ring. Then for all e , the function $\mathfrak{p} \mapsto s_e(R_{\mathfrak{p}})$ is lower semi-continuous.

Proof. This is [EY11, Theorems 3.4 and 4.2], except that R is assumed to be excellent there. However, from the proof it is clear that R/\mathfrak{p} being J-0 for all $\mathfrak{p} \in \operatorname{Spec}(R)$ is enough. \square

Theorem 6.2.6 (cf. [Pol18, Theorem 5.6]). *Let R be a Noetherian \mathbf{F}_p -algebra that satisfies Condition 5.2.2(i)(ii). Assume that either R is Gorenstein, or that R is a quotient of a regular Noetherian ring. Then the function $\mathfrak{p} \mapsto s(R_{\mathfrak{p}})$ is lower semi-continuous.*

Proof. Note that $s(R_{\mathfrak{p}}) \geq 0$ for all \mathfrak{p} . If $s(R_{\mathfrak{p}}) > 0$ for some \mathfrak{p} , then by Fact 6.2.1 and [HL02, Theorem 11], a faithfully flat extension of $R_{\mathfrak{p}}$ is weakly F -regular, hence $R_{\mathfrak{p}}$ is normal, see [HH94, Theorem 3.4(c)]. Since R is J-2, the normal locus of R is open, see [EGA IV₂, Corollaire 6.13.5]. Thus we may assume R normal, in particular locally equidimensional.

Since a Cohen-Macaulay ring is universally catenary [Stacks, Tag 00NM], R satisfies Condition 5.2.2. By Proposition 6.2.4 the function $\mathfrak{p} \mapsto s(R_{\mathfrak{p}})$ is the uniform limit of the functions $\mathfrak{p} \mapsto s_e(R_{\mathfrak{p}})$ which are lower semi-continuous by Lemma 6.2.5, thus $\mathfrak{p} \mapsto s(R_{\mathfrak{p}})$ is lower semi-continuous. \square

Corollary 6.2.7. *Let R be a Noetherian quasi-excellent \mathbf{F}_p -algebra. Assume that R is either Gorenstein or a quotient of a regular Noetherian ring. Then the locus*

$$\{\mathfrak{p} \in \operatorname{Spec}(R) \mid R_{\mathfrak{p}} \text{ is strongly } F\text{-regular}\}$$

is open.

Proof. As seen in the proof of Theorem 6.2.6, R is universally catenary, hence excellent by definition. Thus for $\mathfrak{p} \in \operatorname{Spec}(R)$, $R_{\mathfrak{p}}$ is strongly F -regular if and only if $s(R_{\mathfrak{p}}) > 0$, see [AL02]. \square

Remark 6.2.8. Kevin Tucker informed the author that he was able to prove the openness of the strongly F -regular locus for any quotient of a regular \mathbf{F}_p -algebra via a different method.

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DEPARTMENT OF MATHEMATICS, PRINCETON UNIVERSITY, PRINCETON, NJ 08544-1000, USA

Email address: slyu@math.princeton.edu

URL: <https://www.math.princeton.edu/~slyu>