Triple product L-series and Gross–Kudla–Schoen cycles

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1 Introduction

This paper aims to prove a special case of a generalized Gross–Kudla conjecture [10]. This conjecture relates the height of the modified diagonal cycle on the triple product of Shimura curves and the derivative of the triple product L-series. In their original conjecture, we take three cusp forms f, g, h of weight 2 for $\Gamma_0(N)$ with N square free, and consider the function $F := f \times g \times h$ on $\mathscr{H}^3$, where $\mathscr{H}$ is the upper half plane. There are a triple product L-series $L(s, F)$ as studied by Garrett [6] in the classical setting and by Piatetski-Shapiro and Rallis [29] in the adelic setting. This function is entire and has a functional equation with a center at $s = 2$ and a decomposition of the global root number into a product of local ones:

$$\epsilon(F) = -\prod_{p|N} \epsilon_p(F), \qquad \epsilon_p(F) = -a_p(f)a_p(g)a_p(h) = \pm 1.$$

Assume that the global root number is -1. Then there is a canonically defined Shimura curve X, associated with an indefinite quaternion algebra B, which is nonsplit over a non-archimedean prime p if and only if $\epsilon_p(F) = -1$. There is an F-eigen component $\Delta(F)$ of the diagonal Δ of X^3 as an elements in the Chow group of codimension 2 cycles in X^3 as studied by Gross and Schoen [11]. The conjecture formulated by Gross and Kudla take shape.

$$L'(2, F) = \Omega(F)\langle \Delta(F), \Delta(F)\rangle_{\mathrm{BB}},$$

where $\Omega(F)$ is an explicit positive constant and $\langle \cdot, \cdot \rangle_{\mathrm{BB}}$ is the Beilinson–Bloch height pairing. This formula is an immediate higher dimensional generalization of the Gross–Zagier formula [12].

In this paper, we will give a full generalization of the conjecture to totally real fields and prove the conjecture in the spherical case.

We will consider cuspidal Hilbert modular forms of parallel weight 2 with arbitrary level and Gross–Kudla–Schoen cycles on Shimura curves over totally real number fields. We will formulate a conjecture 1.3.1 regarding automorphic representations. This conjecture is analogous to the central value formula of

Ichino [14]. In this paper, we can prove this conjecture when the representations are unramified; see Theorem 1.3.3. In the following, we will describe our conjectures, theorems, and the main ideas of proofs.

1.1 Shimura curves and abelian varieties

First, let us recall the definition of Shimura curves defined by an incoherent quaternion algebra, and the abelian varieties parameterized by these Shimura varieties, following [33].

Shimura curves

Let F be a number field with adele ring $\mathbb{A} = \mathbb{A}_F$ and let $\widehat{F} = \mathbb{A}_f$ be the ring of finite adeles. Let Σ be a finite set of places of F. Up to isomorphism, there is a unique $\mathbb{A}$-algebra $\mathbb{B}$, free of rank 4 as an $\mathbb{A}$-module, such that for each place v, the *localization* $\mathbb{B}_v := \mathbb{B} \otimes_{\mathbb{A}} F_v$ is isomorphic to $M_2(F_v)$ if $v \notin \Sigma$ and to the unique division quaternion algebra over F_v if $v \in \Sigma$. We call $\mathbb{B}$ *the quaternion algebra over* $\mathbb{A}$ *with ramification set* $\Sigma(\mathbb{B}) := \Sigma$.

If $\#\Sigma$ is even then $\mathbb{B} = B \otimes_F \mathbb{A}$ for a quaternion algebra B over F unique up to F-isomorphisms. In this case, we call $\mathbb{B}$ a *coherent* quaternion algebra. If $\#\Sigma$ is odd, then $\mathbb{B}$ is not the base change of any quaternion algebra over F. In this case, we call $\mathbb{B}$ an *incoherent* quaternion algebra. This terminology is inspired by Kudla's notion of *incoherent collections of quadratic spaces* [19].

Now assume that F is a totally real number field and that $\mathbb{B}$ is an incoherent quaternion algebra over $\mathbb{A}$, totally definite at infinity in the sense that $\mathbb{B}_\tau$ is the Hamiltonian algebra for every archimedean place τ of F. Then we have a projective system X_U of projective curves over F indexed by open subgroups U of $\mathbb{B}_f^\times$. The projective system X is endowed with an action T_x of $x \in \mathbb{B}^\times$ given by "*the right multiplication by* x_f." The action T_x is trivial if and only if $x_f \in \overline{F^\times}$, the closure of $F^\times$ in $\mathbb{B}_f^\times$. Each X_U is just the quotient of X by the action of U.

The induced action of $\mathbb{B}_f^\times$ on the set $\pi_0(X_{\overline{F}})$ of geometrically connected components of X factors through the norm map $q : \mathbb{B}_f^\times \to \mathbb{A}_f^\times$ and makes $\pi_0(X_{\overline{F}})$ a principal homogeneous space over $\overline{F_+^\times}\backslash\mathbb{A}_f^\times$.

Abelian varieties parametrized by Shimura curves

Let A be a simple abelian variety defined over F. We say that A *is parametrized by* X if there is a non-constant morphism $X_U \to A$ over F for some U. By the Eichler–Shimura theory, if A is parametrized by X, then A *is of strict* $\mathrm{GL}(2)$-*type* in the sense that

$$M = \mathrm{End}^0(A) := \mathrm{End}_F(A) \otimes_{\mathbb{Z}} \mathbb{Q}$$

is a field and $\mathrm{Lie}(A)$ is a free module of rank one over $M \otimes_{\mathbb{Q}} F$ by the induced action.

Define

$$\pi_A = \mathrm{Hom}^0_{\xi}(X, A) := \varinjlim_U \mathrm{Hom}^0_{\xi_U}(X_U, A),$$

where $\mathrm{Hom}^0_{\xi_U}(X_U, A)$ denotes the morphisms in $\mathrm{Hom}_F(X_U, A) \otimes_{\mathbb{Z}} \mathbb{Q}$ using the normalized Hodge bundle ξ_U as a base point. Since any morphism $X_U \to A$ factors through the Jacobian variety J_U of X_U, we also have

$$\pi_A = \mathrm{Hom}^0(J, A) := \varinjlim_U \mathrm{Hom}^0(J_U, A).$$

Here $\mathrm{Hom}^0(J_U, A) = \mathrm{Hom}_F(J_U, A)_{\otimes \mathbb{Z}} \mathbb{Q}$. The direct limit of $\mathrm{Hom}(J_U, A)$ defines an integral structure on π_A but we will not use this.

The space π_A admits a natural $\mathbb{B}^\times$-module structure. It is an *automorphic representation of* $\mathbb{B}^\times$ *over* $\mathbb{Q}$. See [33, §3.2]. We will see the natural identity $\mathrm{End}_{\mathbb{B}^\times}(\pi_A) = M$ and that π_A has a decomposition $\pi = \otimes_v \pi_v$ where π_v is an absolutely irreducible representation of $\mathbb{B}_v^\times$ over M. Using the Jacquet–Langlands correspondence, one can define the L-series

$$L(s, \pi) = \prod_v L_v(s, \pi_v) \in M \otimes_{\mathbb{Q}} \mathbb{C}$$

as an entire function of $s \in \mathbb{C}$. Let

$$L(s, A, M) = \prod L_v(s, A, M) \in M \otimes_{\mathbb{Q}} \mathbb{C}$$

be the L-series defined using ℓ-adic representations with coefficients in $M \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$, completed at archimedean places using the Γ-function. Then $L(s, A, M)$ converges absolutely in $M \otimes \mathbb{C}$ for $\mathrm{Re}(s) > 3/2$. The Eichler–Shimura theory asserts that, for almost all finite places v of F, the local L-function of A is given by

$$L_v(s, A, M) = L(s - \frac{1}{2}, \pi_v).$$

Conversely, by the Eichler–Shimura theory and the isogeny theorem of Faltings, if A is of strict GL(2)-type, and if for some automorphic representation π of $\mathbb{B}^\times$ over $\mathbb{Q}$, $L_v(s, A, M)$ is equal to $L(s - 1/2, \pi_v)$ for almost all finite places v, then A is parametrized by the Shimura curve X.

If A is parametrized by X, then so is the dual abelian variety $A^\vee$. Denote by $M^\vee = \mathrm{End}^0(A^\vee)$. There is a canonical isomorphism $M \to M^\vee$ sending a homomorphism $m : A \to A$ to its dual $m^\vee : A^\vee \to A^\vee$.

There is a perfect $\mathbb{B}^\times$-invariant pairing

$$\pi_A \times \pi_{A^\vee} \longrightarrow M$$

given by

$$(f_1, f_2) = \mathrm{vol}(X_U)^{-1}(f_{1,U} \circ f_{2,U}^\vee), \quad f_{1,U} \in \mathrm{Hom}(J_U, A),\ f_{2,U} \in \mathrm{Hom}(J_U, A^\vee),$$

where $f_{2,U}^\vee : A \to J_U$ is the dual of $f_{2,U}$ composed with the canonical isomorphism $J_U^\vee \simeq J_U$. It follows that $\pi_{A^\vee}$ is dual to π_A as representations of $\mathbb{B}^\times$ over M.

1.2 Trilinear cycles on the triple product of abelian varieties

Let A_1, A_2, A_3 be three abelian varieties over a number field F. Let $A = A_1 \times A_2 \times A_3$ be their product. We consider the space $\mathrm{Ch}_1(A)$ of 1-dimensional Chow cycles with $\mathbb{Q}$-coefficients.

Using Mukai–Fourier transformation, we have a decomposition

$$\mathrm{Ch}_1(A) = \bigoplus_s \mathrm{Ch}_1(A, s),$$

where $s = (s_1, s_2, s_3)$ is a triple of non-negative integers, and $\mathrm{Ch}_1(A, (s_1, s_2, s_3))$ consists of cycles x such that under push-forward by multiplication by $k = (k_1, k_2, k_3) \in (\mathbb{Z} \setminus \{0\})^3$ on A:

$$[k]_* x = k^s \cdot x, \qquad k^s := k_1^{s_1} k_2^{s_2} k_3^{s_3}.$$

The cycles with $s = (1, 1, 1)$ are called *trilinear cycles* and denoted by

$$\mathrm{Ch}^{\ell\ell\ell}(A) := \mathrm{Ch}_1(A, (1, 1, 1)).$$

The space $\mathrm{Ch}^{\ell\ell\ell}(A)$ is conjecturally the complement of the subspace generated by cycles supported on the image of $A_i \times A_j \times 0_k$ for some reordering (i, j, k) of $(1, 2, 3)$, where 0_k denote the 0-point on A_k.

Let $L(s, A_1 \boxtimes A_2 \boxtimes A_3)$ denote the L-series attached the triple product of ℓ-adic representation of $\mathrm{Gal}(\bar{F}/F)$ on

$$H^1(A_1, \mathbb{Q}_\ell) \otimes H^1(A_2, \mathbb{Q}_\ell) \otimes H^1(A_3, \mathbb{Q}_\ell).$$

Then it is conjectured that $L(s, A_1 \boxtimes A_2 \boxtimes A_3)$ has a holomorphic continuation on the complex plane. An extension of the Birch and Swinneron-Dyer conjecture or Beilison–Bloch conjecture gives the following:

Conjecture 1.2.1. *The space* $\mathrm{Ch}^{\ell\ell\ell}(A)$ *is finite-dimensional and*

$$\dim_{\mathbb{Q}} \mathrm{Ch}^{\ell\ell\ell}(A) = \mathrm{ord}_{s=2} L(s, A_1 \boxtimes A_2 \boxtimes A_3).$$

Like the Neron–Tate height pairing between points on A and $A^\vee = \mathrm{Pic}^0(A)$, there is a canonical height pairing between $\mathrm{Ch}^{\ell\ell\ell}(A)$ and $\mathrm{Ch}^{\ell\ell\ell}(A^\vee)$ given by the Poincare bundles $\mathscr{P}_i$ on $A_i \times A_i^\vee$ with trivializations on $A_i \times 0$ and $0 \times A_i^\vee$:

$$\langle x, y \rangle := (x \times y) \cdot \widehat{c}_1(\bar{\mathscr{P}}_1) \cdot \widehat{c}_1(\bar{\mathscr{P}}_2) \cdot \widehat{c}_1(\bar{\mathscr{P}}_3), \qquad x \in \mathrm{Ch}^{\ell\ell\ell}(A), \quad y \in \mathrm{Ch}^{\ell\ell\ell}(A^\vee),$$

where $\widehat{c}_1(\bar{\mathscr{P}}_i)$ is the first Chern class of the arithmetic cubic structure $\bar{\mathscr{P}}_i$ of $\mathscr{P}_i$. The right hand of the formula makes sense for all elements $x \in \mathrm{Ch}_1(A)$ and $y \in \mathrm{Ch}_1(A^\vee)$.

Refinement for abelian varieties of strictly GL_2-type

Assume that each A_i is of GL_2-type with endomorphism field $M_i := \mathrm{End}^0(A_i) = \mathrm{End}^0(A_i^\vee)$. Then $M = M_1 \otimes M_2 \otimes M_3$ acts on $\mathrm{Ch}^{\ell\ell\ell}(A)$ and on $\mathrm{Ch}^{\ell\ell\ell}(A^\vee)$ by pushing forward. As we will see in §3.1, these actions are additive and thus make $\mathrm{Ch}^{\ell\ell\ell}(A)$ and $\mathrm{Ch}^{\ell\ell\ell}(A^\vee)$ modules over M. As M is a direct sum of its quotients fields L, $\mathrm{Ch}^{\ell\ell\ell}(A)$ is the direct sum of $\mathrm{Ch}^{\ell\ell\ell}(A, L) := \mathrm{Ch}^{\ell\ell\ell}(A) \otimes_M L$. We can also define the triple product L-series $L(s, A_1 \boxtimes A_2 \boxtimes A_3, L) \in L \otimes \mathbb{C}$ with coefficients in L using Galois representation on

$$H^1(A_1, \mathbb{Q}_\ell) \otimes_{L\otimes\mathbb{Q}_\ell} \otimes H^1(A_2, \mathbb{Q}_\ell) \otimes_{L\otimes\mathbb{Q}_\ell} H^1(A_3, \mathbb{Q}_\ell)$$

where we choose ℓ inert in L.

Conjecture 1.2.2. *The space $\mathrm{Ch}^{\ell\ell\ell}(A)_L$ is finitely generated with*

$$\dim_L \mathrm{Ch}^{\ell\ell\ell}(A, L) = \mathrm{ord}_{s=2}\, \iota L(s, A_1 \boxtimes A_2 \boxtimes A_3, L),$$

where $\iota : L \otimes \mathbb{C} \longrightarrow \mathbb{C}$ is the surjection given by any embedding $L \longrightarrow \mathbb{C}$.

Also, we have a unique height paring with values in L:

$$\langle -, - \rangle_L : \qquad \mathrm{Ch}^{\ell\ell\ell}(A, L) \otimes_L \mathrm{Ch}^{\ell\ell\ell}(A^\vee, L) \longrightarrow L \otimes \mathbb{R}$$

such that

$$\mathrm{Tr}_{L\otimes\mathbb{R}/\mathbb{R}} \langle ax, y \rangle_L = \langle ax, y \rangle, \qquad a \in L, \quad x \in \mathrm{Ch}^{\ell\ell\ell}(A, L), \quad y \in \mathrm{Ch}^{\ell\ell\ell}(A^\vee, L).$$

1.3 Generalized Gross–Kudla conjecture

Now we assume that all A_i are parametrized by a Shimura curve X as before and take a quotient L of $M = M_1 \otimes M_2 \otimes M_3$. For any $f_i \in \pi_{A_i}$, we have a morphism

$$f := f_1 \times f_2 \times f_3 : \quad X \longrightarrow A.$$

We define $f_*(X) \in \mathrm{Ch}_1(A)$ by

$$f_*(X) := \mathrm{vol}(X_U)^{-1} f_{U*}(X) \in \mathrm{Ch}_1(A)$$

if f_i is represented by f_{iU} on X_U. This definition does not depend on the choice of U. Define

$$P_L(f) := f_*(X)^{\ell\ell\ell} \otimes 1 \in \mathrm{Ch}^{\ell\ell\ell}(A, L).$$

Let $\pi_{i,L} = \pi_{A_i} \otimes_{M_i} L$ be the automorphic representation of $\mathbb{B}^\times$ with coefficients in L. Let $\pi_L = \pi_{1,L} \otimes \pi_{2,L} \otimes \pi_{3,L}$ be their product representation of $(\mathbb{B}^\times)^3$. Then by §2.1, $f \mapsto P(f)$ defines a linear map:

$$P_L : \qquad \pi_L \longrightarrow \mathrm{Ch}^{\ell\ell\ell}(A, L).$$

This map is invariant under the action of the diagonal $\Delta(\mathbb{B}^\times)$. Thus it defines an element

$$P_L \in \mathscr{P}(\pi_{A,L}) \otimes_L \mathrm{Ch}^{\ell\ell\ell}(A, L),$$

where

$$\mathscr{P}(\pi_{A,L}) = \mathrm{Hom}_{\Delta(\mathbb{B}^\times)}(\pi_{A,L}, L).$$

Therefore $P_L(f) \neq 0$ for some f only if $\mathscr{P}(\pi_{A,L}) \neq 0$.

By a theorem of Prasad and Loke ([27], [28], [26]), $\mathscr{P}(\pi_{A,L})$ is at most one-dimensional, and it is one-dimensional if and only if the following two conditions both hold:

1. the central characters ω_i of π_i satisfy

$$\omega_1 \cdot \omega_2 \cdot \omega_3 = 1,$$

2. and the ramification $\Sigma(\mathbb{B})$ of $\mathbb{B}$ is equal to

$$\Sigma(A, L) = \left\{ \text{places } v \text{ of } F : \quad \epsilon\left(\frac{1}{2}, \pi_{A,L,v}\right) = -1 \right\}.$$

The next problem is to find a non-zero element α of $\mathscr{P}(\pi_{A,L})$ if it is non-zero. It is more convenient to work with $\mathscr{P}(\pi_L) \otimes \mathscr{P}(\widetilde{\pi}_L)$ where $\widetilde{\pi}_L$ is the contragradient of π_L is given by the product $A^\vee$ of $A_i^\vee$. Decompose $\pi_L = \otimes_v \pi_v$ then we have a decomposition $\mathscr{P}(\pi_L) = \otimes \mathscr{P}(\pi_v)$ where the space $\mathscr{P}(\pi_v)$ is defined analogously. We construct an element α_v in $\mathscr{P}(\pi_v) \otimes \mathscr{P}(\widetilde{\pi}_v)$ for each place v of F by

$$\alpha(f_v \otimes \widetilde{f}_v) := \frac{L(1, \pi_v, ad)}{\zeta_v(2)^2 L(1/2, \pi_v)} \int_{F_v^\times \backslash B_v^\times} (\pi(b) f_v, \widetilde{f}_v) db, \qquad f_v \otimes \widetilde{f}_v \in \pi_v \otimes \widetilde{\pi}_v.$$

main-conj **Conjecture 1.3.1** (Generalized Gross–Kudla conjecture)**.** *Assume* $\omega_1 \cdot \omega_2 \cdot \omega_3 = 1$. *Then we have for any* $f \in \pi_{A,L}$ *and* $\widetilde{f} \in \pi_{A^\vee, L}$,

$$\langle P_L(f), P_L(\widetilde{f}) \rangle = \frac{16 \zeta_F(2)^2}{L(1, \pi_L, ad)} L'(1/2, \pi_L) \cdot \alpha(f, \widetilde{f})$$

as an identity in $L \otimes \mathbb{C}$.

prop-CL **Proposition 1.3.2.** *Under the condition of the conjecture 1.3.1, there is a constant* $\mathscr{L}(\pi)$ *such that for any* $f \in \pi_{A,L}$ *and* $\widetilde{f} \in \pi_{A^\vee, L}$,

$$\langle P_L(f), P_L(\widetilde{f}) \rangle = \mathscr{L}(\pi) \cdot \alpha(f, \widetilde{f})$$

as an identity in $L \otimes \mathbb{C}$.

The main result of this paper is as follows.

Theorem 1.3.3. *With assumption as Conjecture 1.3.1 Let P be the set of rational primes p such that π is ramified over a prime $v \mid p$. Then there are algebraic numbers $c_p \in \bar{\mathbb{Q}} \subset \mathbb{C}$ such that for any $f \in \pi_{A,L}$ and $\widetilde{f} \in \pi_{A^\vee,L}$,*

$$\mathscr{L}(\pi) = \frac{16\zeta_F(2)^2}{L(1,\pi_L,ad)} L'(1/2,\pi_L) + \sum_{p\in P} c_p \log p$$

as an identity in $L \otimes \mathbb{C}$, where $\Omega(\sigma)$ is the Peterson period of the Jacquet–Langlands correspondence of π. In particular, the Conjecture 1.3.1 holds when π is unramified.

Remarks 1.3.1.

1. The theorem implies that $L'(1/2,\pi_L) = 0$ if and only if it is zero for all conjugates of σ.
2. Assume that σ is unitary and take $\widetilde{f} = \bar{f}$. The Hodge index conjecture implies $L'(1/2,\pi_L) \geq 0$. This positivity is a consequence of the Riemann hypothesis.

By the Theorem of Prasad and Loke, we have the following weak form of the conjecture:

An outline of the proof

The basic strategy is analogous in spirit to the proof of the Gross-Zagier formula [12] in our book [33]. We want to compare the analytic kernel function and the geometric one.

By the work of Garrett [6] and Piatetski-Shapiro–Rallis [29], the analytic kernel function can be constructed from the central derivative of an incoherent Siegel–Eisenstein series on Sp_6. Kudla first studied this derivative of the Eisenstein series in [19] and is the analytic side of his conjectured "arithmetic Siegel–Weil formula". The construction of the geometric kernel function is similar to that in the proof of the Gross-Zagier formula in our book [33]. More precisely, we can define some generating functions of Hecke operators. Such generating functions have appeared in Gross–Zagier's paper. Works of Kudla–Millson and Borcherds relate them to the Weil representation. A little extension of our result ([32]) shows that these generating functions are automorphic forms on GL_2. Then the geometric kernel function is given by a specific arithmetic intersection of three such generating functions of Hecke operators.

We reduce the questions to local ones to compare the analytic kernel function with the geometric one. At a non-archimedean place where the Shimura curve has a sound reduction, it is sufficient to use the result of Gross–Keating [9]. At an archimedean home, we can carry out the calculation explicitly. But there are essential difficulties in carrying out the regional analysis explicitly at finite places where the Shimura curve could have a better reduction. Under the assumption of the main theorem, we can overcome these difficulties by choosing some special test functions to define the generating function of Hecke operators.

1.4 Notations

In the following, k denotes a local field of a number field.

- Normalize the absolute value $|\cdot|$ on k as follows:

  It is the usual one if $k = \mathbb{R}$.

  It is the square of the usual one if $k = \mathbb{C}$.

  If k is non-archimedean, it maps the uniformizer to N^{-1}. Here N is the cardinality of the residue field.

- Normalize the additive character $\psi : k \to \mathbb{C}^\times$ as follows:

  If $k = \mathbb{R}$, then $\psi(x) = e^{2\pi i x}$.

  If $k = \mathbb{C}$, then $\psi(x) = e^{4\pi i \mathrm{Re}(x)}$.

  If k is non-archimedean, then it is a finite extension of $\mathbb{Q}_p$ for some prime p. Take $\psi = \psi_{\mathbb{Q}_p} \circ \mathrm{tr}_{k/\mathbb{Q}_p}$. Here the additive character $\psi_{\mathbb{Q}_p}$ of $\mathbb{Q}_p$ is defined by $\psi_{\mathbb{Q}_p}(x) = e^{-2\pi i \iota(x)}$, where $\iota : \mathbb{Q}_p/\mathbb{Z}_p \hookrightarrow \mathbb{Q}/\mathbb{Z}$ is the natural embedding.

- For a reductive algebraic group G defined over a number field F we denote by Z_G its center and by $[G]$ the quotient

$$[G] := Z_G(\mathbb{A})G(F)\backslash G(\mathbb{A}).$$

- We will use measures normalized as follows. We first fix a non-trivial additive character $\psi = \otimes_v \psi_v$ of $F\backslash\mathbb{A}$. Then we will take the self-dual measure dx_v on F_v with respect to ψ_v and take the product measure on $\mathbb{A}$. We will use this measure for the normal unipotent subgroup N of $SL_2(F)$ and $GL_2(F)$. We will take the Haar measure on $F_v^\times$ as $d^\times x_v = \zeta_{F_v}(1)|x_v|^{-1}dx_v$. Similarly, the measure on B_v and $B_v^\times$ are the self-dual measure dx_v with respect to the character $\psi_v(tr(xy^\iota))$ and $d^\times x_v = \zeta_{F_v}(1)|\nu(x_v)|^{-2}dx_v$. If $\mathbb{B}$ is coherent: $\mathbb{B} = B_{\mathbb{A}}$, then we have a decomposition of the Haar measure on $\mathbb{A}^\times\backslash\mathbb{B}^\times$: $dx = \prod dx_v$. We will choose the Tamagawa measure on $SL_2(\mathbb{A}_E)$ defined by an invariant differential form and denote the induced decomposition into a product $dg = \prod_v dg_v$. Then we choose a decomposition $dg = \prod_v dg_v$ of the Tamagawa measure on $\mathbb{G}(\mathbb{A})$ such that locally at every place, it is compatible with the chosen measure on $SL_2(E_v)$.

- For the non-connected group $O(V)$, we will normalize the measure on $O(V)(\mathbb{A})$ such that

$$\mathrm{vol}([O(V)]) = 1.$$

- For the quadratic space $V = (B, \nu)$ associated with a quaternion algebra, we have three groups: $\mathrm{SO}(V)$, $O(V)$, and $\mathrm{GSpin}(V)$. They can be described as follows.

$$\mathrm{GSpin}(V) = \{x, y) \in B^\times \times B^\times | \nu(x) = \nu(y)\}.$$

$$\mathrm{SO}(V) = \mathrm{GSpin}(V)/\Delta(F^\times).$$

  Let μ_2 be the group of order two generated by the canonical involution on B. Then we have a semi-direct product.

$$O(V) = \mathrm{SO}(V) \rtimes \mu_2.$$

  Moreover, by the description above, we have an isomorphism.

$$\mathrm{GSpin}(V) = B^\times \times B^1,$$

  Where B^1 is the kernel of the reduced norm:

$$1 \to B^1 \to B^\times \to F^\times \to 1.$$

  And similarly, we have an isomorphism.

$$\mathrm{SO}(V) = B^\times / F^\times \times B^1.$$

  Then for a local field F, we will choose the measure on B^1, $B^\times/F^\times$ induced from the action we have fixed on $F^\times$ and $B^\times$ via the exact sequences. In this way, we also get a Haar measure on $\mathrm{SO}(V)$. We normalize the measure on $\mu_2(F) = \{\pm 1\}$ such that the total volume is 1. The measure on $O(V)$ is then the product measure.

- $\mathbb{G} = \mathrm{GL}^\circ_{2,E} := \{g \in GL_2(E) | \det(g) \in F^\times\}$.

- We will also identify Sym_3 with the unipotent radical of the Siegel parabolic P of Sp_6:

$$n(b) = \begin{pmatrix} 1 & b \\ & 1 \end{pmatrix}, \quad b \in \mathrm{Sym}_3(\mathbb{A}).$$

  And we denote $[\mathrm{Sym}_3] = \mathrm{Sym}_3(F)\backslash\mathrm{Sym}_3(\mathbb{A})$. And we use the self-dual measure on $Sym_3(\mathbb{A})$ concerning the additive character $\psi \circ tr$ of $Sym_3(\mathbb{A})$. By $\mathrm{Sym}_3(F)_{reg}$, we denote the subset of non-singular elements. For a non-archimedean local field f, denote by $\mathrm{Sym}(\mathscr{O}_F)^\vee$ the dual of $\mathrm{Sym}_3(\mathscr{O}_F)$ with respect to the pairing $(x, y) \mapsto tr(xy)$. For $X, Y \in \mathrm{Sym}_3(F)$, we write $X \sim Y$ if there exists $g \in GL_3(\mathscr{O}_F)$ such that $X = {}^t g Y g$. For $F = \mathbb{R}$, we have a similar notation but with $g \in \mathrm{SO}(3)$.

2 Weil representations and Ichino's formula

This section will review Weil's representation and apply it to the triple product L-series. We will follow the work of Garrett, Piateski-Shapiro–Rallis, Waldspurger, Harris–Kudla, Prasad, and Ichino. The first main result is Theorem 2.3.1 about the integral representation of the triple product L-series using the Eisenstein series from the Weil representation on an adelic quaternion algebra.

When the sign of the functional equation is $+1$, the adelic quaternion algebra is coherent because it comes from a quaternion algebra over a number field. Then we have the central value formula of the triple product L-series due to Ichino (Theorem 2.4.4), a refinement of Jacquet's conjecture proved by Harris and Kudla.

When the sign is -1, then the quaternion algebra is *incoherent*, and the derivative of the Eisenstein series is the kernel function for the result of the triple product L-series (cf. (2.3.10)). Next, we will study the T-th Fourier coefficients for nonsingular T. We show that these coefficients are non-vanishing only if T is represented by elements in a nearby quaternion algebra (cf. (2.5.4)).

2.1 Weil representation and theta liftings

In this subsection, we will review the Weil representation as its its extension to similitudes by Harris and Kudla and normalized Shimuzu lifting by Waldspurger.

Extending Weil representation to similitudes

Let F be a local filed, n a positive integer, and Sp_{2n} the symplectic group with the standard alternating form $J = \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix}$ on F^{2n}. With the standard polarization $F^{2n} = F^n \oplus F^n$, we have two subgroups of Sp_{2n}:

$$M = \left\{ m(a) = \begin{pmatrix} a & 0 \\ 0 & {}^t a^{-1} \end{pmatrix} \middle| a \in \mathrm{GL}_n(F) \right\}$$

and

$$N = \left\{ n(b) = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \middle| b \in \mathrm{Sym}_n(F) \right\}.$$

Note that M, N and J generate the symplectic group Sp_{2n}.

Let $(V, (\cdot,\cdot))$ be a non-degenerate quadratic space of even dimension m with orthogonal group $\mathrm{O}(V)$. Associated to V, there is a character χ_V of $F^\times/F^{\times,2}$ defined by

$$\chi_V(a) = (a, (-1)^{m/2}\det(V))_F$$

where $(\cdot,\cdot)_F$ is the Hilbert symbol of F and $\det(V) \in F^\times/F^{\times,2}$ is the determinant of the moment matrix $Q(\{x_i\}) = \frac{1}{2}((x_i, x_j))$ of any basis $x_1, ..., x_m$ of V.

Let $\mathscr{S}(V^n)$ be the space of Bruhat-Schwartz functions on $V^n = V \otimes F^n$ (for archimedean F, functions corresponding to polynomials in the Fock model). Then the Weil representation $r = r_\psi$ of $Sp_{2n} \times O(V)$ can be realized on $\mathscr{S}(V^n)$ by the following formulae:

$$r(m(a))\Phi(x) = \chi_V(\det(a))|\det(a)|_F^{\frac{m}{2}}\Phi(xa),$$

$$r(n(b))\Phi(x) = \psi(\mathrm{Tr}(bQ(x)))\Phi(x),$$

and

$$r(J)\Phi(x) = \gamma\widehat{\Phi}(x),$$

where γ is an eighth root of unity and $\widehat{\Phi}$ is the Fourier transform of Φ:

$$\widehat{\Phi}(x) = \int_{F^n}\Phi(y)\psi(\sum_i(x_i, y_i))dy$$

for $x = (x_1, ..., x_n) \in V^n$ and $y = (y_1, ..., y_n) \in V^n$.

Now we want to extend r to representations of groups of similitudes. Let GSp_{2n} and $\mathrm{GO}(V)$ be groups of similitudes with similitude homomorphism ν (to save notations, ν will be used for both groups). Consider a subgroup $R = \mathrm{GSp}_{2n} \times_{\mathbb{G}_m} \mathrm{GO}(V)$ of $\mathrm{GSp}_{2n} \times \mathrm{GO}(V)$

$$R = \{(g, h) \in \mathrm{GSp}_{2n} \times \mathrm{GO}(V) | \nu(g) = \nu(h)\}.$$

Then we can identify $\mathrm{GO}(V)$ (resp., Sp_{2n}) as a subgroup of R consisting of $(d(\nu(h)), h)$ where

$$d(\nu) = \begin{pmatrix} 1_n & 0 \\ 0 & \nu \cdot 1_n \end{pmatrix}$$

(resp. $(g, 1)$). We then have isomorphisms

$$R/\mathrm{Sp}_{2n} \simeq \mathrm{GO}(V), \quad R/O(V) \simeq \mathrm{GSp}_{2n}^+$$

where GSp_{2n}^+ is the subgroup of GSp_{2n} with similitudes in $\nu(\mathrm{GO}(V))$.

We then extend r to a representation of R as follows: for $(g, h) \in R$ and $\Phi \in \mathscr{S}(V^n)$,

$$r((g, h))\Phi = L(h)r(d(\nu(g)^{-1})g)\Phi = r(gd(\nu(g)^{-1}))L(h)\Phi$$

where

$$L(h)\Phi(x) = |\nu(h)|_F^{-\frac{mn}{4}}\Phi(h^{-1}x).$$

For F, a number field, we patch every local representation to obtain representations of adelic groups. For $\Phi \in \mathscr{S}(V_{\mathbb{A}})$, we can define a theta series as an automorphic form on $R(\mathbb{A})$:

$$\theta(g, h, \Phi) = \sum_{x \in V^n} r(g, h)\Phi(x), \qquad (g, h) \in R(\mathbb{A}).$$

Theta lifting: local and global

Now we consider the case when $n = 1$ and V is the quadratic space attached to a quaternion algebra B with its reduced norm. Note that $\mathrm{Sp}_2 = \mathrm{SL}_2$ and $\mathrm{GSp}_2 = \mathrm{GL}_2$. And $\mathrm{GL}_2^+(F) = \mathrm{GL}_2(F)$ unless $F = \mathbb{R}$ and B is the Hamilton quaternion in which case $\mathrm{GL}_2^+(\mathbb{R})$ is the subgroup of $\mathrm{GL}_2(\mathbb{R})$ with positive determinants.

We first consider the local theta lifting. For an infinite-dimensional representation σ of $\mathrm{GL}_2(F)$, let π be the representation of $B^\times$ associated by Jacquet-Langlands correspondence and let $\widetilde{\pi}$ be the contragredient of π. Note that we set $\pi \simeq \sigma$ when $B = M_{2\times 2}$.

We have natural isomorphisms between various groups:

$$1 \to \mathbb{G}_m \to B^\times \times B^\times \to \mathrm{GSO}(V) \to 1$$

where $(b_1, b_2) \in B^\times \times B^\times$ acts on B via $(b_1, b_2)x = b_1 x b_2^{-1}$,

$$\mathrm{GO}(V) = \mathrm{GSO}(V) \rtimes \{1, c\}$$

where c acts on B via the canonical involution $c(x) = x^\iota$ and acts on $\mathrm{GSO}(V)$ via $c(b_1, b_2) = (b_2^\iota, b_1^\iota)^{-1}$. Let

$$R' = \mathrm{GSO}(V)_{\mathbb{G}_m}\mathrm{GL}_2 := \{(h, g)) \in \mathrm{GSO}(V) \times \mathrm{GL}_2 | \nu(g) = \nu(h)\}.$$

Proposition 2.1.1 (Shimizu liftings)**.** *There exists an* $\mathrm{GSO}(V) \simeq R'/\mathrm{SL}_2$*-equivariant isomorphism*

loc shimizu (2.1.1) $$(\sigma \otimes r)_{\mathrm{SL}_2} \simeq \pi \otimes \widetilde{\pi}.$$

Proof. Note that this is stronger than Howe's usual duality in the current setting. The result essentially follows from the results on Jacquet-Langlands correspondence. Here we explain why we can replace $\mathrm{GO}(V)$ by $\mathrm{GSO}(V)$. Two ways exist to extend an irreducible representation of $\mathrm{GSO}(V)$ to $\mathrm{GO}(V)$. But only one can participate in the theta correspondence because the representation *sign* of $\mathrm{GO}(V)$ does not occur in the theta correspondence unless $\dim V \leq 2$. $\square$

Let $\mathscr{W}_\sigma = \mathscr{W}_\sigma^\psi$ be the ψ-Whittaker model of σ and let W_φ be a Whittaker function corresponding to φ. Define

$$\begin{aligned} &S : \mathscr{S}(V) \otimes \mathscr{W}_\sigma \to \mathbb{C} \\ &\qquad (\Phi, W) \mapsto S(\Phi, W) = \frac{\zeta(2)}{L(1, \sigma, ad)} \int_{N(F)\backslash \mathrm{SL}_2(F)} r(g)\Phi(1)W(g)dg. \end{aligned}$$

See the normalization of measure in "Notations". By W [31, Lemma 5] the integral is absolutely convergent and defines an element in

$$\mathrm{Hom}_{\mathrm{SL}_2 \times B^\times}(r \otimes \sigma, \mathbb{C})$$

where $B^\times$ is diagonally embedded into $B^\times \times B^\times$, and $S(\Phi, W) = 1$ for unramified data. Since

$$\mathrm{Hom}_{\mathrm{SL}_2 \times B^\times}(r \otimes \sigma, \mathbb{C}) \simeq \mathrm{Hom}_{B^\times}((r \otimes \sigma)_{\mathrm{SL}_2}, \mathbb{C}) \simeq \mathrm{Hom}_{B^\times}(\pi \otimes \widetilde{\pi}, \mathbb{C})$$

and the last space is of one dimensional spanned by the canonical $B^\times$-invariant pairing between π and its (smooth) dual space $\widetilde{\pi}$, we may define a normalized R'-equivariant map θ

theta-normalization

$$\theta : \sigma \otimes r \to \pi \otimes \widetilde{\pi}. \tag{2.1.2}$$

such that

$$S(\Phi, W) = (f_1, f_2)$$

where $f_1 \otimes f_2 = \theta(\Phi \otimes W)$.

Now in the global situation where B is a quaternion algebra defined over a number field, we define the normalized global theta lifting by

$$\theta(\Phi \otimes \varphi)(h) = \frac{\zeta(2)}{2L(1, \sigma, ad)} \int_{\mathrm{SL}_2(F)\backslash \mathrm{SL}_2(\mathbb{A})} \varphi(g_1 g)\theta(g_1 g, h, \Phi) dg_1, \qquad (h, g) \in R'(\mathbb{A}).$$

prop theta decomp

Proposition 2.1.1. *We have a decomposition $\theta = \bigotimes_v \theta_v$ in*

$$\mathrm{Hom}_{R'(\mathbb{A})}(r \otimes \sigma, \pi \otimes \widetilde{\pi}).$$

Proof. It suffices to prove the identity after composing with the tautological pairing $\pi \times \widetilde{\pi} \to \mathbb{C}$. Assume that $f_1 \otimes f_2 \in \pi \otimes \widetilde{\pi}$, $\Phi = \otimes \Phi_v \in \mathscr{S}(V_{\mathbb{A}})$ and $\varphi = \otimes \varphi_v \in \sigma$ satisfy

$$f_1 \otimes f_2 = \theta(\Phi \otimes \varphi).$$

and decomposable. We need to prove

$$(f_1, f_2) = \prod_v S(\Phi_v, \varphi_v).$$

This follows from [14, Prop. 3.1]. We have different normalizations of θ and the map S (essentially the map $B_v^\sharp$ in [14]).

□

subsec local Z

2.2 Local zeta integrals

Let E be a semi-simple algebra over F of dim 3. Consider the symplectic form on the six-dimensional F-vector space E^2:

$$E^2 \otimes E^2 \xrightarrow{\wedge} E \xrightarrow{\mathrm{Tr}} F$$

$$(x, y) \otimes (x', y') \mapsto \mathrm{Tr}_{E/F}(xy' - yx').$$

Let GSp_6 be the group of similitudes relative to this symplectic form, and we define

$$\mathbb{G} = \left\{ g \in \mathrm{GL}_2(E) | \det(g) \in F^\times \right\}.$$

Then the above construction of symplectic form identifies $\mathbb{G}$ with a subgroup of GSp_6.

Let $I(s) = \mathrm{Ind}_P^{\mathrm{GSp}_6} \lambda_s$ be the degenerate principle series of GSp_6. Here, P is the Siegel parabolic subgroup:

$$P = \left\{ \begin{pmatrix} a & * \\ 0 & \nu {}^t a^{-1} \end{pmatrix} \in \mathrm{GSp}_6 \middle| a \in \mathrm{GL}_F(E), \, \nu \in F^\times \right\}$$

and for $s \in \mathbb{C}$, λ_s is the character of P defined by

$$\lambda_s \left(\begin{pmatrix} a & * \\ 0 & \nu {}^t a^{-1} \end{pmatrix} \right) = |\nu|_F^{-3s} |\det(a)|_F^{2s}.$$

We have a $\mathrm{GSp}_6 \times_{\mathbb{G}_m} \mathrm{GO}(B_F))$-intertwining map

$$i: \quad \mathscr{S}(B_E) \to I(0) \tag{2.2.1}$$

sending Φ to $f_\Phi(\cdot, 0)$, where

$$f_\Phi(g, 0) = |\nu(g)|^{-3} r(d(\nu(g))^{-1} g)\Phi(0), \quad g \in \mathrm{GSp}_6(F).$$

We extend it to a standard section $f_{\Phi,s}$ of $I(s)$, called the *Seigel–Weil section* associated to Φ. Let $\Pi(B)$ be the image of the map (2.2.1).

Lemma 2.2.1. *For non-archimedean F, let B, B' be the two (isomorphism classes of) quaternion algebras. Then we have*

$$I(0) = \Pi(B) \oplus \Pi(B'). \tag{2.2.2}$$

Proof. See Harris–Kudla [13], section. 4, (4.4)-(4.7) and Kudla [18], II.1. ☐

Now we assume that F is archimedean.

- If $F = \mathbb{C}$, then one has only one quaternion algebra B over F. In this case, we have

$$I(0) = \Pi(B). \tag{2.2.3}$$

This is proved in Lemma A.1 of Appendix of Harris–Kudla [13].

- If $F = \mathbb{R}$, then one has two quaternion algebras, $B = M_{2\times 2}$ and B' the Hamilton quaternion. The replacement of Lemma 2.2.1 is the following isomorphism (Harris–Kudla [13], (4.8))

$$I(0) = \Pi(B) \oplus \Pi(B') \tag{2.2.4}$$

where $\Pi(B') = \Pi(4,0) \oplus \Pi(0,4)$ is the direct sum of the two spaces associated with the two quadratic spaces obtained by changing signs of the reduced norm on the Hamilton quaternion.

Local zeta integral of triple product

For an irreducible admissible representation σ of $\mathbb{G}$, let $W_\sigma = W_\sigma^\psi$ be the ψ-Whittaker module of σ. The local zeta integral of Garrett ([6]) and Piatetski-Shapiro–Rallis ([29]) is a (family of) linear functional on $I(s) \times W_\sigma$ defined by

eqn local Z

$$Z(s, f, W) = \int_{F^\times N_0 \backslash \mathbb{G}} f_s(\eta g) W(g) dg, \quad (f, W) \in I(s) \times W_\sigma. \tag{2.2.5}$$

See the normalization of measure in "Notations". Here, N_0 is a subgroup of $\mathbb{G}$ defined as

$$N = \left\{ \left(\begin{array}{cc} 1 & b \\ 0 & 1 \end{array} \right) \middle| b \in E, \mathrm{Tr}_{E/F}(b) = 0 \right\},$$

and $\eta \in \mathrm{GSp}_6$ is a representative of the unique open orbit of $\mathbb{G}$ acting on $P\backslash\mathrm{GSp}_6$. The integral $Z(s, f, W)$ is absolutely convergent for $\mathrm{Re}(s) \gg 0$. When the exponent $\Lambda(\sigma) < \frac{1}{2}$ (cf. [14, §2]) of the representation σ (cf. [14, §2]), the integral $Z(0, f, W)$ is absolutely convergent.

non-van

Proposition 2.2.2. *For σ with $\Lambda(\sigma) < \frac{1}{2}$, the local zeta integral $Z(0, f, W)$ defines a non-vanishing linear functional on $I(0) \times W_\sigma$.*

Proof. See [29, Prop. 3.3] and [16, pp. 227]. □

Let π be an irreducible admissible representation of $B_E^\times$ with trivial restriction on $F^\times$. We define the integration of matrix coefficients as follows:

eqn alpha

(2.2.6)

$$\alpha(\phi_1 \otimes \phi_2) := \frac{L(1, \sigma, ad)}{\zeta(2)^2 L(1/2, \sigma)} \int_{F^\times \backslash B^\times} (\pi(b)\phi_1, \phi_2) db, \qquad \phi_1 \otimes \phi_2 \in \pi \otimes \widetilde{\pi}.$$

Let σ be the Jacquet–Langlands correspondence of π to $\mathrm{GL}_2(E)$. Assume that $\Lambda(\sigma) < 1/2$. If f is the Siegel–Weil section f_Φ associated to $\Phi \in \mathscr{S}(B)$, we also write

eqn local Z Phi

$$Z(s, \Phi, \varphi) := Z(s, f_\Phi, W_\varphi), \tag{2.2.7}$$

where $\varphi \mapsto W_\varphi$ is a fixed homomorphism $\sigma \to \mathscr{W}_\sigma$.

Ichino

Proposition 2.2.3 (Ichino [14])**.** *Assume $\Lambda(\sigma) < \frac{1}{2}$. Under the normalization of θ as in 2.1.2, we have*

$$Z(0, \Phi, \varphi) = \mathrm{sgn}(B) \frac{L(1/2, \sigma)}{\zeta_F(2)} \alpha(\theta(\Phi \otimes \varphi)),$$

where $\mathrm{sgn}(B) = 1$ *if B is split and -1 if B is division.*

Proof. This is Proposition 5.1 of Ichino [14]. Notice that our choice of the local Haar measure on $F^\times \backslash B^\times$ differs from that of [14] by $\zeta_F(2)$. □

2.3 Integral representation of triple-product L-series

In this subsection, we review the integral representation of the triple product L-series of Garrett, Piatetski-Shapiro, and Rallis and various improvements of Harris–Kudla. Let F be a number field with adeles $\mathbb{A}$, $\mathbb{B}$ a quaternion algebra over $\mathbb{A}$ with ramification set $\Sigma(\mathbb{B})$, E a cubic semisimple algebra. We write $\mathbb{B}_E := \mathbb{B} \otimes_F E$ the base changed quaternion algebra over $\mathbb{A}_E := \mathbb{A} \otimes_F E$.

Siegel–Eisenstein series

For $\Phi \in \mathscr{S}(\mathbb{B}_E)$, analogous to (2.2.1) we define

$$f_\Phi(g, s) = r(g)\Phi(0)\lambda_s(g),$$

where the character λ_s of P defined as

$$\lambda_s(d(\nu)n(b)m(a)) = |\nu|^{-3s}|\det(a)|^{2s}.$$

and it extends to a function on GSp_6 via Iwasawa decomposition $\mathrm{GSp}_6 = PK$ such that $\lambda_s(g)$ is trivial on K. It satisfies

$$f_\Phi(d(\nu)n(b)m(a)g, s) = |\nu|^{-3s-3}|\det(a)|^{2s+2} f_\Phi(g, s).$$

It thus defines a section, called a Siegel–Weil section, of $I(s) = \mathrm{Ind}_P^{\mathrm{GSp}_6}(\lambda_s)$. Then the Siegel–Eisenstein series is defined to be

eqn S-E (2.3.1)
$$E(g, s, \Phi) = \sum_{\gamma \in P(F)\backslash \mathrm{GSp}_6(F)} f_\Phi(\gamma g, s).$$

This is absolutely convergent when $\mathrm{Re}(s) > 2$. It extends to a meromorphic function of $s \in \mathbb{C}$ and holomorphic at $s = 0$ ([19, Thm. 2.2]).

For $T \in \mathrm{Sym}_3(F)$, we define its T-th Fourier coefficients to be:

eqn E_T (2.3.2)
$$E_T(g, s, \Phi) = \int_{[\mathrm{Sym}_3]} E(n(b)g, s, \Phi)\psi(-Tb)db.$$

(cf. "Notations" and we have shorten $\psi(T)$ for $\psi(\mathrm{Tr}(T))$ if no confusion arises.)

Suppose that $\Phi = \otimes_v \Phi_v$ is decomposable. When T is non-singular, we have a decomposition into a product of local Whittaker functions

eqn E_T=W_T (2.3.3)
$$E_T(g, s, \Phi) = \prod_v W_{T,v}(g_v, s, \Phi_v),$$

where the local Whittaker function is given by

eqn W_T (2.3.4)
$$W_{T,v}(g_v, 0, \Phi_v) = \int_{\mathrm{Sym}_3(F_v)} f_\Phi(wn(b)g, s)\psi(-Tb)db,$$

where

$$w = \begin{pmatrix} & 1_3 \\ -1_3 & \end{pmatrix}.$$

By [19, Prop. 1.4], for non-singular T, the Whittaker function $W_{T,v}(g_v, s, \Phi_v)$ has an entire analytic extension to $s \in \mathbb{C}$. Moreover, under the following "unramified" conditions:

- v is non-archimedean, T is integral with $\det(T) \in \mathscr{O}_{F_v}^{\times}$,
- the maximal fractional ideal of $\mathscr{O}_v$ on which ψ_v is trivial is $\mathscr{O}_{F_v}$,
- Φ_v is the characteristic function of a self-dual lattice Λ_v of V_v,
- $g_v \in K_v = \mathrm{GSp}_6(\mathscr{O}_v)$, the standard maximal compact subgroup of $\mathrm{GSp}_6(F_v)$,

we have [19, Prop. 4.1]:

$$W_{T,v}(g_v, s, \Phi_v) = \zeta_{F_v}(s+2)^{-1}\zeta_{F_v}(2s+2)^{-1}.$$

Rankin triple product L-function

Let σ be a cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_E)$. Let π be associated with Jacquet-Langlands correspondence of σ on $\mathbb{B}_E^{\times}$. Let ω_σ be the central character of σ. We assume that

$$\omega_\sigma|_{\mathbb{A}_F^{\times}} = 1. \tag{2.3.5}$$

Define a finite set of places of F

$$\Sigma(\sigma) = \left\{ v \,\middle|\, \epsilon(\sigma_v, \frac{1}{2}) = -1 \right\}. \tag{2.3.6}$$

Define the global zeta integral as

$$Z(s, \phi, \varphi) = \int_{[\mathbb{G}]} E(g, s, \Phi)\varphi(g)dg, \tag{2.3.7}$$

where $[\mathbb{G}] := \mathbb{A}^{\times}\mathbb{G}(F)\backslash\mathbb{G}(\mathbb{A})$. Recall that the local zeta integral is defined by (2.2.5), (2.2.7).

Theorem 2.3.1 (Piatetski-Shapiro–Rallis [29]). *Assume that $\Phi = \otimes\Phi_v$ is decomposable. For a cusp form $\varphi \in \sigma$ and $\mathrm{Re}(s) \gg 0$ we have an Euler product*

$$Z(s, \Phi, \varphi) = \prod_v Z(s, \Phi_v, \varphi_v) = \frac{L(s+\frac{1}{2}, \sigma)}{\zeta_F(2s+2)\zeta_F(4s+2)} \prod_v \alpha(s, \Phi_v, \varphi_v), \tag{2.3.8}$$

where

$$\alpha(s, \Phi_v, W_{\varphi_v}) := \frac{\zeta_{F_v}(2s+2)\zeta_{F_v}(4s+2)}{L(s+\frac{1}{2}, \sigma_v)} Z(s, \Phi_v, \varphi_v),$$

which equals one for almost all v.

cor kernel L' **Corollary 2.3.2.** *1. The global zeta integral $Z(0,\Phi,\varphi)\neq 0$ only if $\Sigma(\mathbb{B})=\Sigma(\sigma)$ and both have even cardinality. In this case, we have an identity:*

zeta int (2.3.9)
$$\int_{[\mathbb{G}]} E(g,0,\Phi)\varphi(g)dg=\frac{L(\frac{1}{2},\sigma)}{\zeta_F(2)}\prod_v \alpha(\theta(\Phi_v,\varphi_v)).$$

2. If $\Sigma(\mathbb{B})=\Sigma(\sigma)$ is odd, then the global root number $\epsilon(1/2,\sigma)=-1$ and $L(1/2,\sigma)=0$. We have the following integral representation for the central derivative:

eqn kernel E' (2.3.10)
$$\int_{[\mathbb{G}]} E'(g,0,\Phi)\varphi(g)dg=-\frac{L'(\frac{1}{2},\sigma)}{\zeta_F(2)}\prod_v \alpha(\theta(\Phi_v,\varphi_v)).$$

Proof. The condition $\Lambda(\sigma_v)<\frac{1}{2}$ holds if σ is a local component of a cuspidal automorphic representation by the work of Kim–Shahidi [25]. By Proposition 2.2.3, we have

(2.3.11)
$$\alpha(0,\Phi_v,\varphi_v)=\mathrm{sgn}(B_v)\,\zeta_F(2)\,\alpha(\theta(\Phi_v\otimes\varphi_v)).$$

The corollary follows that the integration of matrix coefficients $\alpha:\pi_v\otimes\widetilde{\pi}_v\to\mathbb{C}$ is nonzero if and only if $\epsilon(1/2,\sigma_v)=\mathrm{sgn}(B_v)$.

□

2.4 Ichino's formula

In this subsection, we review a central value formula of Ichino. We assume that $\Sigma(\mathbb{B})$ is even. Let B be a quaternion algebra over F with ramification set $\Sigma(\mathbb{B})$. Then, we write V for the orthogonal space (B,q).

For our purpose, we first recall the Siegel–Weil formula for groups of similitudes. The theta kernel is defined to be, for $(g,h)\in R(\mathbb{A})$,

eqn theta (2.4.1)
$$\theta(g,h,\Phi)=\sum_{x\in B_E} r(g,h)\Phi(x).$$

It is $R(F)$-invariant. The theta integral is the theta lifting of the trivial automorphic form, for $g\in \mathrm{GSp}_6^+(\mathbb{A})$,

eqn I int (2.4.2)
$$I(g,\Phi)=\int_{[\mathrm{O}(B_E)]}\theta(g,h_1h,\Phi)dh_1,$$

where h is any element in $\mathrm{GO}(B_E)$ such that $\nu(h)=\nu(g)$. It does not depend on the choice of h. When $B=M_{2\times 2}$ the integral needs to be regularized. The measure is normalized such that the volume of $[\mathrm{O}(B_E)]$ is one. The function $g\mapsto I(g,\Phi)$ is left invariant under $\mathrm{GSp}_6^+(\mathbb{A})\cap\mathrm{GSp}_6(F)$ and under the center $Z_{\mathrm{GSp}_6}(\mathbb{A})$ of $\mathrm{GSp}_6(\mathbb{A})$.

The following Siegel–Weil formula can be found [13, Thm. 4.2].

thm SW **Theorem 2.4.1** (Siegel–Weil)**.** *The Siegel–Eisenstein series $E(g,s,\Phi)$ is holomorphic at $s=0$ and*

$$E(g,0,\Phi)=2I(g,\Phi),\quad g\in \mathrm{GSp}_6^+(\mathbb{A}). \tag{2.4.3}$$

To eliminate the dependence on the choice of measure on $O(V)(\mathbb{A})$, we shall write it as

$$E(g,0,\Phi)=2(\mathrm{vol}([\mathrm{O}(V)]))^{-1}I(g,\Phi). \tag{2.4.4}$$

Now we deduce a formula for the T-th Fourier coefficient of the Siegel–Eisenstein series.

cor E_T=I_T **Corollary 2.4.2.** *Assume that V is anisotropic and* $\det(T)\neq 0$. *Then for $g\in \mathrm{GSp}_6^+(\mathbb{A})$ we have*

$$E_T(g,0,\Phi)=2\,\mathrm{vol}([\mathrm{O}(V)_{x_0}])\int_{\mathrm{O}(V)(\mathbb{A})/\mathrm{O}(V)_{x_0}(\mathbb{A})} r(g,h)\Phi(h_1^{-1}x_0)\,dh_1,$$

where $h\in \mathrm{GO}(V_{\mathbb{A}})$ has the same similitude as g, $x_0\in V(F)$ is a base point with $Q(x_0)=T$, and $\mathrm{O}(V)_{x_0}\simeq \mathrm{O}(x_0^{\perp})$ is the stabilizer of x_0.

Proof. Put $g_1=d(\nu(g))^{-1}g$. We obtain by Theorem thm SW 2.4.1:

$$\begin{aligned}E_T(g,0,\Phi)&=2\int_{[\mathrm{Sym}_3]}\psi(-Tb)I(n(b)g,\Phi)\,db\\&=2\int_{[\mathrm{Sym}_3]}\psi(-Tb)\int_{[\mathrm{O}(V)]}\sum_{x\in V(F)}|\nu(g)|_{\mathbb{A}}^{-3}r(d(\nu(g)^{-1})n(b)g)\Phi(h^{-1}h_1^{-1}x)\,dh_1db.\end{aligned}$$

Note that $d(\nu(g)^{-1})n(b)g=n(\nu(g)b)d(\nu(g)^{-1})g$. We thus have

$$r(d(\nu(g)^{-1})n(b)g)\Phi(h^{-1}h_1^{-1}x)=\psi(\nu(g)bQ(h^{-1}x))r(g_1)\Phi(h^{-1}h_1^{-1}x)=\psi(bQ(x))r(g_1)\Phi(h^{-1}h_1^{-1}x).$$

Since $[O(V)]$ is compact, we may interchange the order of integrations. Then the integral over $[\mathrm{Sym}_3]$ is zero unless $T=Q(x)$. Since T is non-singular, by Witt theorem, the set of $x\in V(F)^3$ with $Q(x)=T$ is either empty or a single $\mathrm{O}(V)(F)$-orbit. Fix a base point x_0. Then the stabilizer $\mathrm{O}(V)_{x_0}$ of x_0 is isomorphic to $\mathrm{O}(W)$ for the orthogonal complement W of the space spanned by the components of x_0. We now have

$$\begin{aligned}E_T(g,0,\Phi)&=2\int_{[\mathrm{O}(V)]}\sum_{\gamma\in \mathrm{O}(V)(F)/\mathrm{O}(V)_{x_0}(F)} r(g_1)\Phi(h^{-1}h_1^{-1}\gamma^{-1}x_0)dh_1\\&=2\mathrm{vol}([\mathrm{O}(V)_{x_0}])\int_{\mathrm{O}(V)(\mathbb{A})/\mathrm{O}(V)_{x_0}(\mathbb{A})} r(g_1)\Phi(h^{-1}h_1x_0)dh_1.\end{aligned}$$

This completes the proof. □

For non-singular T we define

$$I_T(g,\Phi) = 2\mathrm{vol}([\mathrm{O}(V)_{x_0}]) \int_{\mathrm{O}(V)(\mathbb{A})/\mathrm{O}(V)_{x_0}(\mathbb{A})} r(g,h)\Phi(h_1^{-1}x_0)dh_1. \tag{2.4.5}$$

We have $\mathrm{O}(V) = \mathrm{SO}(V) \rtimes \mu_2$ (cf. Notations) where $\mu_2 \subset \mathrm{O}(V)$ is generated by the canonical involution on the quaternion algebra. When T is non-singular, it is easy to see that $\mathrm{SO}(V)$ is surjective onto $\mathrm{O}(V)/\mathrm{O}(V)_{x_0}$. We then may choose a measure on $\mathrm{O}(V)(\mathbb{A})$ such that it is the product measure of the Tamagawa measure on $\mathrm{SO}(V)(\mathbb{A})$ and the measure on $\mu_2(\mathbb{A})$ such that

$$\mathrm{vol}(\mu_2(\mathbb{A})) = 1.$$

Since the Tamagawa number of $\mathrm{SO}(V)$ is 2, we have

$$\mathrm{vol}([\mathrm{O}(V)]) = \frac{1}{2}\mathrm{vol}(\mathrm{SO}(V)(F)\backslash \mathrm{O}(V)(\mathbb{A})) = \frac{1}{2}\mathrm{vol}([\mathrm{SO}(V)])\mathrm{vol}(\mu_2(\mathbb{A})) = 1,$$

$$\mathrm{vol}(\mu_2(F)\backslash \mu_2(\mathbb{A})) = \frac{\mathrm{vol}(\mu_2(\mathbb{A}))}{|\mu_2(F)|} = \frac{1}{2}.$$

Now we define (a certain orbital integral):

$$I_{T,v}(g_v,\Phi_v) = \int_{\mathrm{SO}(V)(F_v)} r(g_v,h_v)\Phi_v(h_1x_0)\,dh_1, \quad \nu(g_v) = \nu(h_v). \tag{2.4.6}$$

Then we may rewrite (2.4.5), when $\Phi = \otimes_v \Phi_v$ is decomposable,

$$I_T(g,\Phi) = \prod_v I_{T,v}(g_v,\Phi_v), \tag{2.4.7}$$

Moreover, Corollary 2.4.2 can be rewritten as:

$$E_T(g,0,\Phi) = I_T(g,\Phi). \tag{2.4.8}$$

We also need a *local Siegel–Weil* formula for later use.

Proposition 2.4.3. *Suppose that $T \in \mathrm{Sym}_3(F_v)$ is non-singular. Then there is a non-zero constant κ such that for all $g_v \in \mathrm{GSp}_6(F_v)$, $\Phi_v \in \mathscr{S}(V_v^3)$*

$$W_{T,v}(g_v,0,\Phi_v) = \kappa \cdot I_{T,v}(g_v,\Phi_v).$$

In particular, the functional $\Phi_v \mapsto W_{T,v}(1,0,\Phi_v)$ is non-zero if and only if T is represented by V_v.

Proof. It suffices to prove the statement for $g_v = 1$. Consider the space of linear functionals ℓ on $\mathscr{S}(V_v^3)$ that satisfy

$$\ell(r(n(b))\Phi_v) = \psi(Tb)\ell(\Phi_v).$$

Then by [19, Prop. 1.2], this space is spanned by $\Phi_v \mapsto I_{T,v}(1,\Phi_v)$ (whose definition depends on the normalization of the measure $d\mu_{T,v}$). Since $\Phi_v \mapsto W_{T,v}(1,0,\Phi_v)$ also satisfies this relation, it defines a multiple of the linear functional $I_{T,v}(1,\cdot)$ above. The multiple can be chosen to be non-zero by [19, Prop. 1.4 (ii)]. $\square$

We now state the central value formula due to Ichino [14, Theorem 1.1]. Note that we will not use this formula in this paper. However, the proofs of this and our main theorem are parallel. Considering the diagonal embedding $B^\times \hookrightarrow B_E^\times$, we define the trilinear period:

$$P_\pi(f) = \int_{F^\times \backslash B^\times(\mathbb{A})} f(b)db, \quad f \in \pi.$$

central value

Theorem 2.4.4. *For $f = \otimes_v f_v \in \pi, \widetilde{f} = \otimes_v \widetilde{f}_v \in \widetilde{\pi}$, we have*

$$P_\pi(f)P_{\widetilde{\pi}}(\widetilde{f}) = \frac{1}{2^c}\frac{\zeta_E(2)}{\zeta_F(2)}\frac{L(\frac{1}{2},\sigma)}{L(1,\sigma,ad)}\alpha(f,\widetilde{f}).$$

Here the constant c is $3, 2,$ and 1 respectively if $E = F \oplus F \oplus F$, $E = F \oplus K$ for a quadratic K, and a cubic field extension E of F respectively.

Remark 2.4.1. The formula is trivial if the global root number is -1. Therefore, the primary purpose of this paper is to study the case where the global root number is -1.

decomp

2.5 Derivatives of Eisenstein series

Now we fix an *incoherent* quaternion algebra $\mathbb{B}$ over $\mathbb{A}$ with ramification set Σ. We assume that $\mathbb{B}$ has definite $\mathbb{B}_v$ at archimedean places. We consider the Eisenstein series $E(g, s, \Phi)$ for $\Phi \in \mathscr{S}(\mathbb{B}^3)$. We always take Φ_∞ to be standard Gaussian. In this case this Eisenstein series vanishes at $s = 0$ as observed by Kudla [19, Thm. 2.2(ii)]. As we now discuss, vanishing a non-singular T-th Fourier coefficient is easier to see.

For $T \in \mathrm{Sym}_3(F)_{reg}$, let $\Sigma(T)$ be the set of places over which T is anisotropic. Then $\Sigma(T)$ has even cardinality. By Prop. 2.4.3, the vanishing order of the T-th Fourier coefficient $E_T(g, s, \Phi)$ at $s = 0$ is at least (also cf. [19, Coro. 5.3])

$$|\Sigma \cup \Sigma(T)| - |\Sigma \cap \Sigma(T)|.$$

Since $|\Sigma|$ is odd, we see that $E_T(g, s, \Phi)$ always vanishes at $s = 0$. Furthermore, its derivative does not vanish only if Σ and $\Sigma(T)$ is nearby: they differ by precisely one place v. Thus we define

$$\Sigma(v) = \begin{cases} \Sigma \setminus \{v\} & \text{if } v \in \Sigma \\ \Sigma \cup \{v\} & \text{otherwise} \end{cases} \tag{2.5.1}$$

When $\Sigma(T) = \Sigma(v)$, the derivative is given by

$$E'_T(g, 0, \Phi) = W'_{T,v}(g_v, 0, \Phi_v) \cdot \prod_{w \neq v} W_{T,w}(g_w, 0, \Phi_w). \tag{2.5.2}$$

We thus obtain a decomposition of $E'(g,0,\Phi)$ according to the difference of $\Sigma(T)$ and Σ:

$$E'(g,0,\Phi)=\sum_v E'_v(g,0,\Phi)+E'_{\mathrm{sing}}(g,0,\Phi), \tag{2.5.3}$$

where

eqn E'v

$$E'_v(g,0,\Phi):=\sum_{\Sigma(T)=\Sigma(v)} E'_T(g,0,\Phi), \tag{2.5.4}$$

and

$$E'_{\mathrm{sing}}(g,0,\Phi)=\sum_{\det(T)=0} E'_T(g,0,\Phi).$$

A weak intertwining property

In the case where Σ is odd, the functional $\Phi\mapsto E'(g,0,\Phi)$ is not equivariant under the action of $\mathrm{Sp}_6(\mathbb{A})$. Instead, we have a weak intertwining property:

prop wk inter

Proposition 2.5.1. *Let $\mathscr{A}_0$ be the image of $\Pi(B_{\mathbb{A}})$ under the map $f\mapsto E(g,0,f)$ for all quaternion algebra B over F. Then for any $h\in\mathrm{Sp}_6(\mathbb{A})$, $f\in I(0)$, the function*

$$\mathrm{Sp}_6(\mathbb{A})\ni g\mapsto E'(gh,0,f)-E'(g,0,r(h)f)$$

belongs to $\mathscr{A}_0$.

Proof. Let $\alpha(s,h)(g)=\alpha(s,g,h)=\frac{1}{s}(|\frac{\delta(gh)}{\delta(g)}|^s-1)$, $s\neq 0$. Then it extends to an entire function of s and is left $P_{\mathbb{A}}$-invariant. Now for $\mathrm{Re}(s)\gg 0$, we have

$$E(gh,s,f)-E(g,s,r(h)f)=sE(g,s,\alpha(s,h)r(h)f)$$

Now note that $g\to\alpha(s,h)r(h)f(g)\delta(g)^s$ defines a holomorphic section of $I(s)$. Hence the Eisenstein series $E(g,s,\alpha(s,h)r(h)f)$ is holomorphic at $s=0$ since any holomorphic section of $I(s)$ is a finite linear combination of the standard section with holomorphic coefficients. This implies the desired assertion.

$\square$

3 Trilinear cycles and generating series

In this section, we construct the geometric kernel function for $\Phi\in\mathscr{S}(\mathbb{B}^3)$ where $\mathbb{B}$ is an incoherent totally definite quaternion algebra over a totally real field F. We will first prove the spectral decomposition of 1-cycles and additivity under the action of endomorphisms using the Fourier–Mukai transform. Then we review the generating series of Hecke operators and its modularity (Proposition prop mod Z 3.3.1) following our previous paper Y-Z-Z [32]. The main conjecture can then be reformulated as a kernel identity between the derivative of the Eisenstein series and the geometric kernel associated with Φ, see Conjecture kernel-identity ??. Finally, we introduce arithmetic Hodge classes and arithmetic Hecke operators, which gives a decomposition of the geometric kernel function to a sum of local heights and singular pairings.

3.1 Trilinear 1-cyles

S3.1

In this subsection, we would like to prove the essential facts in the introduction. Let $A = A_1 \times A_2 \times A_3$ be a product of three simple abelian varieties over a field F. First, we want to decompose $\mathrm{Ch}_1(A)$ into eigenspaces under push forwards. For $k = (k_1, k_2, k_3) \in (\mathbb{Z} \setminus \{0\})^3$, we have a multiplication $[k]$ on A by component-wise multiplication by k_i.

Lemma 3.1.1. *We have a decomposition*

$$\mathrm{Ch}_1(A) = \sum_s \mathrm{Ch}(A, s),$$

where $s = (s_1, s_2, s_3)$ is triple of non-negative integers, and $\mathrm{Ch}_1(A, s)$ is subspace of cycles x such that

$$[k]_* x = k^s \cdot x, \qquad k^s := k_1^{s_1} k_2^{s_2} k_3^{s_3}.$$

Proof. We prove the lemma for general cycles $\mathrm{Ch}^*(A)$ using Fourier–Mukai transform. Let $A^\vee = A_1^\vee \times A_2^\vee \times A_3^\vee$ be the product of duals. Let $\mathscr{P}_i$ be the Poincare bundle on $A_i \times A_i^\vee$, and use the same notation for their pull and-back on A. Then $\mathscr{P} = \boxtimes \mathscr{P}_i$ is the Poincare bundle on $A \times A^\vee$. Define a Fourier transform on Chow groups with rational coefficients by

$$\mathscr{F}: \quad \mathrm{Ch}^*(A) \longrightarrow \mathrm{Ch}^*(A^\vee), \qquad \mathscr{F}(x) = q_*(p^* x \cdot e^{c_1(\mathscr{P})})$$

$$\mathscr{F}^\vee: \quad \mathrm{Ch}^*(A^\vee) \longrightarrow \mathrm{Ch}^*(A), \qquad \mathscr{F}^\vee(y) = p_*(q^* y \cdot e^{c_1(\mathscr{P})})$$

where p and q are projections of $A \times A^\vee$ onto A and $A^\vee$ respectively. Then these two operators are almost inverse to each other:

$$\mathscr{F}^\vee \circ \mathscr{F} = [-1]^g [-1]_A^*, \qquad \mathscr{F} \circ \mathscr{F}^\vee = [-1]^g [-1]_{A^\vee}^*$$

where $g = \dim A$.

From these identities, it follows that any $x \in \mathrm{Ch}^*(A)$ has an expansion

$$x = \sum_{t_1, t_2, t_3 \geq 0} p_* \left(c_1(\mathscr{P}_1)^{t_1} \cdot c_1(\mathscr{P}_2)^{t_2} \cdot c_1(\mathscr{P}_3)^{t_3} \cdot q^*(y_{t_1, t_2, t_3}) \right)$$

with $y_{t_1, t_2, t_3} \in \mathrm{Ch}^*(A^\vee)$. It is easy to see that each term on the right-hand side is an eigenvector under pull-back $[k]^*$ with eigenvalue $k_1^{t_1} t_2^{t_2} t_3^{t_3}$. Since $[k]_*[k]^* = k_1^{2g_1} k_2^{2g_2} k_3^{2g_3}$, it follows that each term is also an eigenvector under $[k]_*$ with eigenvalue $k_1^{2g_1 - t_1} k_2^{2g_2 - t_2} k_3^{2g_3 - t_3}$. $\square$

Let $\mathrm{Ch}^{\ell\ell\ell}(A)$ denote $\mathrm{Ch}_1(A,(1,1,1))$ and call it the space of *trilinear one-cycles*. We want to study the $\mathrm{End}(A_i)$ action on this space.

Lemma 3.1.2. *For each i, let $\phi_i, \psi_i \in \mathrm{End}(A_i)$ and let $\phi = (\phi_1, \phi_2, \phi_3)$ and $\psi = (\psi_1, \psi_2, \psi_3)$ be the induced element in $\mathrm{End}(A)$. Then for any $x \in \mathrm{Ch}^{\ell\ell\ell}(A)$,*

$$(\phi + \psi)_* x = \prod_i (\phi_{i*} + \psi_{i*})x.$$

Proof. We use functoriality of Mukai–Fourier transform: if $\varphi : B \longrightarrow C$ is a morphism of abelian varieties, then for any $z \in \mathrm{Ch}^*(B)$,

$$\mathscr{F}(\varphi_* z) = \varphi^{\vee *}\mathscr{F}(z),$$

where $\varphi^\vee : C^\vee \longrightarrow B^\vee$ is the dual morphism.

Let $y = \mathscr{F}(x) \in \mathrm{Ch}^*(A^\vee)$. Applying the above formula for $\phi : A \longrightarrow A$, the equality in the lemma is equivalent to

$$(\phi^\vee + \psi^\vee)^* y = \prod_i (\phi_i^{\vee *} + \psi_i^{\vee *})y.$$

By the same formula and the assumption, we have that x and y are trilinear under pull-back morphism:

$$[k]^* y = k_1 k_2 k_3 y.$$

From the proof of the previous lemma, y has an expression as

$$y = q_* \left(c_1(\mathscr{P}_1) \cdot c_1(\mathscr{P}_2) \cdot c_1(\mathscr{P}_3) \cdot p^*(z)\right)$$

for some $z \in \mathrm{Ch}^*(A)$. It follows that the following identity holds:

$$(\phi^\vee + \psi^\vee)^* y = q_* \left((\widetilde{\phi}_1^\vee + \widetilde{\psi}_1^\vee)^* c_1(\mathscr{P}_1) \cdot (\widetilde{\phi}_2^\vee + \widetilde{\psi}_2^\vee)^* c_1(\mathscr{P}_2) \cdot (\widetilde{\phi}_3^\vee + \widetilde{\psi}_3^\vee)^* c_1(\mathscr{P}_3) \cdot p^*(z) \right)$$

where for each i, $\widetilde{\phi}_i^\vee$ and $\widetilde{\psi}_i^\vee$ denote the endomorphism of $A_i \times A_i^\vee$ induced by ϕ_i and ψ_i on the second factor. Thus we are reduced to prove the following identity for each i:

$$(\widetilde{\phi}_i^\vee + \widetilde{\psi}_i^\vee)^* c_1(\mathscr{P}_i) = (\widetilde{\phi}_1^{\vee *} + \widetilde{\psi}_1^{\vee *}) c_1(\mathscr{P}_i)$$

This follows from the additivity of $\mathscr{P}_i$ in the second variable: Let μ_i, β_i, and γ_i denote morphisms induced by addition and two projections in the second and third variable:

$$A \times A^\vee \times A^\vee \longrightarrow A \times A^\vee,$$

then

$$\mu_i^* \mathscr{P}_i = \beta_i^* \mathscr{P} + \gamma_i^* \mathscr{P}.$$

Composing the above identity with φ^* where φ is the morphism defined by

$$\varphi : \qquad A \times A^\vee \longrightarrow A \times A^\vee \times A^\vee : \qquad (x, y) \longrightarrow (x, \phi_i(x), \psi_i(x)),$$

Then we obtain the desired identity:

$$(\widetilde{\phi}_i^\vee + \widetilde{\psi}_i^\vee)^* c_1(\mathscr{P}_i) = (\widetilde{\phi}_1^{\vee *} + \widetilde{\psi}_1^{\vee *}) c_1(\mathscr{P}_i).$$

$\square$

By the second lemma, we see that the action of $\mathrm{End}(A_i)$ on $\mathrm{Ch}^{\ell\ell\ell}(A)$ makes $\mathrm{Ch}^{\ell\ell\ell}(A)$ a module over

$$\mathrm{End}^{\ell\ell\ell}(A) := \mathrm{End}(A_1) \otimes \mathrm{End}(A_2) \otimes \mathrm{End}(A_3).$$

Recall that we have defined the height pairing of trilinear 1-cycles:

$$\langle -, - \rangle : \quad \mathrm{Ch}^{\ell\ell\ell}(A) \otimes_{\mathbb{Q}} \mathrm{Ch}^{\ell\ell\ell}(A^\vee) \longrightarrow \mathbb{C}, \qquad \langle x, y \rangle := (x \times y) \prod_i \widehat{c}_1(\bar{\mathscr{P}}_i).$$

Let $\phi \mapsto \phi^\vee$ denote the anti-isomorphism induced by duality:

$$\mathrm{End}^{\ell\ell\ell}(A) \longrightarrow \mathrm{End}^{\ell\ell\ell}(A^\vee).$$

By definition and projection formula, we have

Lemma 3.1.3. *For $\phi \in \mathrm{End}^{\ell\ell\ell}(A)$, $x \in \mathrm{Ch}^{\ell\ell\ell}(A)$, $y \in \mathrm{Ch}^{\ell\ell\ell}(A^\vee)$, we have*

$$\langle \phi x, y \rangle = \langle x, \phi^\vee y \rangle.$$

Moreover, if $\phi = \phi_1 \otimes \phi_2 \otimes \phi_3$ is a pure tensor, then both terms in the above are equal to

$$(x \times y) \prod_i \widetilde{\phi}_i^* \widehat{c}_1(\bar{\mathscr{P}}_i) = (x \times y) \prod_i \widetilde{\phi}_i^{\vee *} \widehat{c}_1(\bar{\mathscr{P}}_i),$$

where $\widetilde{\phi}_i$ and $\widetilde{\phi}_i^\vee$ are endomorphisms of $A \times A^\vee$ induced by ϕ_i and $\phi_i^\vee$.

Now we assume that each A_i is of GL(2)-type with $\mathrm{End}^0(A_i) = M_i$. Then $M = \otimes_i M_i$ acts linearly on $\mathrm{Ch}^{\ell\ell\ell}(A)$. We can define a height pairing

$$\langle -, - \rangle_M : \quad \mathrm{Ch}^{\ell\ell\ell}(A) \otimes_M \mathrm{Ch}^{\ell\ell\ell}(A^\vee) \longrightarrow M \otimes \mathbb{C}$$

such that for any $a \in M$, $x \in \mathrm{Ch}^{\ell\ell\ell}(A)$, $y \in \mathrm{Ch}^{\ell\ell\ell}(A^\vee)$,

$$\mathrm{Tr}_{M \otimes \mathbb{C}/\mathbb{C}}(a \langle x, y \rangle_M) = \langle ax, y \rangle = \langle x, ay \rangle.$$

We need the formula to express this pairing. For this we consider $\mathrm{Pic}^-(A_i \times A_i^\vee)$ of line bundles on $A_i \times A_i^\vee$ with trivializations at $0 \times A_i^\vee$ and $A_i \times 0$. Then we have actions of M_i on $\mathrm{Pic}^-(A_i \times A_i^\vee)_{\mathbb{Q}}$ by pulling back which makes $\mathrm{Pic}^-(A_i \times A_i^\vee)$ a free vector space over M_i of dimension one generated by

Poincare bundle $\mathscr{P}_i$. This induces an action of M on $\prod_i \mathrm{Pic}^-(A_i \times A_i^\vee)$. Using the same formulation; we can define an intersection paring with coefficients in $M \otimes \mathbb{C}$:

$$(x \times y) \cdot \prod_{M,i} \widehat{c}_1(\bar{\mathscr{P}}_i) \in M \otimes \mathbb{C}$$

such that for each $a \in M$,

$$\mathrm{Tr}_{M\otimes\mathbb{C}/\mathbb{C}} \left\{ a((x \times y) \prod_{M,i} \widehat{c}_1(\bar{\mathscr{P}}_i)) \right\} = (x \times y) \cdot a^* \prod_i \widehat{c}_1(\bar{\mathscr{P}}_i).$$

By the above lemma and projection formula, we have

lem ht formula **Lemma 3.1.4.** *For $x \in \mathrm{Ch}^{\ell\ell\ell}(A)$, $y \in \mathrm{Ch}^{\ell\ell\ell}(A^\vee)$,*

$$\langle x, y \rangle_M = (x \times y) \prod_M \widehat{c}_1(\bar{\mathscr{P}}_i).$$

Remark 3.1.1. Let C be a curve over F. Gross and Schoen have constructed height pairings $\langle -, - \rangle_{GS}$ on the space $\mathrm{Ch}_1^{00}(C^3)$ of one-cycles homologous to 0. Let $\phi : C \longrightarrow J$ an embedding into its Jacobian defined by a divisor of degree 1 on C. Then one has to map into one cycle on J^3 homologous to 0:

$$\phi_* : \qquad \mathrm{Ch}_1^{00}(C^3) \longrightarrow \mathrm{Ch}_1^{00}(J^3).$$

It can be shown that

$$\langle x, y \rangle_{GS} = \langle \phi_* x^{\ell\ell\ell}, \phi_* y^{\ell\ell\ell} \rangle.$$

The advantage of using $\mathrm{Ch}^{\ell\ell\ell}(J^3)$ over $\mathrm{Ch}_1^{00}(J^3)$ is that the previous is a module over $\mathrm{End}^{\ell\ell\ell}(J^3)$. How to prove this module structure on $\mathrm{Ch}_1^{00}(J^3)$ needs to be clarified. Conjecturally these two spaces are isomorphic to each other.

3.2 Gross–Kudla–Schoen cycles

Now go back to the setting in the introduction. Let X be a Shimura curve over a totally real field associated with an incoherent quaternion algebra $\mathbb{B}$. Let $A = A_1 \times A_2 \times A_3$ be a product of three simple abelian varieties over F parametrized by X, and L a quotient field of $\otimes_i \mathrm{End}^0(A_i)$. Let $f_i \in \pi_{A_i,L}$ and $g_i \in \pi_{A_i^\vee,L}$ and define $f = \prod_i f_i$ and $g = \prod_i g_i$ are morphisms from X to A and $A^\vee$ respectively. Then we have Gross–Kudla–Schoen cycles

$$P_L(f) = (f_* X)^{\ell\ell\ell} \otimes 1 \in \mathrm{Ch}^{\ell\ell\ell}(A, L), \qquad P_L(g) = (g_* X)^{\ell\ell\ell} \otimes 1 \in \mathrm{Ch}^{\ell\ell\ell}(A^\vee, L).$$

Consider the height pairing:

$$\langle P_L(f), P_L(g) \rangle_L \in L \otimes \mathbb{C}.$$

We want to express this height pairing as an intersection number on $X \times X$.

By Lemma lem ht formula 3.1.4 and the projection formula we have

eqn triple pair
$$\langle P_L(f), P_L(g)\rangle_M = (f_*X \times g_*X)\prod_{L,i}\widehat{c}_1(\bar{\mathscr{P}}_i) = \prod_{L,i}\widehat{c}_1(\bar{\mathscr{L}}(f_i, g_i)), \tag{3.2.1}$$

where
$$\mathscr{L}(f_i, g_i) := (f_i, g_i)^*\mathscr{P}_i \in \mathrm{Pic}^-(X\times X)_{\mathbb{Q}}$$
is endowed with the admissible metric. We must describe the bundle $\mathscr{L}(f_i, g_i)$ directly.

The space $\mathrm{Pic}^-(X\times X)_{\mathbb{Q}}$ is a subspace of correspondences on X with an action by $(\mathbb{B}^\times)^2$. It is closed under convolution and thus has a ring structure. This ring has an action T on representations π_B for any simple abelian variety B parametrized by X by obvious way: if $\mathscr{L}\in \mathrm{Pic}^-(X\times X)$ and $f\in \mathrm{Hom}(X, B)$ realized at some level U, then $\mathrm{T}(\mathscr{L})\cdot f$ will bring $x\in X_U$ to
$$f_*(c_1(\mathscr{L}|_{x\times X_U}))\in B.$$
As a representation of $(\mathbb{B}^\times)^2$, we have
$$\mathrm{Pic}^-(X\times X)_{\mathbb{Q}} = \bigoplus_B \pi_B\otimes_{M_B}\pi_{B^\vee},$$
where the sum runs through the set of isogenous classes of simple abelian varieties B parametrized by X, and $M_B = \mathrm{End}^0(B)$. The main result of this subsection is the following:

lem triple pair

Lemma 3.2.1. *Let B be a simple abelian variety parametrized by X with endomorphism field M_B. Then for $\alpha\in\pi_B$, $\beta\in\pi_{B^\vee}$,*
$$(\alpha,\beta)^*\mathscr{P}_B = \alpha\otimes\beta.$$

Proof. We need to check the identity by applying to the representations π_C for any abelian variety C parametrized by X. Let $\gamma\in\pi_C$. Assume α,β,γ are all realized on some X_U. Then for any $x\in X_U$,
$$\mathrm{T}((\alpha,\beta)^*\mathscr{P}_B)\gamma(x) = \gamma_*((\alpha,\beta)^*\mathscr{P}_B|_{x\times X_U}) = \gamma_*\beta^*\mathscr{P}_{\alpha(x)\times B^\vee}.$$
Notice that
$$t\mapsto \gamma_*\beta^*\mathscr{P}_{t\times B^\vee}$$
defines a morphism $B\longrightarrow C$. Thus this vanishes if C and B are not isogenous. Now we assume that $B = C$. By definition, this morphism is simply the multiplication by $(\gamma,\beta)\in M_B$. Thus we have shown that
$$\mathrm{T}((\alpha,\beta)^*\mathscr{P}_B)\gamma = (\gamma,\beta)\alpha.$$
This completes the proof. ∎

3.3 Generating series of Hecke correspondences

Let $\mathbb{V}$ denote the orthogonal space $\mathbb{B}$ with quadratic form q. Recall that $\mathscr{S}(\mathbb{V})$ carries an extended Weil representation of

$$\mathscr{R} = \left\{(b_1, b_2, g) \in \mathbb{B}^\times \times \mathbb{B}^\times \times \mathrm{GL}_2(\mathbb{A}) : \quad q(b_1 b_2^{-1}) = \det g\right\}$$

by

$$r(h, g)\Phi(x) = |q(h)|^{-1} r(d(\det(g))^{-1} g)\Phi(h^{-1}x).$$

For $\alpha \in F_+^\times \backslash \mathbb{A}_f^\times$, let M_α denote the union

$$M_\alpha = \coprod_{a \in \pi_0(X)} X_a \times X_{a\alpha}.$$

This is a Shimura subvariety of $X \times X$ stabilized by the subgroup $\mathrm{GSpin}(\mathbb{V})$ of $\mathbb{B}^\times \times \mathbb{B}^\times$ of elements with the same norms. Define the group of cocycles:

$$\mathrm{Ch}^1(M_\alpha) := \varinjlim_{U_1} \mathrm{Ch}^1(M_{\alpha, U_1}),$$

where U_1 runs through the open and compact subgroups of $\mathrm{GSpin}(\mathbb{V})$. For an $h \in \mathbb{B}^\times \times \mathbb{B}^\times$, the pull-back morphism $\mathrm{T}(h)$ of right multiplication defines an isomorphism

$$\mathrm{T}(h) : \quad \mathrm{Ch}^1(M_\alpha) \longrightarrow \mathrm{Ch}^1(M_{\alpha\nu(h)^{-1}}).$$

Using Kudla's generating series and the modularity proved in [32], for each $\Phi \in \mathscr{S}(\mathbb{V})$ and $g \in \mathrm{GL}_2(F)_+ \backslash \mathrm{GL}_2(\mathbb{A})_+$, we will construct an element

$$Z(g, \Phi) \in \mathrm{Ch}^1(M_{\det g})$$

such that for any $(g', h') \in \mathscr{R}$,

$$Z(g, r(g', h')\Phi) = \mathrm{T}(h') Z(gg', \Phi).$$

Hecke correspondences

For any double coset UxU of $U \backslash \mathbb{B}_f^\times / U$, we have a Hecke correspondence

$$Z(x)_K \in Z^1(X_U \times X_U)$$

defined as the image of the morphism

$$(\pi_{U \cap xUx^{-1}, U}, \pi_{U \cap x^{-1}Ux, U} \circ \mathrm{T}_x) : \qquad Z_{U \cap xUx^{-1}} \longrightarrow X_U^2.$$

In terms of complex points at a place of F as above, the Hecke correspondence $Z(x)_U$ takes

$$(z, g) \longmapsto \sum_i (z, gx_i)$$

for points on $X_{U,\tau}(\mathbb{C})$ represented by $(z, g) \in \mathscr{H}^\pm \times \mathbb{B}_f$ where x_i are representatives of UxU/U.

Notice that this cycle is supported on the component $M_{\nu(x)^{-1}}$ of $X \times X$.

Hodge class

On $X \times X$, one has a *Hodge bundle* $\mathscr{L}_K \in \mathrm{Pic}(X \times X) \otimes \mathbb{Q}$ defined as

$$\mathscr{L}_K = \frac{1}{2}(p_1^*\mathscr{L} + p_2^*\mathscr{L}).$$

Generating Function

Write $M = M_1$, which has an action by $\mathrm{GSpin}(\mathbb{V})$. For any $x \in \mathbb{V}$ and an open and compact subgroup K of $\mathrm{GSpin}(\mathbb{V})$ of the form $U \times U$, let us define a cycle $Z(x)_K$ on M_K as follows. This cycle is non-vanishing only if $q(x) \in F^\times$ or $x = 0$. If $q(x) \in F^\times$, then we define $Z(x)_K$ to be the Hecke operator UxU defined in the last subsection. If $x = 0$, then we define $Z(x)_K$ to be the push-forward of the Hodge class on the subvariety M_α which is the union of connected components $X_a \times X_a$ with $a \in \pi_0(X)$. Let $\widetilde{K} = \mathrm{O}(F_\infty) \cdot K$ act on $\mathbb{V}$.

For $\Phi \in \mathscr{S}(\mathbb{V})^{\widetilde{K}}$, we can form a generating series

$$Z(\Phi) = \sum_{x \in \widetilde{K} \backslash \mathbb{V}} \Phi(x) Z(x)_K.$$

It is easy to see that this definition is compatible with pull-back maps in Chow groups in the projection $M_{K_1} \longrightarrow M_{K_2}$ with $K_1 \subset K_2$. Thus it defines an element in the direct limit $\mathrm{Ch}^1(M)_\mathbb{Q} := \lim_K \mathrm{Ch}^1(M_K)$ if it is absolutely convergent. We extend this definition to $\mathscr{S}(\mathbb{V})$ by projection

$$\mathscr{S}(\mathbb{V}) \longrightarrow \mathscr{S}(\mathbb{V})^{\mathrm{O}(F_\infty)}, \qquad \Phi \longrightarrow \widetilde{\Phi} := \int_{\mathrm{O}(F_\infty)} r(g)\Phi dg,$$

where dg is the Haar measure on $\mathrm{O}(F_\infty)$ with volume 1.

For $g \in \mathrm{SL}_2(\mathbb{A})$, define

$$Z(g, \Phi) = Z_{r(g)\Phi} \in \mathrm{Ch}^1(M).$$

By our previous paper [32], this series is absolutely convergent and is modular for $\mathrm{SL}_2(\mathbb{A})$:

$$Z(\gamma g, \Phi) := Z(g, \Phi), \qquad \gamma \in \mathrm{SL}_2(F). \tag{3.3.1}$$

Moreover, for any $h \in \mathbb{H}$,

$$Z(g, r(h)\Phi) = \mathrm{T}(h) Z(g, \Phi). \tag{3.3.2}$$

where $\mathrm{T}(h)$ denotes the pull-back morphism on $\mathrm{Ch}^1(M)$ by right translation of h_f.

Let $\mathrm{GL}_2(\mathbb{A})^+$ denote subgroup of $\mathrm{GL}_2(\mathbb{A})$ with totally positive determinant at archimedean places. For $g \in \mathrm{GL}_2(\mathbb{A}_F)^+$, define

$$Z(g, \Phi) = \mathrm{T}(h)^{-1} Z(r(g, h)\Phi) \in \mathrm{Ch}^1(M_{\det g}),$$

where h is an element in $\mathbb{B}^\times \times \mathbb{B}^\times$ with norm $\det g$. By (3.3.1), the definition here does not depend on the choice of h. It is easy to see that this cycle satisfies the property

$$Z(g, r(g_1, h_1)\Phi) = T(h_1)Z(gg_1, \Phi), \qquad (g, h) \in \mathscr{R}.$$

The following is the modularity of $Z(g, \Phi)$:

prop mod Z

Proposition 3.3.1. *The cycle $Z(g, \Phi)$ is automorphic for $\mathrm{GL}_2(\mathbb{A})^+$: for any $\gamma \in \mathrm{GL}_2(F)^+$, $g \in \mathrm{GL}_2(\mathbb{A})$,*

$$Z(\gamma g, \Phi) = Z(g, \Phi).$$

Moreover, the minus part $Z(g, \Phi)^-$ is cuspidal.

Proof. Let $\gamma \in \mathrm{GL}_2(F)^+$ it suffices to show

$$\mathrm{T}(\alpha h)^{-1} Z(r(\gamma g, \alpha h)\Phi) = \mathrm{T}(h)^{-1} Z(r(g, h)\Phi),$$

where (γ, α) and (g, h) are both elements in $\mathscr{R}$. This is equivalent to

$$\mathrm{T}(\alpha)^{-1} Z(r(\gamma g, \alpha h)\Phi) = Z(r(g, h)\Phi)$$

and then to

$$\mathrm{T}(\alpha)^{-1} Z(r(\gamma, \alpha)\Phi) = Z(\Phi)$$

with $r(g, h)\Phi$ replaced by Φ. Write $\gamma_1 = d(\gamma)^{-1}\gamma$. By definition, the left-hand side is equal to

$$\begin{aligned}
\mathrm{T}(\alpha)^{-1} Z(L(\alpha) r(\gamma_1)\Phi) &= \sum_{x \in \widetilde{K} \backslash \mathbb{V}} r(\gamma_1)\Phi(\alpha^{-1}x)\rho(\alpha)^{-1} Z(x)_K \\
&= \sum_{x \in K \cdot \mathrm{O}(F_\infty) \backslash \mathbb{V}} r(\gamma_1)\Phi(\alpha^{-1}x) Z(\alpha^{-1}x)_K \\
&= \sum_{x \in \widetilde{K} \backslash \mathbb{V}} r(\gamma_1)\Phi(x) Z(x)_K \\
&= Z(r(\gamma_1)\Phi) = Z(\Phi).
\end{aligned}$$

For the minus part, we notice that the constant term of $Z(g, \Phi)$ which is a multiple of $\mathscr{L}_K$. Thus the constant term of $Z^-(g, \Phi)$ vanishes by definition. $\square$

Notice that the natural embedding $\mathrm{GL}_2(\mathbb{A}_F)^+ \longrightarrow \mathrm{GL}_2(\mathbb{A}_F)$ gives bijective map

$$\mathrm{GL}_2(F)^+ \backslash \mathrm{GL}_2(\mathbb{A}_F)^+ \xrightarrow{\sim} \mathrm{GL}_2(F) \backslash \mathrm{GL}_2(\mathbb{A}_F).$$

Thus we can define $Z(g, \Phi)$ for $g \in \mathrm{GL}_2(\mathbb{A}_F)$ by

$$Z(g, \Phi) = Z(\gamma g, \Phi)$$

for some $\gamma \in \mathrm{GL}_2(F)$ such that $\gamma g \in \mathrm{GL}_2^+(\mathbb{A}_F)$. Then $Z(g, \Phi)$ is automorphic for $\mathrm{GL}_2(\mathbb{A})$.

3.4 Geometric theta lifting

Let σ be an irreducible cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A})$ of parallel weight 2. For any $\varphi \in \sigma$, $\alpha \in F_+^\times \backslash \mathbb{A}_f^\times$, we define

$$Z_\alpha(\Phi \otimes \varphi) := \int_{\mathrm{SL}_2(F)\backslash \mathrm{SL}_2(\mathbb{A})} Z(g_1 g, \Phi)\varphi(g_1 g) dg_1 \in \mathrm{Ch}^1(M_\alpha),$$

where $g \in \mathrm{GL}_2(\mathbb{A})$ with determinant equal to α. Then it is easy to see that by [33], Theorem 3.5.2, we have the following identity:

$$(3.4.1) \qquad Z(\Phi \otimes \varphi) = \frac{L(1, \pi, \mathrm{ad})}{2\zeta_F(2)} \mathrm{T}(\theta(\Phi \otimes \varphi)).$$

The collection $(Z_\alpha(\Phi \otimes \varphi))$ defines an element

$$(Z_\alpha(\Phi \otimes \varphi)) \in \prod_\alpha \mathrm{Ch}^1(M_\alpha).$$

It is easy to see that this element is invariant under open compact subgroup $U \times U$ of $\mathbb{B}^\times \times \mathbb{B}^\times$. An element gives

$$Z(\Phi \otimes \varphi) \in \mathrm{Ch}^1(X \times X), \quad \Phi \in \mathscr{S}(\mathbb{V}).$$

Kernel identity

For a $\Phi = \otimes \Phi_i \in \mathscr{S}(\mathbb{V}^3)$, we can define an automorphic form on $\mathrm{GL}_2(\mathbb{A})^3$ by

$$(3.4.2) \qquad \Theta^-(g, \Phi) = \widehat{Z}^-(g_1, \Phi_1) \cdot \widehat{Z}^-(g_2, \Phi_2) \cdot \widehat{Z}^-(g_3, \Phi_3),$$

where the right-hand side is the intersection of the admissible class extending the projection $Z(g_i, \Phi)^- \in \mathrm{Pic}^-(Y \times Y)$ of $Z(g_i, \Phi)$. By Proposition 3.3.1, this a cusp form

Proposition 3.4.1. *The Conjecture 1.3.1 is equivalent to that $\Theta^-(g, \Phi)$ is the projection of $-2E'(\cdot, 0, \Phi)$ in the space of cups forms of parallel weight 2, i.e., the following identity for any cusp form φ for of parallel weight 2 for $\mathrm{GL}_2(\mathbb{A})^3$:*

$$(-2E'(\cdot, 0, \Phi), \varphi) = (\Theta(\cdot, \Phi), \varphi).$$

Proof. After decomposing the space of cusp forms into irreducible representations, we may assume that $\varphi \in \sigma$ for some irreducible representations.

By Corollary 2.3.2, the left-hand side of the kernel identity is

$$\frac{2L'(1/2, \sigma)}{\zeta_F(2)} \alpha(\theta(\Phi \otimes \varphi).$$

If $\theta(\Phi \otimes \varphi) = \bigotimes_i (f_i \otimes g_i)$, by (3.4.1) the right-hand side is

$$\frac{L(1, \pi, \mathrm{ad})}{8\zeta_F(2)^3} \prod_i \widehat{c}_1(\widehat{Z}(f_i, g_i)).$$

By the formula (3.2.1) and Lemma 3.2.1, we have

$$\prod_i \widehat{c}_1(\widehat{Z}(f_i, g_i)) = \langle P(f), P(g) \rangle.$$

Thus both sides multiples of $\alpha(\Phi \otimes \varphi)$. If either side is non-zero, then $\Sigma(\sigma) = \Sigma(\mathbb{B})$. Therefore Conjecture 1.3.1 follows. $\square$

For the actual computation, we may replace $\widehat{Z}(g, \Phi_i)^-$ by arithmetic classes extending $Z(g, \Phi_i)$. In fact, since each $Z(g, \Phi_i)$ will fix class $\mathrm{Pic}^\xi(X)$, it is in the space

$$\pi_1^* \mathrm{Pic}^\xi(X) \otimes \mathrm{Ch}^0(X) + \mathrm{Ch}^0(X) \otimes \pi_2^* \mathrm{Pic}^\xi(X) + \mathrm{Pic}^-(X \times X).$$

Thus we have a decomposition

$$Z(g, \Phi_i) = Z_1^\xi(g, \Phi_i) + Z_2^\xi(g, \Phi_i) + Z^-(g, \Phi_i).$$

It is easy to see that both $Z^\xi(g, \Phi_i)$ are Eisenstein series with values in Hodge cycles. Now for each $\alpha \in \mathrm{Pic}^\xi(X)$, fix an arithmetic extension $\widehat{\alpha}$. Then the above decomposition defines an arithmetic extension $\widehat{Z}(g, \Phi_i)$. Now we define

$$\Theta(g, \Phi) = \widehat{Z}(g_1, \Phi_1) \cdot \widehat{Z}(g_2, \Phi_2) \cdot \widehat{Z}(g_3, \Phi_3). \tag{3.4.3}$$

Then the difference $\widehat{Z}(g, \Phi) - \widehat{Z}^-(g, \Phi)$ is *Eisenstein* in the sense that it is a sum of forms which is Eisenstein for at least one variable g_i. It follows that it has zero inner product with cusp forms. Thus we have the following equivalent form of the above Conjecture 1.3.1:

Remark 3.4.1. Unlike the formalism $\Phi_i \mapsto Z(g, \Phi_i)$ which is equivariant under the action of $\mathbb{B}^\times \times \mathbb{B}^\times$, the formalism $\Phi_i \mapsto \widehat{Z}(g, \Phi_i)$ is not $\mathbb{B}^\times \times \mathbb{B}^\times$ equivariant in general.

3.5 Arithmetic Hodge class and Hecke operators

In this section, we want to introduce an arithmetic Hodge class and the arithmetic Hecke operators. The construction depends on the choice of integral models, which depends on a maximal order $\mathscr{O}_{\mathbb{B}}$ of $\mathbb{B}$ we fix here.

Moduli interpretation at an archimedean place

Let U be an open and compact subgroup of $\mathscr{O}_{\mathbb{B}}^\times$. Let τ be an archimedean place of F. Write B a quaternion algebra over F with ramification set $\Sigma \setminus \{\tau\}$. Fix an isomorphism $\mathbb{B}^\tau \simeq B \otimes \mathbb{A}^\tau$. Recall that from [35, §5.1], the curve X_U parameterizes isomorphism classes of triples $(V, h, \bar{\kappa})$ where

1. V is a free B-module of rank 1;

2. h is an embedding $\mathbb{S} \longrightarrow \mathrm{GL}_B(V_\mathbb{R})$ which has weight -1 at τ_1, and trivial component at τ_i for $i > 1$, where $\tau_1 := \tau, \tau_2, \cdots, \tau_g$ are all archimedean places of F;

3. $\bar{\kappa}$ is a class in $\mathrm{Isom}(\widehat{V}_0, \widehat{V})/U$, where $V_0 = B$ as a left B-module.

The Hodge structure h defines a Hodge decomposition on $V_{\tau,\mathbb{C}}$:

$$V_{\tau,\mathbb{C}} = V^{-1,0} + V^{0,-1}.$$

By Hodge theory, the tangent space of Y at a point (V, h, κ) is given by

$$\mathscr{L}(V)_\tau = \mathrm{Hom}_B(V^{-1,0}, V_\mathbb{C}/V^{-1,0}) = \mathrm{Hom}_B(V^{-1,0}, V^{0,-1}).$$

Since the complex conjugation on $V_\mathbb{C}$ switches two factors $V^{-1,0}$ and $V^{0,-1}$, one has a canonical identification

$$\mathscr{L}(V)_\tau \otimes \overline{\mathscr{L}(V)_\tau} = \mathrm{Hom}_B(V^{-1,0}, V^{-1,0}) = \mathbb{C}.$$

This identification defines a Hermitian norm on $\mathscr{L}(V)_\tau$.

Lemma 3.5.1. *Let $\delta(V)$ denote the one dimensional vector space over F generated by symbol $\delta(v)$ for $v \in V$ with relation $\delta(bv) = \nu(b)\delta(v)$. Then we have a canonical isomorphism:*

$$\mathscr{L}(V) = \delta(V) \otimes_{F,\tau} \det(V_\mathbb{C}^{-1,0})^\vee.$$

Proof. There is a pairing $\psi : V \otimes V \longrightarrow \delta(V)$ defined by

$$\psi(u, v) := \frac{1}{2}(\delta(u + v) - \delta(u) - \delta(v)).$$

Let $B^\times$ act on this space by multiplication by $\nu : B^\times \longrightarrow F^\times$. Then we have

$$\psi \in \mathrm{Hom}_{B^\times}(V \otimes V, \delta(V)).$$

This pairing is compatible with Hodge structures when $\delta(V)$ is equipped with action weight $(-1, -1)$. Thus on $V_{\tau,\mathbb{C}}$, the above pairing has isotropic spaces $V^{-1,0}$ and $V^{0,-1}$ and defines bilinear $B_\mathbb{C}^\times$-equivariant pairing

$$V^{-1,0} \otimes V^{0,-1} \longrightarrow \delta(V)_\mathbb{C}.$$

On the other hand, the wedge product defines a $B_\mathbb{C}^\times$-pairing

$$V^{-1,0} \otimes V^{0,-1} \longrightarrow \det(V^{-1,0}),$$

where the later space is equipped with an action $\nu : B^\times \longrightarrow F^\times$. The above two pairings define canonical identifications:

$$V^{0,-1} = \delta(V)_{\tau,\mathbb{C}} \otimes \mathrm{Hom}_{B^\times}(V^{-1,0}, \mathbb{C}),$$
$$V^{-1,0} = \det(V^{-1,0}) \otimes \mathrm{Hom}_{B^\times}(V^{-1,0}, \mathbb{C}).$$

Thus we have

$$\mathscr{L}(V)_\tau = \mathrm{Hom}(V^{-1,0}, V^{0,-1}) = \delta(V)_{\tau,\mathbb{C}} \otimes \det(V^{-1,0})^\vee.$$

∎

Modular interpretation at a finite place

Let v be a finite place. Recall from [35, §5.3], the prime to v-part of $(\widehat{V}_U, \bar{\kappa})$ extends to an étale system over $\mathscr{X}_U$, the canonical integral model of X_U. The v-part extends to a system of special divisible $\mathscr{O}_{\mathbb{B}_v}$-module of dimension 2, height 4, with Drinfeld level structure:

$$(\mathscr{A}, \bar{\alpha})$$

with an identification

$$\kappa_v(\mathscr{O}_v) \simeq \mathrm{T}_v(\mathscr{A}),$$

where $\mathrm{T}_v(\mathscr{A})$ is the Tate module of $\mathscr{A}$ for prime v.

The Lie algebra of the formal part $\mathscr{A}^0$ of $\mathscr{A}$ defines a two-dimensional vector bundle $\mathrm{Lie}(\mathscr{A})$ on $\mathscr{X}_U$. The tangent space of X_U is canonically identified with $\mathscr{L}_v := \delta(V)_{\mathscr{O}_F} \otimes \mathrm{Lie}(\mathscr{A})^\vee$. The level structure defines an integral structure on $\delta(V)$ at place v. Thus $\mathscr{L}_v$ has an integral structure by the tensor product.

If v is not split in $\mathbb{B}$, then $\mathscr{O}_{\mathbb{B}_v}$ is the unique maximal order in $\mathbb{B}_v$, and the integral structure on $\mathscr{L}$ is unique. This can also be seen from the group $\mathscr{A}$ being formal and supersingular. Any isogeny $\varphi : \mathscr{A}_x \longrightarrow \mathscr{A}_y$ of two such $\mathscr{O}_{\mathbb{B}_v}$-modules representing two points x and y on $\mathscr{X}_U$ smooth over $\mathscr{O}_v$ induces an isomorphism of $\mathscr{O}_v$-modules:

$$\mathscr{L}(\mathscr{A}) \simeq \mathscr{L}(\mathscr{B}).$$

If v is split in $\mathbb{B}$, then we may choose an isomorphism $\mathscr{O}_{\mathbb{B}_v} = M_2(\mathscr{O}_v)$. Then the divisible module $\mathscr{A}$ is a direct sum $\mathscr{E} \oplus \mathscr{E}$ where $\mathscr{E}$ is a divisible $\mathscr{O}_F$-module of dimension 1 and height 2. Then we have an isomorphism

$$\mathscr{L} = \mathrm{Lie}(\mathscr{E})^{\otimes -2} \otimes \det \mathrm{T}_v(\mathscr{A}).$$

Let x be an ordinary $\mathscr{O}_v$-point of $\mathscr{Y}_U$ then we have an formal–étale decomposition

$$0 \longrightarrow \mathscr{E}_x^0 \longrightarrow \mathscr{E} \longrightarrow \mathscr{E}^{et} \longrightarrow 0.$$

This induces an isomorphism

$$\mathscr{L}_x = (\mathrm{Lie}(\mathscr{E})^\vee \otimes \mathrm{T}_v(\mathscr{E}^0))^{\otimes 2} \otimes (\mathrm{T}_v(\mathscr{E}_x^{et}) \otimes \mathrm{T}_v(\mathscr{E}_x^0)^\vee).$$

The first part does not depend on the level structure, but the second part does. If $\varphi : \mathscr{E}_x \longrightarrow \mathscr{E}_y$ is an isogeny of orders a, b on the formal and etale parts, respectively, then it has order $b - a$ for the bundles $\mathscr{L}_x \longrightarrow \mathscr{L}_y$.

Admissible arithmetic classes

Combining the above, we have introduced an arithmetic structure $\widehat{\mathscr{L}}$ for $\mathscr{L}$. The roots of this define an arithmetic structure on elements of Hodges classes $\mathrm{Pic}^\xi(X)$. We denote the resulting groups of arithmetic classes as $\widehat{\mathrm{Pic}}^\xi(X)$.

Unlike $\mathrm{Pic}^{\xi}(X)$, the group $\widehat{\mathrm{Pic}}^{\xi}(X)$ is not invariant under the action of $\mathbb{B}^{\times}$ but invariant under $\mathscr{O}_{\mathbb{B}}^{\times}$. We normalize the metric of $\widehat{\xi}$ at one archimedean place such that on each connected component of any X_U

$$\widehat{\xi}^2 = 0.$$

Now for any class $\alpha \in \mathrm{Ch}^1(X_{U,a} \times X_{U,b})$ in some irreducible component of $X \times X$ in a finite level which fixes ξ by both push-forward and pull-back, we can attach a class $\widehat{\alpha}$ such that if $\alpha = \alpha^- + d_2\pi_1^*\xi_1 + d_1\pi_2^*\xi_2$ with $\alpha \in \mathrm{Pic}^-(X \times X)$ and $\xi_i \in \mathrm{Pic}^{\xi}(X)$, then we have

$$\widehat{\alpha} = \widehat{\alpha}^- + d_2\pi_1^*\widehat{\xi}_1 + d_1\pi_2\widehat{\xi}_2.$$

We call such *a* L*-lifting of* α. This agrees with the notion of "L-liftings" in [37, Corollary 2.5.7], Corollary for polarization $\widehat{\mathscr{L}} = \widehat{\xi}_1 + \widehat{\xi}_2 + c$, where c is positive number making $\widehat{\mathscr{L}}$ ample. The following properties can characterize such a class:

1. admissibility: for any point $(p_1, p_2) \in X_{U,a} \times X_{U,b}$, the induced arithmetic classes $\widehat{\alpha}_1 := \widehat{\alpha}|_{p_1 \times U_b}$ and $\widehat{\alpha}_2 := \widehat{\alpha}_{X_a \times p_2}$ on $X_{U,a}$ or $X_{U,b}$ are $\widehat{\xi}$-admissible in the sense that $\widehat{\alpha}_i - \deg \alpha_i \widehat{\xi}$ has curvature 0 at all archimedean places and zero intersection with vertical cycles.

2. rigidity: $\widehat{\alpha} \cdot \pi_1^*\widehat{\xi}_1 \cdot \pi_2^*\widehat{\xi}_2 = 0$.

The class $\alpha \mapsto \widehat{\alpha}$ extends to the whole group $\mathrm{Ch}^1(X \times X)$.

Arithmetic Hecke operators

Let Z be a Hecke correspondence as a divisor in $X_U \times X_U$. We want to construct canonical arithmetic lifting $\widehat{Z} \in \widehat{Z}^1(X_U \times X_U)$ so that its class in $\widehat{\mathrm{Ch}}^1(X_U \times X_U)$ is the L-lifting of the class $[Z] \in \mathrm{Ch}^1(X_U \times X_U)$. For the construction of $\widehat{Z}$, we first construct Arakelov lifting $\widehat{Z}^{\mathrm{Ar}} = (Z, g^{\mathrm{Ar}})$ as in our recent paper [37], where g^{Ar} is admissible with integral 0 against $c_1(\widehat{\xi}_1) \cdot c_1(\widehat{\xi}_2)$ meausure on each fiber. The difference $[\widehat{Z}] - [\widehat{Z}^{\mathrm{Ar}}] \in \mathrm{Ch}^1$ is a class $C \in \widehat{\mathrm{Pic}}(F)$ such that

$$\deg C = -\widehat{\xi}_1|_Z \cdot \widehat{\xi}_2|_Z = \frac{1}{2}(\widehat{\xi}_1 - \widehat{\xi}_2)|_Z^2.$$

Notice that $\xi_1|_Z$ and $\xi_2|_Z$ are canonically isomorphic. Thus $(\widehat{\xi}_1 - \widehat{\xi}_2)|_Z$ is canonically represented by a vertical cycles $\sum_v G_v$ supported over the fiber F_v of $X_U \times X_U$ where U is not maxial. Thus can define

$$\widehat{Z} = \widehat{Z}^{\mathrm{Ar}} + \sum_v (G_v^2) F_v$$

where $G_v^2 \in \mathbb{Q}$ is the geometric intersection of G_v.

Let p_1, p_2 be the two projections of Z onto X_U. Then p_i's have the same degree called d, and there is a canonical isomorphism $p_1^*\xi \to p_2^*\xi$ of line bundles (with fractional power). This induces isomorphisms,

$$Z_*\xi_1 \simeq d\,\xi_2, \qquad Z^*\xi_2 = \xi_1.$$

Proposition 3.5.1. *The above isomorphisms induce isometries:*

$$\widehat{Z}_*\widehat{\xi}_1 \simeq d\,\widehat{\xi}_2, \qquad \widehat{Z}^*\widehat{\xi}_2 = d\widehat{\xi}_1.$$

Proof. By similarity, we need only prove the first identity. Since $\widehat{Z}$ is $\widehat{\xi}$-admissible, $\widehat{Z}_*\xi_1$ is also admissible. Thus we have constant $C \in \widehat{\mathrm{Pic}}(F)$ cycles such that $\widehat{Z}_*\widehat{\xi}_1 = d\widehat{\xi}_2 + C$. Now we intersect with $\widehat{\xi}_2$ to obtain

$$\deg C = \widehat{Z}_*\widehat{\xi}_1 \cdot \widehat{\xi}_2 = \widehat{Z} \cdot \widehat{\xi}_1 \cdot \widehat{\xi}_2 = 0.$$

□

First decomposition

With the construction of cycles as above, we can decompose the intersection as follows

$$\Theta(g, \Phi) := \widehat{Z}(g_1, \Phi_1) \cdot \widehat{Z}(g_2, \Phi_2) \cdot \widehat{Z}(g_3, \Phi_3).$$

First, this intersection is non-trivial only if all g_i have the same norm. In this case we have one $h \in \mathbb{B}^\times \times \mathbb{B}^\times$ such that

$$Z(g_i, \Phi_i) = \mathrm{T}(h) Z(r(g_i, h)\Phi_i).$$

Thus we have that

$$\widehat{Z}(g_1, \Phi_1) \cdot \widehat{Z}(g_2, \Phi_2) \cdot \widehat{Z}(g_3, \Phi_3) = \widehat{Z}(r(g_1, h)\Phi_1) \cdot \widehat{Z}(r(g_2, h)\Phi_2) \cdot \widehat{Z}(r(g_3, h)\Phi_3).$$

Assume that each $r(g_i, h_i)\Phi_i$ is invariant under K. In this case, this intersection number is given by

$$\Theta(g, \Phi) = \sum_{(x_1, x_2, x_3) \in (\widetilde{K} \backslash \mathbb{V})^3} r(g, h)\Phi(x_1, x_2, x_3) \widehat{Z}(x_1)_K \cdot \widehat{Z}(x_2)_K \cdot \widehat{Z}(x_3)_K.$$

We write $\Theta(g, \phi)_{sing}$ for the partial sum where $Z(x_i)$ has a non-empty intersection at the generic fiber. Then the remaining terms can be decomposed into local intersections. Thus we have a decomposition

eqn 1st decomp (3.5.1)
$$\Theta(g, \Phi) = \Theta(g, \Phi)_{\mathrm{sing}} + \sum_v \Theta(g, \Phi)_v.$$

4 Fourier expansions of Eisenstein series

As we have seen in section 2.5, we need to study Fourier coefficients of the derivative of Eisenstein series for Schwartz function $\Phi \in \mathscr{S}(\mathbb{B}^3)$ on an incoherent (adelic) quaternion algebra $\mathbb{B}$ over the adeles $\mathbb{A}$ of a number field F.

For non-singular coefficients, we want to compute them directly. The computation is known in the unramified Siegel–Weil section at a non-archimedean place, and we will recall the results. Then we compute the archimedean Whittaker integrals.

For singular coefficients, we give some criteria for the vanishing property. First of all, we show that if locally at two places Φ is supported on elements in $\mathbb{B}^3$ whose components are linearly independent, then $E'_T(g, 0, \Phi) = 0$ for singular T (cf. Proposition 4.3.3). Then we show that $E_T(e, s, \Phi) = 0$ if Φ is *k-regularly supported* for large k (cf. Proposition 4.4.3). These two facts together imply that $E'(g, 0, \Phi)$ has only non-zero Fourier coefficients at non-singular T with $\Sigma(T) = \Sigma(v)$ for those unramified v for suitable Φ (cf. (4.5.1)). By Theorem A.0.1 of Yifeng Liu, we conjecture that we can always make such a choice (cf. Conjecture) such that the local triple zeta integral does not vanish.

4.1 Nonarchimedeanl local Whittaker integral

Now we recall some results about the local Whittaker integral and local density.

Let F be a non-archimedean local field with integer ring $\mathscr{O}$ whose residue field is of *odd* characteristic p. All results in this subsection hold for $p = 2$. For simplicity of exposition, we only record the results for odd p. Let ϖ be a uniformizer and $q = |\mathscr{O}/(\varpi)|$ be the cardinality of the residue field. Assume further that the additive character ψ is unramified.

Let $V = B = M_2(F)$ with the quadratic form $q = \det$. Let Φ_0 the characteristic function of $M_2(\mathscr{O})$. Let $T \in \mathrm{Sym}_3(\mathscr{O})^\vee$ (cf. "Notations"). It is a fact that $W_T(e, s, \Phi_0)$ is a polynomial of q^{-s}.

To describe the formula, we recall the definition of several invariants of $T \in \mathrm{Sym}_3(\mathscr{O})^\vee$. Suppose that $T \sim \mathrm{diag}[u_i \varpi^{a_i}]$ with $a_1 < a_2 < a_3 \in \mathbb{Z}$, $u_i \in \mathscr{O}^\times$. Then we define $\xi(T)$ to be the Hilbert symbol $\left(\frac{-u_1u_2}{\varpi}\right) = (-u_1u_2, \varpi)$ if $a_1 \equiv a_2 \,(mod\, 2)$ and $a_2 < a_3$, otherwise zero. Note that this does not depend on the choice of the uniformizer ϖ.

Firstly, we have a formula for the central value of Whittaker integral $W_T(e, 0, \Phi_0)$.

Proposition 4.1.1. *The Whittaker function at $s = 0$ is given by*

$$W_T(e, 0, \Phi_0) = \zeta_F(2)^{-2}\beta(T)$$

where

1. *When T is anisotropic, we have*

$$\beta = 0.$$

2. *When T is isotropic, we have three cases*

   (a) *If $a_1 \neq a_2 \mod 2$, we have*

$$\beta(T) = 2\left(\sum_{i=0}^{a_1}(1+i)q^i + \sum_{i=a_1+1}^{(a_1+a_2-1)/2}(a_1+1)q^i\right).$$

   (b) *If $a_1 \equiv a_2 \mod 2$ and $\xi = 1$, we have*

$$\beta(T) = 2\left(\sum_{i=0}^{a_1}(i+1)q^i + \sum_{i=a_1+1}^{(a_1+a_2-2)/2}(a_1+1)q^i\right) + (a_1+1)(a_3-a_2+1)q^{(a_1+a_2)/2}.$$

   (c) *If $a_1 \equiv a_2 \mod 2$ and $\xi = -1$, we have*

$$\beta(T) = 2\left(\sum_{i=0}^{a_1}(i+1)q^i + \sum_{i=a_1+1}^{(a_1+a_2-2)/2}(a_1+1)q^i\right) + (a_1+1)q^{(a_1+a_2)/2}.$$

The second result we need is a formula for the central derivative $W'_T(e,0,\Phi_0)$.

local W derivative

Proposition 4.1.2. *We have*

$$W'_T(e,0,\Phi_0) = \log q \cdot \zeta_F(2)^{-2}\nu(T),$$

where $\nu(T)$ is given as follows. Let $T \sim \mathrm{diag}[t_1,t_2,t_3]$ with $a_i = \mathrm{ord}(t_i)$ in the order $a_1 \leq a_2 \leq a_3$.

1. *If $a_1 \neq a_2 \mod 2$, we have*

$$\nu(T) = \sum_{i=0}^{a_1}(1+i)(3i-a_1-a_2-a_3)q^i + \sum_{i=a_1+1}^{(a_1+a_2-1)/2}(a_1+1)(4i-2a-1-a_2-a_3)q^i.$$

2. *If $a_1 \equiv a_2 \mod 2$, we must have $a_2 \neq a_3 \mod 2$. In this case, we have*

$$\begin{aligned}\nu(T) = &\sum_{i=0}^{a_1}(i+1)(3i-a_1-a-2-a_3)q^i \\ &+ \sum_{i=a_1+1}^{(a_1+a_2-2)/2}(a_1+1)(4i-2a-1-a_2-a_3)q^i \\ &- \frac{a_1+1}{2}(a_3-a_2+1)q^{(a_1+a_2)/2}.\end{aligned}$$

prop W_T Phi'

Proposition 4.1.3. *Let Φ'_0 be the characteristic function of $\mathscr{O}_D^3$ where $\mathscr{O}_D$ is the maximal order of the division quaternion algebra D. Then we have for all anisotropic $T \in \mathrm{Sym}_3(\mathscr{O})^\vee$:*

$$W_T(e,0,\Phi'_0) = -2q^{-2}(1+q^{-1})^2.$$

For the proof of the three propositions above, we refer to [1, Chap. 15, 16] where a key ingredient is a result in [17] on the local representation density for Hermitian forms.

Proposition 4.1.4. *Let Φ'_0 be the characteristic function of maximal order $\mathscr{O}_D$ of the division quaternion algebra D. Then we have for all anisotropic $T \in \mathrm{Sym}_3(\mathscr{O})^\vee$:*

$$I_T(e, \Phi'_0) = \mathrm{vol}(\mathrm{SO}(V')),$$

where the left-hand side is defined by (2.4.6).

Proof. A prior we know that $I_T(e, \Phi'_0)$ is a constant multiple of $W_T(e, 0, \Phi'_0)$. Take any $x \in \mathscr{O}_D^3$ with moment T. Then it is easy to see that $h \cdot x$ is still in $\mathscr{O}_D^3$ for all $h \in \mathrm{SO}(V')$. This completes the proof. $\square$

4.2 Archimedean Whittaker integral

We want to compute the Whittaker integral $W_T(g, s, \Phi)$ when $F = \mathbb{R}$, $B = \mathbb{H}$ is the Hamiltonian quaternion algebra,

$$\Phi_\infty(x) = \Phi(x) = e^{-2\pi tr(Q(x))}, \quad x \in B^3 = \mathbb{H}^3. \tag{4.2.1}$$

and the additive character

$$\psi(x) = e^{2\pi i x}, \quad x \in \mathbb{R}.$$

Let K_∞ be the maximal compact subgroup of $\mathrm{Sp}_6(\mathbb{R})$:

$$K_\infty = \left\{ \begin{pmatrix} x & y \\ -y & x \end{pmatrix} \in \mathrm{Sp}_6(\mathbb{R}) \middle| x + yi \in U(3) \right\}.$$

Denote by χ_m the character of K_∞

$$\chi_m \begin{pmatrix} x & y \\ -y & x \end{pmatrix} = \det(x + yi)^m.$$

Then the Siegel–Weil section attached to Φ transform by the character χ_2 under the action of K_∞ (cf. [10], [19]).

Lemma 4.2.1. *Let $g = n(b)m(a)k \in \mathrm{Sp}_6(\mathbb{R})$ be the Iwasawa decomposition. Then we have when $\mathrm{Re}(s) \gg 0$:*

$$W_T(g, s, \Phi) = \chi_2(k)\psi(Tb)\lambda_s(m({}^ta^{-1}))|\det(a)|^4 W_{{}^taTa}(e, s, \Phi).$$

Proof. This follows the invariance under K_∞ and the property of Siegel–Weil section.

$\square$

Thus it suffices to consider only the identity element $g = e$ of $\mathrm{Sp}_6(\mathbb{R})$. It is easy to obtain a formula for $\lambda_s(wn(u))$ to see that

$$W_T(e, s, \Phi) = \int_{\mathrm{Sym}_3(\mathbb{R})} \psi(-Tu) \det(1 + u^2)^{-s} r(wn(u))\Phi(0)du.$$

Lemma 4.2.2. *When* $\mathrm{Re}(s) \gg 0$*, we have*

eqn W_T 1

$$W_T(e, s, \Phi) = -\int_{\mathrm{Sym}_3(\mathbb{R})} \psi(-Tu) \det(1 + iu)^{-s} \det(1 - iu)^{-s-2}du, \tag{4.2.2}$$

where we have the usual convention $i = \sqrt{-1}$.

Proof. Let $u = {}^t k a k$ be the Cartan decomposition where $a = \mathrm{diag}(u_1, u_2, u_3)$ is diagonal and $k \in \mathrm{SO}(3)$. Then it is easy to see that $n(u) = m(k)^{-1}n(a)m(k)$ and $wm(k)^{-1} = m(-k^{-1})w$. Note that $\det(k) = 1$ and $\chi_2(m(k)) = 1$. We obtain:

$$r(wn(u))\Phi(0) = r(wn(a))\Phi(0).$$

By definition, we have

$$r(wn(a))\Phi(0) = \gamma(\mathbb{H}, \psi) \int_{\mathbb{H}^3} \psi(aQ(x))\Phi(x)dx,$$

where, for our choice, the Weil constant is

$$\gamma(\mathbb{H}, \psi) = -1.$$

Therefore we have

$$r(wn(a))\Phi(0) = -\prod_{j=1}^{3} \int_{\mathbb{H}} e^{\pi(iu_j - 1)q(x_j)}dx_j.$$

This is equal to constant times.

$$\prod_{j}^{3} \frac{1}{(1 - iu_j)^2} = \det(1 - iu)^{-2}.$$

To recover the constant, we let $u = 0$ and note that

$$r(w)\Phi(0) = \chi_2(w)\Phi(0) = -\Phi(0) = -1.$$

We thus obtain that

$$r(wn(u))\Phi(0) = r(wn(a))\Phi(0) = -\det(1 - iu)^{-2}.$$

Since $\det(1 + u^2) = \det(1 - iu)\det(1 + iu)$, the lemma now follows. $\square$

Following Shimura ([30, pp.274]), we introduce an integral for $g, h \in \mathrm{Sym}_n(\mathbb{R})$ and $\alpha, \beta \in \mathbb{C}$

eqn def eta
$$\eta(g,h;\alpha,\beta) = \int_{x>\pm h, x\in \mathrm{Sym}_n(\mathbb{R})} e^{-gx} \det(x+h)^{\alpha-2} \det(x-h)^{\beta-2} dx, \tag{4.2.3}$$

which is convergent when $g > 0$ and $\mathrm{Re}(\alpha), \mathrm{Re}(\beta) > \frac{n}{2}$. Here $x > \pm h$ means that $x + h > 0$ and $x - h > 0$; and in the rest of this section, for simplicity, we will write, for a (square) matrix g:

$$e^g = e^{\mathrm{Tr}(g)}. \tag{4.2.4}$$

Here we point out that the measure dx in [30] is the Euclidean measure viewing $\mathrm{Sym}_n(\mathbb{R})$ as $\mathbb{R}^{n(n+1)/2}$ naturally. *This measure is not self-dual but only up to a constant* $2^{n(n-1)/4}$. In the following, we always use the Euclidean measure as [30] does. For two elements $h_1, h_2 \in \mathrm{Sym}_n(\mathbb{R})$, by $h_1 \sim h_2$ we mean that $h_1 = kh_2k^{-1}$ for some $k \in O(n)$.

We recall a formula in [30, (1.16)]. Let $z \in \mathrm{Sym}_n(\mathbb{C})$ with $\mathrm{Re}(z) > 0$, then we have for $s \in \mathbb{C}$ with $\mathrm{Re}(s) > \frac{n-1}{2}$,

siegel
$$\int_{\mathrm{Sym}_n(\mathbb{R})_+} e^{-\mathrm{Tr}(zx)} \det(x)^{s-\frac{n+1}{2}} dx = \Gamma_n(s) \det(z)^{-s}, \tag{4.2.5}$$

where the "higher" Gamma function is defined as

$$\Gamma_n(s) = \pi^{\frac{n(n-1)}{4}} \Gamma(s)\Gamma(s-\frac{1}{2})...\Gamma(s-\frac{n-1}{2}).$$

For instance, when $n = 1$, we have when $\mathrm{Re}(z) > 0$ and $\mathrm{Re}(s) > 0$

$$\int_{\mathbb{R}_+} e^{-zx} x^{s-1} dx = \Gamma(s) z^{-s}.$$

Tth-arch
Lemma 4.2.3. *When* $\mathrm{Re}(s) > 1$, *we have*

$$W_T(e,s,\Phi) = \kappa(s)\Gamma_3(s+2)^{-1}\Gamma_3(s)^{-1}\eta(2\pi, T; s+2, s)$$

where

eqn def kappa
$$\kappa(s) = -2^{9/2}\pi^{6s+6}. \tag{4.2.6}$$

Proof. Consider

$$f(x) = \begin{cases} e^{-vx}\det(x)^{s-\frac{n+1}{2}} & x > 0, \\ 0 & \text{otherwise.} \end{cases}$$

Applying (4.2.5) to $z = v + 2\pi i u$ for $u, v \in \mathbb{R}$, we obtain when $\mathrm{Re}(s) > \frac{n-1}{2}$,

$$\widehat{f}(u) = \Gamma_n(s) \det(v + 2\pi i u)^{-s}.$$

Take the inverse Fourier transformation; we obtain a formula that we will use several times later:

fou-inv (4.2.7)

$$\int_{\mathrm{Sym}_n(\mathbb{R})} e^{2\pi i u x} \det(v+2\pi i u)^{-s} du = \begin{cases} \frac{1}{2^{n(n-1)/2}\Gamma_n(s)} e^{-vx} \det(x)^{s-\frac{n+1}{2}} & x>0, \\ 0 & \text{otherwise.} \end{cases}$$

By (4.2.5) for $n=3$, we may rewrite (4.2.2)

$$W_T(e,s,\Phi) = -\frac{\pi^{3s+6}}{\Gamma_3(s+2)} \int_{\mathrm{Sym}_3(\mathbb{R})} e^{-2\pi i T u} \det(1+iu)^{-s} \int_{\mathrm{Sym}_3(\mathbb{R})_+} e^{-\pi(1-iu)x} \det(x)^s dx 2^{3/2} du.$$

Here du is changed to the Euclidean measure and the constant multiple $2^{3/2}$ comes from the ratio between the self-dual measure and the Euclidean one. Interchange the order of the two integrals

$$-2^{3/2}\frac{\pi^{3s+6}}{\Gamma_3(s+2)} \int_{\mathrm{Sym}_3(\mathbb{R})_+} e^{-\pi x} \det(x)^s \left(\int_{\mathrm{Sym}_3(\mathbb{R})} e^{2\pi i u(\frac{1}{2}x-T)} \det(1+iu)^{-s} du \right) dx.$$

By (4.2.7) for $n=3$, we obtain

$$\begin{aligned} & -2^{3/2}\frac{\pi^{3s+6}}{\Gamma_3(s+2)} \int_{x>0, x>2T} e^{-\pi x} \det(x)^s \frac{(2\pi)^6}{2^3\Gamma_3(s)} e^{-2\pi(\frac{x}{2}-T)} \det(2\pi(\frac{x}{2}-T))^{s-2} dx \\ = & -2^{9/2}\frac{\pi^{6s+6}}{\Gamma_3(s+2)\Gamma_3(s)} \int_{x>0,x>2T} e^{-2\pi(x-T)} \det(x)^s \det(x-2T)^{s-2} dx. \end{aligned}$$

Finally we may substitute $x \mapsto T+x$ to complete the proof. □

To compute the integral η-integral (4.2.3) in an inductive way, we recall the "higher" *confluent hypergeometric function* ([30, pp.280,(3.2)]). Let $\mathrm{Sym}_n(\mathbb{C})_+$ be the subset of z with $\mathrm{Re}(z)>0$. Then for $z \in \mathrm{Sym}_n(\mathbb{C})_+$, we define

zeta n (4.2.8)
$$\zeta_n(z,\alpha,\beta) = \int_{\mathrm{Sym}_n(\mathbb{R})_+} e^{-zx} \det(x+1)^{\alpha-\frac{n+1}{2}} \det(x)^{\beta-\frac{n+1}{2}} dx.$$

Shimura first introduced the analytic continuation:

zeta infinity **Lemma 4.2.4** (Shimura)**.** *For $z \in \mathrm{Sym}_n(\mathbb{C})$ with $\mathrm{Re}(z)>0$, the integral $\zeta_n(z;\alpha,\beta)$ is absolutely convergent for $\alpha \in \mathbb{C}$ and $\mathrm{Re}(\beta) > \frac{n-1}{2}$. And the function*

$$\omega(z,\alpha,\beta) := \Gamma_n(\beta)^{-1}\det(z)^\beta \zeta_n(z,\alpha,\beta)$$

can be extended to a holomorphic function of $(\alpha,\beta) \in \mathbb{C}^2$.

Proof. See [30, Thm. 3.1]. □

The following proposition gives an inductive way to compute the Whittaker integral $W_T(e,s,\Phi)$, or equivalently $\eta(2\pi,T;s+2,s)$. To simplify notations, we use w' to denote the transpose of w if no confusion arises.

ind of infinity

Proposition 4.2.5. *Assume that sign*$(T) = (p, q)$ *with* $p + q = 3$ *so that we have* $4\pi T \sim \mathrm{diag}(a, -b)$ *for* $a \in \mathbb{R}_+^p, b \in \mathbb{R}_+^q$. *Let* $t = \mathrm{diag}(a, b)$. *Then we have*

$$\eta(2\pi, T; s+2, s) = 2^{6s} e^{-t/2} |\det(T)|^{2s} \xi(T, s),$$

where

$$\xi(T, s) = \int_M e^{-(aW + bW')} \det(1+W)^{2s} \zeta_p(ZaZ, s+2, s - \frac{3-p}{2}) \times \zeta_q(Z'bZ', s, s + \frac{q+1}{2})\, dw,$$

where $M = \mathbb{R}_q^p$, $W = w \cdot w'$, $W' = w'w$, $Z = (1+W)^{1/2}$ *and* $Z' = (1+W')^{1/2}$.

Proof. We may assume that $4\pi T = kt'k^{-1}$ where $k \in O(3)$ and $t' = \mathrm{diag}(a, -b)$. Then it is easy to see that.

$$\eta(2\pi, T; s+2, s) = \eta(2\pi, t'/(4\pi); s+2, s) = |\det(T)|^{2s} \eta(t/2, 1_{p,q}; s+2, s)$$

where $1_{p,q} = \mathrm{diag}(1_p, -1_q)$. By [30, p.289, (4.16),(4.18),(4.24)], we have

$$\eta(2\pi, T; s+2, s) = 2^{6s} e^{-t/2} |\det(T)|^{2s} \xi(T, s).$$

This completes the proof. □

Corollary 4.2.6. *Suppose that sign*$(T) = (p, q)$ *with* $p+q = 3$. *Then* $W_T(e, s, \Phi)$ *is holomorphic at* $s = 0$ *with vanishing order*

$$\mathrm{ord}_{s=0} W_T(e, s, \Phi) \geq [\frac{q+1}{2}].$$

Proof. By Proposition 4.2.5, we know that

$$W_T(e, s, \Phi) \sim \frac{\Gamma_p(s - \frac{3-p}{2})\Gamma_q(s + \frac{q+1}{2})}{\Gamma_3(s+2)\Gamma_3(s)} \int_F e^{-(aW + bW^*)} \det(1+W)^{2s} \times \frac{1}{\Gamma_p(s - \frac{3-p}{2})} \zeta_p(ZaZ; s+2, s - \frac{3-p}{2}) \frac{1}{\Gamma_q(s + \frac{q+1}{2})} \zeta_q(Z'bZ'; s, s + \frac{q+1}{2}) dw$$

where "$\sim$" means up to nowhere vanishing the entire function. Lemma 4.2.4 implies that the latter two factors in the integral are entire functions. Thus we obtain that.

$$\mathrm{ord}_{s=0} W_T(e, s, \Phi) \geq \mathrm{ord}_{s=0} \frac{\Gamma_p(s - \frac{3-p}{2})\Gamma_q(s + \frac{q+1}{2})}{\Gamma_3(s+2)\Gamma_3(s)} = [\frac{q+1}{2}].$$

□

Remark 4.2.1. 1. The same argument also applies to higher rank Whittaker integral. More precisely, let V be the $n+1$-dimensional positive definite quadratic space and Φ_0 be the standard Gaussian $e^{-2\pi \mathrm{Tr}(x,x)}$ on V^n. Then for T non-singular, we have

$$\mathrm{ord}_{s=0} W_T(e, s, \Phi_0) \geq \mathrm{ord}_{s=0} \frac{\Gamma_p(s - \frac{n-p}{2})\Gamma_q(s + \frac{q+1}{2})}{\Gamma_n(s + \frac{n+1}{2})\Gamma_n(s)} = [\frac{n-p+1}{2}] = [\frac{q+1}{2}].$$

And it is easy to see that when $T > 0$ (namely, represented by V), $W_T(e, 0, \Phi_0)$ is non-vanishing. One immediate consequence is that $W_T(e, s, \Phi_0)$ vanishes with order precisely one at $s = 0$ only if the quadratic space with signature $(n-1, 2)$ represents T. We will see by concrete computation for $n = 3$ that the formula above actually gives the exact order of vanishing at $s = 0$. It should be true for general n, but we have yet to try to verify this.

prop WT T>0

Proposition 4.2.7. *When $T > 0$, we have*

$$W_T(e, 0, \Phi) = \kappa(0)\Gamma_3(2)^{-1}e^{-2\pi T},$$

*where $\kappa(s)$ is defined by (*eqn def kappa *4.2.6).*

Proof. Near $s = 0$, we have

$$\begin{aligned}
&\eta(2\pi, T; s+2, s) \\
=&e^{-2\pi T} \int_{x>0} e^{-2\pi x} \det(x + 2T)^s \det(x)^{s-2} dx \\
=&e^{-2\pi T} \left(\int_{x>0} e^{-2\pi x} \det(2T)^s \det(x)^{s-2} dx + O(s) \right) \\
=&e^{-2\pi T} \left(\det(2T)^s (2\pi)^{-3s} \Gamma_3(s) + O(s) \right).
\end{aligned}$$

Note that $\Gamma_3(s) = \pi^{3/2}\Gamma(s)\Gamma(s - \frac{1}{2})\Gamma(s - 1)$ has a double pole at $s = 0$ and $\Gamma_3(s + 2)$ is non-zero at $s = 0$. The desired result follows immediately. $\square$

4.3 Singular coefficients

In this subsection, we deal with the singular part $E'_{sing}(g, 0, \Phi)$ of the Siegel-Eisenstein series on $G = \mathrm{GSp}_6$. We will write $G_r = \mathrm{Sp}_{2r}$, and P the Siegel parabolic of Sp_6 (not G).

Definition 4.3.1. *For a place v of F, we define the open subset $\mathbb{B}^3_{v,\mathrm{sub}}$ (resp. $\mathbb{B}^3_{v,\mathrm{reg}}$) of $\mathbb{B}^3_v$ to be all $x \in \mathbb{B}^3_v$ such that the components of x generate a dimension 3 subspace of $\mathbb{B}_v$ (resp. with non-degenerate moment matrix).*

Note that $\mathscr{S}(\mathbb{B}^3_{v,\mathrm{sub}})$ is $P(F_v)$-stable under the action defined by the Weil representation.

big cell

Lemma 4.3.2. *For a place v, if a Siegel–Weil section $f_{\Phi,s} \in I(s)$ is associated to $\Phi \in \mathscr{S}_0(\mathbb{B}^3_{v,sub})$, then $f_{\Phi,s}$ is supported in the open cell Pw_0P for all s.*

Proof. By definition we have $f_{\Phi,s}(g) = r(g)\Phi(0)\lambda_s(g)$. Thus it suffices to prove $supp(f_{\Phi,0}) \subset Pw_0P$. Note that by the Bruhat decomposition $G = \coprod_i Pw_iP$, it suffices to prove $r(Pw_iP)\Phi(0) = 0$ for $i = 1, 2, 3$. Since $\mathscr{S}(\mathbb{B}^3_{v,sub})$ is $P(F_v)$-stable, it suffices to prove $r(w_i)\Phi(0) = 0$ for $i = 1, 2, 3$. By

$$r(w_i)\Phi(0) = \gamma \int_{\mathbb{B}^{n-i}} \Phi(0, ...0, x_{i+1}, ..., x_3)dx_{i+1}...dx_3$$

for a certain eighth-root of unity γ, and since

$$\Phi(0, ..., 0, x_{i+1}, ..., x_3) \equiv 0$$

when $i \geq 1$, we complete the proof. $\square$

singular coe

Proposition 4.3.3. *For an integer $k \geq 1$, fix non-archimedean (distinct) places $v_1, v_2, ..., v_k$. Let $\Phi = \otimes_v \Phi_v \in \mathscr{S}(\mathbb{B}^3)$ with $\mathrm{supp}(\Phi_{v_i}) \subset \mathbb{B}^n_{v_i,\mathrm{sub}}$ (i=1,2,...,k). Let $g \in G(\mathbb{A})$ satisfy $g_{v_i} \in P(F_{v_i}), (i = 1, 2, ..., k)$. Then for singular T, the vanishing order $\mathrm{ord}_{s=0}E_T(g, s, \Phi)$ is at least k. In particular, when T is singular, we have $E_T(g, 0, f) = 0$ if $k \geq 1$, and $E'_T(g, 0, \Phi) = 0$ if $k \geq 2$.*

Proof. We deduce this from some results of Kudla–Rallis ([21], [22]). Suppose $\mathrm{rank}(T) = 3 - r$ with $r > 0$. We may write $T = {}^t\gamma T'\gamma$, $T' = \begin{pmatrix} 0 & \\ & \beta \end{pmatrix}$ for some $\beta \in \mathrm{GL}_{3-r}$ and $\gamma \in \mathrm{GL}_3$. We have

$$E_T(g, s, \Phi) = E_{T'}(m(\gamma)g, s, \Phi), \quad g \in G(\mathbb{A}).$$

Since $m(\gamma) \in P(F_{v_i})$, it suffices to prove the assertion for

$$T = \begin{pmatrix} 0 & \\ & \beta \end{pmatrix}$$

with $\beta \in \mathrm{GL}_{3-r}$ non-singular.

For $\mathrm{Re}(s) \gg 0$, the T-th Fourier coefficient is a sum

$$\begin{aligned} E_T(g, s, \Phi) &= \int_{[N]} \sum_{P(F)\backslash G(F)} f_{\Phi,s}(\gamma ng)\psi_{-T}(n)dn \\ &= \int_{[N]} \sum_{i=0}^{3} \sum_{\gamma \in P\backslash Pw_iP} f_{\Phi,s}(\gamma ng)\psi_{-T}(n)dn, \end{aligned}$$

where for $i = 0, 1, 2, 3$,

$$w_i := \begin{pmatrix} 1_i & & & \\ & & & 1_{3-i} \\ & & 1_i & \\ & -1_{3-i} & & \end{pmatrix}.$$

By Lemma 4.3.2, $f_{\Phi_v}(\gamma n_v g_v, s) \equiv 0$ for $\gamma \in Pw_iP, i > 0$, $v \in \{v_1, ..., v_k\}$ and $g_v \in P(F_v)$. Thus for g as in the desired statement, only the open cell has nonzero contribution in the coefficients

$$E_T(g, s, \Phi) = \int_{N_{\mathbb{A}}} f_{\Phi,s}(w_0 ng)\psi_{-T}(n)dn.$$

This is exactly the Whittaker functional $W_T(g, s, \Phi) = W_T(e, s, r(g)\Phi)$.

Let $i : G_{3-r} \to G_3$ be the standard embedding defined by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} 1_r & & & \\ & a & & b \\ & & 1_r & \\ & c & & d \end{pmatrix}.$$

Then this induces a map by restriction: $i^* : I(s) \to I^{3-r}(s+\frac{r}{2})$ to the degenerate principal series on G_{3-r}. We now denote by f the Siegel–Weil section f_Φ. Let $M(s) = \prod_v M_v(s) : I(s) \to I(-s)$ be the intertwining operator (cf. [21, §4]).

Lemma 4.3.4. *Let $E_\beta(g, s, i^*M(s)f)$ denote the β-Fourier coefficient of the Eisensetin series on $G_{3-r}(\mathbb{A})$ defined by section $i^*M(s)f$. Then we have*

$$W_T(e, s, f) = E_\beta(e, -s + \frac{r}{2}, i^*M(s)f).$$

Proof. By [22, (4.13)–(4.18)] we have

$$W_T(e, s, f) = E_\beta(e, s - \frac{r}{2}, i^*U(s)f),$$

where $U(s) = U_r(s)$ is [22, (4.14)]. By the functional equation, we have

$$W_T(e, s, f) = E_\beta(e, -s + \frac{r}{2}, M(s - \frac{r}{2}) \circ i^*U(s)f).$$

By the relation ([22, (4.19)]),

$$M(s - \frac{r}{2}) \circ i^*U(s) = i^*M(s),$$

we obtain

$$W_T(e, s, f) = E_\beta(e, -s + \frac{r}{2}, i^*M(s)f).$$

This completes the proof. □

Since $\det(\beta) \neq 0$, this last lemma shows that we have an Euler product when $\mathrm{Re}(s) \gg 0$,

$$W_T(e, s, f) = \prod_v W_{\beta,v}(e, -s + \frac{r}{2}, i^*M_v(s)f_v).$$

By the theory of intertwining operator ([21, §4, (4.7)]), there exist certain Artian L-functions $a_v(s), b_v(s)$ such that, for a finite set S outside which f^S is spherical,

$$M(s)f(s) = \frac{a(s)}{b(s)} \left(\bigotimes_{v \in S} \frac{b_v(s)}{a_v(s)} M_v(s) f_v(s)) \right) \otimes f^S(-s).$$

By [21, Lemma 4.2, Prop. 4.3,(4.10)], for a local Siegel–Weil section f_v, $\frac{b_v(s)}{a_v(s)} M_v(s) f_v$ is holomorphic at $s = 0$ and there is a non-zero constant λ_v independent of f such that

$$\frac{b_v(s)}{a_v(s)} M_v(s) f_v(s))|_{s=0} = \lambda_v f_v(0).$$

Thus there is a certain Artian L-function $\Lambda_{3-r,v}(s)$ (cf. [22, (1.14)]) such that

$$\begin{aligned}
&W_T(e, s, f) = \prod_v W_{\beta,v}(e, -s + \frac{r}{2}, i^* M_v(s) f_v) \\
=&\Lambda_{3-r}(-s + \frac{r}{2}) \frac{a(s)}{b(s)} \prod_{v \in S'_\beta} \frac{1}{\Lambda_{3-r,v}(-s + \frac{r}{2})} W_{\beta,v}(e, -s + \frac{r}{2}, i^*(\frac{b_v(s)}{a_v(s)} M_v(s) f_v)) \\
=&\Lambda_{3-r}(-s + \frac{r}{2}) \frac{a(s)}{b(s)} \prod_{v \in S'_\beta} A_{\beta,v}(s, f),
\end{aligned}$$

where S_β is the set of all primes such that outside S_β, f_v is the spherical vector, ψ_v is unramified and $\mathrm{ord}_v(\det(\beta)) = 0$.

Since $\mathrm{ord}_{s=0} \Lambda_{3-r,v}(-s + \frac{r}{2}) = 0$ (cf. [22, (1.14)]), $\frac{b_v(s)}{a_v(s)} M_v(s) f_v$ is holomorphic and $W_\beta(e, s, f)$ extends to an entire function, we know that $A_{\beta,v}(s, f)$ is holomorphic at $s = 0$. We have a formula

$$A_{\beta,v}(0, f) = \frac{\lambda_v}{\Lambda_{3-r,v}(0)} W^{3-r}_{\beta,v}(e, \frac{r}{2}, i^* f_v(0)).$$

Lemma 4.3.5. *Define a linear functional*

$$\begin{aligned}
\iota : \mathscr{S}(\mathbb{B}_v^3) &\to \mathbb{C} \\
\Phi_v &\mapsto A_{\beta,v}(0, f_{\Phi_v}).
\end{aligned}$$

Then, we have $\iota(r(n(b))\Phi_v) = \psi_{v,T}(b)\iota(\Phi_v)$, *i.e.,* $\iota \in \mathrm{Hom}_N(\mathscr{S}(\mathbb{B}_v^3), \psi_T)$.

Proof. It is straightforward to check that

$$W_{\beta,v}(e, -s + \frac{r}{2}, i^*(M_v(s) r(n(b)) f_v)) = \psi_T(b) W_{\beta,v}(e, -s + \frac{r}{2}, i^* M_v(s) f_v).$$

Thus, the linear functional $f_s \mapsto A_{\beta,v}(s, f)$ defines an element in $\mathrm{Hom}_N(I(s), \psi_T)$. In particular, when $s = 0$, the composition ι of $A_{\beta,v}$ with the G-intertwining map $\mathscr{S}(\mathbb{B}_v^3) \to I(0)$ defines a linear functional in $\mathrm{Hom}_N(\mathscr{S}(\mathbb{B}_v^3), \psi_T)$. □

Then the map ι factors through the ψ_T-twisted Jacquet module $\mathscr{S}(\mathbb{B}_v^3)_{N,T}$ (i.e., the maximal quotient of $\mathscr{S}(\mathbb{B}^3)$ on which N acts by character ψ_T). Thus by the following result of Rallis (cf. K-R-3 [22, Lemma 2.3]), ι is trivial on $\mathscr{S}(\mathbb{B}_{v,\mathrm{sub}}^3)$ when T is singular:

Lemma 4.3.6. *Let $\Omega_{T,v}$ be the (closed) subset of $\mathbb{B}_v^3$ of elements x with moment $Q(x) = T$. Then the map $\mathscr{S}(\mathbb{B}_v^3) \to \mathscr{S}(\mathbb{B}_v^3)_{N,T}$ can be realized as the restriction $\mathscr{S}(\mathbb{B}_v^3) \to \mathscr{S}(\Omega_{T,v})$.*

Now since the restriction of Φ_{v_i} to $\Omega_{T,v}$ is zero if $\Phi_{v_i} \in \mathscr{S}(\mathbb{B}_{v_i,\mathrm{reg}}^3)$, and since $\mathrm{ord}_{s=0}\frac{a(s)}{b(s)} = 0$, we conclude that

$$\mathrm{ord}_{s=0} W_T(e, s, f_\Phi) \geq k.$$

For a general $g \in G(\mathbb{A})$, we have

$$\begin{aligned} W_T(g, s, f_\Phi) =& W_T(e, s, r(g)\Phi) \\ =& \Lambda_{n-r}(-s + \frac{r}{2})\frac{a(s)}{b(s)} \prod_{v \in S'_{\beta,g}} A_{\beta,v}(s, r(g_{v_i})\Phi_v), \end{aligned}$$

where $S_{\beta,g}$ is a finite set of place that depends also on g. Since $\mathscr{S}(\mathbb{B}_v^3) \to I(0)$ is $G(F_v)$-equivariant, we have $A_{\beta,v}(0, r(g_v)f_v) = \iota(r(g_v)\Phi_v)$. Since $g_{v_i} \in P(F_{v_i})$, we have $r(g_{v_i})\Phi_{v_i} \in \mathscr{S}(\mathbb{B}_{v_i,sub}^3)$ and $A_{\beta,v_i}(0, r(g_{v_i})f_{v_i}) = 0$ by the same argument above. This completes the proof of Proposition singular coe 4.3.3. ∎

4.4 Functions with regular support

Let F be a non-archimedean field. Let B be a quaternion algebra over F. Recall that we have the moment map

$$Q : B^3 \longrightarrow \mathrm{Sym}_3(F).$$

defn-ram func **Definition 4.4.1.** *Let k be an integer. A function $\Phi \in \mathscr{S}(B_{reg}^3)$ is "k-regularly supported" if it satisfies the condition that $Q(\mathrm{supp}(\Phi)) + p^{-k}\mathrm{Sym}_n(\mathscr{O}) \subseteq Q(B_{\mathrm{reg}}^3)$.*

Even though it looks that such functions are exceptional, in fact generate $\mathscr{S}(B_{reg}^3)$ under the action of a very small subgroup of Sp_6.

Lemma 4.4.2. *Let k be any fixed integer. Then $\mathscr{S}(B_{reg}^3)$ is generated by all k-regularly supported functions under the action of elements $m(aI_3) \in \mathrm{Sp}_6$ for all $a \in F^\times$.*

Proof. Without loss of generality, we can assume that k is even and that $\Phi = 1_U \in \mathscr{S}(B_{reg}^3)$ is the characteristic function of some open compact set $U \subseteq B^3$. Then $Q(U)$ is an compact open subset of $\mathrm{Sym}_3(F)_{\mathrm{reg}}$. Let

$$\mathbb{Z}_+^3 = \{(a_1, a_2, a_3) \in \mathbb{Z}^3 | a_1 \leq a_2 \leq a_3\}.$$

Then the "elementary divisors." defines a map $\delta : b \in \mathrm{Sym}_3(F) \to (a_1, a_2, a_3) \in \mathbb{Z}_+^3$. One can check that it is locally constant on $\mathrm{Sym}_3(F)_{reg}$. Hence the composition of this map and the moment map Q is also locally constant on B^3_{reg}. In particular, this gives a partition of U into a disjoint union of finitely many open subsets. So we can assume that $\delta \circ Q$ is constant on U, say, $\delta \circ Q(U) = \{(a_1, a_2, a_3)\}$.

Consider $m(aI_3)\Phi$, a certain multiple of 1_{aU}. Choose $a = p^{-A}$ for some integer $A > 1 + a_1 + (a_2 - a_1) + (a_3 - a_1)$. Then we can prove that such $1_{p^{-A-k/2}U}$ is k-regularly supported. It suffices to prove that, for any $x \in U$ and $t \in \mathrm{Sym}_3(\mathscr{O})$, $Q(p^{-A-k/2}x) + p^{-k}t$ belongs to $Q(B^3_{\mathrm{reg}})$. Note that

$$Q(p^{-A-k/2}x) + p^{-k}t = p^{-k-2A+2[\frac{a_1-1}{2}]}(Q(p^{-[\frac{a_1-1}{2}]}x) + p^{2A-2[\frac{a_1-1}{2}]}t).$$

Now $Q(p^{-[\frac{a_1-1}{2}]}x) \in \mathrm{Sym}_3(\mathscr{O})$. It is well-known that for $T \in \mathrm{Sym}_3(\mathscr{O})_{\mathrm{reg}}$, T and $T + p^{2+\det(T)}T'$ for any $T' \in \mathrm{Sym}_3(\mathscr{O})$ define isomorphic integral quadratic forms of rank n. Equivalently, $T + p^{2+\det(T)}T' = {}^t\gamma T \gamma$ for some $\gamma \in \mathrm{GL}_3(\mathscr{O})$. Now it is easy to see that $Q(p^{-A-k/2}x) + p^{-k}t \in Q(B^3_{\mathrm{reg}})$. $\square$

We have the following pleasant property: a k-regularly supported for large k.

an of ram whittaker **Proposition 4.4.3.** *Suppose that $\Phi \in \mathscr{S}(B^3_{\mathrm{reg}})$ is k-regularly supported for a sufficiently large k (depending on the conductor of the additive character ψ). Then we have*

$$W_T(e, s, \Phi) \equiv 0$$

for regular $T \notin Q(B^3_{\mathrm{reg}})$ and any $s \in \mathbb{C}$. In particular, for such T,

$$W_T(e, 0, \Phi) = W'_T(e, 0, \Phi) = 0.$$

Proof. When $\mathrm{Re}(s) \gg 0$, we have

$$\begin{aligned} W_T(e, s, \Phi) &= \gamma(V, \psi) \int_{\mathrm{Sym}_3(F)} \psi(-b(T - Q(x))) \int_{B^3} \Phi(x)\delta(wn(b))^s \, dx \, db \\ &= \gamma(V, \psi) c_v \int_{\mathrm{Sym}_3(F)} \psi(b(T' - T))\delta(wn(b))^s I_{T'}(\Phi) \, db \, dT', \end{aligned}$$

where c_v is a suitable non-zero constant and $I_{T'}(\Phi) = I_{T'}(e, \Phi)$ is defined by eqn I_T v (2.4.6). Then $T' \mapsto I_{T'}(\Phi)$ defines a function in $\mathscr{S}(\mathrm{Sym}_3(F)_{\mathrm{reg}})$ for our choice of Φ. As a function of $b \in \mathrm{Sym}_3(F)$, $\delta(wn(b))$ is invariant under the translation of $\mathrm{Sym}_3(\mathscr{O})$. It follows that

$$\int_{\mathrm{Sym}_3(F)} \psi(bt)\delta(wb)^s db = \left(\int_{\mathrm{Sym}_3(\mathscr{O})} \psi(xt) dx \right) \sum_{b \in \mathrm{Sym}_3(F)/\mathrm{Sym}_3(\mathscr{O})} \psi(bt)\delta(wb)^s,$$

which is zero unless $t \in p^{-k}\mathrm{Sym}^3(\mathscr{O})$ for some k depending on the conductor of the additive character ψ.

Therefore the nonzero contribution to the integral comes from $T' - T \in p^{-k}\mathrm{Sym}^3(\mathscr{O})$ and $I_{T'}(\Phi) \neq 0$. The assumption in the proposition forces that T' is not in $Q(\mathrm{supp}(\Phi))$. But this in turn implies that $I_{T'}(\Phi) = 0$! Therefore, for a k-regularly supported Φ, we have $W_T(e, s, \Phi) \equiv 0$ for $\mathrm{Re}(s) \gg 0$, and hence by analytic continuation, for all $s \in \mathbb{C}$. This completes the proof.

□

[1]

4.5 Holomorphic projection

From now on, we will choose $\Phi = \otimes_v \Phi_v$ such that Φ_v to be k-regularly supported for sufficiently higher k when v is in a finite set S of finite places with at least two elements, and Φ_v is spherical for each finite $v \notin S$. And we always choose the standard Gaussian at all archimedean places. Then for $g \in \mathbb{G}(\mathbb{A}^S)$, we have by Proposition singular cwan of ram whittaker 4.3.3 and 4.4.3,

eqn E' S

$$E'(g, 0, \Phi) = \sum_{v \notin S} \sum_{\Sigma(T) = \Sigma(v)} E'_T(g, 0, \Phi), \tag{4.5.1}$$

where the sum runs over v outside S and nonsingular T. In this section, we study the holomorphic projection of $E'(g, 0, \Phi)$ (restricted to $\mathbb{G}(\mathbb{A})$).

Firstly let us try to study the holomorphic projection for a cusp form φ on $\mathrm{GL}_2(\mathbb{A})$. Fix a non-trivial additive character ψ of $F\backslash\mathbb{A}$, say $\psi = \psi_0 \circ \mathrm{Tr}_{F/\mathbb{Q}}$ with ψ_0 the standard additive character on $\mathbb{Q}\backslash\mathbb{A}_{\mathbb{Q}}$, and let W be the corresponding Whittaker function:

$$W_\varphi(g) = \int_{F\backslash\mathbb{A}} \varphi(n(b)g)\psi(-b)db.$$

Then φ has a Fourier expansion

$$\varphi(g) = \sum_{a \in F^\times} W_\varphi\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} g\right).$$

We say that φ is *holomorphic of weight* 2, if $W_\Phi = W_\infty \cdot W_f$ has a decomposition with W_∞ satisfying the following properties:

eqn W infty

$$W_\infty(g) = \begin{cases} ye^{2\pi i(x+iy)}e^{2i\theta} & \text{if } y > 0 \\ 0 & \text{otherwise} \end{cases} \tag{4.5.2}$$

for the decomposition of $g \in \mathrm{GL}_2(\mathbb{R})$:

$$g = z\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}.$$

[1] Shall we call such Φ S-admissible/nice/exceptional??

For any Whittaker function W of $\mathrm{GL}_2(\mathbb{A})$ which is holomorphic of weight 2 as above with $W_f(g_f)$ compactly supported modulo $Z(\mathbb{A}_f)N(\mathbb{A}_f)$, the Poinaré series is defined as follows:

$$\varphi_W(g) := \lim_{t\to 0+} \sum_{\gamma\in Z(F)N(F)\backslash G(F)} W(\gamma g)\delta(\gamma g)^t, \quad G = \mathrm{GL}_2,$$

where

$$\delta(g) = |a_\infty/d_\infty|, \qquad g = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} k, \qquad k \in K,$$

where K is the standard maximal compact subgroup of $\mathrm{GL}_2(\mathbb{A})$. Let φ be a cusp form and assume that both W and φ have the same central character. Then we can compute their inner product as follows:

eqn W inn prod

$$\begin{aligned}(\varphi, \varphi_W) &= \int_{Z(\mathbb{A})\mathrm{GL}_2(F)\backslash \mathrm{GL}_2(\mathbb{A})} \varphi(g)\overline{\varphi}_W(g)dg \\ &= \lim_{t\longrightarrow 0} \int_{Z(\mathbb{A})N(F)\backslash \mathrm{GL}_2(\mathbb{A})} \varphi(g)\overline{W}(g)\delta(g)^t dg \\ &= \lim_{t\longrightarrow 0} \int_{Z(\mathbb{A})N(\mathbb{A})\backslash \mathrm{GL}_2(\mathbb{A})} W_\varphi(g)\overline{W}(g)\delta(g)^t dg. \end{aligned} \tag{4.5.3}$$

Let φ_0 be the holomorphic projection of φ in the space of holomorphic forms of weight 2. Then we may write

$$W_{\Phi_0}(g) = W_\infty(g_\infty)W_{\varphi_0}(g_f)$$

with W_∞ as in (4.5.2). Then (4.5.3) is a product of integrals over finite places and integrals at infinite places:

$$\int_{Z(\mathbb{R})N(\mathbb{R})\backslash \mathrm{GL}_2(\mathbb{R})} |W_\infty(g_\infty)|^2 dg = \int_0^\infty y^2 e^{-4\pi y} dy/y^2 = (4\pi)^{-1}.$$

In other words, we have

eqn W prod

$$(\varphi, \varphi_W) = (4\pi)^{-g} \int_{Z(\mathbb{A}_f)N(\mathbb{A}_f)\backslash \mathrm{GL}_2(\mathbb{A}_f)} W_{\varphi_0}(g_f)\overline{W}(g_f)dg_f. \tag{4.5.4}$$

As $\overline{W}$ can be any Whittaker function with compact support modulo $Z(\mathbb{A}_f)N(\mathbb{A}_f)$, the combination of (4.5.3) and (4.5.4) gives

lem holo proj

Lemma 4.5.1. *Let φ be a cusp form with trivial central character at each infinite place. Then the holomorphic projection φ_0 of φ has Whittaker function $W_\infty(g_\infty)W_{\varphi_0}(g_f)$ with $W_{\varphi_0}(g_f)$ given as follows:*

$$W_{\varphi_0}(g_f) = (4\pi)^g \lim_{t\longrightarrow 0+} \int_{Z(F_\infty)N(F_\infty)\backslash \mathrm{GL}_2(F_\infty)} W_\varphi(g_\infty g_f)\bar{W}_\infty(g_\infty)\delta(g_\infty)^t dg_\infty.$$

For more details, see [33, §6.4, 6.5].

Now we calculate the holomorphic projection of $E'(g,0,\Phi)$. By Lemma 4.5.1, we need to calculate the integral

$$\alpha_s(T) := \int_{\mathbb{R}_+^3} W_T'(\Phi, \begin{pmatrix} y^{1/2} & \\ & y^{-1/2} \end{pmatrix}, 0) \det(y)^{1+s} e^{-2\pi T y} \frac{dy}{\det(y)^2}, \tag{4.5.5}$$

where $y = \mathrm{diag}(y_1, y_2, y_3)$ and $T \in \mathrm{Sym}_3(\mathbb{R})$ with positive diagonal $\mathrm{diag}(T) = t = \mathrm{diag}(t_1, t_2, t_3)$.

Note that when $t > 1$ and $\mathrm{Re}(s) > -1$, we have an integral representation of the Legendre function of the second kind:

$$Q_s(t) = \int_{\mathbb{R}_+} \frac{du}{(t + \sqrt{t^2-1} \cosh\, u)^{1+s}} = \frac{1}{2} \int_1^\infty \frac{(x-1)^s dx}{x^{1+s}(\frac{t-1}{2}x + 1)^{1+s}}.$$

The admissible pairing at the Archimedean place will be given by the constant term at $s = 0$ of (the regularized sum of, cf. [33, §8.1]) $Q_s(1 + 2s_{\gamma x}(z)/q(x))$.

Consider another closely related function for $t > 1, \mathrm{Re}(s) > -1$:

$$P_s(t) := \frac{1}{2} \int_1^\infty \frac{dx}{x(\frac{t-1}{2}x + 1)^{1+s}}.$$

Then obviously, we have

$$Q_0(t) = P_0(t).$$

One may use any one of the three functions (i.e., Ei, Q_s and P_s) to construct Green's functions. As Theorem 5.1.1 The function Ei is the right choice to match the analytic kernel function, while the admissible pairing requires using Q_s. The following proposition relates Ei to P_s and hence to Q_s by coincidence $Q_0 = P_0$.

Proposition 4.5.1. *Let $x \in M_{2,\mathbb{R}}^3$ such that $T = T(x)$ is non-singular and has positive diagonal. Then we have*

$$\alpha_s(T) = \det(t)^{-1} \left(\frac{\Gamma(s+1)}{(4\pi)^{1+s}} \right)^3 \int_{D_\pm} \eta_s(x_1) * \eta_s(x_2) * \eta_s(x_3),$$

where

$$\eta_s(x, z) := P_s\left(1 + 2\frac{s_x(z)}{q(x)}\right)$$

defines a Green's function of Z_x.

Proof. First by the definition (4.5.5) we have

$$\alpha_s(T) = \int_{\mathbb{R}_+^3} \det(\sqrt{y})^2 W'_{\sqrt{y}T\sqrt{y}}(\Phi, e, 0) \det(y) e^{-2\pi T y} \det(y)^s \frac{dy}{y^2},$$

which is equal to

$$\int_{\mathbb{R}_+^3} W'_{\sqrt{y}T\sqrt{y}}(\Phi, e, 0)e^{-2\pi Ty}\det(y)^s dy.$$

If we modify $x \in M_{2,\mathbb{R}}^3$ with moment $T = T(x)$ to a new $x' = (x_i')$ with $x_i' = x_i/q(x_i)^{1/2}$, we have $T(x') = t^{-\frac{1}{2}}Tt^{-\frac{1}{2}}$ (so that the diagonal are all 1). By Theorem 5.1.1 we have (after substitution $y \to yt$)

$$\alpha_s(T) = \det(t)^{-1-s}\int_{\mathbb{R}_+^3} \Lambda(y^{\frac{1}{2}}T(x')y^{\frac{1}{2}})e^{-4\pi y}\det(y)^s dy.$$

By the definition of $\Lambda(T)$, this is the same as

$$\det(t)^{-1-s}\int_{\mathbb{R}_+^3}\left\{*_{i=1}^3\eta(y_i^{\frac{1}{2}}x_i'; z), 1\right\}_{D_\pm} e^{-4\pi y}\det(y)^s dy$$

where we write $\eta(x; z) = \eta(s_x(z))$ and

$$\left\{*_{i=1}^3\eta(y_i^{\frac{1}{2}}x_i'; z), 1\right\}_{D_\pm} = \int_{D_\pm} *_{i=1}^3\eta(y_i^{\frac{1}{2}}x_i'; z).$$

We can interchange the star product and integral over y to obtain

$$\alpha_s(T) = \det(t)^{-1-s}\left\{*_{i=1}^3\int_{\mathbb{R}_+}\eta(y_i^{\frac{1}{2}}x_i; z)e^{-4\pi y}y_i^s dy_i, 1\right\}_{D_\pm}.$$

Now we compute the inner integral:

$$\begin{aligned}
&\int_{\mathbb{R}_+}\eta(y^{\frac{1}{2}}x; z)e^{-4\pi y}y^s dy\\
=&\int_{\mathbb{R}_+}\mathrm{Ei}(-4\pi y s_x(z))e^{-4\pi y}y^s dy\\
=&\int_{\mathbb{R}_+}\int_1^\infty e^{-4\pi y s_x(z)u}\frac{1}{u}du e^{-4\pi y}y^s dy\\
=&\frac{\Gamma(s+1)}{(4\pi)^{1+s}}\int_1^\infty\frac{1}{u(1+s_x(z)u)^{1+s}}du\\
=&\frac{\Gamma(s+1)}{(4\pi)^{1+s}}P_s(1+2s_x(z)).
\end{aligned}$$

(Also cf. [33, §8.1].) This completes the proof. □

Based on the decomposition of $E'(g, 0, \Phi)$ in §2.5, we have a decomposition of its holomorphic projection, denoted by $E'(g, 0, \Phi)_{hol}$:

decomp hol (4.5.6) $$E'(g, 0, \Phi)_{hol} = \sum_v E'(g, 0, \Phi)_{v,hol},$$

and

$$E'(g, 0, \Phi)_{v,hol} = \sum_{T, \Sigma(T)=\Sigma(v)} E_T'(g, 0, \Phi)_{hol}.$$

5 Local triple height pairings

In this section, we want to compute the local triple-height pairings of Hecke operators at archimedean places, the unramified places, and reduce Conjecture 1.3.1 to some local conjecture.

For archimedean places, we introduce Green functions for Hecke correspondences and compute their star product. The central technical part is to relate the star product to the archimedean Whittaker function in §4.

For unramified places, we first study the modular interpretation of Hecke operators and reduce the question to the work of Gross–Keating on deforming endomorphisms of formal groups.

For ramified places v, we made a conjecture about the local intersections, which will imply Conjecture 1.3.1. In particular, Conjecture 1.3.1 holds if there is no ramified places.

In this section, we assume that $\Phi = \otimes \Phi_v$ is a pure tensor so that for for at least two places v, Φ_v is k-regularly supported for sufficient large k.

5.1 Archimedean height

Now let $B = \mathbb{H}$ be the Hamilton quaternion and let Φ be the standard Gaussian (4.2.1). Let $B' = M_{2,\mathbb{R}}$ be the matrix algebra. Let $D_\pm$ be the union of $\mathscr{H}_\pm^2$ and $\mathscr{H}_+ = \mathscr{H}$ ($\mathscr{H}_-$, resp.) is the upper (lower, resp.) half-plane. Let $x = (x_1, x_2, x_3) \in B'^3$ with non-singular moment matrix $Q(x)$ and let $g_i = g_{x_i}$ be a Green's function of Z_{x_i}, the special divisor on $D_\pm$ defined by x_i. Define the star product

$$\Lambda(x) = \int_{D_\pm} g_1 * g_2 * g_3. \tag{5.1.1}$$

Then $\Lambda(x)$ depends only on the moment $Q(x) \in \mathrm{Sym}_3(\mathbb{R})$ (with signature either $(1,2)$ or $(2,1)$ since B' has signature $(2,2)$). Hence we write it as $\Lambda(\frac{1}{4\pi} Q(x))$ (note that we need to shift it by a multiple 4π).

We will consider Green's function of logarithmic singularity, which we call *pre-Green function* since it does not give the admissible Green's function. Their difference will be discussed later.

Now we specify our choice of pre-Green functions. For $x \in B'$ we define a function $D_\pm = \mathscr{H}_\pm^2 \to \mathbb{R}_+$ defined by

$$s_x(z) := q(x_z) = 2\frac{(x,z)(x,\bar z)}{(z,\bar z)}.$$

In terms of coordinates $z = \begin{pmatrix} z_1 & -z_1 z_2 \\ 1 & -z_2 \end{pmatrix}$ and $x = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, we have

$$s_x(z) = \frac{(-a z_2 + d z_1 - b + c z_1 z_2)(-a \bar z_2 + d \bar z_1 - b + c \bar z_1 \bar z_2)}{-(z_1 - \bar z_1)(z_2 - \bar z_2)}. \tag{5.1.2}$$

We will consider the pre-Green function of Z_x on D given by

$$g_x(z) := \eta(s_x(z)),$$

where we recall that

$$\eta(t) = \mathrm{Ei}(-t) := -\int_1^\infty e^{-tu}\frac{du}{u}.$$

In the following, we want to compute the star product for a non-singular moment $4\pi T = Q(x)$.

thm W'T arch

Theorem 5.1.1. *For $T \in \mathrm{Sym}_3(\mathbb{R})$ with signature either $(1,2)$ or $(2,1)$, we have*

$$W'_{T,\infty}(e, 0, \Phi) = \frac{\kappa(0)}{2\Gamma_3(2)} e^{-2\pi T}\Lambda(T).$$

In particular, everything depends only on the eigenvalues of T (a priori not obvious).

Proof. $\square$

Comparison

Assume that $\tau|\infty$ and we want to relate the archimedean height at τ to the global τ-Fourier coefficient (eqn E'v 2.5.4) of the Eisenstein series. Recall that the generating function is defined for $g \in \mathrm{GL}_2^+(\mathbb{A})$

$$Z(g, \Phi) = \sum_{x \in \widehat{V}/K} r(g_{1f})\Phi(x)Z(x)_K W_{T(x)}(g_\infty),$$

Where the sum runs over all admissible classes. For our fixed embedding $\tau : F \hookrightarrow \mathbb{C}$ we have an isomorphism of $\mathbb{C}$-analytic varieties (as long as K is neat):

$$Y^{an}_{K,\tau} \simeq G(F)\backslash D \times G(\mathbb{A}_f)/K \cup \{\text{cusp}\}$$

where, for short, $G = G(\tau)$ is the nearby group.

For $x = x_i \in V, i = 1, 2, 3$, we define a Green function as follows: for $[z, h'] \in G(F)\backslash D \times G(\mathbb{A}_f)/K$

$$g_{x,hK}([z, h']) = \sum_{\gamma \in G(F)/G_x(F)} \gamma^*[\eta(s_x(z))1_{G_x(\widehat{F})hK}(h')].$$

For an admissible class $x \in \widehat{V}$, we will denote its Green function by g_x. Note that this is *not* the right choice of the Green function. We will get the right one when we come to the holomorphic projection of the analytic kernel function. Therefore we denote

$$(Z(x_1, h_1)_K \cdot Z(x_2, h_2)_K \cdot Z(x_3, h_3)_K)_{\mathrm{Ei},\infty} := g_{x_1,h_1K} * g_{x_2,h_2K} * g_{x_3,h_3K},$$

where Ei is to indicate the current choice of Green functions.

cf infty **Theorem 5.1.2.** *Let $\tau|\infty$ and $g = (g_1, g_2, g_3) \in \mathbb{G}_{\mathbb{A}} = \mathrm{GL}_2^{+,3}(\mathbb{A})$. Assume that Φ_v is supported on non-singular locus at some finite place v. Then the archimedean contribution*

$$(Z(g_1, \Phi_1) \cdot Z(g_2, \Phi_2) \cdot Z(g_3, \Phi_3))_{\mathrm{Ei},\infty} = -2E'_v(g, 0, \Phi),$$

where $E'_v(g, 0, \Phi)$ is defined by (eqn E'v 2.5.4).

Proof. First we consider $g = (g_1, g_2, g_3) \in \mathrm{SL}_2^3(\mathbb{A})$. Afterward, we extend this to $\mathrm{GL}_2^+(\mathbb{A})$.

By definition, the left-hand side is given by

$$Z(g, \Phi)_\infty = \mathrm{vol}(\widetilde{K}) \sum_{x=(x_i)\in(K\backslash\widehat{V})^3} \Phi(x) W_{T(x_\infty)}(g_\infty) \left(\int_{G(F)\backslash D_\pm \times G(\mathbb{A}_f)/K} *_{i=1}^3 g_{x_i}(z, h') d[z, h'] \right),$$

where the sum is over all admissible classes.

Note that

$$\gamma^*[\eta(s_x(z) 1_{G_x(\widehat{F})hK}(h'))] = \eta(s_{\gamma^{-1}x}(z) 1_{G_{\gamma^{1}x}(\widehat{F})\gamma^{-1}hK}(h')).$$

For a fixed triple (x_i), the integral is nonzero only if there exists a $\gamma \in G(F)$ such that

$$\gamma h' \in G_{\gamma_i^{-1}x_i}(\widehat{F})\gamma_i^{-1} h_i K \Longleftrightarrow \gamma_i^{-1} h_i \in G_{\gamma_i^{-1}x_i}(\widehat{F})\gamma h' K.$$

Observe that the sum in the admissible classes can be written as $x_i \in G(F)\backslash V(F)$ and $h_i \in G_{x_i}(\widehat{F})\backslash G(\widehat{F})/K$. Here we denote for short $V = V(\tau)$ that is the nearby quadratic space ramified at $\Sigma(\tau)$. Thus we may combine the sum $x_i \in G(F)\backslash V(F)$ with $\gamma_i \in G(F)/G_{x_i}(F)$ and combine the sum over $\gamma \in G(F)$ with the quotient $G(F)\backslash D_\pm \times G(\mathbb{A}_f)/K$:

$$\mathrm{vol}(\widetilde{K}) \sum_{x\in G(F)\backslash V(F)^3} \left(\int_{h'\in G(\widehat{F})/K} \Phi(h'x) dh' \right) \left(\int_{D_\pm} *_{i=1}^3 \eta_{x_i}(z) dz \right).$$

Here we have used the fact that $G_x = \{1\}$ if $T(x)$ is non-singular and we are assuming that Φ_v is supported in the non-singular locus at some finite place v.

Therefore we have

$$Z(g, \Phi)_\infty = \sum_T \mathrm{vol}(\mathrm{SO}(\mathbb{B}_\infty)) e^{-2\pi T} \Lambda(T) I_T(g^\infty, \Phi^\infty), \tag{5.1.3}$$

where the sum is over all non-singular T with $\Sigma_T = \Sigma(\tau)$, namely those non-singular T represented by the nearby quaternion $B(\tau)$.

Similar to the unramified p-adic case, we compare this with the derivative of the Eisenstein series for a regular T:

Kudla formula infty

$$E'_T(g, 0, \Phi) = \frac{W'_T(g_\infty, 0, \Phi_\infty)}{W_T(g_\infty, 0, \Phi'_\infty)} E_T(g, 0, \Phi^\infty \otimes \Phi'_\infty), \tag{5.1.4}$$

where Φ'_∞ is any test function on V'^3_∞ which makes $W_T(g_\infty, 0, \Phi'_\infty)$ nonvanishing. We may also rewrite

eqn height nifty

$$Z(g,\Phi)_\infty = \sum_T \frac{\mathrm{vol}(\mathrm{SO}(\mathbb{B}_\infty))e^{-2\pi T}\Lambda(T)}{I_T(g_\infty, \Phi'_\infty)} I_T(g, \Phi^\infty \otimes \Phi'_\infty). \tag{5.1.5}$$

Like the p-adic case, we may reduce the desired equality to $g = e$, which we assume now.

We need to evaluate the constant. Note that by the local Siegel–Weil Prop. prop local SW 2.4.3, the ratio

$$\frac{W_T(e, 0, \Phi_v)}{I_T(\Phi_v)}$$

(whenever the denominator is non-zero) does not depend on Φ_v, T ($\det(T) \neq 0$), but only on the measure on $\mathrm{SO}(V_v)$ (and, of course, ψ_v). Let $c_{v,+}$ ($c_{v,+}$, resp.) be this ratio for the quaternion algebra over F_v that is split (division, resp.). We now use the Siegel–Weil formula of Kudla–Rallis to show that (under our choice of measures)

$$a_v := \frac{c_{v,+}}{c_{v,-}} = \pm 1.$$

Indeed, fix two distinct places v_1, v_2. Choose a global quaternion algebra B split at v_1, v_2. Let $B(v_1, v_2)$ be the quaternion algebra that differs from B only at v_1, v_2. Note that our choice of measures on the orthogonal groups associated with all quaternion algebras ensures we always get Tamagawa measures on the adelic points. Compare the Siegel–Weil (we may choose B anisotropic to apply) for B and $B(v_1, v_2)$:

$$a_{v_1} a_{v_2} = 1.$$

But v_1, v_2 are arbitrary, we conclude that a_v is independent of v and hence $a_v^2 = 1$.

From §4.2 Prop. prop WT T>0 4.2.7, we have for $T > 0$

$$W_{T,\infty}(e, 0, \Phi_\infty) = \kappa(0)\Gamma_3(2)^{-1}e^{-2\pi T},$$

where $\kappa(0) < 0$. It is easy to see that

$$I_{T,\infty}(e, \Phi_\infty) = \mathrm{vol}(\mathrm{SO}(\mathbb{B}_\infty))e^{-2\pi T}.$$

Hence, we have

$$c_{\infty,-} = \frac{\kappa(0)\Gamma_3(2)^{-1}}{\mathrm{vol}(\mathrm{SO}(\mathbb{B}_\infty))} < 0.$$

On the other hand, it is not hard to see that $c_{\infty,+}$ is positive, so we have

$$c_{\infty,+} = -c_{\infty,-} = -\frac{\kappa(0)\Gamma_3(2)^{-1}}{\mathrm{vol}(\mathrm{SO}(\mathbb{B}_\infty))}.$$

Now note that $I_T(g, \Phi^\infty \otimes \Phi'_\infty) = E_T(g, 0, \Phi^\infty \otimes \Phi'_\infty)$, and by Theorem thm W'T arch 5.1.1:

$$W'_T(g_\infty, 0, \Phi_\infty) = \frac{\kappa(0)}{2\Gamma_3(2)} e^{-2\pi T} \Lambda(T).$$

Hence the ratio of eqn Kudla formula infty 5.1.4 over the T-th term of eqn height nifty 5.1.5 is given by

$$\frac{\kappa(0)\Gamma_3(2)^{-1}}{2\mathrm{vol}(\mathrm{SO}(\mathbb{B}_\infty))} \cdot \frac{1}{c_{\infty,+}} = -\frac{1}{2}.$$

This completes the proof.

□

cf arch **Theorem 5.1.3.** *Let τ be an archimedean place. Then for $g \in \mathbb{G}$ with $g_w = 1$ for $w \in S_f$ at each place where Φ_v is not unramified,*

$$\Theta(g, \Phi)_\tau = -2E'(g, 0, \Phi)_{\tau, hol}.$$

Proof. The holomorphic projection changes $E'_T(g, 0, \Phi)$ only when $\Sigma(T) = \Sigma(v)$ and v is an archimedean place, in which case we have a formula for $g_\infty = e$:

$$E'_T(g, 0, \Phi)_{hol} = W_T(g_\infty)\, m_v(T)\, W_{T,f}(g_f, 0, \Phi_f),$$

where $m(T)$ is the star product defined by $P_s(1 + 2s_x(z)/q(x))$ for x with moment T. The general g_∞ can be recovered by the transformation rule under Iwasawa decomposition. Then all equalities above are valid for $g \in \mathbb{G}$ with $g_v = 1$ when $v \in S$, the finite set of non-archimedean places outside which Φ_v is unramified.

Under the assumption, all singular coefficients vanish on both sides. For the non-singular coefficients, the right choice of Green's function is the regularized limit of Q_s as $s \to 0$. Since $P_s - Q_s$ is holomorphic and equal to zero when $s = 0$, by the same argument of [YZZ-GZ 33, §8.1], we may use P_s in the Green's function and then take the regularized limit. Then the result follows from Theorem cf infty 5.1.2 and Proposition prop star prod 4.5.1. □

5.2 Modular interpretation of Hecke operators

In this section, we would like to study the reduction of Hecke operators. Let U be an open and compact subgroup of $\mathbb{B}_f^\times$ and let $K = U \times_{\mathbb{A}_f^\times} U$. For an $x \in \mathbb{V}$ with (totally) the positive norm in F, the cycle $Z(x)_K$ is the graph of the Hecke operator given by the cost UxU. Namely, $Z(x)_K$ is the correspondence defined by maps:

$$Z(x)_K \simeq Y_{U \cap xUx^{-1}} \longrightarrow X_U \times_{F_U} X_U.$$

Moduli interpretation at an archimedean place

First, let us give some moduli interpretation of Hecke operators at an archimedean place τ. Let $B = B(\tau)$ be the nearby quaternion algebra. If we decompose $UxU = \coprod_i x_i U$, then $Z(x)_K$ as a correspondence sends one object $(V, h, \bar{\kappa})$ to sum of $(V, h, \overline{\kappa x_i})$. In other words, we may write abstractly,

$$Z(x)_K(V, h, \bar{\kappa}) = \sum_i (V_i, h_i, \bar{\kappa}_i), \tag{5.2.1}$$

where the sum is over the isomorphism class of $(V_i, h_i, \bar{\kappa}_i)$ such that there is an isomorphism $y_i : (V_i, h_i) \longrightarrow (V, h)$ such that the induced diagram is commutative:

$$\begin{tikzcd} \widehat{V}_0 \arrow[r, "\kappa_i"] \arrow[d, "x_i"] & \widehat{V}_i \arrow[d, "\widehat{y}_i"] \\ \widehat{V}_0 \arrow[r, "\kappa"] & \widehat{V}. \end{tikzcd} \tag{5.2.2}$$

Assume that $x_i = u_i x v_i$. Replacing κ and κ_i by equivalent classes $\kappa \circ u_i$ and $\kappa_i \circ v_i^{-1}$, we may assume that $x_i = x$. Thus the subvariety $Z(x)_K$ of M_K parameterizes the triple:

$$(V_1, h_1, \bar{\kappa}_1), \qquad (V_2, h_2, \bar{\kappa}_2), \quad y,$$

where the first two are objects as described above for $\bar{\kappa}_1$ and $\bar{\kappa}_2$ level structures modulo $U_1 := U \cap xUx^{-1}$ and $U_2 = U \cap x^{-1}Ux$ respectively, and $y : (V_2, h_2) \longrightarrow (V_1, h_1)$ is an isomorphism of Hodge structures such that the diagram

$$\begin{tikzcd} \widehat{V}_0 \arrow[r, "\kappa_2"] \arrow[d, "x"] & \widehat{V}_2 \arrow[d, "\widehat{y}"] \\ \widehat{V}_0 \arrow[r, "\kappa_1"] & \widehat{V}_1 \end{tikzcd} \tag{5.2.3}$$

is commutative.

Now we want to describe the above moduli interpretation with an integral Hodge structure concerning a maximal open compact subgroup of the form $\widehat{\mathscr{O}}_B^\times = \mathscr{O}_{\mathbb{B}}^\times$ containing U, where $\mathscr{O}_B$ is a maximal order of B. Let $V_{0,\mathbb{Z}} = \mathscr{O}_B$ as an $\mathscr{O}_B$-lattice in V_0. Then for any triple $(V, h, \bar{\kappa})$ we obtain a triple $(V_\mathbb{Z}, h, \bar{\kappa})$ with $V_\mathbb{Z} = \kappa(V_{0\mathbb{Z}})$ which satisfies the analogous properties as above. M_U parameterizes such integral triples. The Hecke operator $Z(x)_K$ has the following expression:

$$Z(x)_K(V_\mathbb{Z}, h, \bar{\kappa}) = \sum_i (V_{i\mathbb{Z}}, h_i, \kappa_i)$$

where $V_{i\mathbb{Z}} = \kappa_i(V_{0\mathbb{Z}})$. We can't replace terms in the above diagram by integral lattices as y_i and x_i only define a Quasi-isogeny:

$$y_i \in \mathrm{Hom}_{\mathscr{O}_B}(V_{i\mathbb{Z}}, V_{\mathbb{Z}}) \otimes F, \qquad x_i \in \widehat{B} = \mathrm{End}_{\mathscr{O}_B}(\widehat{V}_{0,\mathbb{Z}}) \otimes F.$$

When U is sufficiently small, we have universal objects $(V_U, h, \bar{\kappa})$, $(V_{U,\mathbb{Z}}, h, \bar{\kappa})$. We will also consider the divisible $\mathscr{O}_B$-module $\widetilde{V}_U = \widehat{V}_U / \widehat{V}_{U,\mathbb{Z}}$. The subvariety $Z(x)_K$ also has a universal object $y : V_{U_2} \longrightarrow V_{U_1}$.

Let us return to curves X_U over F. Though the rational structure V at a point on X_U does not make sense, the local system $\widehat{V}$ and $\widehat{V}_{\mathbb{Z}}$ make sense as $\mathbb{B}_f$ and $\mathscr{O}_{\mathbb{B}_f}$ modules respectively. The Hecke operator parameterizes the morphism $\widehat{y} : \widehat{V}_{U_2} \longrightarrow \widehat{V}_{U_1}$.

Modular interpretation at a finite place v

We would like to give a moduli interpretation for the Zariski closure $\mathscr{Z}(x)_K$ of $Z(x)_K$. The isogeny $y : \widehat{V}_{U_2} \longrightarrow \widehat{V}_{U_1}$ induces a quasi-isogeny on divisible $\mathscr{O}_{\mathbb{B}_f}$-modules. For prime to v-part, this is the same as over generic fiber. We need to describe the quasi-isogeny in standard modules. First, assume that $U_v = \mathscr{O}_{\mathbb{B}_v}^\times$ is maximal.

If v is not split in $\mathbb{B}$, then $U_{1v} = U_{2v} = U_v$. Thus, the condition on y_v on the generic fiber requires an order equal to $\mathrm{ord}(\nu(x))$. Hence $\mathscr{Z}(x)_K$ parameterizes the quasi-isogeny of pairs whose order at v has order x. Here we refer to [35, §5.3] for the notion of quasi-isogeny as quasi-isogeny of the divisible module, which can be lifted to the generic fiber.

If v is split in $\mathbb{B}$, then we may choose an isomorphism $\mathscr{O}_{\mathbb{B}_v} = M_2(\mathscr{O}_v)$. Then the formal module $\mathscr{A}$ is a direct sum $\mathscr{E} \oplus \mathscr{E}$ where $\mathscr{E}$ is a divisible $\mathscr{O}_F$-module of dimension 1 and height 2. By replacing x with an element in $U_v x U_v$ we may assume that x_v is diagonal: $x_v = \begin{pmatrix} \varpi^c & \\ & \varpi^d \end{pmatrix}$ with $c, d \in \mathbb{Z}$ and $c \leq d$. It is clear that the condition on y on the generic fiber is a composition of a scalar multiplication by ϖ_v^c (as a quasi-isogeny) and an isogeny with kernel isomorphic to the cyclic module $\mathscr{O}_v/\varpi^{d-a}\mathscr{O}_v$. Thus the scheme $\mathscr{Z}(x)_K$ parameterizes quasi-isogenies f of geometric points of type (c, d) in the following sense:

1. the v-component $\varpi^{-c} y_v : \mathscr{E}_2 \longrightarrow \mathscr{E}_1$ is an isogeny;

2. the kernel of $\varpi^{-c} y_v$ is cyclic of order $d-c$ in the sense that it is the image of a homomorphism $\mathscr{O}_v/\varpi^{d-c} \longrightarrow \mathscr{E}_2$.

We also call such a quasi-isogeny of *type* (c, d). Notice that the number c, d can be defined without reference to U_v. Indeed, c is the minimal integer such that $\varpi^{-c} x_v$ is integral over $\mathscr{O}_v$ and that $c + d = \mathrm{ord}(\det x_v)$.

5.3 Supersingular points on Hecke correspondences

For a geometric point in M_K with formal object $\mathscr{E}_1, \mathscr{E}_2$, by Serre–Tate theory, the formal neighborhood $\mathscr{D}$ is the product of universal deformations $\mathscr{D}_i$ of $\mathscr{E}_i$. The divisor of $\mathscr{Z}(x)_K^{ss}$ in this neighborhood is defined as the sum of the universal deformation of quasi-isogenies. In the following, we want to study the behaviors of this divisor in a formal neighborhood of a pair of supersingular points on M_K when $U = U_v U^v$ with U_v maximal.

Supersingular points on $\mathscr{X}_U$ and $\mathscr{M}_K$

Recall that by [35, §5.4], all supersingular points on X_U are isogenous to each other. Fix one of the supersingular point P_0 representing the triple $(\mathscr{A}_0, \widetilde{V}_0^v, \bar{\kappa}_0^v)$. Let $B = \mathrm{End}^0(P_0)$ which is the nearby quaternion algebra $B(v)$ over F. We may use κ_0 to identify $\widetilde{V}_0$ with $\widehat{V}_0/\widehat{V}_{0\mathbb{Z}}$. The groups $(B \otimes \mathbb{A}_f^v)^\times$ and $(\mathbb{B}_f^v)^\times$ both act on $\widetilde{V}_0$. We may use κ_0 to identify them. In this way, the set $\mathscr{X}_U^{ss}$ of supersingular point is identified with

$$\mathscr{Y}_{U,v}^{ss} = B_0 \backslash (B \otimes \mathbb{A}_f^v)^\times / U^v$$

so that the element $g \in (B \otimes \mathbb{A}_f^v)^\times$ represents the triple

$$(\mathscr{A}_0, \widehat{V}_0^v, gU^v),$$

where B_0 denotes the subgroup of $B^\times$ of elements with order 0 at v.

The supersingular points on $\mathscr{M}_K$ will be represented by a pairs of elements in $(B \otimes \mathbb{A}_f^v)^\times$ with the same norm. Thus we can describe the set of supersingular points on $\mathscr{M}_K$ using orthogonal space $V = (B, q)$ and the Spin similitudes:

$$H = \mathrm{GSpin}(V) = \{(g_1, g_2) \in B^\times, \quad \nu(g_1) = \nu(g_2)\},$$

which acts on V by

$$(g_1, g_2)x = g_1 x g_2^{-1}, \qquad g_i \in B^\times, x \in V.$$

We then have a bijection

$$\mathscr{M}_{K,v}^{ss} \simeq H(F)_0 \backslash H(\mathbb{A}_f^v)/K^v.$$

Supersingular points on $\mathscr{Z}(x)_K$

The set $\mathscr{Z}(x)_{K,v}^{ss}$ of supersingular points on the cycle $\mathscr{Z}(x)_K$ represents the isogeny $y : P_2 \longrightarrow P_1$ of two supersingular points of level $U_1 = U \cap xUx^{-1}$ and $U_2 = U \cap x^{-1}Ux$. In terms of triples as above, $\mathscr{Z}(x)_K^{ss}$ represents equivalent classes of the triples (g_1, g_2, y) of elements $g_i \in (B \otimes \mathbb{A}_f^v)^\times / U_i$ and $y \in B^\times$ with the following properties

$$g_1^{-1} y^v g_2 = x^v, \qquad \mathrm{ord}_v(\det(x_v)) = \mathrm{ord}_v(q(y_v)). \tag{5.3.1}$$

Two triples (g_1, g_2, y) and (g_1', g_2', y') are equivalent if there are $\gamma_1, \gamma_2 \in B_0^\times$ such that

gammagf (5.3.2) $$\gamma_i g_i = g_i', \qquad \gamma_1 y \gamma_2^{-1} = y'.$$

By (5.3.1), the norms of g_1 and g_2 are in the same class modulo $F_+^\times$. By (5.3.2), we may modify them so that they have the same norm. Thus in terms of the group H, we may rewrite condition (5.3.1) as

xgf (5.3.3) $$x^v = g^{-1} y^v, \qquad g = (g_1, g_2) \in H(\mathbb{A}_f^v).$$

This equation is always solvable in g, y for given x. Indeed, since the norm of x is positive, we have an element $y \in B$ with the same norm as x. Then there is a $g \in H(\mathbb{A}_f^v)$ such that $x = g^{-1} y^v$ in $\widehat{V}^v$. In summary, we have shown the following description of $\mathscr{Z}(x)_{K,v}^{ss}$:

Lemma 5.3.1. *Let (y, g) be a solution to (5.3.3) and H_y be the stabilizer of y. Then we have*

$$\begin{aligned}\mathscr{Z}(x)_{K,v}^{ss} =& H(F)_0 \backslash H(F)_0 (H_y(\mathbb{A}_f^v) g) K^v / K^v \\ \simeq& H_y(F)_0 \backslash H_y(\mathbb{A}_f^v) / K_y,\end{aligned}$$

where $K_y := H_y(\mathbb{A}_f^v) \cap g K^v g^{-1}$.

Supersingular formal neighborhood on Hecke operators

Let $\mathscr{H}_v$ be the universal deformation of $\mathscr{A}_0$. Then the union of universal deformation of supersingular points is given by

$$\widehat{\mathscr{Y}_U^{ss}} := B_0 \backslash \mathscr{H}_v \times (B \otimes \mathbb{A}_f^v)^\times / U^v.$$

Notice that $\mathscr{H}_v$ is a formal scheme over $\mathscr{O}_v^{\mathrm{ur}}$. Thus the formal completion of $\mathscr{M}_K$ along its supersingular points is given by

$$\widehat{\mathscr{M}_K^{ss}} := H(F)_0 \backslash \mathscr{D}_v \times H(\mathbb{A}_f^v) / K^v.$$

where $\mathscr{D}_v = \mathscr{H}_v \widehat{\otimes}_{\mathscr{O}_v^{\mathrm{ur}}} \mathscr{H}_v$. Let $\mathscr{D}_y(c, d)$ be the divisor of $\mathscr{D}$ defined by universal deformation of y of type (c, d).

lem CZ(x) **Lemma 5.3.2.** *Let H_y be the stabilizer of y. Then for any $g \in H(\mathbb{A}_f^v)$, the formal neighborhood of $\mathscr{Z}(x)_{K,v}^{ss}$ is given by*

$$\begin{aligned}\widehat{\mathscr{Z}}(x)_K^{ss} =& H(F)_0 \backslash H(F)_0 (\mathscr{D}_y(c, d) \times H_y(\mathbb{A}_f^v) g) K^v / K^v \\ \simeq& H_y(F) \backslash \mathscr{D}_f(c, d) \times H_y(\mathbb{A}_f^v) / K_y,\end{aligned}$$

where $K_y = H_y(\mathbb{A}_f^v) \cap g K^v g^{-1}$.

5.4 Local intersection at an unramified place

In this subsection, we want to study the local intersection at a finite place v which is split in $\mathbb{B}$.

Reduction to a local calculation We still work with the group $\mathbb{H} = \mathrm{GSpin}(\mathbb{V})$. Let x_1, x_2, x_3 be three vectors in $K\backslash\mathbb{V}_f$ such that the cycles $\mathscr{Z}(x_i)_K$ intersects properly in the integral model $\mathscr{M}_K$ of M_K. This means that there are no $k_i \in K$ such that the space

$$\sum F k_i x_i$$

is one or two-dimensional with totally positive norms.

First, let us consider the case where U_v is maximal. We want to compute the intersection index at a geometric point (P_1, P_2) in the special fiber over a finite prime v of F. The non-zero intersection of the three cycles will provide three quasi-isogenies $y_i : P_2 \longrightarrow P_1$ with type determined by x_i's. Notice that P_1 is ordinary (resp. supersingular) if and only if P_2 is ordinary (resp. supersingular).

If they both are ordinary, then we have canonical liftings $\widetilde{P}_i$ to CM points on the generic fiber. Since

$$\mathrm{Hom}(P_1, P_2) = \mathrm{Hom}(\widetilde{P}_1, \widetilde{P}_2),$$

all y_i can be also lifted to quasi-isogenies of $\widetilde{y}_i : \widetilde{P}_2 \longrightarrow \widetilde{P}_1$. This will contradict the assumption that the three cycles $Z(x_i)_K$ have no intersection on the generic fiber. It follows that all P_i's are supersingular points.

Now let us assume that all P_i's are supersingular. Then we have the nearby quaternion algebra $B = B(v)$ and quadratic space (V, q) as before. By Lemma lem CZ(x) 5.3.2, we know that $\mathscr{Z}(x_i)_K^{ss}$ has an extension

$$\widehat{\mathscr{Z}(x_i)}_K^{ss} = H_f(F)\backslash \mathscr{D}_{y_i}(c_i, d_i) \times H_{y_i}(\mathbb{A}_f^v)/K_{y_i}$$

on the formal neighborhood of supersingular points:

$$\widehat{\mathscr{M}_K^{ss}} = H(F)_0\backslash \mathscr{D} \times H(\mathbb{A}_f^v)/K^v.$$

Here $c_i, d_i \in \mathbb{Z}$ such that $\begin{pmatrix} \varpi^{c_i} & \\ & \varpi^{d_i} \end{pmatrix} \in U_v x_{iv} U_v$, and $(y_i, g_i) \in B \times H(\mathbb{A}_f^v)$ such that $g_i^{-1}(y_i) = x_i^v$ in $\mathbb{V}_f^v$. If these three have a nontrivial intersection at a supersingular point represented by $g \in H(F)_0\backslash H(\mathbb{A}_f^v)/K^v$, then we can write $g_i = g k_i$ with some $k_i \in K^v$. The intersection scheme $\mathscr{Z}(k_1x_1, k_2x_2, k_3x_3)_K$ is represented by

$$\mathscr{Z}(k_1x_1, k_2x_2, k_3x_3)_K = [\mathscr{D}_{y_1}(c_1, d_1) \cdot \mathscr{D}_{y_2}(c_2, d_2) \cdot \mathscr{D}_{y_3}(c_3, d_3) \times g]$$

on $\mathscr{D}$, here $y = (y_i) \in V^3$ and $c = (c_i), d = (d_i) \in \mathbb{Z}^3$. As this intersection is proper, the space generated by y_i's is three-dimensional and positive definite. Notice that $g \in H(\mathbb{A}_f^v)/K^v$ is completely determined by the condition $g^{-1}y_i \in K^v x_i^v$. Thus we have that the total intersection at supersingular points is given by

$$\mathscr{Z}(x_1)_K \cdot \mathscr{Z}(x_2)_K \cdot \mathscr{Z}(x_3)_K := \sum_{kx^v \in K^v \backslash (Kx_1^v, Kx_2^v, Kx_3^v)} \deg \mathscr{Z}(k_1x_1, k_2x_2, k_3x_3)_K,$$

where the sum runs through cosets such that $k_i x_i^v$ generated a subspace of dimension 3.

In the following, we let us compute the intersection at v for cycles $\mathscr{Z}(\Phi_i)$ for $\Phi_i \in \mathscr{S}(\mathbb{V})$. Assume that $\Phi_i(x) = \Phi_i^v(x^v)\Phi_{iv}(x_v)$. By the above discussion, we see that the total supersingular intersection is given by

eqn deg Z(Phi)

$$\begin{aligned} \mathscr{Z}(\Phi_1) \cdot \mathscr{Z}(\Phi_2) \cdot \mathscr{Z}(\Phi_3) =& \mathrm{vol}(\widetilde{K}) \prod_{i=1}^{3} \sum_{x_i \in \widetilde{K} \backslash \mathbb{V}} \Phi_i(x_i) \mathscr{Z}(x_i)_K \\ =& \mathrm{vol}(\widetilde{K}) \sum_{x^v \in \widetilde{K}^3 \backslash (\mathbb{V}^v)_+^3} \sum_{x_v \in K_v^3 \backslash (\mathbb{V}_v)_{x^v}^3} \Phi(x) \deg \mathscr{Z}(x)_K \\ =& \mathrm{vol}(\widetilde{K}) \sum_{x^v \in \widetilde{K}^v \backslash (\mathbb{V}^v)_+^3} \Phi^v(x^v) m(x^v, \Phi_v), \qquad (5.4.1) \end{aligned}$$

where $(\widehat{V})_+^3$ denote the set of elements $x^v \in (\widehat{V}^v)^3$ such that the moment matrix of x_i^v as a symmetric elements in $M_3(\mathbb{A}_f^v)$ takes entries in F_+, $(V_v)_{x^v}^3$ denote the set of elements (x_{i_v}) with norm equal to the norms of (x_i^v), and

eqn def m(Phi)

$$m(x^v, \Phi_v) = \sum_{x_v \in K_v^3 \backslash (\mathbb{V}_v)_{x^v}^3} \Phi_v(x_v) \deg \mathscr{Z}(x^v, x_v)_K. \tag{5.4.2}$$

We note that the volume factor $\mathrm{vol}(\widetilde{K})$ is a product of the volume of the image of K_v in $\mathrm{SO}(V_v)$ concerning the Tamagawa measure (cf. Notations). Consequently, and by definition, it also includes the archimedean factor $\mathrm{vol}(\mathrm{SO}(\mathbb{B}_\infty))$.

To compare the above with the theta series, let us rewrite the intersection (eqn deg Z(Phi) 5.4.3) in terms of the quadratic space $V = B$. Notice that every x^v can be written as $x^v = g^{-1}(y)$ with $y \in (V)_+^3$ of elements with non-degenerate moment matrix. Thus we have

eqn deg Z(Phi)

$$\mathscr{Z}(\Phi_1) \cdot \mathscr{Z}(\Phi_2) \cdot \mathscr{Z}(\Phi_3) = \mathrm{vol}(\widetilde{K}) \sum_{y \in H(F) \backslash V_+^3} \sum_{g \in H(\mathbb{A}^v)/\widetilde{K}^v} \Phi^v(g^{-1}y) m(y, \Phi_v)_K, \tag{5.4.3}$$

where for $y \in (V_v)^3$

eqn def m(y,Phi)

$$m(y, \Phi_v) = \sum_{x_v \in K_v^3 \backslash (\mathbb{V}_v)_{x^v}^3} \Phi_v(x_v) \deg \mathscr{Z}(y, x_v)_K. \tag{5.4.4}$$

This is a *pseudo-theta series* (cf. [33]) if $m(\cdot, \Phi_v)$ has no singularity over $y \in (V_v)^3$.

The intersection formula of Gross–Keating. In the following, we want to deduce a formula of the intersection number (??) using the work of Gross–Keating [9]. For an element $y \in B_v$ with the integral norm, let $\mathscr{T}_y$ denote the universal deformation divisor on $\mathscr{D}$ of the isogeny $y : \mathscr{A} \longrightarrow \mathscr{A}$. We extend this definition to arbitrary y by setting $\mathscr{T}_y = 0$ if y is not integral. Then we have the following relation:

$$\mathscr{D}_y(c, d) = \mathscr{T}_{\varpi^{-c}y} - \mathscr{T}_{\varpi^{-c-1}y}.$$

Indeed, for any $y \in \varpi \mathscr{O}_B$, there is an embedding from $\mathscr{T}_{y/\varpi}$ to $\mathscr{T}_y$ by taking any deformation $\varphi : \mathscr{E}_1 \longrightarrow \mathscr{E}_2$ to $\varpi\varphi$. The complement is exactly the deformation with the cyclic kernel. It follows that $\deg \mathscr{Z}(y, x_v)$ is an alternative sum of intersection of Gross–Keating's cycles:

$$\deg \mathscr{Z}(y, x_v)_K = \sum_{\epsilon_i \in \{0,1\}} (-1)^{\epsilon_1+\epsilon_2+\epsilon_3} \mathscr{T}_{\varpi^{-c_1-\epsilon_1}y_1} \cdot \mathscr{T}_{\varpi^{-c_2-\epsilon_2}y_2} \cdot \mathscr{T}_{\varpi^{-c_3-\epsilon_3}y_3}$$

thm GK **Theorem 5.4.1** (Gross–Keating, [9]). *Let $\Phi_v = 1_{\mathscr{O}^3_{B,v}}$ be the characteristic function of $\mathscr{O}^3_{B,v}$. Then for $y \in (V_v)^3_{reg}$, the intersection number $\mathscr{T}_{y_1} \cdot \mathscr{T}_{y_2} \cdot \mathscr{T}_{y_3}$ and $m(y, \Phi_v)$ depends only on the moment $T = Q(y)$ and*

$$m(y, \Phi_v) = \nu(Q(y)),$$

Where the ν-invariant is defined as in Prop. 4.1.2.

cor G-K **Corollary 5.4.2.** *Let $g_v \in \mathbb{G}(F_v)$ and $\Phi_v = 1_{\mathscr{O}^3_{B,v}}$. Then for $y \in (V_v)^3_{reg}$, the intersection number $m(y, r(g_v)\Phi_v)$ depends only on the moment $T = Q(y)$ and is thus denoted by $m_T(r(g_v)\Phi_v)$, and we have*

$$W'_{T,v}(g_v, 0, \Phi_v) = \zeta_v(2)^{-2} m_T(r(g_v)\Phi_v). \tag{5.4.5}$$

Proof. By Gross–Keating theorem 5.4.1 and Prop. 4.1.2, this is true when $g_v = e$ is the identity element. We will reduce the general g_v to this known case.

Suppose that

$$g_v = d(\nu)n(b)m(a)k$$

for b, a are both diagonal matrices and k in the standard maximal compact subgroup of $\mathbb{G}$. Then it is easy to see that the Whittaker function obeys the rule:

$$W'_{T,v}(g_v, 0, \Phi_v) = \psi(\nu T b)|\nu|^{-3}|\det(a)|^2 W'_{\nu a T a}(e, 0, \Phi_v).$$

On the intersection side, from Theorem 5.4.1 we know that $\deg \mathscr{Z}(y, x_v)_K := \deg \mathscr{Z}_T(x_v)_K$ depends only on $T = Q(y)$. We have a similar formula:

$$\begin{aligned} m(y, r(g)\Phi_v) =& |\nu|^{-3} \sum_{x_v} r(g_1)\Phi_v(h_v x_v) \deg \mathscr{Z}_T(x_v)_K \\ =& \psi_{\nu T}(b)|\nu|^{-3}|\det a|^2 \sum_{x_v} \Phi_v(x_v a) \deg \mathscr{Z}_{\nu T}(x_v)_K, \end{aligned}$$

where $h_v \in \mathrm{GO}(V_v)$ with $\nu(h_v) = \nu^{-1}$, and the last sum runs over all x_v with norm $\nu \cdot \mathrm{diag}(T)$.

By our definition of cycles, for diagonal matrix a, we have

$$\deg \mathscr{Z}_{\nu T}(x) = \deg \mathscr{Z}_{\nu a T a}(xa).$$

It follows that

$$m_T(r(g)\Phi_v) = \psi_{\nu T}(b)|\nu|^{-3}|\det a|^2 m_{\nu a T a}(\Phi_v).$$

This completes the proof.

$\square$

Comparison

In this subsection, we will relate the global v-Fourier coefficient (2.5.4) of the analytic kernel function with the local intersection of triple Hecke correspondences when the Shimura curve has a good reduction at v.

Recall that we have a decomposition of $E'(g, 0, \Phi)$ according to the difference of Σ_T and Σ:

$$E'(g, 0, \Phi) = \sum_v E'_v(g, 0, \Phi), \tag{5.4.6}$$

where

$$E'_v(g, 0, \Phi) = \sum_{\Sigma_T = \Sigma(v)} E'_T(g, 0, \Phi). \tag{5.4.7}$$

On the intersection part, we have an analogous decomposition

$$\Theta(g, \Phi) = \Theta(g, \Phi)_{\mathrm{sing}} + \sum_v \Theta(g, \Phi)_v, \tag{5.4.8}$$

and each $\Theta(g, \Phi)_v$ has a part $\mathscr{Z}(g, \Phi)_v$ of intersection of horizontal cycles.

cf good v

Theorem 5.4.3. *Let v be a finite place such that Φ_v is the characteristic function of $\mathscr{O}^3_{\mathbb{B}_v}$. Then for $g = (g_1, g_2, g_3) \in \mathbb{G}$ such that $g_{i,v} = 1$ for $v \in S$, we have an equalities*

$$(\mathscr{Z}(g_1, \Phi_1) \cdot \mathscr{Z}(g_2, \Phi_2) \cdot \mathscr{Z}(g_3, \Phi_3))_v = -2E'_v(g, 0, \Phi)$$

and

$$\Theta(g, \Phi)_v = -2E'_v(g, 0, \phi).$$

Proof. Since X_U has a smooth model $\mathscr{X}_{U,v}$ over v, the restriction of $\widehat{Z}(g_i, \Phi_i)$ over $\mathscr{O}_v$ is equal to $\mathscr{Z}(g_i, \Phi_i) + c(g_i, \Phi_i)V$. This implies

$$\Theta(g, \Phi) = \mathscr{Z}(g_1, \Phi_1) \cdot \mathscr{Z}(g_2, \Phi_2) \cdot \mathscr{Z}(g_3, \Phi_3).$$

Thus the second equality follows from the first one.

By our choice of Φ_S , there is no self-intersection in $\mathscr{Z}(g_1, \Phi_1) \cdot \mathscr{Z}(g_2, \Phi_2) \cdot \mathscr{Z}(g_3, \Phi_3)_v$:

$$\begin{aligned}
&(\mathscr{Z}(g_1, \Phi_1) \cdot \mathscr{Z}(g_2, \Phi_2) \cdot \mathscr{Z}(g_3, \Phi_3))_v \\
=& \sum_{x^v \in (\widetilde{K}^v)^3 \backslash (\mathbb{V}^v)^3_+} r(g^v)\Phi^v(x^v) m(x^v, r(g_v)\Phi_v) \\
=& \sum_{\Sigma(T)=\Sigma(v)} \prod_{w \neq v} \int_{(\mathbb{B}_v^3)_T} r(g^v)\Phi_w(x_w) dx_w \cdot m_T(r(g_v)\Phi_v),
\end{aligned}$$

where

$$m_T(\Phi_v) = \sum_{x_v \in K_v^3 \backslash (\mathbb{B}_v)^3_{\mathrm{diag}(T)}} \Phi_v(x_v) \deg \mathscr{Z}_T(x_v)_K,$$

where the sum is over elements of $\mathbb{B}_v^3$ with norms equal to diagonal of T, and the cycle $\mathscr{Z}_T(x_v)$ is equal to $\mathscr{Z}(x^v, x_v)$ with $x^v \in (\mathbb{V}^v)$ with *non-singular* moment matrix T.

In summary, the intersection number is given by

$$\sum_T \mathrm{vol}(K_v) I_T(g^v, \Phi^v) m_T(r(g_v)\Phi_v). \tag{5.4.9}$$

We need to compare this with the derivative of the Eisenstein series. We invoke the formula of Kudla ([K97][19]):

eqn Kudla formula
$$E_T'(g, 0, \Phi) = \frac{W_T'(g, 0, \Phi_v)}{W_T(g, 0, \Phi_v')} E_T(g, 0, \Phi^v \otimes \Phi_v'). \tag{5.4.10}$$

Under our choice of measures, by Siegwl–Weil, we have

$$E_T(g, 0, \Phi^v \otimes \Phi_v') = I_T(g, \Phi^v \otimes \Phi_v').$$

We therefore have

$$\begin{aligned}
E_T'(g, 0, \Phi) =& \frac{W_T'(g, 0, \Phi_v)}{W_T(g, 0, \Phi_v')} I_T(g, \Phi^v \otimes \Phi_v') \\
=& W_T'(g, 0, \Phi_v) \frac{I_{T,v}(g_v, \Phi_v')}{W_T(g, 0, \Phi_v')} I_T(g^v, \Phi^v).
\end{aligned}$$

Note that $\frac{I_{T,v}(g_v, \Phi_v')}{W_T(g,0,\Phi_v')}$ is a constant independent of T, g, Φ_v'. By Corollary [cor G-K] 5.4.2,

$$E_T'(g, 0, \Phi) = \zeta_v(2)^{-2} m_T(r(g_v)\Phi_v) \frac{I_{T,v}(e, \Phi_v')}{W_T(e, 0, \Phi_v')} I_T(g^v, \Phi^v).$$

It suffices to prove that

$$\zeta_v(2)^{-2}\frac{I_{T,v}(e,\Phi'_v)}{W_T(e,0,\Phi'_v)} = -\frac{1}{2}\mathrm{vol}(K_v).$$

Now the nearby quaternion B is non-split at v. And we have

$$I(e,\Phi'_v) = \mathrm{vol}(\mathrm{SO}(B_v)).$$

So we need to show

$$\frac{\mathrm{vol}(\mathrm{SO}(B_v))}{\mathrm{vol}(K_v)} = -\frac{1}{2}\zeta_v(2)^2 W_T(e,0,\Phi'_v).$$

It is easy to see that (cf. [1, Chap. 16, §3.5]):

$$\frac{\mathrm{vol}(\mathrm{SO}(B_v))}{\mathrm{vol}(K_v)} = \frac{1}{(q-1)^2}.$$

Indeed, we have an isomorphism (cf. Notations)

$$\mathrm{SO}(B) \simeq B^\times/F^\times \times B^1.$$

We now may compute the ratio for a non-archimedean v:

$$\frac{\mathrm{vol}(GL_2(\mathscr{O}_v))}{\mathrm{vol}(\mathscr{O}_{B_v}^\times)} = \frac{\zeta_v(1)^{-1}\zeta_v(2)^{-1}}{\zeta_v(2)^{-1}}\cdot\frac{\mathrm{vol}(M_2(\mathscr{O}_v))}{\mathrm{vol}(\mathscr{O}_{B_v})} = (q-1).$$

Moreover, we have

$$\frac{\mathrm{vol}(GL_2(\mathscr{O}_v))}{\mathrm{vol}(\mathscr{O}_{B_v}^\times)} = \frac{\mathrm{vol}(\mathrm{SL}_2(\mathscr{O}_v))}{\mathrm{vol}(B_v^1)}.$$

By Prop. 4.1.3, we also have

$$\zeta_v(2)^2 W_T(e,0,\Phi'_v) = -\frac{2}{(q-1)^2}.$$

This completes the proof.

□

5.5 Local intersection at ramified places

Let $\mathscr{E}(g,\Phi)$ denote the holomorphic projection of $E'(g,0,\Phi)$. Then by Proposition 3.4.1, the Conjecture 1.3.1 is equivalent to the identity

$$\Theta^-(g,\Phi) = -2\mathscr{E}(g,\Phi).$$

By Theorem 5.1.3 and 5.4.3,

$$\Theta(g,\Phi) + 2\mathscr{E}(g,\Phi) = \Theta(g,\Phi)_{\mathrm{sing}} + \sum_{v\in S}(\Theta(g,\Phi)_v + 2\mathscr{E}(g,\Phi)_v),$$

where S is the set of finite places of F where Φ_v is not spherical. Thus the conjecture is equivalent to

$$\Theta(g,\Phi)-\Theta^-(g,\Phi)=\Theta(g,\Phi)_{\rm sing}+\sum_{v\in S}(\Theta(g,\Phi)_v+2\mathscr{E}(g,\Phi)_v).$$

Notice that the left-hand side is an explicit Eisenstein series for $\mathbb{G}$. This leads to the following speculation:

Conjecture 5.5.1. *For each $v\in S$, $\Theta(g,\Phi)_v+2\mathscr{E}(g,\Phi)_v$ is the nonsingular part of an Eisenstein series for $\mathbb{G}$. In other words, there is an Eisenstein series $G(g,\Phi)_v$ with Fourier coefficients $G_{t_1,t_2,t_3}(g,\Phi)_v$ such that it Fourier expansion takes*

$$\Theta(g,\Phi)_v+2\mathscr{E}(g,\Phi)_v=\sum_{t_1t_2t_3\neq 0}G_{t_1,t_2,t_3}(g,\Phi).$$

This conjecture implies Conjecture main-conj 1.3.1. At this moment, we can prove the following weak form:

prop-alg **Proposition 5.5.2.** *For each prime p, there is a modular form f_p with algebraic coefficients in $\bar{\mathbb{Q}}$ such that*

$$\Theta(g,\Phi)+2\mathscr{E}(g,\Phi)=\sum_{p\in P}f_p(g)\log p.$$

Proof. Since modular forms on $\mathbb{G}$ are determined by their non-singular coefficients, it suffices to show there there are modular forms f with non-singular coefficients given by nonsingular part is given by

$$f_p^*:=\frac{1}{\log p}\sum_{v|p}\Theta(g,\Phi)_v+2\mathscr{E}(g,\Phi)_v.$$

It is clear that f_p^* has algebraic coefficients, and we have an expansion

$$\Theta(g,\Phi)+2\mathscr{E}(g,\Phi)=\Theta(g,\Phi)_{\rm sing}+\sum_p f_p^*\log p.$$

It is clear that $\mathscr{E}(g,\Phi)$ and $\Theta(g,\Phi)$ are both in the finite-dimensional space A of holomorphic modular forms of parallel weight 2 of $\mathrm{GL}_2(\mathbb{A}^\infty)^3$ some level $U=\otimes_vU_v$ where U_v is maximal for $v\notin S$.

Now we consider the action on A by the Hecke algebra

$$\mathbb{T}=\otimes_{v\notin S}\mathbb{Q}[U_v\backslash\mathrm{GL}_2(F_v)^3/U_v].$$

This action is semi-simple. Thus we have a decomposition into eigenforms:

$$\Theta(g,\Phi)+2\mathscr{E}(g,\Phi)=\sum_i f_i.$$

For each $v \notin S$, the coefficient $(t_1, t_2, t_3) = (\pi_v^{n_1}, \pi_v^{n_2}, \pi_v^{n_3})$ has the form

$$a(\pi_v^{n_1}, \pi_v^{n_2}, \pi_v^{n_3}) = \sum_i c_i \prod_{j=1}^{3} \frac{\alpha_{ij}^{n_1} - \beta_{ij}^{n_1}}{\alpha_{ij} - \beta_{ij}}$$

where $c_i, \alpha_{ij}, \beta_{ij}$ are non-zero complex numbers. It is easy to see that the vector space $\sum_{n_1,n_2,n_3} \bar{\mathbb{Q}} a(\pi_v^{n_1}, \pi_v^{n_2}, \pi_v^{n_3})$ over $\bar{\mathbb{Q}}$ is finite-dimensional if and only if $\alpha_{ij}, \beta_{ij} \in \bar{\mathbb{Q}}$. This shows that f_i are multiples of algebraic forms φ_i: $f_i = c_i \varphi_i$. By comparing the Fourier coefficient and linear independence of $\log p$ over $\bar{\mathbb{Q}}$, we see that $c_i \in \sum_p \bar{\mathbb{Q}} \log p$. This shows that each f_p^* is a linear combination over $\bar{\mathbb{Q}}$ of nonsingular parts of φ_i. This finishes the proof of the Proposition. $\square$

Proof of Theorem main-thm *1.3.3*. Let v_1, v_2 be two places prime to P. Then we apply the non-vanishing theorem of Yifeng Liu to find a Schwartz function $\Phi \in \mathscr{S}(\mathbb{V}^3)$ so that $\alpha(\theta(\Phi \otimes \varphi) \neq 0$ for some $\varphi \in \sigma$ and that Φ_v is k-regularly supported for sufficiently large k. By density of algebraic functions, we may assume that Φ is algebraic. Now apply prop-alg 5.5.2 for the set $S' = S \cup \{v_1, v_2\}$ and the set P' of primes under S to get algebraic modular forms f_p for each $p \in P'$ so that

$$\Theta(g, \Phi) + 2\mathscr{E}(g, \Phi) = \sum_{p \in P'} f_p(g) \log p.$$

Now we do pairing with φ to obtain

$$(\Theta(-, \Phi), \varphi) + (2\mathscr{E}(-, \Phi), \varphi) = \sum_{p \in P'} (f_p, \varphi) \log p.$$

Assume that $\theta(\Phi \otimes \varphi) = f \otimes \widetilde{f} \in \pi \otimes \widetilde{\pi}$, then as in the proof of Proposition prop-kernel 3.4.1, the left-hand side is

$$\frac{2L'(1/2, \sigma)}{\zeta_F(2)} \alpha(\theta(\Phi \otimes \varphi)) + \frac{L(1, \pi, \mathrm{ad})}{8\zeta_F(2)^3} \langle P(f), P(\widetilde{f}) \rangle.$$

For the right hand side (f_p, φ) is same as the projection $f_{p,\sigma}$ in f_p with φ. Thus it is in $\bar{\mathbb{Q}}\Omega(\sigma)$. This shows that we have $d_p \in \bar{\mathbb{Q}}_p$ such that

$$\frac{2L'(1/2, \sigma)}{\zeta_F(2)} \alpha(\theta(\Phi \otimes \varphi)) + \frac{L(1, \pi, \mathrm{ad})}{8\zeta_F(2)^3} \langle P(f), P(\widetilde{f}) \rangle = \sum_{p \in P'} \Omega(\sigma) d_p \log p.$$

Rewrite this as

$$\langle P(f), P(\widetilde{f}) \rangle = \left(\frac{8L'(1/2, \sigma)\zeta_F(2)^2}{L(1, \pi, \mathrm{ad})} + \frac{\zeta_F(2)^3 \Omega(\sigma)}{L(1, \pi, \mathrm{ad})} \sum_{p \in P'} c_p \log p \right) \alpha(f, \widetilde{f}),$$

where $c_p = -8d_p / \alpha(f, \widetilde{f}) \in \bar{\mathbb{Q}}$.

As $\alpha(f, \widetilde{f}) = \alpha(\theta(\Phi \otimes \varphi)) \neq 0$, comparison with Proposition prop-CL 1.3.2 gives

$$\mathscr{L}(\pi) = \frac{8L'(1/2, \sigma)\zeta_F(2)^2}{L(1, \pi, \mathrm{ad})} + \frac{\zeta_F(2)^3\Omega(\sigma)}{L(1, \pi, \mathrm{ad})} \sum_{p \in P'} c_p \log p.$$

Notice that $\mathscr{L}(\pi)$ does not depend on the choice of v_1, v_2. This shows the sum is in fact a sum over P. ☐

A Test functions with trilinear zeta integrals with regular support

Yifeng Liu

-vanishing of local

Theorem A.0.1. *Assume that* $\mathrm{Hom}_{\mathbb{G}}(\mathscr{S}(V^3) \times \sigma, \mathbb{C}) \neq 0$. *Then the local zeta integral* $Z(0, f, W)$ *is non-zero for some choice of* $W \in \mathscr{W}(\sigma, \psi)$ *and* $f \in \Pi(B)$ *attached to* $\Phi \in \mathscr{S}(V^3_{reg})$.

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