Triple product L-series and Gross–Kudla–Schoen cycles

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1 Introduction

The aim of this paper is to prove a formula conjectured by B. Gross and S. Kudla in [11] which relates the heights of modified diagonal cycles on the triple products of Shimura curves and the derivative of the triple product L-series. More precisely, for three cusp forms f, g, h of weight 2 for a congruent subgroup $\Gamma_0(N)$ of $\mathrm{SL}_2(\mathbb{Z})$ with N square free, we may consider the function $F := f \times g \times h$ on $\mathscr{H}^3$. There is a triple product L-series $L(s, F)$ as studied by Garrett [7] in classical setting and by Piatetski-Shapiro and Rallis [32] in adelic setting. This function is entire and has functional equation with center at $s = 2$ and a decomposition of the global into a product of local signs:

$$\epsilon(F) = \prod_{p\leq\infty} \epsilon_p(F), \qquad \epsilon_p(F) = \pm 1.$$

Assume that the global sign is -1, then there is canonically defined a Shimura curve X defined by some congruent subgroup of $B^\times$ where B is an indefinite quaternion algebra B which is nonsplit over a non-archimedean prime p if and only if $\epsilon_p(F) = -1$. There is an F-eigen component $\Delta(F)$ of the diagonal Δ of X^3 as an elements in the Chow group of codimension 2 cycles in X^3 as studied by Gross and Schoen [13]. The conjecture formulated by Gross and Kudla takes the shape

$$L'(2,F) = \Omega(F)\langle\Delta(F),\Delta(F)\rangle_{\mathrm{BB}},$$

where $\Omega(F)$ is an explicit positive constant and $\langle\cdot,\cdot\rangle_{\mathrm{BB}}$ is the Beilinson–Bloch height pairing. This formula is an immediate higher dimensional generalization of the Gross–Zagier formula [14].

The objective in this paper is more general than that considered by Gross and Kudla. In fact, we will consider cuspidal Hilbert modular forms of parallel weight 2 and Gross–Schoen cycles on Shimura cures over totally real number fields. We will formulate a conjecture 1.2.3 in terms of automorphic representations and linear forms. This conjecture is analogous to a central value formula of Ichino [16]. In this paper, we can prove this conjecture under some assumption on ramifications, see Theorem 1.2.4. In the following we will describe our conjecture, theorem, and the main idea of proof.

1.1 Shimura curves and abelian varieties

1.1.1 Incoherent quaternion algebras and Shimura curves

Let F be a number field with adele ring $\mathbb{A}_F$ and let $\mathbb{A}_f$ be the ring of finite adeles. Let Σ be a finite set of places of F. Up to isomorphism, let $\mathbb{B}$ be the unique $\mathbb{A}$-algebra, free of rank 4 as an $\mathbb{A}$-module, whose localization $\mathbb{B}_v := \mathbb{B}\otimes_{\mathbb{A}} F_v$ is isomorphic to $M_2(F_v)$ if $v\notin\Sigma$ and to the unique division quaternion algebra over F_v if $v\in\Sigma$. We call $\mathbb{B}$ *the quaternion algebra over* $\mathbb{A}$ *with ramification set* $\Sigma(\mathbb{B}) := \Sigma$.

If $\#\Sigma$ is even then $\mathbb{B} = B\otimes_F \mathbb{A}$ for a quaternion algebra B over F unique up to an F-isomorphism. In this case, we call $\mathbb{B}$ a *coherent* quaternion algebra. If $\#\Sigma$ is odd, then $\mathbb{B}$ is not the base change of any quaternion algebra over F. In this case, we call $\mathbb{B}$ an *incoherent* quaternion algebra. This terminology is inspired by Kudla's notion of *incoherent collections of quadratic spaces*.

Now assume that F is a totally real number field and that $\mathbb{B}$ is an incoherent quaternion algebra over $\mathbb{A}$, totally definite at infinity in the sense that $\mathbb{B}_\tau$ is the Hamiltonian algebra for every archimedean place τ of F.

For each open compact subgroup U of $\mathbb{B}_f^\times := (\mathbb{B}\otimes_{\mathbb{A}}\mathbb{A}_f)^\times$, we have a (compactified) Shimura curve X_U over F. For any embedding $\tau: F\hookrightarrow\mathbb{C}$, the complex points of X_U at τ forms a Riemann surface as follows:

$$X_{U,\tau}(\mathbb{C}) \simeq B(\tau)^\times\backslash\mathscr{H}^\pm\times\mathbb{B}_f^\times/U\cup\{\text{cusps}\}.$$

Here $B(\tau)$ is the unique quaternion algebra over F with ramification set $\Sigma\setminus\{\tau\}$, $\mathbb{B}_f$ is identified with $B(\tau)_{\mathring{A}_f}$ as an $\mathring{A}_f$-algebra, and $B(\tau)^\times$ acts on $\mathscr{H}^\pm$ through an isomorphism $B(\tau)_\tau\simeq M_2(\mathbb{R})$. The set $\{\text{cusps}\}$ is non-empty if and only if $F=\mathbb{Q}$ and $\Sigma=\{\infty\}$.

For any two open compact subgroups $U_1\subset U_2$ of $\mathbb{B}_f\times$, one has a natural surjective morphism

$$\pi_{U_1,U_2}: X_{U_1}\to X_{U_2}.$$

Let X be the projective limit of the system $\{X_U\}_U$. It is a regular scheme over F, locally noetherian but not of finite type. In terms of the notation above, it has a uniformization

$$X_\tau(\mathbb{C}) \simeq B(\tau)^\times \backslash \mathscr{H}^\pm \times \mathbb{B}_f^\times / D \cup \{\text{cusps}\}.$$

Here D denotes the closure of $F\times$ in $\mathbb{A}_f^\times$. If $F = \mathbb{Q}$, then $D = F^\times$. In general, D is much larger than $F^\times$.

The Shimura curve X is endowed with an action T_x of $x \in \mathbb{B}^\times$ given by "*right multiplication by x_f.*" The action T_x is trivial if and only if $x_f \in D$. Each X_U is just the quotient of X by the action of U. In terms of the system $\{X_U\}_U$, the action gives an isomorphism $\mathrm{T}_x : X_{xUx^{-1}} \to X_U$ for each U.

The induced action of $\mathbb{B}_f^\times$ on the set $\pi_0(X_{U,\overline{F}})$ of geometrically connected components of X_U factors through the norm map $q : \mathbb{B}_f^\times \to \mathbb{A}_f^\times$ and makes $\pi_0(X_{U,\overline{F}})$ a principal homogeneous space over $F_+^\times \backslash \mathbb{A}_f^\times / q(U)$. There is a similar description for X.

1.1.2 Hodge classes

The curve X_U has a *Hodge class* $L_U \in \mathrm{Pic}(X_U)_\mathbb{Q}$. It is the line bundle whose global sections are holomorphic modular forms of weight two. The system $L = \{L_U\}_U$ is a direct system in the sense that it is compatible under the pull-back via the projection $\pi_{U_1,U_2} : X_{U_1} \to X_{U_2}$.

Here are some basic explicit descriptions. If X_U is a modular curve, which happens exactly when $F = \mathbb{Q}$ and $\Sigma = \{\infty\}$, then L_U is linearly equivalent to some linear combination of cusps on X_U. If $F \neq \mathbb{Q}$ or $\Sigma \neq \{\infty\}$, then X_U has no cusps and L_U is isomorphic to the canonical bundle of X_U over F for sufficiently small U.

For each component $\alpha \in \pi_0(X_{U,\overline{F}})$, denote by $L_{U,\alpha} = L_U|_{X_{U,\alpha}}$ the restriction to the connected component $X_{U,\alpha}$ of $X_{U,\bar{F}}$ corresponding to α. It is also viewed as a divisor class on X_U via push-forward under $X_{U,\alpha} \to X_U$. Denote by $\xi_{U,\alpha} = \dfrac{1}{\deg L_{U,\alpha}} L_{U,\alpha}$ *the normalized Hodge class on* $X_{U,\alpha}$, and by $\xi_U = \sum_\alpha \xi_{U,\alpha}$ *the normalized Hodge class on* X_U.

We remark that $\deg L_{U,\alpha}$ is independent of α since all geometrically connected components are Galois conjugate to each other. It follows that $\deg L_{U,\alpha} = \deg L_U / |F_+^\times \backslash \mathbb{A}_f^\times / q(U)|$. The degree of L_U can be further expressed as the volume of X_U.

For any open compact subgroup U of $\mathbb{B}_f^\times$, define

$$\mathrm{vol}(X_U) := \int_{X_{U,\tau}(\mathbb{C})} \frac{dxdy}{2\pi y^2}.$$

Here the measure $\dfrac{dxdy}{2\pi y^2}$ on $\mathscr{H}$ descends naturally to a measure on $X_{U,\tau}(\mathbb{C})$ via the complex uniformization for any $\tau : F \hookrightarrow \mathbb{C}$. It can be shown that $\deg L_U = \mathrm{vol}(X_U)$. In particular, the volume is always a positive rational number.

For any $U_1 \subset U_2$, the projection $\pi_{U_1,U_2} : X_{U_1} \to X_{U_2}$ has degree

$$\deg(\pi_{U_1,U_2}) = \mathrm{vol}(X_{U_1})/\mathrm{vol}(X_{U_2}).$$

It follows from the definition. Because of this, we will often use $\mathrm{vol}(X_U)$ as a normalizing factor.

1.1.3 Abelian varieties parametrized by Shimura curves

Let A be a simple abelian variety defined over F. We say that A *is parametrized by* X if there is a non-constant morphism $X_U \to A$ over F for some U. By the Eichler–Shimura theory, if A is parametrized by X, then A *is of strict GL(2)-type* in the sense that

$$M = \mathrm{End}^0(A) := \mathrm{End}_F(A) \otimes_{\mathbb{Z}} \mathbb{Q}$$

is a field and $\mathrm{Lie}(A)$ is a free module of rank one over $M \otimes_{\mathbb{Q}} F$ by the induced action.

Define

$$\pi_A = \mathrm{Hom}^0_\xi(X, A) := \varinjlim_U \mathrm{Hom}^0_{\xi_U}(X_U, A),$$

where $\mathrm{Hom}^0_{\xi_U}(X_U, A)$ denotes the morphisms in $\mathrm{Hom}_F(X_U, A) \otimes_{\mathbb{Z}} \mathbb{Q}$ using ξ_U as a base point. More precisely, if ξ_U is represented by a divisor $\sum_i a_i x_i$ on $X_{U,\overline{F}}$, then $f \in \mathrm{Hom}_F(X_U, A) \otimes_{\mathbb{Z}} \mathbb{Q}$ is in π_A if and only if $\sum_i a_i f(x_i) = 0$ in $A(\overline{F})_{\mathbb{Q}}$.

Since any morphism $X_U \to A$ factors through the Jacobian variety J_U of X_U, we also have

$$\pi_A = \mathrm{Hom}^0(J, A) := \varinjlim_U \mathrm{Hom}^0(J_U, A).$$

Here $\mathrm{Hom}^0(J_U, A) = \mathrm{Hom}_F(J_U, A)_{\otimes \mathbb{Z}} \mathbb{Q}$. The direct limit of $\mathrm{Hom}(J_U, A)$ defines an integral structure on π_A but we will not use this.

The space π_A admits a natural $\mathbb{B}^\times$-module structure. It is an *automorphic representation of* $\mathbb{B}^\times$ *over* $\mathbb{Q}$. We will see the natural identity $\mathrm{End}_{\mathbb{B}^\times}(\pi_A) = M$ and that π_A has a decomposition $\pi = \otimes_M \pi_v$ where π_v is an absolutely irreducible representation of $\mathbb{B}_v^\times$ over M. Using the Jacquet–Langlands correspondence, one can define L-series

$$L(s, \pi) = \prod_v L_v(s, \pi_v) \in M \otimes_{\mathbb{Q}} \mathbb{C}$$

as an entire function of $s \in \mathbb{C}$. Let

$$L(s, A, M) = \prod L_v(s, A, M) \in M \otimes_{\mathbb{Q}} \mathbb{C}$$

be the L-series defined using ℓ-adic representations with coefficients in $M \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$, completed at archimedean places using the Γ-function. Then $L(s, A, M)$ converges absolutely in $M \otimes \mathbb{C}$ for $\mathrm{Re}(s) > 3/2$. The Eichler–Shimura theory asserts that, for almost all finite places v of F, the local L-function of A is given by

$$L_v(s, A, M) = L(s - \frac{1}{2}, \pi_v).$$

Conversely, by the Eichler–Shimura theory and the isogeny theorem of Faltings, if A is of strict GL(2)-type, and if for some automorphic representation π of $\mathbb{B}^\times$ over $\mathbb{Q}$, $L_v(s, A, M)$ is equal to $L(s - 1/2, \pi_v)$ for almost all finite places v, then A is parametrized by the Shimura curve X.

If A is parametrized by X, then the dual abelian variety $A^\vee$ is also parametrized by X. Denote by $M^\vee = \mathrm{End}^0(A^\vee)$. There is a canonical isomorphism $M \to M^\vee$ sending a homomorphism $m : A \to A$ to its dual $m^\vee : A^\vee \to A^\vee$.

There is a perfect $\mathbb{B}^\times$-invariant pairing

$$\pi_A \times \pi_{A^\vee} \longrightarrow M$$

given by

$$(f_1, f_2) = \mathrm{vol}(X_U)^{-1}(f_{1,U} \circ f_{2,U}^\vee), \quad f_{1,U} \in \mathrm{Hom}(J_U, A),\ f_{2,U} \in \mathrm{Hom}(J_U, A^\vee)$$

where $f_{2,U}^\vee : A \to J_U$ is the dual of $f_{2,U}$ composed with the canonical isomorphism $J_U^\vee \simeq J_U$. It follows that $\pi_{A^\vee}$ is dual to π_A as representations of $\mathbb{B}^\times$ over M. Replacing $A^\vee$ in the above construction, then we get a perfect $\mathbb{B}^\times$-invariant pairing

$$\pi_A \times \pi_{A^\vee} \longrightarrow \mathrm{Hom}^0(A^\vee, A),$$

where the M acts on the right hand side through its action on A, and the $\mathbb{B}^\times$ acts via the central character ω_{π_A} of π_A. So we will denote

$$\omega_A := \mathrm{Hom}^0(A^\vee, A).$$

In the case that A is an elliptic curve, we have $M = \mathbb{Q}$ and π_A is self-dual. For any morphism $f \in \pi_A$ represented by a direct system $\{f_U\}_U$, we have

$$(f, f) = \mathrm{vol}(X_U)^{-1} \deg f_U.$$

Here $\deg f_U$ denotes the degree of the finite morphism $f_U : X_U \to A$.

1.2 Trilinear cycles on the triple product of abelian varieties

1.2.1 Trilinear cycles on triple product of abelian varieties

Let A_1, A_2, A_3 be three abelian varieties defined over a number field F. Let $A = A_1 \times A_2 \times A_3$ be denote their product. We consider the space $\mathrm{Ch}_1(A)$ of 1-dimensional Chow cycles with $\mathbb{Q}$-coefficients.

Using Mukai–Fourier transformation, we have a decomposition

$$\mathrm{Ch}_1(A) = \oplus_s \mathrm{Ch}_1(A, s),$$

where $s = (s_1, s_2, s_3)$ are non-negative integers, and $\mathrm{Ch}_1(A, s_1, s_2, s_3)$ consists of cycles x such that under push-forward of by multiplication by $k = (k_1, k_2, k_3) \in \mathbb{Z}^3$ on A:

$$[k]_* x = k^s \cdot x, \qquad k^s := k_1^{s_1} k_2^{s_2} k_3^{s_3}.$$

If s has non-trivial contribution in the decomposition, then it is known that $|s| = s_1 + s_2 + s_3 \geq 2$, and conjectured that $|s| \leq 3$. When $|s| = 3$ the cycles are homologically trivial. Further more, the cycles with $s = (1,1,1)$ are conjecturally the complement of the subspace generated by cycles supported on the image of $A_i \times A_j \times 0_k$ for some reordering (i,j,k) of $(1,2,3)$, where 0_k denote the 0-point on A_k. Using Mukai–Fourier transfer, the group $\mathrm{Ch}_1(A,(1,1,1))$ can be further defined as the group of *trilinear* cycles, namely those cycles $z \in \mathrm{Ch}_1(A)$ satisfying the following equations:

$$m_i^* z = p_i^* z + q_i^* z, \qquad i = 1,2,3 \tag{1.2.1}$$

where m_i, p_i, q_i are respectively the addition map, first projection, and second projection on the i-th factor:

$$m_i, p_i, q_i : \qquad A_i \times A_i \times A_j \times A_k \longrightarrow A_i \times A_j \times A_k, \qquad \{i,j,k\} = \{1,2,3\}.$$

We will denote

$$\mathrm{Ch}_1^{\ell\ell\ell}(A) := \mathrm{Ch}_1(A,(1,1,1)).$$

Let $L(s, A_1 \boxtimes A_2 \boxtimes A_3)$ denote the L-series attached the triple product of ℓ-adic representation of $\mathrm{Gal}(\bar{F}/F)$ on

$$H^1(A_1, \mathbb{Q}_\ell) \otimes H^1(A_2, \mathbb{Q}_\ell) \otimes H^1(A_3, \mathbb{Q}_\ell).$$

Then it is conjectured that $L(s, A_1 \boxtimes A_2 \boxtimes A_3)$ has homomorphic continuation on whole complex plane. An extension of the Birch and Swinneron-Dyer or Beilison–Bloch conjecture gives the following:

Conjecture 1.2.1. *The space group $\mathrm{Ch}_1^{\ell\ell\ell}(A)$ is finitely generated with rank given by*

$$\dim \mathrm{Ch}_1^{\ell\ell\ell}(A) = \mathrm{ord}_{s=2} L(s, A_1 \boxtimes A_2 \boxtimes A_3).$$

Like Neron–Tate height pairing between points on A and $A^\vee = \mathrm{Pic}^0(A)$, there is a canonical height pairing between $\mathrm{Ch}_1^{\ell\ell\ell}(A)$ and $\mathrm{Ch}_1^{\ell\ell\ell}(A^\vee)$ given by Poincare bundles $\mathscr{P}_i$ on $A_i \times A_i^\vee$ with trivializations on $A_i \times 0$ and $0 \times A_i^\vee$:

$$\langle x, y \rangle := (x \times y) \cdot \widehat{c}_1(\bar{\mathscr{P}}_1) \cdot \widehat{c}_1(\bar{\mathscr{P}}_2) \cdot \widehat{c}_1(\bar{\mathscr{P}}_3), \qquad x \in \mathrm{Ch}_1^{\ell\ell\ell}(A), \quad y \in \mathrm{Ch}_1^{\ell\ell\ell}(A^\vee),$$

where $\widehat{c}_1(\bar{\mathscr{P}}_i)$ is the first Chern class of arithmetic cubic structure $\bar{\mathscr{P}}_i$ of $\mathscr{P}_i$. This paring can also defined using Tate's iteration formula analogous to the Neron–Tate height pairing. The right hand of the formula makes sense for all elements in $\mathrm{Ch}_1(A)$ and vanishes on $\mathrm{Ch}_1^{\ell\ell\ell}(A)$.

1.2.2 Refinement for abelian varieties of strictly GL_2-type

Assume that A_i are strictly GL_2-type over fields $M_i := \mathrm{End}^0(A_i)$. Let $M = M_1 \otimes M_2 \otimes M_3$. Then M acts on $\mathrm{Ch}_1^{\ell\ell\ell}(A)$ by push forward and on $\mathrm{Ch}_1^{\ell\ell\ell}(A^\vee)$ by duality. Using equation 1.2.1, one can show that these actions are linear. As M is a direct sum of its quotients fields L, $\mathrm{Ch}_1^{\ell\ell\ell}(A)$ is the direct sum of $\mathrm{Ch}_1^{\ell\ell\ell}(A, L) := \mathrm{Ch}_1^{\ell\ell\ell}(A) \otimes_M L$. We can also define the triple product L-series $L(s, A_1 \boxtimes A_2 \boxtimes A_3, L) \in L \otimes \mathbb{C}$ with coefficients in L using Galois representation on

$$H^1(A_1, \mathbb{Q}_\ell) \otimes_{L \otimes \mathbb{Q}_\ell} \otimes H^1(A_2, \mathbb{Q}_\ell) \otimes_{L \otimes \mathbb{Q}_\ell} H^1(A_3, \mathbb{Q}_\ell)$$

where we choose ℓ inert in L.

Conjecture 1.2.2. *The space group* $\mathrm{Ch}_1^{\ell\ell\ell}(A)_L$ *is finitely generated with rank given by*

$$\dim_L \mathrm{Ch}_1^{\ell\ell\ell}(A, L) = \mathrm{ord}_{s=2}\iota L(s, A_1 \boxtimes A_2 \boxtimes A_3, L),$$

where $\iota : L \otimes \mathbb{C} \longrightarrow \mathbb{C}$ *is the surjection given by any embedding* $L \longrightarrow \mathbb{C}$.

Also we have a unique height paring with coefficient in M:

$$\langle -, - \rangle_L : \qquad \mathrm{Ch}_1^{\ell\ell\ell}(A, L) \otimes_L \mathrm{Ch}_1^{\ell\ell\ell}(A^\vee, L) \longrightarrow L \otimes \mathbb{R}$$

such that

$$\mathrm{tr}_{L\otimes\mathbb{R}/\mathbb{R}}(ax, y)_L) = \langle ax, y \rangle, \qquad a \in L, \quad x \in \mathrm{Ch}_1^{\ell\ell\ell}(A, L), \quad y \in \mathrm{Ch}_1^{\ell\ell\ell}(A^\vee, L).$$

1.2.3 Gross–Kudla–Shoen cycles

Now we assume that A_i are parametrized by a Shimura curve X as before. Let $M_i = \mathrm{End}^0(A_i)$ and L a quotient of $M_1 \otimes M_2 \otimes M_3$. For any $f_i \in \pi_{A_i}$, we have a morphism

$$f := f_1 \times f_2 \times f_3 : \quad X \longrightarrow A.$$

We define $f_*(X) \in \mathrm{Ch}_1(A)$ by

$$f_*(X) := \mathrm{vol}(X_U)^{-1} f_{U*}(X) \in \mathrm{Ch}_1(A)$$

if f_i is represented by f_{iU} on X_U. It is clear that this definition does not depend on the choice of U. Define

$$P_L(f) := f_*(X)^{\ell\ell\ell} \otimes 1 \in \mathrm{Ch}_1^{\ell\ell\ell}(A, L).$$

Let $\pi_i = \pi_{A_i} \otimes_{M_i} L$ be the automorphic representation of $\mathbb{B}^\times$ with coefficients in L. Let $\pi_L = \pi_1 \otimes \pi_2 \otimes \pi_3$ be their triple representation of $(\mathbb{B}^\times)^3$. Then by equation 1.2.1 $f \mapsto P(f)$ defines a linar map:

$$P_L : \qquad \pi_L \longrightarrow \mathrm{Ch}_1^{\ell\ell\ell}(A, L).$$

It is clear that this map is invariant under the action of the diagonal $\Delta(\mathbb{B}^\times)$; thus it defines an element

$$P_L \in \mathscr{P}(\pi_{A,L}) \otimes_L \mathrm{Ch}_1^{\ell\ell\ell}(A, L)$$

where

$$\mathscr{P}(\pi_{A,L}) = \mathrm{Hom}_{\Delta(\mathbb{B}^\times)}(\pi_{A,L}, L).$$

Thus $P_L(f) \neq 0$ for some f only if $\mathscr{P}(\pi_{A,L}) \neq 0$.

By the following Theorem 1.4.1, $\mathscr{P}(\pi_{A,L})$ is at most one dimensional, and it is one-dimensional if and only if the central characters ω_i of π_i satisfy

$$\omega_1 \cdot \omega_2 \cdot \omega_3 = 1$$

and the ramification $\Sigma(\mathbb{B})$ of $\mathbb{B}$ is equal to

$$\Sigma(A, L) = \left\{\text{places } v \text{ of } F : \quad \epsilon\left(\frac{1}{2}\pi_{A,L,v}\right) = -1\right\}.$$

The next problem is to find a non-zero element α of $\mathscr{P}(\pi_{A,L})$ if it is non-zero. It is more convenient to work with $\mathscr{P}(\pi_L) \otimes \mathscr{P}(\widetilde{\pi}_L)$ where $\widetilde{\pi}_L$ is the contragradient of π_L is by the product $A^\vee$ of $A_i^\vee$. Decompose $\pi_L = \otimes_v$ then we have decomposition $\mathscr{P}(\pi_L) = \otimes \mathscr{P}(\pi_v)$ where the space $\mathscr{P}(\pi_v)$ is defined analogously. We will construct element α_v in $\mathscr{P}(\pi_v) \otimes \mathscr{P}(\widetilde{\pi}_v)$ for each place v of F by

$$\alpha(f_v \otimes \widetilde{f}_v) := \frac{L(1, \pi_v, ad)}{\zeta_v(2)^2 L(1/2, \pi_v)} \int_{F_v^\times \backslash B_v^\times} (\pi(b) f_v, \widetilde{f}_v) db, \qquad f_v \otimes \widetilde{f}_v \in \pi_v \otimes \widetilde{\pi}_v.$$

Conjecture 1.2.3. *Assume $\omega_1 \cdot \omega_2 \cdot \omega_3 = 1$. For any $f_1 \in \pi_{A,L}$ and $f_2 \in \pi_{A^\vee,L}$,*

$$\langle P_L(f_1), P_L(f_2)\rangle = \frac{8\zeta_F(2)^2}{L(1, \pi_L, ad)} L'(1/2, \pi_L) \cdot \alpha(f_1, f_2)$$

as an identity in $L \otimes \mathbb{C}$.

Theorem 1.2.4. *The conjecture is true under the assumption that $\mathbb{B}$ has at least two finite place not split over F and that π is unramified over the places which is split in $\mathbb{B}$.*

Remarks 1.2.1.
1. it is conjectured that the theorem is true without the assumption in the theorem; we plan to treat other case in future;
2. the theorem implies that $L'(1/2, \pi_L) = 0$ if and only if it is zero for all conjugates of σ;
3. assume that σ is unitary, then we take $\widetilde{f} = \bar{f}$. The Hodge index conjecture implies $L'(1/2, \pi_L) \geq 0$. This is an consequence of the Riemann Hypothesis.

1.3 Application to adjoint and exterior products

1.3.1 Adjoint product

Assume that $A_1 = A_2^\vee$ and that M_1 and M_2 are identified in L via the dual map $M_1 \mapsto M_2$. Let $\varphi : A_1 \longrightarrow A_2$ be any polarization. Define an involution $s \in \mathrm{End}^0(A_1 \times A_2)$ by

$$s(x, y) = (\varphi^{-1} y, \varphi(x)).$$

Then s induced an involution on $\mathrm{Ch}_1^{\ell\ell\ell}(A)$ which does not depend on the choice of φ. Decompose $\mathrm{Ch}_1^{\ell\ell\ell}(A)$ as a direct sum of $\pm$ eigen spaces. The Beilonson–Bloch conjecture in this case gives

$$\dim \mathrm{Ch}_1^{\ell\ell\ell}(A, L)^- = \mathrm{ord}_{s=1/2} L(s, \mathrm{Ad}(A_1)^2 \boxtimes A_3, L),$$

and

$$\dim \mathrm{Ch}_1^{\ell\ell\ell}(A, L)^+ = \mathrm{ord}_{s=1/2} L(s, A_3, L).$$

In view of the usual BSD

$$\dim A_3(F)_L = \mathrm{ord}_{s=1/2} L(s, A_3, L),$$

we will define a homomorphism:

$$\alpha : \quad \mathrm{Ch}_1^{\ell\ell\ell}(A, L)^+ \longrightarrow A(F)_L,$$

$$z \mapsto \frac{1}{2} p_{3*}(z \cdot \pi_{12}^* \mathscr{P}_{12})) \in \mathrm{Ch}_0(A_3)_L^{\deg=0} = A_3(F)_L.$$

One can prove that the Neron–Tate height paring are related by:

$$\langle z_1, z_2 \rangle = 2\langle \alpha z_1, \alpha z_2 \rangle.$$

For $f \in \pi$, we denote

$$P_L^{A_3}(f) := \alpha P_L(f) \in A_3(F)_L.$$

Since $\mathscr{P}(\widetilde{\pi}) := \mathrm{Hom}_{\mathbb{B}^\times}(\widetilde{\pi}, \mathbb{C})$ is one dimensional, it is given by a sign $\epsilon(s) = \pm 1$. By work of Prasad [30],

$$\epsilon(s) = \epsilon(\mathrm{Ad}(\pi_1) \otimes \pi_3).$$

Corollary 1.3.1. *Assume that $\pi_1 = \pi_2^\vee$ and $\epsilon(s) = -1$, then $\epsilon(\pi_3) = 1$ and*

$$P_L(f) \in \mathrm{Ch}_1^{\ell\ell\ell}(A, L)^-, \qquad f \in \pi$$

and for any $f_1 \in \pi$, $f_2 \in \widetilde{\pi}$,

$$\langle P_L(f_1), P_L(f_2) \rangle = \frac{8\zeta_F(2)^2 L(1/2, \pi_3)}{L(1, \pi, ad)} L'(1/2, \mathrm{Ad}\pi_1 \otimes \pi_3).$$

Corollary 1.3.2. *Assume that $\pi_1 = \pi_2^\vee$ and $\epsilon(s) = 1$, then $\epsilon(\pi_3) = -1$ and*

$$P_L(f) \in \mathrm{Ch}_1^{\ell\ell\ell}(A, L)^+, \qquad f \in \pi$$

and for any $f_1 \in \pi$ and $f_2 \in \widetilde{\pi}$,

$$\langle P_L^{A_3}(f_1), P_L^{A_3}(f_2)_L \rangle = \frac{4\zeta_F(2)^2 L(1/2, \mathrm{Ad}\pi_1 \otimes \pi_3)}{L(1, \pi, ad)} L'(1/2, \pi_3)\alpha(f_1, f_2).$$

Here is a simple formula for $P_L^{A_3}(f)$ for $f = (f_1, f_2, f_3)$ whose first two components satisfy $f_2 = \varphi \circ f_1$ where $\varphi : A_1 \longrightarrow A_1^\vee = A_2$ is a polarization associated to an ample and symmetric line bundle $\mathscr{L}$:

$$\varphi(x) = \mathrm{T}_x^* \mathscr{L} \otimes \mathscr{L}^{-1}.$$

In this case

$$P_L^{A_3}(f) = \sum (f_{3*} f_1^* c_1(\mathscr{L})) \in A_3(F)_L$$

where the sum is the group addition map $\mathrm{Ch}_0(A_3)_L \longrightarrow A_3(F)_L$.

1.3.2 Symmetric product

Finally we assume that $A_1 = A_2 = A_3$ with $M_i = L$ and with trivial central characters. Then the permutation group $\mathscr{S}_3$ acts on A^3 and then $\mathrm{Ch}_1^{\ell\ell\ell}(A)$ and decompose it into subspaces according three irreducible representations of $\mathscr{S}_3$

$$\mathrm{Ch}_1^{\ell\ell\ell}(A, L) = \mathrm{Ch}_1^{\ell\ell\ell}(A, L)^+ \oplus \mathrm{Ch}_1^{\ell\ell\ell}(A, L)^- \oplus \mathrm{Ch}_1^{\ell\ell\ell}(A, L)^0$$

where $\mathrm{Ch}_1^{\ell\ell\ell}(A, L)^+$ is the space of invariants under $\mathscr{S}_3$, and $\mathrm{Ch}_1^{\ell\ell\ell}(A, L)^-$ is the space where $\mathscr{S}_3$ acts as sign function, and $\mathrm{Ch}_1^{\ell\ell\ell}(A, L)^0$ is the space where $\mathscr{S}_3$ is acts as a direct sum of the unique 2 dimensional representation. Then the Beilinson–Bloch conjecture gives

$$\dim \mathrm{Ch}_1^{\ell\ell\ell}(A, L)^+ = 0$$

$$\dim \mathrm{Ch}_1^{\ell\ell\ell}(A, L)^- = \mathrm{ord}_{s=2} L(s, \mathrm{Sym}^3 A_1, L)$$
$$\dim \mathrm{Ch}_1^{\ell\ell\ell}(A, L)^0 = 2\mathrm{ord}_{s=1} L(s, A_1, L).$$

The action of $\mathscr{S}_3$ on $\mathscr{P}(\pi)$ is either trivial or given by sign function. By Prasad's theorem we have:

Corollary 1.3.3. *Assume that $A_1 = A_2 = A_3$ with trivial central character. Let $L = M_i$.*

1. *If $\epsilon(\mathrm{Sym}^3 \sigma_1) = 1$ or $\epsilon(\sigma_1) = -1$, then*

$$P_L(f) = 0, \qquad f \in \pi$$

2. *If $\epsilon(\mathrm{Sym}^3) = -1$ and $\epsilon(\sigma_1) = 1$, then*

$$P_L(f) \in \mathrm{Ch}_1^{\ell\ell\ell}(A)^-,$$

and for any $f_1 \in \pi$ and $f_2 \in \widetilde{\pi}$,

$$\langle P_L(f_1), P_L(f_2) \rangle = \frac{8\zeta_F(2)^2 L(1/2, \pi_1)^2}{L(1, \pi, ad)} L'(1/2, \mathrm{Sym}^3 \pi_1).$$

1.4 Local linear forms over local fields

Let F be a local field and E a cubic semisimple algebra over F. More precisely, E can be taken as one of the following:

- $F \oplus F \oplus F$,
- $F \oplus K$ for a quadratic field extension K of F, and
- a cubic field extension E of F.

Let B be a quaternion algebra over F. Thus B is isomorphic to either the matrix algebra $M_2(F)$ or the division quaternion algebra D (unique up to isomorphism). We define the sign $\epsilon(B)$ of B as 1 if $B \simeq M_2(F)$ and -1 if $B \simeq D$. Let π be a admissible representation of $B_E^\times$ and σ its Jacquet–Langlands correspondence on $\mathrm{GL}_2(E)$. Assume that the central character ω of π has trivial restriction to $F^\times$

$$\omega|_{F^\times} = 1.$$

Consider the space of linear functionals invariant under the subgroup $B^\times$ of $B_E^\times$:

$$\mathscr{P}(\pi) := \mathrm{Hom}_{B^\times}(\pi, \mathbb{C}).$$

By the following result of Prasad and Loke, this space is determined by the local root number

$$\epsilon(\sigma) := \epsilon(\frac{1}{2}, \sigma, \psi \circ \mathrm{tr}_{E/F}) \in \{\pm 1\}.$$

(The definition here does not depend on the choice of the non-trivial character ψ of F.)

Theorem 1.4.1. *[Prasad, Loke [28, 27]] The space $\mathscr{P}(\pi)$ is at most one dimensional. Moreover it is non-zero if and only if*

$$\epsilon(\sigma) = \epsilon(B).$$

Now assume that π is tempered or a local component of an irreducible unitary cuspidal automorphic representation, then the following integration of matrix coefficients with respect to a Haar measure on $F^\times \backslash B^\times$ is absolutely convergent by Ichino [16]:

$$I(f \otimes \widetilde{f}) := \int_{F^\times \backslash B^\times} (\pi(b) f, \widetilde{f}) db, \qquad f \otimes \widetilde{f} \in \pi \otimes \widetilde{\pi}$$

This integration defines an element I in $\mathscr{P}(\pi) \otimes \mathscr{P}(\widetilde{\pi})$ which is invariant under $B^\times \times B^\times$, i.e., an element in

$$\mathscr{P}(\pi) \otimes \mathscr{P}(\widetilde{\pi}) = \mathrm{Hom}_{B^\times \times B^\times}(\pi \otimes \widetilde{\pi}, \mathbb{C}).$$

One can show that this linear form is nonzero if and only if $\mathscr{P}(\pi) \neq 0$. Moreover, we may evaluate the integral in the following spherical case:

1. E/F and π are unramified, f and $\widetilde{f}$ are spherical vectors such that $(f, \widetilde{f}) = 1$;

2. the measure dg is normalized such that the volume of the maximal compact subgroup of $B^\times$ is one.

In this case, one can show that the integration is given by

$$\frac{\zeta_E(2)}{\zeta_F(2)} \frac{L(1/2, \sigma)}{L(1, \sigma, ad)}.$$

See Ichino [16, Lem. 2.2]. Thus we can define a normalized linear form

$$\alpha \in \mathscr{P}(\pi) \otimes \mathscr{P}(\widetilde{\pi})$$

$$\alpha := \frac{\zeta_F(2)}{\zeta_E(2)} \frac{L(1, \sigma, ad)}{L(1/2, \sigma)} I.$$

If π is tempered and unitary then this pairing induces a positive hermitian form on $\mathscr{P}(\pi)$. We remark that the linear form depends only on a choice of the Haar measure on $F^\times \backslash B^\times$.

1.5 Global linear forms

Let F be a number field with ring of adeles $\mathbb{A}$ and E a cubic semisimple algebra over F. We start with an irreducible (unitary) cuspidal automorphic representation σ of $\mathrm{GL}_2(\mathbb{A}_E)$. In [32], Piatetski-Shapiro and Rallis defined an eight dimensional representation r_8 of the L-group of the algebraic group $Res_F^E GL_2$. Thus we have a Langlands L-series $L(s, \sigma, r_8)$ which we abbreviate as $L(s, \sigma)$ in this paper. When $E = F \oplus F \oplus F$ and $\sigma = \sigma_1 \otimes \sigma_2 \otimes \sigma_3$, this L-function is the Rankin type triple product L-function. When E is a field, the L-function $L(s, \sigma)$ is the Asai L-function of σ for the cubic extension E/F. Without confusion, we will simply denote the L-function by $L(s, \sigma)$.

Assume that the central character ω of σ is trivial when restricted to $\mathbb{A}^\times$

$$\omega|_{\mathbb{A}^\times} = 1.$$

Then we have a functional equation

$$L(s, \sigma) = \epsilon(s, \sigma) L(1 - s, \sigma).$$

And the global root number $\epsilon(1/2, \sigma) \in \{\pm 1\}$. For a fixed non-trivial additive character ψ of $F \backslash \mathbb{A}$, we have a decomposition

$$\epsilon(s, \sigma,) = \prod_v \epsilon(s, \sigma_v, \psi_v).$$

The local root number $\epsilon(1/2, \sigma_v, \psi_v) \in \{\pm 1\}$ does not depend on the choice of ψ_v. Thus we have a well-defined (finite) set of places of F:

$$\Sigma = \{v : \qquad \epsilon(1/2, \sigma_v, \psi_v) = -1.\}$$

Let $\mathbb{B}$ be a quaternion algebra over $\mathbb{A}$ which is obtained from $M_2(\mathbb{A})$ with $M_2(F_v)$ replaced by D_v if $v \in \Sigma$. Let $\pi = \otimes_v \pi_v$ be the admissible representation of $\mathbb{B}_E^\times$ such that π_v is the Jacquet–Langlands correspondence of σ_v. Define

$$\mathscr{P}(\pi) := \mathrm{Hom}_{\mathbb{B}^\times}(\pi, \mathbb{C}).$$

Then we have

$$\mathscr{P}(\pi) = \otimes_v \mathscr{P}(\pi_v).$$

Fix a Haar measure $db = \otimes db_v$ on $\mathbb{A}^\times \backslash \mathbb{B}^\times$ then we have integral of matrix coefficients α_v for each place v.

If Σ is even then $\mathbb{B}$ is *coherent* in the sense that it is the base change a quaternion algebra B over F:

$$\mathbb{B} = B \otimes_F \mathbb{A}.$$

In this case π is automorphic and the periods integrals over diagonal will define an element $P_\pi \in \mathscr{P}(\pi)$ and the Ichino formula will give an expression for $\langle P_\pi, P_\pi \rangle$ in terms of $L(1/2, \sigma)$.

If Σ is odd, the $\mathbb{B}$ is *incoherent* in the sense that such a B does not exit. In this case, $\epsilon(1/2, \sigma) = -1$, the central value $L(\frac{1}{2}, \sigma) = 0$ as forced by the functional equation, and we are led to consider the first derivative $L'(\frac{1}{2}, \sigma)$. In this case π is no longer an automorphic representation. Instead, heights of certain cohomologically trivial cycles will provide an invariant linear form P_π whose heights will be given in terms of $L'(1/2, \sigma)$.

We will need to impose certain constraints as follows:

1. F is a totally real field.

2. $E = F \oplus F \oplus F$ is split. We may thus write $\sigma = \sigma_1 \otimes \sigma_2 \otimes \sigma_3$ where each σ_i is a cuspidal automorphic representation of $GL_2(\mathbb{A})$. In this case, the condition on the central character of σ can be rewritten as

$$\omega_1 \cdot \omega_2 \cdot \omega_3 = 1.$$

3. For $i = 1, 2, 3$ and $v|\infty$, all $\sigma_{i,v}$ are discrete of weight 2. It follows that the odd set Σ must contain all archimedean places.

1.6 Ichino's formula

Assume that the global root number is 1. Then $|\Sigma|$ is even. In this case, $\mathbb{B}$ is the base change $B_{\mathbb{A}}$ of a quaternion algebra B over F, and π is an irreducible cuspidal automorphic representation of $\mathbb{B}_E^\times$. Thus we may view elements in π and $\widetilde{\pi}$ as functions on $B_E^\times \backslash \mathbb{B}_E^\times$ with duality given by Tamagawa measures. As the central characters of π (resp. $\widetilde{\pi}$) is trivial when restricted to $\mathbb{A}^\times$, we can define an element $P_\pi \in \mathscr{P}(\pi)$ by periods integral:

$$P_\pi(f) := \int_{Z(\mathbb{A})B^\times \backslash \mathbb{B}^\times} f(b)db.$$

Here the Haar measure is normalized as Tamagawa measure. Jacquet's conjecture says that $\ell_\pi \neq 0$ if and only if $L(1/2, \sigma) \neq 0$. this conjecture has been proved by Harris and Kudla [15] for the split case, Prasad and Schulze-Pillot [31] in the general case. A refinement of Jacquet's conjecture is the following formula due to Ichino:

Theorem 1.6.1 (Ichino [16]). *For each $f_1 \in \pi$ and $f_2 \in \widetilde{\pi}$,*

$$P_\pi(f_1) \cdot P_{\widetilde{\pi}}(f_2) = \frac{1}{2^c} \frac{\zeta_E(2)}{\zeta_F(2)} \frac{L(1/2, \sigma)}{L(1, \sigma, ad)} \cdot \alpha(f_1, f_2).$$

Here the constant c is $3, 2,$ and 1 respectively if $E = F \oplus F \oplus F$, $E = F \oplus K$ for a quadratic K, and a cubic field extension E of F respectively.

Here if the RHS use the measure db_v on $F^\times \backslash B_v^\times$, then we require that $db = \prod_v db_v$.

1.7 Strategy of proof

The strategy of proof of the height formula will be analogous in spirit to the proof of Gross-Zagier formula [14]. Basically it contains the analytic and geometric sides and the comparison between them. Instate of newforms theory, we will make use of representation of adelic groups and linear forms in the same sprit of our recent work of Gross–Zagier formula [36].

First of all, we notice that our conjecture 1.2.3 is an identity between two liner functionals in

$$\mathscr{P}(\pi)\otimes\mathscr{P}(\widetilde{\pi})\subset \mathrm{Hom}(\pi\otimes\widetilde{\pi},\mathbb{C}).$$

The reduced norm on $\mathbb{B}_E$ defines an orthogonal form with values in $\mathbb{A}_E=\mathbb{A}^3$. Thus we have a Weil representation on $R=G(\mathrm{SL}_2(\mathbb{A}_E)\times \mathrm{SO}(\mathbb{V}_E))$ on the space $\mathscr{S}(\mathbb{B}_E)$ of Schwartz functions. We can define a Shimizu lifting

$$\theta:\quad \sigma\otimes r\longrightarrow \pi\otimes\widetilde{\pi}$$

by a decomposition $\theta=\otimes\theta_v$ and a normalized local theta lifting in 2.1.2:

$$\theta_v:\quad \sigma_v\otimes r_v\to\pi_v\otimes\widetilde{\pi}_v. \tag{1.7.1}$$

Via Shimizu's lifting, the height formula can be expressed as an identity of two functionals ℓ_1 and ℓ_2 in

$$\mathrm{Hom}_{\mathrm{SL}_2(\mathbb{A}_E)}(\sigma\otimes r,\mathbb{C}).$$

For each ℓ_i, we will construct a kernel function

$$k_i\in \mathrm{Hom}_{\mathrm{SL}_2(\mathbb{A}_E)}(r, C^\infty(\mathrm{GL}_2(F)\backslash \mathrm{GL}_2(\mathbb{A})))$$

to represent ℓ in the sense

$$\ell_i(\varphi\otimes\phi)=\int_{\mathrm{SL}_2(E)\backslash\mathrm{SL}_2(\mathbb{A})}\varphi(g)k_i(g,\phi)dg.$$

Thus at the end, we need only to prove an identity $k_1=k_2$ of two kernel functions.

The kernel function for analytic side is given by the derivative of the restriction of a Siegel–Eisenstein series By the work of Garrett and Piatetski-Shapiro-Rallis. More precisely, consider $\mathbb{B}_E$ as an orthogonal space over $\mathbb{A}$ via trace $\mathbb{A}_E\to\mathbb{A}$. One can associate to $\phi\in\mathscr{S}(\mathbb{B}_E)$ the Siegel-Eisenstein series $E(s,g,\phi)$. Due to the incoherence, $E(s,g,\phi)$ vanishes at $s=0$. We obtain an integral representation

$$\int_{[\mathbb{G}]}E'(g,0,f_\Phi)\varphi(g)dg=-\frac{L'(1/2,\sigma)}{\zeta_F(2)}\prod_v m(\theta(\Phi\otimes\varphi). \tag{1.7.2}$$

In this method we obtain $E'(g,0,f_\Phi)$ as a kernel function. This kind of Siegel-Eisenstein series has been studied extensively. In particular, its first derivative was firstly studied by Kudla in [21]. It is natural to consider its Fourier expansion:

$$E'(g,0,f_\Phi)=\sum_{T\in\mathrm{Sym}_3(F)}E'_T(g,0,f_\Phi).$$

For nonsingular $T \in \mathrm{Sym}_3(F)$, we have an Euler expansion as a product of local Whittaker functions (for $\mathrm{Re}(s) \gg 0$)

$$E_T(g, s, f_\Phi) = \prod_v W_{T,v}(g, s, \Phi).$$

It is known that the Whittaker functional $W_{T,v}(g, s, \Phi)$ can be extended to an entire function on the complex plane for the s-variable and that $W_{T,v}(g, 0, \Phi)$ vanishes if T cannot be represented as moment matrix of three vectors in the quadratic space B_v. This motivates the following definition. For $T \in \mathrm{Sym}_3(F)_{reg}$ (here "reg" meaning that T is regular), let $\Sigma(T)$ be the set of places over which T is anisotropic. Then $\Sigma(T)$ has even cardinality and the vanishing order of $E_T(g, s, \Phi)$ at s is at least

$$|\{v : T \text{ is not representable in } B_v\}| = |\Sigma \cup \Sigma(T)| - |\Sigma \cap \Sigma(T)|.$$

Since $|\Sigma|$ is odd, $E_T(g, s, \Phi)$ always vanishes at $s = 0$. And its derivative is non-vanishing only if Σ and $\Sigma(T)$ is nearby: they differ by precisely one place v, i.e., only if $\Sigma(T) = \Sigma(v)$ with

$$\Sigma(v) = \begin{cases} \Sigma \setminus \{v\} & \text{if } v \in \Sigma \\ \Sigma \cup \{v\} & \text{otherwise} \end{cases}$$

Moreover when $\Sigma(T) = \Sigma(v)$, the derivative is given by

$$E'_T(g, 0, \Phi) = \prod_{w \neq v} W_{T,w}(g_w, 0, \Phi_w) \cdot W'_{T,v}(g_v, 0, \Phi_v).$$

We thus obtain a decomposition of $E'(g, 0, \Phi)$ according to the difference of Σ_T and Σ:

$$E'(g, 0, \Phi) = \sum_v E'_v(g, 0, \Phi) + E'_{sing}(g, 0, \Phi) \tag{1.7.3}$$

where

$$E'_v(g, 0, \Phi) = \sum_{\Sigma_T = \Sigma(v)} E'_T(g, 0, \Phi) \tag{1.7.4}$$

and

$$E'_{sing}(g, 0, \Phi) = \sum_{T, det(T)=0} E'_T(g, 0, \Phi).$$

Moreover, the local Whittaker functional $W'_{T,v}(g, 0, \Phi_v)$ is closely related to the evaluation of local density. In the spherical case (i.e., $B_v = M_2(F_V)$ is split, ψ_v is unramified, Φ_v is the characteristic function of the maximal lattice $M_2(\mathscr{O}_v)^3$), $W'_{T,v}(g, 0, \Phi_v)$ has essentially been calculated by Katsurada ([19]).

Now two difficulties arise:

1. The vanishing of singular Fourier coefficients (parameterized by singular $T \in \mathrm{Sym}_3(F)$) are not implied by local reason. Hence it is hard to evaluate the first derivative E'_T for singular T.

2. The explicit calculation of $W'_{T,v}(e, 0, \Phi_v)$ for a general Φ_v seems to be extremely complicated.

The solution is to utilize the uniqueness of linear form (note that we have a lot of freedom to choose appropriate Φ) and to focus on certain very special Φ_v. More precisely, define the open subset $B^3_{v,reg}$ of B^3_v to be all $x \in B^3_v$ such that the components of x generates a non-degenerate subspace of B_v of dimension 3. Then we can prove

1. If Φ_v is supported on $B^3_{v,reg}$ for $v \in S$ where S contains at least two finite places, then for singular T and $g \in \mathbb{G}(\mathbb{A}^S)$, we have

$$E'_T(g, v, \Phi) = 0.$$

2. If the test function Φ_v is "regular at a sufficiently higher order" (see Definition 6.2.1), we have for all non-singular T with $\Sigma_T = \Sigma(v)$ and $g \in \mathbb{G}(\mathbb{A}^v)$:

$$E'_T(g, 0, \Phi) = 0.$$

To conclude the discussion of analytic kernel function, we choose Φ_v to be a test function "regular at a sufficiently higher order" for $v \in S$ where S is a set of finite places with at least two elements such that any finite place outside S is spherical. And we always choose the standard Gaussian at all archimedean places. Then for $g \in \mathbb{G}(\mathbb{A}^S)$, we have

$$E'(g, 0, \Phi) = \sum_v E'_v(g, 0, \Phi) \tag{1.7.5}$$

where the sum runs over v outside S and

$$E'_v(g, 0, \Phi) = \sum_{T, \Sigma(T) = \Sigma(v)} E'_T(g, 0, \Phi) \tag{1.7.6}$$

where the sum runs over nonsingular T.

Moreover, we can have a decomposition of its holomorphic projection, denoted by $E'(g, 0, \Phi)_{hol}$. And it has a decomposition

$$E'(g, 0, \Phi)_{hol} = \sum_v \sum_{T, \Sigma(T) = \Sigma(v)} E'_T(g, 0, \Phi)_{hol} \tag{1.7.7}$$

where we only change $E'_T(g, 0, \Phi)$ to $E'_T(g, 0, \Phi)_{hol}$ when $\Sigma(T) = \Sigma(v)$ for v an archimedean place. So similarly we may define $E'_v(g, 0, \Phi)_{hol}$.

This yields an analytic kernel function of the central derivative $L'(\frac{1}{2}, \sigma)$ for all three possibilities of the cubic algebra E.

Now we describe the geometric kernel function under the further assumptions appeared in the beginning of the last subsection. The construction of geometric kernel function is similar to that in the proof of Gross-Zagier formula. More precisely, for $\Phi \in \mathscr{S}(\mathbb{B}_f)$ we can define a generating function of Hecke operators, denoted by $Z(\Phi)$ (see Section 3). Such generating functions have appeared in Gross-Zagier's paper. Works of Kudla-Millson and Borcherds first relate it to the Weil representation. A little extension of our result ([35]) shows that $Z(\Phi)$ is a modular form on $GL_2(\mathbb{A})$. Thus it is natural to consider the generating function for a triple $\Phi = \otimes_i \Phi_i \in \mathscr{S}(\mathbb{B}_f^3)$ fixed by U^6 for a compact open $U \subset B_f^\times$ valued in the correspondences on Y_U^3. The kernel function for geometric side is given by

$$Z(g, \Phi, \Delta_\xi) := \langle \Delta_{U,\xi}, Z(g, \Phi)\Delta_{U,\xi} \rangle, \quad g \in \mathrm{GL}_2^3(\mathbb{A})$$

where Δ_ξ is the projection of the diagonal Δ_U of Y_U^3 in $\mathrm{Ch}^2(Y_U)^{00}$.

Now the main ingredient of our proof is the following weak form of an arithmetic Siegel–Weil formula:

$$-E'(g, 0, \Phi) \equiv Z(g, \Phi, \Delta_\xi), \quad g \in \mathbb{G}(\mathbb{A})$$

where "$\equiv$" means modulo all forms on $\mathbb{G}(\mathbb{A})$ that is perpendicular to σ. Note that this is parallel to the classical Siegel–Weil formula in the coherent case

$$E(g, 0, \Phi) = 2I(g, \Phi).$$

The replacement of "$=$" by "$\equiv$" should be necessary due to representation theory reason.

To make local computation, we will define arithmetic generating series $\widehat{Z}(g_i, \Phi_i)$ with generic fiber $Z(g_i, \Phi_i)$ on the product $Y_U \times Y_U$ and their triple product

$$Z(g, \Phi, \Delta) = \widehat{Z}(g_1, \Phi_1) \cdot \widehat{Z}(g_2, \Phi_2) \cdot \widehat{Z}(g_3, \Phi_3)$$

and show that

$$Z(g, \Phi, \Delta_\xi) \equiv Z(g, \Phi, \Delta).$$

It follows that we have a decomposition to a sum of local heights:

$$Z(g, \Phi, \Delta) \equiv \sum_v Z(g, \Phi, \Delta)_v$$

where the intersection takes place on certain "good" model of Y_U^2.

Under our assumption that for $v \notin \Sigma$, U_v is maximal and the Shimura curve Y_U has good reduction at v. The work of Gross-Keating ([10]) essentially implies that for $g \in \mathbb{G}(\mathbb{A}^S)$:

$$Z(g, \Phi, \Delta)_v \equiv -E'_v(g, 0, \Phi).$$

And when $v|\infty$, using the complex uniformization we may construct the Green current. And we prove that the contribution from the main diagonal to the archimedean height in the intersection is equal to $E'_v(g, 0, \Phi)_{hol}$ (1.7.7).

Finally, when v is a finite place in Σ, then we use Cerednik–Drinfeld uniformization to show that

$$Z(g, \Phi, \Delta)_v \equiv 0.$$

Under our assumption of Φ, we have the same conclusion that $E'_v = 0$ in this case.

1.8 Notations

In the following, k denotes a local field of a number field.

- Normalize the absolute value $|\cdot|$ on k as follows:

  It is the usual one if $k = \mathbb{R}$.

  It is the square of the usual one if $k = \mathbb{C}$.

  If k is non-archimedean, it maps the uniformizer to N^{-1}. Here N is the cardinality of the residue field.

- Normalize the additive character $\psi : k \to \mathbb{C}^\times$ as follows:

  If $k = \mathbb{R}$, then $\psi(x) = e^{2\pi i x}$.

  If $k = \mathbb{C}$, then $\psi(x) = e^{4\pi i \mathrm{Re}(x)}$.

  If k is non-archimedean, then it is a finite extension of $\mathbb{Q}_p$ for some prime p. Take $\psi = \psi_{\mathbb{Q}_p} \circ \mathrm{tr}_{k/\mathbb{Q}_p}$. Here the additive character $\psi_{\mathbb{Q}_p}$ of $\mathbb{Q}_p$ is defined by $\psi_{\mathbb{Q}_p}(x) = e^{-2\pi i \iota(x)}$, where $\iota : \mathbb{Q}_p/\mathbb{Z}_p \hookrightarrow \mathbb{Q}/\mathbb{Z}$ is the natural embedding.

- For a reductive algebraic group G defined over a number field F we denote by Z_G its center and by $[G]$ the quotient
$$[G] := Z_G(\mathbb{A})G(F)\backslash G(\mathbb{A}).$$

- We will use measures normalized as follows. We first fix a non-trivial additive character $\psi = \otimes_v \psi_v$ of $F\backslash\mathbb{A}$. Then we will take the self-dual measure dx_v on F_v with respect to ψ_v and take the product measure on $\mathbb{A}$. We will use this measure for the standard unipotent subgroup N of $SL_2(F)$ and $GL_2(F)$. We will take the Haar measure on $F_v^\times$ as $d^\times x_v = \zeta_{F_v}(1)|x_v|^{-1}dx_v$. Similarly, the measure on B_v and $B_v^\times$ are the self-dual measure dx_v with respect to the character $\psi_v(tr(xy^\iota))$ and $d^\times x_v = \zeta_{F_v}(1)|\nu(x_v)|^{-2}dx_v$. If $\mathbb{B}$ is coherent: $\mathbb{B} = B_\mathbb{A}$ then we have a decomposition of the Haar measure on $\mathbb{A}^\times\backslash\mathbb{B}^\times$: $dx = \prod dx_v$. We will choose the Tamagawa measure on $SL_2(\mathbb{A}_E)$ defined by an invariant differential form and denote the induced decomposition into a product $dg = \prod_v dg_v$. Then we choose a decomposition $dg = \prod_v dg_v$ of the Tamagawa measure on $\mathbb{G}(\mathbb{A})$ such that locally at every place it is compatible with the chosen measure on $SL_2(E_v)$.

- For the non-connected group $O(V)$, we will normalize the measure on $O(V)(\mathbb{A})$ such that
$$\mathrm{vol}([O(V)]) = 1.$$

- For the quadratic space $V = (B, \nu)$ associated to a quaternion algebra, we have three groups: $SO(V)$, $O(V)$ and $\mathrm{GSpin}(V)$. They can be described as follows.
$$\mathrm{GSpin}(V) = \{x, y) \in B^\times \times B^\times | \nu(x) = \nu(y)\}.$$

$$SO(V) = \mathrm{GSpin}(V)/\Delta(F^\times).$$

Let μ_2 be the group of order two generated by the canonical involution on B. Then we have a semi-direct product

$$O(V) = \mathrm{SO}(V) \rtimes \mu_2.$$

Moreover, by the description above, we have an isomorphism

$$\mathrm{GSpin}(V) = B^\times \times B^1,$$

where B^1 is the kernel of the reduced norm:

$$1 \to B^1 \to B^\times \to F^\times \to 1.$$

And similarly, we have an isomorphism

$$SO(V) = B^\times/F^\times \times B^1.$$

Then for a local field F, we will choose the measure on B^1, $B^\times/F^\times$ induced from the measure we have fixed on $F^\times$ and $B^\times$ via the exact sequences. In this way, we also get a Haar measure on $SO(V)$. We normalize the measure on $\mu_2(F) = \{\pm 1\}$ such that the total volume is 1. The measure on $O(V)$ is then the product measure.

- $$\mathbb{G} = \mathrm{GL}^\circ_{2,E} := \{g \in GL_2(E) | det(g) \in F^\times\}.$$

- We will also identify Sym_3 with the unipotent radical of the Siegel parabolic of Sp_6:

$$n(b) = \begin{pmatrix} 1 & b \\ & 1 \end{pmatrix}, \quad b \in \mathrm{Sym}_3(\mathbb{A}).$$

And we denote $[\mathrm{Sym}_3] = \mathrm{Sym}_3(F)\backslash\mathrm{Sym}_3(\mathbb{A})$. And we use the self-dual measure on $Sym_3(\mathbb{A})$ with respect to the additive character $\psi \circ tr$ of $Sym_3(\mathbb{A})$. We denote by $\mathrm{Sym}_3(F)_{reg}$ the subset of non-singular elements. For a non-archimedean local field f, denote by $\mathrm{Sym}(\mathscr{O}_F)^\vee$ the dual of $\mathrm{Sym}_3(\mathscr{O}_F)$ with respect to the pairing $(x, y) \mapsto tr(xy)$. For $X, Y \in \mathrm{Sym}_3(F)$, we write $X \sim Y$ if there exists $g \in GL_3(\mathscr{O}_F)$ such that $X =^t gYg$. For $F = \mathbb{R}$, we have similar notation but with $g \in SO(3)$.

2 Weil representations and Ichino's formula

In this section, we will review Weil representation and apply it to triple product L-series. We will follow work of Garrett, Piateski-Shapiro–Rallis, Waldspurger, Harris–Kudla, Prasad, and Ichino etc. The first main result is Theorem 2.3.1 about integral representation of the triple product L-series using Eisenstein series from the Weil representation on an adelic quaternion algebra.

When the sign of the functional equation is $+1$, then the adelic quaternion algebra is coherent in the sense that it comes form a quaternion algebra over number field, then our main result is the special value formula of Ichino Theorem 2.4.3.

When the sign is -1, then the quaternion algebra is *incoherent*, and the derivative of the Eisenstein series is the kernel function for the derivative of L-series, see formula (2.3.7). We will study the non-singular Fourier coefficients T. We show that these coefficients are non-vanishing only if T is represented by elements in $\mathbb{B}$ if we remove one factor at a place v, see formula (2.5.2).

2.1 Weil representation and theta liftings

In this subsection, we will review the Weil representation as its its extension to similitudes by Harris and Kudla, and normalized Shimuzu lifting by Waldspurger.

Extending Weil representation to similitudes

Let F be a local filed. Let n be a positive integer and let Sp_{2n} be the symplectic group with the standard alternating form $J = \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix}$ on F^{2n}. With the standard polarization $F^{2n} = F^n \oplus F^n$, we have two subgroups of Sp_{2n}:

$$M = \left\{ m(a) = \begin{pmatrix} a & 0 \\ 0 & {}^t a^{-1} \end{pmatrix} \middle| a \in \mathrm{GL}_n(F) \right\}$$

and

$$N = \left\{ n(b) = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \middle| b \in \mathrm{Sym}_n(F) \right\}.$$

Note that M, N and J generate the symplectic group Sp_{2n}.

Let $(V, (\cdot,\cdot))$ be a non-degenerate quadratic space of even dimension m. Associated to V there is a character χ_V of $F^\times/F^{\times,2}$ defined by

$$\chi_V(a) = (a, (-1)^{m/2}\det(V))_F$$

where $(\cdot,\cdot)_F$ is the Hilbert symbol of F and $\det(V) \in F^\times/F^{\times,2}$ is the determinant of the moment matrix $Q(\{x_i\}) = \frac{1}{2}((x_i, x_j))$ of any basis $x_1, ..., x_m$ of V. Let $\mathrm{O}(V)$ be the orthogonal group.

Let $\mathscr{S}(V^n)$ be the space of Bruhat-Schwartz functions on $V^n = V \otimes F^n$ (for archimedean F, functions corresponding to polynomials in the Fock model). Then the Weil representation $r = r_\psi$ of $Sp_{2n} \times O(V)$ can be realized on $\mathscr{S}(V^n)$ by the following formulae:

$$r(m(a))\Phi(x) = \chi_V(\det(a))|\det(a)|_F^{\frac{m}{2}}\Phi(xa),$$

$$r(n(b))\Phi(x) = \psi(tr(bQ(x)))\Phi(x)$$

and

$$r(J)\Phi(x) = \gamma\widehat{\Phi}(x)$$

where γ is an eighth root of unity and $\widehat{\Phi}$ is the Fourier transformation of Φ:

$$\widehat{\Phi}(x) = \int_{F^n} \Phi(y)\psi(\sum_i x_i y_i)dy$$

for $x = (x_1, ..., x_n)$ and $y = (y_1, ..., y_n)$.

Now we want to extend r to representations of groups of similitudes. Let GSp_{2n} and $\mathrm{GO}(V)$ be groups of similitudes with similitude homomorphism ν (to save notations, ν will be used for both groups). Consider a subgroup $R = G(\mathrm{Sp}_{2n} \times \mathrm{O}(V))$ of $\mathrm{GSp}_{2n} \times \mathrm{GO}(V)$

$$R = \{(g, h) \in \mathrm{GSp}_{2n} \times \mathrm{GO}(V) | \nu(g) = \nu(h)\}.$$

Then we can identify $\mathrm{GO}(V)$ (resp., Sp_{2n}) as a subgroup of R consisting of $(d(\nu(h)), h)$ where

$$d(\nu) = \begin{pmatrix} 1_n & 0 \\ 0 & \nu \cdot 1_n \end{pmatrix}$$

(resp. $(g, 1)$). We then have isomorphisms

$$R/\mathrm{Sp}_{2n} \simeq \mathrm{GO}(V), \quad R/O(V) \simeq \mathrm{GSp}_{2n}^+$$

where GSp_{2n}^+ is the subgroup of GSp_{2n} with similitudes in $\nu(\mathrm{GO}(V))$.

We then extend r to a representation of R as follows: for $(g, h) \in R$ and $\Phi \in \mathscr{S}(V^n)$,

$$r((g, h))\Phi = L(h)r(d(\nu(g)^{-1})g)\Phi$$

where

$$L(h)\Phi(x) = |\nu(h)|_F^{-\frac{mn}{4}}\Phi(h^{-1}x).$$

For F a number field, we patch every local representation to obtain representations of adelic groups. For $\Phi \in \mathscr{S}(V_{\mathbb{A}})$ we can define a theta series as an automorphic form on $R(\mathbb{A})$:

$$\theta(g, h, \Phi) = \sum_{x \in V} r(g, h)\Phi(x), \qquad (g, h) \in R(\mathbb{A}).$$

Theta lifting: local and global

In this subsection, we consider the case when $n = 1$ and V is the quadratic space attached to a quaternion algebra B with its reduced norm. Note that $\mathrm{Sp}_2 = \mathrm{SL}_2$ and $\mathrm{GSp}_2 = \mathrm{GL}_2$. And $\mathrm{GL}_2^+(F) = \mathrm{GL}_2(F)$ unless $F = \mathbb{R}$ and B is the Hamilton quaternion in which case $\mathrm{GL}_2^+(\mathbb{R})$ is the subgroup of $\mathrm{GL}_2(\mathbb{R})$ with positive determinants.

We first consider the local theta lifting. For an infinite-dimensional representation σ of $\mathrm{GL}_2(F)$, let π be the representation of $B^\times$ associated by Jacquet-Langlands correspondence and let $\widetilde{\pi}$ be the contragredient of π. Note that we set $\pi = \sigma$ when $B = M_{2\times 2}$.

We have natural isomorphisms between various groups:

$$1 \to \mathbb{G}_m \to B^\times \times B^\times \to \mathrm{GSO}(V) \to 1$$

where $(b_1, b_2) \in B^\times \times B^\times$ acts on B via $(b_1, b_2)x = b_1 x b_2^{-1}$,

$$\mathrm{GO}(V) = \mathrm{GSO}(V) \rtimes \{1, c\}$$

where c acts on B via the canonical involution $c(x) = x^\iota$ and acts on $\mathrm{GSO}(V)$ via $c(b_1, b_2) = (b_2^\iota, b_1^\iota)^{-1}$, and

$$R' = \{(h, g)) \in \mathrm{GSO}(V) \times \mathrm{GL}_2 | \nu(g) = \nu(h)\}.$$

Proposition 2.1.1 (Shimizu liftings)**.** *There exists an* $\mathrm{GSO}(V) \simeq R'/\mathrm{SL}_2$*-equivariant isomorphism*

$$(\sigma \otimes r)_{\mathrm{SL}_2} \simeq \pi \otimes \widetilde{\pi}. \tag{2.1.1}$$

Proof. Note that this is stronger than the usual Howe's duality in the present setting. The result essentially follows from results on Jacquet-Langlands correspondence. Here we explain why we can replace $\mathrm{GO}(V)$ by $\mathrm{GSO}(V)$. In fact, there are exactly two ways to extend an irreducible representation of $\mathrm{GSO}(V)$ to $\mathrm{GO}(V)$. But only one of them can participate the theta correspondence due to essentially the fact that the *sign* of $\mathrm{GO}(V)$ does not occur in the theta correspondence unless $\dim V \leq 2$. $\square$

Let $\mathscr{W}_\sigma = \mathscr{W}_\sigma^\psi$ be the ψ-Whittaker model of σ and let W_φ be a Whittaker function corresponding to φ. Define

$$\begin{aligned} S : \mathscr{S}(V) \otimes \mathscr{W}_\sigma &\to \mathbb{C} \\ (\Phi, W) \mapsto S(\Phi, W) &= \frac{\zeta(2)}{L(1, \sigma, ad)} \int_{N(F)\backslash \mathrm{SL}_2(F)} r(g)\Phi(1)W(g)dg. \end{aligned}$$

See the normalization of measure in "Notations". The integral is absolutely convergent by Lemma 5 of [34] and defines an element in

$$\mathrm{Hom}_{\mathrm{SL}_2 \times B^\times}(r \otimes \sigma, \mathbb{C})$$

where $B^\times$ is diagonally embedded into $B^\times \times B^\times$. The factor before the integral is chosen so that $S(\Phi, W) = 1$ when everything is unramified. Since

$$\mathrm{Hom}_{\mathrm{SL}_2 \times B^\times}(r \otimes \sigma, \mathbb{C}) \simeq \mathrm{Hom}_{B^\times}((r \otimes \sigma)_{\mathrm{SL}_2}, \mathbb{C}) \simeq \mathrm{Hom}_{B^\times}(\pi \otimes \widetilde{\pi}, \mathbb{C})$$

and the last space is of one dimensional spanned by the canonical $B^\times$-invariant pairing between π and its (smooth) dual space $\widetilde{\pi}$, we may define a normalized R'-equivariant map θ

$$\theta : \sigma \otimes r \to \pi \otimes \widetilde{\pi}. \tag{2.1.2}$$

such that

$$S(\Phi, W) = (f_1, f_2)$$

where $f_1 \otimes f_2 = \theta(\Phi \otimes W)$.

Now in the global situation where B is a quaternion algebra defined over a number field, we define the normalized global theta lifting by

$$\theta(\Phi \otimes \varphi)(h) = \frac{\zeta(2)}{2L(1, \sigma, ad)} \int_{\mathrm{SL}_2(F) \backslash \mathrm{SL}_2(\mathbb{A})} \varphi(g_1 g) \theta(g_1 g, h, \Phi) dg_1, \qquad (h, g) \in R'(\mathbb{A}).$$

Proposition 2.1.1. *With definition as above, we have a decomposition $\theta = \bigotimes \theta_v$ in*

$$\mathrm{Hom}_{R'(\mathbb{A})}(r \otimes \sigma, \pi \otimes \widetilde{\pi}).$$

Proof. It suffices to prove the identity after composing with the tautological pairing on $\pi \times \widetilde{\pi}$. More precisely, let $f_1, \otimes f_2 \in \pi \otimes \widetilde{\pi}$ be an element in a cuspidal representatio, $\Phi \in \mathscr{S}(V_\mathbb{A})$ and $\varphi \in \sigma$ so that

$$f_1 \otimes f_2 = \theta(\Phi \otimes \varphi).$$

Assume everything is decomposable, we want to compute (f_1, f_2) in terms of local terms in

$$\Phi = \otimes \Phi_v \in r = \otimes r_v, \quad \varphi = \otimes \varphi_v \in \sigma = \otimes \sigma_v.$$

Then what we need to prove is

$$(f_1, f_2) = \prod_v S(\Phi_v, \varphi_v).$$

This follows from a result of Waldspurger (see. [16, Prop. 3.1]). Note that we have different normalizations of θ and the map S (which is essentialy the map $B_v^\sharp$ in [16]).

□

2.2 Trilinear form

In this subsection, we review a tri-linear form following Garrett, Piatetski-Shapiro and Rallis, Prasad and Loke, and Ichino.

Consider the symplectic form on the six-dimensional space E^2:

$$E^2 \otimes E^2 \xrightarrow{\wedge} E \xrightarrow{\mathrm{tr}} F$$

$$(x,y) \otimes (x',y') \mapsto \mathrm{tr}_{E/F}(xy' - yx').$$

where the first map is by taking wedge product and the second one is the trace map from E to F. Let GSp_6 be the group of similitudes relative to this symplectic form. In this way, we see that elements in $\mathrm{GL}_2(E)$ with determinants in $F^\times$ belong to GSp_6. So we define

$$\mathbb{G} = \left\{ g \in \mathrm{GL}_2(E) | \det(g) \in F^\times \right\}.$$

and identify it with a subgroup of GSp_6.

Let $I(s) = \mathrm{Ind}_P^{\mathrm{GSp}_6} \lambda_s$ be the degenerate principle series of GSp_6. Here, P is the Siegel parabolic subgroup:

$$P = \left\{ \begin{pmatrix} a & * \\ 0 & \nu {}^t a^{-1} \end{pmatrix} \in \mathrm{GSp}_6 | a \in \mathrm{GL}_F(E), \nu \in F^\times \right\}$$

and for $s \in \mathbb{C}$, λ_s is the character of P defined by

$$\lambda_s \left(\begin{pmatrix} a & * \\ 0 & \nu {}^t a^{-1} \end{pmatrix} \right) = |\det(a)|_F^{2s} |\nu|_F^{-3s}.$$

For an irreducible admissible representation σ of $\mathbb{G}$, let $W_\sigma = W_\sigma^\psi$ be the ψ-Whittaker module of σ.

There is a $G(\mathrm{Sp}_6 \times O(B_F))$-intertwining map

$$i: \quad \mathscr{S}(B_E) \to I(0) \tag{2.2.1}$$

$$\Phi \mapsto f_\Phi(\cdot, 0)$$

where for $g \in \mathrm{GSp}_6$,

$$f_\Phi(g,0) = |\nu(g)|^{-3} r(d(\nu(g))^{-1} g)\Phi(0).$$

We extend it to a standard section $f_{\Phi,s}$ of $I(s)$ and called the *Seigel–Weil section* associated to Φ.

Let $\Pi(B)$ be the image of the map (2.2.1). Similarly, for B', we can define $\Pi(B')$ for the unique quaternion algebra B' over F not equivalent to B.

Lemma 2.2.1. *For nonarchimedean F,*

$$I(0) = \Pi(B) \oplus \Pi(B'). \tag{2.2.2}$$

Proof. See Harris–Kudla [15], section. 4, (4.4)-(4.7) and Kudla [20], II.1. □

Now we treat the case when F is archimedean.

If $F = \mathbb{C}$, then one has only one quaternion algebra B over F. In this case we have

$$I(0) = \Pi(B). \tag{2.2.3}$$

This is proved in Lemma A.1 of Appendix of Harris–Kudla [15].

If $F = \mathbb{R}$, then one has two quaternion algebras, $B = M_{2\times 2}$ and B' the Hamilton quaternion. The replacement of Lemma 2.2.1 is the following isomorphism Harris–Kudla ([15], (4.8))

$$I(0) = \Pi(B) \oplus \Pi(B') \tag{2.2.4}$$

where $\Pi(B') = \Pi(4, 0) \oplus \Pi(0, 4)$ where the two spaces are associated to the two quadratic spaces obtained by changing signs of the reduced norm on the Hamilton quaternion.

Local zeta integral of triple product

The local zeta integral of Garrett ([7]) and Piatetski-Shapiro and Rallis ([32]) is a (family of) linear functional on $I(s) \times W_\sigma$ defined as

$$Z(s, f, W) = \int_{F^\times N_0\backslash\mathbb{G}} f_s(\eta g)W(g)dg, \quad (f, W) \in I(s) \times W_\sigma.$$

See the normalization of measure in "Notations". Here, N_0 is a subgroup of $\mathbb{G}$ defined as

$$N = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} | b \in E, \mathrm{tr}_{E/F}(b) = 0 \right\},$$

and $\eta \in \mathrm{GSp}_6$ is a representative of the unique open orbit of $\mathbb{G}$ acting on $P\backslash\mathrm{GSp}_6$. The integral is absolutely convergent for $\mathrm{Re}(s) \gg 0$. And the integral $Z(0, f, W)$ is absolutely convergent when the exponent $\Lambda(\sigma) < \frac{1}{2}$. This condition holds if σ is a local component of a cuspidal automorphic representation by the work of Kim–Shahidi [26]. If f is the image of a $\Phi \in \mathscr{S}(B)$, we also write $Z(s, f, W)$ as $Z(s, \Phi, W)$.

Proposition 2.2.2. *For σ with $\Lambda(\sigma) < \frac{1}{2}$, $Z(f, W) := Z(0, f, W)$ defines a non-vanishing linear functional on $I(0) \times W_\sigma$.*

Proof. See [32, Prop. 3.3] and [18, pp. 227]. But we will reprove this later in the proof of Theorem 6.3.1. □

Local tri-linear forms

Let π be an irreducible admissible representation of $B_E^\times$ with trivial restriction on $F^\times$. Let σ be the Langlands correspondence to $\mathrm{GL}_2(E)$. Assume that $\Lambda(\sigma) < 1/2$.

Proposition 2.2.3 (Ichino [16]). *Under the normalization of θ as in 2.1.2, we have*

$$Z(\Phi, \varphi) = (-1)^{\epsilon(B)} \frac{L(1/2, \sigma)}{\zeta_F(2)} m(\theta(\Phi \otimes \varphi)).$$

Proof. This is Proposition 5.1 of Ichino [16]. Notice that our choice of the local Haar measure on $F^\times \backslash B^\times$ differs from that of [16] by $\zeta_F(2)$. □

Proposition 2.2.4. *Assume that π is unitary.*

1. *One has the following positivity $I(f, \bar{f}) \geq 0$ for $f \in \pi$.*

2. *Moreover, the following are equivalent:*

   (a) $m(\pi) = 1$.

   (b) Z does not vanish on $\sigma \otimes \Pi(B)$.

   (c) I does not vanish on π.

Proof. The first one follows essentially from a theorem of He [17]. We need to prove the second one. Obviously, $c) \Rightarrow a)$. The previous proposition implies that $b) \Leftrightarrow c)$. We are left to prove $a) \Rightarrow b)$. Let B' be the (unique) quaternion algebra non-isomorphic to B and π' the Jacquet–Langlands correspondence on $(B'_E)^\times$ of σ. By the dichotomy, $\mathrm{Hom}_{(B')^\times}(\pi', \mathbb{C}) = 0$, and thus $Z = I = 0$ identically for B'. First we assume that F is non-archimedean. Then by the direct sum decomposition $I(0) = \Pi(B) \oplus \Pi(B)$ and the non-vanishing of Z on $I(0) \otimes \sigma$, we conclude that Z does not vanish on $\Pi(B) \otimes \sigma$. If F is archimedean, we only need to consider F is real. The assertion is trivial if B is the Hamilton quaternion since then $B^\times / F^\times \simeq \mathrm{SO}(3)$ is compact. We assume that $B = M_{2\times 2, \mathbb{R}}$. Then one can use the same argument as above. □

2.3 Integral representation of L-series

In this subsection, we review integral representation of triple product L-series of Garrett, Piatetski-Shapiro and Rallis, and various improvements of Harris–Kudla. Let F be a number field with adeles $\mathbb{A}$, $\mathbb{B}$ a quaternion algebra over $\mathbb{A}$, E a cubic semisimple algebra. We write $\mathbb{B}_E := \mathbb{B} \otimes_F E$ the base changed quaternion algebra over $\mathbb{A}_E := \mathbb{A} \otimes_F E$.

Eisenstein series

For $\Phi \in \mathscr{S}(\mathbb{B}_E)$, we define

$$f_\Phi(g, s) = r(g)\Phi(0)\lambda_s(g)$$

where the character λ_s of P defined as

$$\lambda_s(d(\nu)n(b)m(a)) = |\nu|^{-3s} |\det(a)|^{2s}.$$

and it extends to a function on GSp_6 via Iwasawa decomposition $\mathrm{GSp}_6 = PK$ such that $\lambda_s(g)$ is trivial on K. It satisfies

$$f_\Phi(d(\nu)n(b)m(a)g, s) = |\nu|^{-3s-3} |\det(a)|^{2s+2} f_\Phi(g, s).$$

It thus defines a section, called a Siegel–Weil section, of $I(s) = \mathrm{Ind}_P^{\mathrm{GSp}_6}(\lambda_s)$. Then the Siegel–Eisenstein series is defined to be

$$E(g, s, \Phi) = \sum_{\gamma \in P(F)\backslash \mathrm{GSp}_6(F)} f_\Phi(\gamma g, s).$$

This is absolutely convergent when $\mathrm{Re}(s) > 2$. It extends to a meromorphic function of $s \in \mathbb{C}$ and holomorphic at $s = 0$ ([21, Thm. 2.2]).

For $T \in \mathrm{Sym}_3(F)$, we define its T-th Fourier coefficients to be:

$$E_T(g, s, \Phi) = \int_{[\mathrm{Sym}_3]} E(n(b)g, s, \Phi)\psi(-Tb)db. \tag{2.3.1}$$

(cf. "Notations" and we have shorten $\psi(T)$ for $\psi(tr(T))$ without confusion.)

Suppose $\Phi = \otimes_v \Phi_v$ is decomposable. When T is non-singular, we have a decomposition into a product of local Whittaker functions

$$E_T(g, s, \Phi) = \prod_v W_{T,v}(g_v, s, \Phi_v),$$

where the local Whittaker function is given by

$$W_{T,v}(g_v, 0, \Phi_v) = \int_{\mathrm{Sym}_3(F_v)} f_\Phi(wn(b)g, s)\psi(-Tb)db$$

where

$$w = \begin{pmatrix} & 1_3 \\ -1_3 & \end{pmatrix}.$$

By [21, Prop. 1.4], for non-singular T, the Whittaker function $W_{T,v}(g_v, s, \Phi_v)$ has an entire analytic extension to $s \in \mathbb{C}$. Moreover, under the following "unramified" conditions:

- v is non-archimedean, T is integral with $det(T) \in \mathscr{O}_{F_v}^\times$,
- The maximal ideal of F_v on which ψ_v is trivial is $\mathscr{O}_{F_v}$,
- V_v has a self-dual lattice Λ and Φ_v is the characteristic function of Λ_v,
- $g_v \in K_v$ the standard maximal compact subgroup of $\mathrm{GSp}(F_v)$,

we have [21, Prop. 4.1]:

$$W_{T,v}(g_v, s, \Phi_v) = \zeta_{F_v}(s+2)^{-1}\zeta_{F_v}(2s+2)^{-1}$$

Rankin triple L-function

Let σ be a cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_E)$. Let π be the associated to Jacquet-Langlands correspondence of σ on $\mathbb{B}_E^\times$. Let ω_σ be the central character of σ. We assume that

$$\omega_\sigma|_{\mathbb{A}_F^\times} = 1. \tag{2.3.2}$$

Define a finite set of places of F

$$\Sigma(\sigma) = \left\{ v | \epsilon(\sigma_v, \frac{1}{2}) = -1 \right\}. \tag{2.3.3}$$

Define the zeta integral as

$$Z(s, \phi, \varphi) = \int_{[\mathbb{G}]} E(g, s, \Phi)\varphi(g)dg \tag{2.3.4}$$

where $[\mathbb{G}] = \mathbb{A}^\times \mathbb{G}(F) \backslash \mathbb{G}(\mathbb{A})$.

Theorem 2.3.1 (Piatetski-Shapiro–Rallis [32]). *Assume that $\Phi = \otimes \Phi_v$ is decomposable. For a cusp form $\varphi \in \sigma$ and $\mathrm{Re}(s) \gg 0$ we have an Euler product*

$$Z(s, \Phi, \varphi) = \prod_v Z(s, \Phi_v, \varphi_v) = \frac{L(s + \frac{1}{2}, \sigma)}{\zeta_F(2s+2)\zeta_F(4s+2)} \prod_v \alpha(s, \Phi_v, \varphi_v) \tag{2.3.5}$$

where

$$\alpha(s, \Phi_v, W_{\varphi_v}) = \frac{\zeta_{F_v}(2s+2)\zeta_{F_v}(4s+2)}{L(s+\frac{1}{2}, \sigma_v)} Z(s, \Phi_v, \varphi_v)$$

which equals one for almost all v.

For σ_v a local component of an irreducible cuspidal automorphic representation, by Kim-Shahidi's work we have $\lambda(\sigma_v) < 1/2$. Hence the local zeta integral is absolutely convergent for all v at $s = 0$. At $s = 0$, the local zeta integral has already appeared earlier in this paper:

$$Z(\Phi_v, W_{\varphi_v}) = \int_{F^\times N_0 \backslash \mathbb{G}} f_{\Phi_v, s}(\eta g) W_{\varphi_v}(g) dg.$$

This constant is non-zero only if $\epsilon(\mathbb{B}_v) = \epsilon(\sigma_v, \frac{1}{2})$. By 2.2.3, the normalization local constant becomes

$$\alpha(0, \Phi_v, \varphi_v) = (-1)^{\mathrm{sgn}(B)} \zeta_F(2) m(\theta(\Phi_v \otimes \varphi_v). \tag{2.3.6}$$

Thus the global $Z(\Phi, \varphi, 1/2) \neq 0$ only if $\Sigma(\mathbb{B}) = \Sigma(\sigma)$ and both of them have even cardinality. In this case we have an identity:

$$\int_{[\mathbb{G}]} E(g, 0, \Phi)\varphi(g)dg = \frac{L(\frac{1}{2}, \sigma)}{\zeta_F(2)} \prod_v m(\theta(\Phi_v, \varphi_v)) \tag{2.3.7}$$

If $\Sigma = \Sigma(\sigma)$ is odd, $L(1/2, \sigma) = 0$. We have the following representation for the derivative:

$$\int_{[\mathbb{G}]} E'(g, 0, \Phi)\varphi(g)dg = -\frac{L'(\frac{1}{2}, \sigma)}{\zeta_F(2)} \prod_v m(\theta(\Phi_v, \varphi_v)) \tag{2.3.8}$$

2.4 Ichino's formula

In this subsection, we review a special value formula of Ichino. We assume that Σ is even. Let B be a quaternion algebra with ramification set Σ. We write V for the orthogonal space (B, q).

Siegel-Weil for similitudes

The theta kernel is defined to be, for $(g, h) \in R(\mathbb{A})$,

$$\theta(g, h, \Phi) = \sum_{x \in B_E} r(g, h)\Phi(x).$$

It is $R(F)$-invariant. The theta integral is the theta lifting of the trivial automorphic form, for $g \in \mathrm{GSp}_6^+(\mathbb{A})$,

$$I(g, \Phi) = \int_{[O(B_E)]} \theta(g, h_1 h, \Phi) dh$$

where h_1 is any element in $\mathrm{GO}(B_E)$ such that $\nu(h_1) = \nu(g)$. It dees not depend on the choice of h_1. When $B = M_{2\times 2}$ the integral needs to be regularized. The measure is normalized such that the volume of $[O(B_E)]$ is one.

$I(g, \Phi)$ is left invariant under $\mathrm{GSp}_6^+(\mathbb{A}) \cap \mathrm{GSp}_6(F)$ and trivial under the center $Z_{\mathrm{GSp}_6}(\mathbb{A})$ of GSp_6.

The following Siegel-Weil formula can be found [15, Thm. 4.2].

Theorem 2.4.1 (Siegel-Weil). *$E(g, s, \Phi)$ is holomorphic at $s = 0$ and*

$$E(g, 0, \Phi) = 2I(g, \Phi). \tag{2.4.1}$$

To eliminate the dependence of the choice of measure on $O(V)(\mathbb{A})$, we rewrite it as

$$E(g, 0, \Phi) = 2(\mathrm{vol}([O(V)]))^{-1} I(g, \Phi). \tag{2.4.2}$$

Now we deduce a formula for the T-th Fourier coefficient of the Siegel–Eisenstein series.

Corollary 2.4.2. *Assume that V is anisotropic and $det(T) \neq 0$. Then for $g \in \mathrm{GSp}^+(\mathbb{A})$ we have*

$$E_T(g, 0, \Phi) = 2\mathrm{vol}([O(V)_{x_0}]) \int_{O(V)(\mathbb{A})/O(V)_{x_0}(\mathbb{A})} r(g, h)\Phi(h_1^{-1} x_0) dh_1,$$

where $h \in \mathrm{GO}(V_\mathbb{A})$ has the same similitude as g, $x_0 \in V(F)$ is a base point with $Q(x_0) = T$, and $O(V)_{x_0} \simeq O(1)$ is the stabilizer of x_0.

Proof. $g_1 = d(\nu(g))^{-1} g$, We obtain by Siegel-Weil for similitudes:

$$\begin{aligned} E_T(g, 0, \Phi) &= 2 \int_{[\mathrm{Sym}_3]} \psi(-Tb) I(n(b)g, \Phi) db \\ &= 2 \int_{[\mathrm{Sym}_3]} \psi(-Tb) \int_{[O(V)]} \sum_{x \in V(F)} |\nu(g)|_\mathbb{A}^{-3} r(d(\nu(g)^{-1})n(b)g)\Phi(h^{-1} h_1^{-1} x) dh_1 db. \end{aligned}$$

Note that $d(\nu(g)^{-1})n(b)g = n(\nu(g)b)d(\nu(g)^{-1})g$. We thus have

$$r(d(\nu(g)^{-1})n(b)g)\Phi(h^{-1}h_1^{-1}x) = \psi(\nu(g)bQ(h^{-1}x))r(g_1)\Phi(h^{-1}h_1^{-1}x) = \psi(bQ(x))r(g_1)\Phi(h^{-1}h_1^{-1}x).$$

Since $[O(V)]$ is compact, we may interchange the order of integration. Then the integral is zero unless $T = Q(x)$. Since T is non-singular, by Witt theorem, the set of $x \in V(F)$ with $Q(x) = T$ is either empty or a single $O(V)(F)$-orbit. Fix a base point x_0. Then the stabilizer $O(V)_{x_0}$ of x_0 is isomorphic to $O(W)$ for the orthogonal complement W of the space spanned by the components of x_0. We now have

$$\begin{aligned} E_T(g, 0, \Phi) &= 2\int_{[O(V)]} \sum_{\gamma \in O(V)(F)/O(V)_{x_0}(F)} r(g_1)\Phi(h^{-1}h_1^{-1}\gamma^{-1}x_0)dh_1 \\ &= 2\mathrm{vol}([O(V)_{x_0}]) \int_{O(V)(\mathbb{A})/O(V)_{x_0}(\mathbb{A})} r(g_1)\Phi(h^{-1}h_1x_0)dh_1. \end{aligned}$$

This completes the proof. $\square$

We now define for non-singular T

$$I_T(g, \Phi) = 2\mathrm{vol}([O(V)_{x_0}]) \int_{O(V)(\mathbb{A})/O(V)_{x_0}(\mathbb{A})} r(g, h)\Phi(h_1^{-1}x_0)dh_1. \tag{2.4.3}$$

Or equivalently, let Ω_T be the set of elements x with $Q(x) = T$ and we may endow $\Omega_T(\mathbb{A})$ an $O(V)$-invariant measure denoted by $\mu_T(x)$ by identifying with $O(V)(\mathbb{A})/O(V)_{x_0}(\mathbb{A}) \cdot x_0$. So we have

$$I_T(g, \Phi) = 2\mathrm{vol}([O(V)_{x_0}]) \int_{\Omega_T(\mathbb{A})} r(g, h)\Phi(x)d\mu_T(x).$$

We also do so locally to define an $O(V)$-invariant measure to define

$$I_{T,v}(g_v, \Phi_v) = \int_{\Omega_T(F_v)} r(g_v, h_v)\Phi_v(x)d\mu_{T,v}(x)$$

and we then have

$$I_T(g, \Phi) = 2\mathrm{vol}([O(V)_{x_0}]) \prod_v I_{T,v}(g_v, \Phi_v),$$

when $\Phi = \otimes_v \Phi_v$ is decomposable.

We may identify it with μ_2 as an algebraic group. and therefore $O(V) = SO(V) \rtimes \mu_2$ (cf. Notations) where $\mu_2 \subset O(V)$ is generated by the canonical involution on the quaternion algebra. When T is non-singular, it is easy to see that $SO(V)$ is surjective onto $O(V)/O(V)_{x_0}$. We then may choose a measure on $O(V)(\mathbb{A})$ such that it is the product measure of the Tamagawa measure on $SO(V)(\mathbb{A})$ and the measure on $\mu_2(\mathbb{A})$ such that

$$\mathrm{vol}(\mu_2(\mathbb{A})) = 1.$$

Since the Tamagawa number of $SO(V)$ is 2, we have

$$\mathrm{vol}([O(V)]) = \frac{1}{2}\mathrm{vol}(SO(V)(F)\backslash O(V)(\mathbb{A})) = \frac{1}{2}\mathrm{vol}([SO(V)])\mathrm{vol}(\mu_2(\mathbb{A})) = 1.$$

Now we have

$$\mathrm{vol}(\mu_2(F)\backslash\mu_2(\mathbb{A})) = \frac{\mathrm{vol}(\mu_2(\mathbb{A}))}{|\mu_2(F)|} = \frac{1}{2}.$$

So we may rewrite

$$I_T(g,\Phi) = \prod_v I_{T,v}(g_v,\Phi_v), \tag{2.4.4}$$

where the local factor is a certain orbital integral:

$$I_{T,v}(g_v,\Phi_v) = \int_{SO(V)(F_v)} \Phi_v(hx_0)dh. \tag{2.4.5}$$

Moreover, the Siegel–Weil formula implies that

$$E_T(g,0,\Phi) = I_T(g,\Phi). \tag{2.4.6}$$

For later use we also need what we may call a *local Siegel–Weil*.

Proposition 2.4.3. *Suppose that $T \in \mathrm{Sym}_3(F_v)$ is non-singular. Then there is a non-zero constant κ such that for all $g_v \in \mathrm{GSp}_6(F_v)$, $\Phi_v \in \mathscr{S}(V_v^3)$*

$$W_{T,v}(g_v,0,\Phi_v) = \kappa \cdot I_{T,v}(g_v,\Phi_v).$$

In particular, the functional $\Phi_v \mapsto W_{T,v}(1,0,\Phi_v)$ is non-zero if and only if T is represented by V_v.

Proof. It suffices to prove the statement for $g_v = 1$. Consider the space of linear functionals ℓ on $\mathscr{S}(V_v^3)$ that satisfy

$$\ell(r(n(b))\Phi_v) = \psi(Tb)\ell(\Phi_v).$$

Then by [21, Prop. 1.2], this space is spanned by $\Phi_v \mapsto I_{T,v}(1,\Phi_v)$ (whose definition depends on the normalization of the measure $d\mu_{T,v}$). Since $\Phi_v \mapsto W_{T,v}(1,0,\Phi_v)$ also satisfies this relation, it defines a multiple of the linear functional $I_{T,v}(1,\cdot)$ above. The multiple can be chosen to be non-zero by [21, Prop. 1.4 (ii)]. □

Special value formula

Theorem 2.4.4 (Ichino [16])**.** *Let $dg = \prod_v dg_v$ be the Tamagawa measure on $B_F^\times \setminus B_{\mathbb{A}}^\times$. For $f = \otimes_v f_v \in \pi$, $\widetilde{f} = \otimes_v \widetilde{f}_v \in \widetilde{\pi}$, we have*

$$P_\pi(f)P_{\widetilde{\pi}}(\widetilde{f}) = \frac{1}{2^c}\frac{\zeta_E(2)}{\zeta_F(2)}\frac{L(\frac{1}{2},\sigma)}{L(1,\sigma,ad)}m(f,\widetilde{f}).$$

Here the constant c is $3, 2,$ and 1 respectively if $E = F\oplus F\oplus F$, $E = F\oplus K$ for a quadratic K, and a cubic field extension E of F respectively.

Proof. Without loss of generality we may assume that $f \otimes \widetilde{f} = \theta(\Phi \otimes \varphi)$ is the normalized theta lifting. Then we have

$$P_\pi(f)P_{\widetilde{\pi}}(\widetilde{f}) = \int_{[B^\times]\times[B^\times]} \theta(\Phi \otimes \varphi)(x,y)dxdy.$$

We recall some results of Harris–Kudla. When the measures are normalized such that the volume of $[B^\times]$, $[GO(B_E)]$ and $[\mathbb{G}]$ are equal to one, Harris–Kulda [15] proved that the seesaw identity, the uniqueness of Prasad and Loke together give

$$\int_{[B^\times]\times[B^\times]} \theta(\Phi \otimes \varphi)(x,y)dxdy = C\int_{[\mathbb{G}]} I(g,\Phi)\varphi(g)dg,$$

where the constant

$$C = \frac{1}{2^c}\frac{\zeta_E(2)}{L(1,\sigma,Ad)}$$

is used in the normalization of the theta lifting. Together with Sigel–Weil formula, we have

$$\int_{[B^\times]\times[B^\times]} \theta(\Phi \otimes \varphi)(x,y)dxdy = \frac{1}{2}C\int_{[\mathbb{G}]} E(g,0,\Phi)\varphi(g)dg.$$

To allow us to change measures, we may rewrite the formula as

$$P_\pi(f)P_{\widetilde{\pi}}(\widetilde{f}) = \mathrm{vol}([B^\times])^2(\mathrm{vol}([\mathbb{G}]))^{-1}\frac{1}{2}C\int_{[\mathbb{G}]} E(g,0,\Phi)\varphi(g)dg.$$

Now with our choice of Tamagawa measures, we have $\mathrm{vol}([B^\times]) = \mathrm{vol}([\mathbb{G}]) = 2$ and hence

$$P_\pi(f)P_{\widetilde{\pi}}(\widetilde{f}) = C\int_{[\mathbb{G}]} E(g,0,\Phi)\varphi(g)dg.$$

By (2.3.6), this implies that

$$P_\pi(f)P_{\widetilde{\pi}}(\widetilde{f}) = I(\theta(\Phi,\varphi)) = C\frac{L(\frac{1}{2},\sigma)}{\zeta_F(2)}\prod_v m(\theta_v(\Phi_v,\varphi_v)) \tag{2.4.7}$$

Since $\theta = \bigotimes_v \theta_v$, plugging in the value of C we obtain

$$P_\pi(f)P_{\widetilde{\pi}}(\widetilde{f}) = \frac{1}{2^c}\frac{\zeta_E(2)}{\zeta_F(2)}\frac{L(\frac{1}{2},\sigma)}{L(1,\pi,ad)}\prod_v m(f_v,\widetilde{f}_v).$$

□

We have the following consequence on the special values of triple product L-series:

Theorem 2.4.5. *Let F be an number field and E/F be a cubic semisimple algebra. For any cuspidal automorphic representation σ of $\mathrm{GL}_2(\mathbb{A}_E)$ with central character $\omega|_{\mathbb{A}^\times} = 1$, we have*

1. *(Positivity)*

$$L(\frac{1}{2},\sigma)\geq 0$$

2. *(Jacquet's conjecture) $L(\frac{1}{2},\sigma)\neq 0$ if and only if there exists (uniquely determined) quaternion algebra B over F such that the period*

$$\int_{[B^\times]} f(b)db\neq 0$$

for some $f\in\Pi_{B,E}$, the Jacquet-Langlands correspondence of σ.

Proof. These trivially follow from local results above and the global period formula

$$\frac{|\int_{[B^\times]}f(b)db|^2}{(f,f)_{Pet}}=C\cdot L(\frac{1}{2},\sigma)\prod_v\alpha_v(f_v,f_v)$$

where $C>0$ is an explicit real number and α is proportional to I_v by a positive multiple such that $\alpha_v=1$ for almost all v. $\square$

Remark 2.4.1. The non-vanishing and positivity of the matrix coefficient integral is conjectured to be true for all pair $(SO(n),SO(n+1))$ in the refinement of Gross-Prasad conjecture by Ichino-Ikeda. One consequence of the non-vanishing and positivity (together with the global period formula) is the positivity of the central value of L-function.

2.5 Derivatives of Eisenstein series

Fix an *incoherent* quaternion algebra $\mathbb{B}$ over $\mathbb{A}$ with ramification set Σ. We assume that $\mathbb{B}$ has totally positive component $\mathbb{B}_v$ at archimedean places. We consider the Eisenstein series $E(g,s,\Phi)$ for $\Phi\in\mathscr{S}(\mathbb{B}^3)$. We always take Φ_∞ to be standard Gaussian. In this case this Eisenstein series vanishes at $s=0$ as observed by Kudla [21, Thm. 2.2(ii)]. The vanishing of a non-singular T-th Fourier coefficient is easier to see as we now discuss.

For $T\in\mathrm{Sym}_3(F)_{reg}$, let $\Sigma(T)$ be the set of places over which T is anisotropic. Then $\Sigma(T)$ has even cardinality. By Prop. 2.4.3, the vanishing order of the T-th Fourier coefficient $E_T(g,s,\Phi)$ at $s=$ is at least

$$|\Sigma\cup\Sigma(T)|-|\Sigma\cap\Sigma(T)|.$$

Also cf. [21, Coro. 5.3]. Since $|\Sigma|$ is odd, $E_T(g,s,\Phi)$ always vanishes at $s=0$. And its derivative is non-vanishing only if Σ and $\Sigma(T)$ is nearby: they differ by precisely one place v. Thus we define

$$\Sigma(v)=\begin{cases}\Sigma\setminus\{v\} & \text{if } v\in\Sigma\\ \Sigma\cup\{v\} & \text{otherwise}\end{cases}$$

When $\Sigma(T)=\Sigma(v)$, the derivative is given by

$$E'_T(g,0,\Phi)=\prod_{w\neq v}W_{T,w}(g_w,0,\Phi_w)\cdot W'_{T,v}(g_v,0,\Phi_v).$$

We thus obtain a decomposition of $E'(g,0,\Phi)$ according to the difference of $\Sigma(T)$ and Σ:

$$E'(g,0,\Phi) = \sum_v E'_v(g,0,\Phi) + E'_{sing}(g,0,\Phi) \tag{2.5.1}$$

where

$$E'_v(g,0,\Phi) = \sum_{\Sigma(T)=\Sigma(v)} E'_T(g,0,\Phi) \tag{2.5.2}$$

and

$$E'_{sing}(g,0,\Phi) = \sum_{T,\,\det(T)=0} E'_T(g,0,\Phi).$$

Weak Intertwining property

In the case where Σ is odd, the formulation $\Phi \mapsto E'(g,0,\Phi)$ is not equivariant under the action of $Sp_6(\mathbb{A})$. The following gives a weak intertwining property:

Proposition 2.5.1. *Let $\mathscr{A}(G)_0$ be the image of $\Pi(B_{\mathbb{A}})$ under the map $f \mapsto E(g,0,f)$ for all quaternion algebra B over F. Then for any $h \in G(\mathbb{A})$, $f \in I(0)$, the function*

$$g \mapsto E'(gh,0,f) - E'(g,0,r(h)f) \in \mathscr{A}_0.$$

Proof. Let $\alpha(s,h)(g) = \alpha(s,g,h) = \frac{1}{s}(|\frac{\delta(gh)}{\delta(g)}|^s - 1), s \neq 0$. Then it obviously extends to an entire function of s and it is left $P_{\mathbb{A}}$-invariant. Now for $\mathrm{Re}(s) \gg 0$, we have

$$E(gh,s,f) - E(g,s,r(h)f) = sE(g,s,\alpha(s,h)r(h)f)$$

Now note that the section $g \to \alpha(s,h)r(h)f(g)\delta(g)^s$ is a holomorphic section of $I(s)$. Hence the Eisenstein series $E(g,s,\alpha(s,h)r(h)f)$ is holomorphic at $s=0$ since any holomorphic section of $I(s)$ is a finite linear combination of standard section with holomorphic coefficients. This implies the desired assertion.

Similarly one can prove the $(\mathscr{G},K)$-intertwining if v_1 is archimedean. We skip this and refer to [23]. □

3 Gross–Schoen cycles and generating series

In this section we construct the geometric kernel function for $\Phi \in \mathscr{S}(\mathbb{B}^3)$ where $\mathbb{B}$ is an incoherent totally definite quaternion algebra over a totally real field F. We first give a decomposition 3.1.1 for codimension 2 cycles on the triple curves and their Bloch-Beilinson height pairing following the Gross–Schoen [13]. Then we define the direct and inverse limits of these cycles on the product of Shimura curves as representation of Hecke operators. This allows us to reformulate our main conjecture 1.2.3 in terms of Hecke operators acts on modified diagonal P of Gross and Schoen, see Conjecture 3.2.1. Then we review the generating series of Hecke operators and its modularity on product of Shimura curves associate to $\mathbb{B}$, see Proposition 3.3.1 following our previous paper [35]. The main conjeture can then be further reformulated as a kernel identity betweenthe derivative of Eisenstein series and geometric kernel associate Φ, see Conjecture 3.5.1. Finally, we introduce arithematic Hodge classes and arithemtic Hecke operators which gives a decomposition of the geometric kernel function to a sum of local heights and singular parirings.

3.1 Cycles on triple product of curves

Decomposition of cycles

Let C_i $(i = 1, 2, 3)$ be three smooth, projective, and connective curves over a number field k. Let $V = C_1 \times C_2 \times C_3$. We want to study the group of codimension 2 cycles $\mathrm{Ch}^2(V)$. First of all, let us define a filtration as follows:

$$\mathrm{Ch}^2(V) \supset \mathrm{Ch}^{2,1}(V) \supset \mathrm{Ch}^{2,2}(V) \supset \mathrm{Ch}^{2,3}(V).$$

For $\mathrm{Ch}^{2,1}(V)$, we consider the class map

$$\mathrm{Ch}^2(V) \longrightarrow H^4(\bar{V}, \mathbb{Q}_\ell)$$

where the right hand side denotes the ℓ-adic cohomology of $V := V_{\bar{k}}$. Let $\mathrm{Ch}^{2,1}(V)$ and $N^2(V)$ denote the kernel and image respectively then we have an exact sequence

$$0 \longrightarrow \mathrm{Ch}^{2,1}(V) \longrightarrow \mathrm{Ch}^2(V) \longrightarrow N^2(V) \longrightarrow 0.$$

The space $N^2(V)$ has dimension predict by Tate's conjecture.

For $\mathrm{Ch}^{2,2}(V)$, we consider the Kunneth decomposition of

$$H^4(\bar{V}, \mathbb{Q}_\ell) = \oplus_{i+j+k=4} H^i(\bar{C}_1) \otimes H^j(\bar{C}_2) \otimes H^k(\bar{C}_k).$$

It is immediate that a class in $N^2(V)$ is zero if and only if it has zero projection to all $H^2(C_i \times C_j)$. Thus we define a subgroup $\mathrm{Ch}^{2,2}(V)$ consisting of elements in $\mathrm{Ch}^2(V)$ with 0 projection to all $\mathrm{Ch}^1(V_i \times V_j)$. The quotient $\mathrm{Ch}^{2,1}(V)/\mathrm{Ch}^{2,2}(V)$ is isomorphic to the direct sum of homologically trivial cycles on $C_i \times C_j$:

$$\bigoplus_{i<j} \mathrm{Ch}^1(C_i \times C_j)^0 = \otimes_i \mathrm{Pic}^0(C_i)^{\oplus 2}$$

where we have used the identity

$$\mathrm{Ch}^1(C_i \times C_j)^0 \simeq \mathrm{Pic}^0(C_i) \oplus \mathrm{Pic}^0(C_j).$$

These groups are finitely generated and taking cared by the BSD conjecture on curves. See Zhang [39], Lemma 5.1.2.

The last subspace $\mathrm{Ch}^{2,3}(V)$ is defined to be generated by elements of the form

$$\pi_i^*\alpha_i \cdot \pi_j^*\alpha_j, \qquad i < j, \quad \alpha_i \in \mathrm{Pic}^0(C_i).$$

By a conjecture of Beilinson–Bloch, this subgroup is finite.

In the following we would like to give a spliting for the above filtration by choosing classes $e_i \in \mathrm{Pic}^1(C_i)_{\mathbb{Q}}$ with rational coefficient and degree 1:

$$e_i = \sum a_{ij} p_j, \qquad \sum_j a_{ij} \deg p_j = 1.$$

For each $i < j$, let $\mathrm{Pic}^-(C_i \times C_j)$ denote the class α in $\mathrm{Pic}(C_i \times C_j)_{\mathbb{Q}}$ such that

$$\pi_{i*}(\alpha \cdot \pi_j^* e_j) = 0, \qquad \pi_{j*}(\alpha \cdot \pi_i^* e_i) = 0.$$

By Zhang [39], Lemma 2.2.1, we have a decomposition given by canonical maps:

$$\mathrm{Pic}^0(C_i \times C_j)_{\mathbb{Q}} = \pi_i^*\mathrm{Pic}^0(C_i)_{\mathbb{Q}} \oplus \pi_j^*\mathrm{Pic}^0(C_j)_{\mathbb{Q}}.$$

$$\mathrm{NS}(C_i \times C_j)_{\mathbb{Q}} \simeq \mathbb{Q}\pi_i^* e_i \oplus \mathbb{Q}\pi_j^* e_j \oplus \mathrm{Pic}^-(C_i \otimes C_j)_{\mathbb{Q}}.$$

By Kunneth formula, the class $N^2(V)$ is generated by $\pi_i^* e_i \otimes \mathrm{NS}(C_j \times C_k)_{\mathbb{Q}}$, it follows the isomorphism given by canonical maps:

$$\mathrm{Ch}^2(V)_{\mathbb{Q}}/\mathrm{Ch}^{2,1}(V)_{\mathbb{Q}} = \bigoplus_{i<j} \left(\mathbb{Q}\pi_i^* e_i \cdot \pi_j^* e_j \oplus \mathbb{Q}\mathrm{Pic}^-(C_i \times C_j) \cdot \pi_k^* e_k\right).$$

Similarly, by the proof in Zhang [39], Lemma 5.1.2,

$$\mathrm{Ch}^{2,1}(V)_{\mathbb{Q}}/\mathrm{Ch}^{2,2}(V)_{\mathbb{Q}} = \bigoplus_{i \neq j} \mathrm{Pic}^0(C_i)_{\mathbb{Q}} \cdot \pi_j^* e_j.$$

Finally, we define $\mathrm{Ch}^2(V)^{00}$ to be subgroup of $\mathrm{Ch}^{2,2}(V)$ consists of elements α such that

$$\pi_{ij*}(\alpha \cdot \pi_k^* e_k) = 0, \qquad i < j$$

in $\mathrm{Ch}^2(C_i \times C_j)$. Then the canonical map gives an isomorphism:

$$\mathrm{Ch}^{2,2}(V)_{\mathbb{Q}}/\mathrm{Ch}^{2,3}(V)_{\mathbb{Q}} \simeq \mathrm{Ch}^2(V)^{00}.$$

In summary, we have a decomposition

$$\mathrm{Ch}^2(V)_{\mathbb{Q}} = \mathrm{Ch}^2(V)^{00}_{\mathbb{Q}} \oplus \bigoplus_{ij} \mathrm{Pic}^0(C_i)_{\mathbb{Q}} e_j \oplus \bigoplus_{i<j} \left(\pi^* \mathrm{Pic}^0(C_i) \cdot \pi_j^* \mathrm{Pic}^0(C_j) \oplus \mathbb{Q} e_i e_j \oplus \mathrm{Pic}^-(C_i \times C_j)_{\mathbb{Q}} e_k\right). \tag{3.1.1}$$

For each i, let T_i be the subspace of $\alpha \in \mathrm{Ch}^1(C_i \times C_i)$ consisting of elements α such that both α_* and α^* fixes the line $\mathbb{Q}e_i$. Then

$$T_i = \mathbb{Q}(e_i \times C_i) + \mathbb{Q}(C_i \times e_i) + \mathrm{Pic}^-(C_i \times C_i).$$

Under the composition, this algebra is isomorphic to the direct sum

$$T_i = \mathbb{Q} \oplus \mathbb{Q} \oplus \mathrm{End}(J_i)_{\mathbb{Q}} \tag{3.1.2}$$

where J_i is the Jacobian of C_i. The actions of T_i's on $\mathrm{Ch}^*(C_i)$ by pulling back preserve the decomposition

$$\mathrm{Ch}^*(C_i)_{\mathbb{Q}} = \mathrm{Ch}^0(C_i)_{\mathbb{Q}} \oplus \mathbb{Q}e_i \oplus \mathrm{Pic}^0(C_i)_{\mathbb{Q}}.$$

Consequently T_i's act on $\mathrm{Ch}^2(V)_{\mathbb{Q}}$ by pullbacks and fix the decomposition 3.1.1. The subspace $\mathrm{Ch}^2(V)^{00}$ is the subspace where all $C_i \times e_i$ and $e_i \times C_i$ acts as 0.

For a codimension 2 cycle Z on V with projection Z_{ij} on $C_i \times C_j$ and $m_k C_k$ on C_k, its decomposition can be performed as follows. First of all, its projection onto $\mathrm{Ch}^2(V)^{2,2}$ is given by

$$Z^{2,2} = Z - \sum_{i<j} Z_{ij} e_k + \sum_k m_k e_i e_j.$$

It has projection

$$Z^{2,3} := \sum_{i,j} \pi_{ij}^* \pi_{ij*}(Z \cdot \pi_k^* e_k)$$

in $\mathrm{Ch}^{2,3}(V)$ and

$$Z^{00} = Z^{2,2} - Z^{2,3}(V)$$

in $\mathrm{Ch}^2(V)^{00}$.

The cycle Z_{ij} has projection Z_{ij}^- on $\mathrm{Pic}^-(C_i \times C_j)$ given by

$$Z_{ij}^- = Z_{i,j} - C_i \times Z_{i,j}(e_i) - Z_{ij}^*(e_j) \times C_j.$$

The cycle $Z_{ij}(e_i)$ has projection on $\mathrm{Pic}^0(C_j)$ given by

$$Z_{ij}(e_i) - \deg Z_{ij}(e_i) e_j.$$

Height pairing

In the following we want to define a height pairing on homologically trivial cycles $\mathrm{Ch}^{2,1}(V)$ following Bloch [4], Beilinson [2, 3], Gillet–Soulé [9], and Gross and Schoen [13].

Let $\mathscr{V}$ be a regular and integral model of V over a the ring of integers $\mathscr{O}_{k'}$ for some finite extension k' of k. For example, we may choose k' such that $\mathscr{C}_i$ has a regular and semistable model $\mathscr{C}_i$ over $\mathscr{O}_{k'}$. Then we may blow-up

$$\mathscr{V}' := \mathscr{C}_1 \times_{\mathscr{O}_{k'}} \mathscr{C}_2 \times_{\mathscr{O}_{k'}} \mathscr{C}_3$$

successively at irreducible components over singular fiber with any given order (cf. [13]). Then for each cycle $Z \in \mathrm{Ch}^{2,1}(V)$ we are going to construct an arithmetic cycle $\widehat{Z} = (\widetilde{Z}, g_Z)$ where

- $\widetilde{Z}$ is an extension of Δ_e over $\mathscr{V}$ over $\mathrm{Spec}\mathscr{O}_k$;
- g_Z is a Green's current on the complex manifold $V(\mathbb{C})$ of the complex variety $V \otimes_{\mathbb{Q}} \mathbb{C}$ for the cycle Z: g_Z is a current on $V(\mathbb{C})$ of degree $(1,1)$ with singularity supported on $Z(\mathbb{C})$ such that the curvature equation holds:

  $$\frac{\partial\bar{\partial}}{\pi i} g_Z = \delta_{Z(\mathbb{C})}.$$

  Here the right hand side denotes the Dirac distribution attached to the cycle $Z(\mathbb{C})$ when integrating with forms of degree $(2,2)$ on $V(\mathbb{C})$.

The height pairing of two cycle Z_1, Z_2 in $\mathrm{Ch}^{2,1}(V)$ is defined to

$$\langle Z_1, Z_2 \rangle = \frac{1}{[k' : k]} \widehat{Z}_1 \cdot \widehat{Z}_2.$$

It is clear that this definition does not depend on the choice of $\widehat{Z}$'s.

Before construction of cycles $\widehat{Z}$, we want to also show that this definition does not depend on the choice of k' and model $\mathscr{V}$ using de Jong's alteration. In fact for any regular, projective, flat scheme, $\mathscr{X}$ over $\mathbb{Z}$ (not necessarily geometrically connected), let $\widehat{\mathrm{Ch}}(\mathscr{X})^0$ denote the quotient of groups of cycles $\widehat{Z}$ which are numerically trivial over all fibers modulo the subgroup of cycles which vanishes on the generic fiber. Then we want show that this group depends only on the generic fiber X of $\mathscr{X}$. More generally, if X and Y are two projective varieties over $\mathbb{Q}$ with regular models $\mathscr{X}$ and $\mathscr{Y}$, and a generically finite morphism $f : X \longrightarrow Y$, then we can define maps extending the corresponding maps over generic fiber:

$$f_* : \widehat{\mathrm{Ch}}(\mathscr{X})^0 \longrightarrow \widehat{\mathrm{Ch}}(\mathscr{Y})^0, \qquad f^* : \widehat{\mathrm{Ch}}(\mathscr{Y})^0 \longrightarrow \widehat{\mathrm{Ch}}(\mathscr{Y})^0$$

such that $f_* \circ f^* = \deg f$.

To define these maps, first we notice that the Zariski closure $\mathscr{X}'$ of the graph of f in $X \times Y$ is another model of X dominating both $\mathscr{X}$ and $\mathscr{Y}$. By de Jong's alteration, there is a regular scheme $\mathscr{X}''$ dominating $\mathscr{X}'$: this we have generically finite morphisms:

$$\alpha : \mathscr{X}'' \longrightarrow \mathscr{X}, \qquad \beta : \mathscr{X}'' \longrightarrow \mathscr{Y}.$$

Now we define

$$f_* := \frac{1}{[\mathscr{X}'':\mathscr{X}]}\beta_*\circ\alpha^*, \qquad f^* = \frac{1}{[\mathscr{X}'':\mathscr{X}]}\alpha_*\circ\beta^*.$$

Now we go back to construction of $\widehat{Z}$. From the decomposition above, we see that $\mathrm{Ch}^{2,1}(V)$ is generated by cycles of the forms in two types:

- $\pi_{ij}^*D_{ij}\cdot\pi_k^*D_k$ with D_{ij} a divisor on $C_i\times C_j$ and D_k a divisor on C_k of degree 0;
- $C^{2,1}$ where C is a curve on V which has finite degree under each projection $\pi_i: V\longrightarrow C_i$.

If Z is of first type, then over a semistable model $\mathscr{C}_i$, D_i can be extend to an arithmetic model $\widehat{D}_i$ which has degree 0 on each component in the special fiber and curvature 0 at all archimedean places. Let $\widehat{\pi_{ij}^*D_{ij}}$ be any integral model of $\pi_{ij}^*D_{ij}$ on $\mathscr{Y}$. Then we may define

$$\widehat{Z} = \widehat{\pi_{ij}^*D_{ij}}\cdot\pi_k^*\widehat{D}_i.$$

If $Z = C^{2,1}$ is of second type, we consider the morphism

$$p: \quad C\times C\times C\longrightarrow V.$$

Let e be base point e on C which exits if k is replaced by a finite extension, and let Δ_e be the modified diagonal cycle defined in Gross–Schoen [13]. In our terminology, $\Delta_e = \Delta^{2,1}$ for the diagonal Δ on $C\times C\times C$ and base (e,e,e). Then $Z - p_*\Delta_e$ is of first type. Thus to construct $\widehat{Z}$ it suffices to construct $\widehat{\Delta}_e$ on a model $\mathscr{Y}$ of $C\times C\times C$ which has been done by Gross–Schoen [13].

In the following, we want to recall a triple product for Gross–Schoen cycles Δ_e. Let $t_i\in T$ $(=1,2,3)$ be three classes fixing $\mathbb{Q}e$ defined as in 3.1.2, and let t_i^- be its projection on $\mathrm{Pic}^-(C\times C)$. Then by Lemma 2.2.3 in [39], each t_i^- can be extended into a unique so called admissible arithmetic class $\widehat{t_i^-}$ such that following holds:

1. $\widehat{t_i^-}$ has zero intersection with components in the fibers over closed points for the two projects $C\times C\longrightarrow C$;
2. $\widehat{t_i^-}$ is trivial on $\{e\}\times C$ and on $C\times\{e\}$.

By formula (2.3.5) in [39], we have the following formula

$$\langle\Delta_e,(t_1\times t_2\times t_3)^*\Delta_e\rangle = \widehat{t_1^-}\cdot\widehat{t_2^-}\cdot\widehat{t_3^-} \tag{3.1.3}$$

where the right hand side is the arithmetic intersection numbers on $C\times C$.

3.2 Cycles on product of Shimura curves

In the following, we want to prove the decomposition formula given in the introduction.

First decomposition

Recall that Shimura curve Y is a projective limit of the curves Y_U which is a disjoint union of curves $Y_{U,\alpha}$ parameterized by $\alpha \in \pi_0(Y_U)$. Let $\xi_{U,\alpha}$ denote the Hodge class on $Y_{U,\alpha}$ of degree 1. Let $\mathrm{Pic}^\xi(Y_U)$ denote the subspace of $\mathrm{Pic}(Y_U)_\mathbb{Q}$ generated by $\xi_{U,i}$'s. Then we have a decomposition

$$\mathrm{Pic}(Y_U)_\mathbb{Q} = \mathrm{Pic}^0(Y_U)_\mathbb{Q} \oplus \mathrm{Pic}^\xi(Y_U)_\mathbb{Q}.$$

Let us define $\mathrm{Ch}^2(X_{U_E})^{00}$ to be the direct sum of the corresponding group for the product of components of X_U. Then the decomposition 3.1.1 implies

$$\begin{aligned}\mathrm{Ch}^2(X_{U_E})_\mathbb{Q} =& \mathrm{Ch}^2(X_{U_E})^{00}_\mathbb{Q} \oplus \bigoplus_{i\neq j} \pi_i^*\mathrm{Pic}^0(Y_U)_\mathbb{Q}\cdot \pi_j^*\mathrm{Pic}^\xi(Y_U)_\mathbb{Q}\cdot \pi_k^*\mathrm{Ch}^0(Y_U)\oplus \\ & \bigoplus_{i<j} \pi_i^*\mathrm{Pic}^0(Y_U)\cdot \pi_j^*\mathrm{Pic}^0(Y_U)\cdot \mathrm{Ch}^0(Y_U)\oplus \\ & \bigoplus_{i<j} \pi_i^*\mathrm{Pic}^\xi(Y_U)_\mathbb{Q}\cdot \pi_j^*\mathrm{Pic}^\xi(Y_U)\cdot \pi_k^*\mathrm{Ch}^0(Y_U)\oplus \\ & \bigoplus_{i<j} \pi_{ij}^*\mathrm{Pic}^-(Y_U\times Y_U)_\mathbb{Q}\cdot \pi_k^*\mathrm{Pic}^\xi(Y_U)_\mathbb{Q}.\end{aligned} \tag{3.2.1}$$

Taking direct limit gives

$$\begin{aligned}\mathrm{Ch}^2(X)_\mathbb{Q} =& \mathrm{Ch}^2(X)^{00}_\mathbb{Q} \oplus \bigoplus_{i\neq j} \pi_i^*\mathrm{Pic}^0(Y)_\mathbb{Q}\cdot \pi_j^*\mathrm{Pic}^\xi(Y)_\mathbb{Q}\cdot \pi_k^*\mathrm{Ch}^0(Y)\oplus \\ & \bigoplus_{i<j} \pi_i^*\mathrm{Pic}^0(Y)\cdot \pi_j^*\mathrm{Pic}^0(Y)\cdot \mathrm{Ch}^0(Y)\oplus \\ & \bigoplus_{i<j} \pi_i^*\mathrm{Pic}^\xi(Y)_\mathbb{Q}\cdot \pi_j^*\mathrm{Pic}^\xi(Y)\cdot \pi_k^*\mathrm{Ch}^0(Y)\oplus \\ & \bigoplus_{i<j} \pi_{ij}^*\mathrm{Pic}^-(Y\times Y)_\mathbb{Q}\cdot \pi_k^*\mathrm{Pic}^\xi(Y)_\mathbb{Q}.\end{aligned} \tag{3.2.2}$$

Hecke algebra

It remains to decompose $\mathrm{Ch}^2(X)$. First we use Hecke operators to reduced to finite level.

Recall that the Hecke algebra $\mathscr{H}_E := C_c^\infty(\mathbb{B}_E^\times/D_E)$ consists of smooth and compactly-supported functions $\Phi : \mathbb{B}_E^\times/D_E \to \mathbb{C}$. Its multiplication is given by the convolution

$$(\Phi_1 * \Phi_2)(h) := \int_{\mathbb{B}_E^\times/D_E} \Phi_1(h')\Phi_2(h'^{-1}h)dh'.$$

For any smooth representation (V,ρ) of $\mathbb{B}_E^\times/D_E$, there is a representation

$$\mathscr{H} \longrightarrow \mathrm{End}(V)$$

given by

$$\rho(f)v = \int_{\mathbb{B}_E^\times/D_E} f(g)\rho(g)v dg.$$

Fix an open compact subgroup U_E of $\mathbb{B}_E^\times/D_E$. Denote

$$\mathscr{H}_{U_E} = C_c^\infty(U\backslash\mathbb{B}_E^\times/D_E U_E) := \{\Phi \in C_c^\infty(\mathbb{B}_E^\times/D_E) : \Phi(U_E h U_E) = \Phi(h),\ \forall h \in \mathbb{B}_E^\times\}.$$

It is a subalgebra of $\mathscr{H}_E$ whose multiplicative unit is the characteristic function $\mathrm{vol}(U_E)^{-1}1_U$. For any smooth representation V of $\mathbb{B}_E^\times/D_E$, the action of $\mathscr{H}_{U_E}$ stabilizes V^{U_E}, the subspace of vectors in V fixed by U_E. The study of decomposition of V under $\mathbb{B}_E^\times/D_E$ is equivalent to the study of the representation of V^{U_E} under $\mathscr{H}_{E,U_E}$. In particular, the correspondence $\sigma \longrightarrow \sigma^{U_E}$ is bijection between the set of irreducible representations of $\mathbb{B}_E^\times/D_E$ with nontrivial U_E-invariants and the set of nonzero irreducible representations of $\mathscr{H}_{E,U_E}$.

We may define the similar algebras $\mathscr{H}$ and $\mathscr{H}_U$ of functions on $\mathbb{B}^\times/D$. Then we have

$$\mathscr{H}_E = \mathscr{H}^{\otimes 3}, \qquad \mathscr{H}_{E,U_E} = \mathscr{H}_U^{\otimes 3}$$

for $U_E = U^3$.

For each $\Phi \in \mathscr{H}_E$, lets us define

$$\mathrm{T}(\Phi) := \int_{\mathbb{B}^\times/D} \Phi(x)\mathrm{T}(x) \in \mathrm{End}(\mathrm{Ch}^2(X)_{\mathbb{C}}).$$

For U_E a compact and open subgroup of $\mathbb{B}_E^\times/D_E$, then $\mathrm{Ch}^2(X_{U_E})_{\mathbb{C}}$ has actions by $\mathscr{H}_{U_E} = \mathscr{H}_U^{\otimes 3}$. It is clear that every element in each component $\mathscr{H}_U$ fixes the base class ξ_U. Thus the actions fix the decomposition 3.2.1 and 3.2.2 and factor through the quotient

$$\mathscr{H}_U \longrightarrow \mathrm{End}(\mathrm{Ch}^0(Y_U)_{\mathbb{Q}}) \oplus \mathrm{End}(\mathrm{Pic}^\xi(Y_U)_{\mathbb{Q}}) \oplus \mathrm{End}(\mathrm{Pic}^0(Y_U)_{\mathbb{Q}}).$$

The actions of $\mathscr{H}_U$ on $\mathrm{Ch}^0(Y_U)$ and $\mathrm{Pic}^\xi(Y_U)$ are both factor through actions on functions on $\pi^0(Y_U)$, and the action of $\mathscr{H}_U$ on $\mathrm{Pic}^0(Y_U)$ factor through its action on the space of cusp forms. Thus the right hand side of the above quotient is isomorphic to

$$\mathscr{H}_U' := \bigoplus_\rho \rho^U \otimes \widetilde{\rho}^U$$

where ρ runs through automorphic characters and automorphic cuspidal representations of parallel weight 2. Since the right hand side is of finite dimensional, any representation V of $\mathscr{H}_U'$ will have decomposition

$$V = \bigoplus_\rho \mathrm{Hom}_{\mathscr{H}_U'}(\rho^U, V) \otimes \rho^U.$$

Applying these to $\mathrm{Ch}^2(X_{U_E}$ and $\mathrm{Ch}^2(X)$, we obtain decompositions

$$\mathrm{Ch}^2(X_{U_E})_{\mathbb{C}} = \bigoplus_\rho \mathrm{Ch}^2(\rho) \otimes \rho^U,$$

$$\mathrm{Ch}^2(X)_{\mathbb{C}} = \bigoplus_{\rho} \mathrm{Ch}^2(\rho) \otimes \rho$$

where $\rho = \rho_1 \otimes \rho_2 \otimes \rho_3$ runs through automorphic representations of $\mathbb{B}_E^\times$ such that each ρ_i is either one dimensional or cuspidal of parallel weight 2. The sum over cuspidal ρ gives

$$\mathrm{Ch}^2(X)^{00} = \bigoplus_{\rho:\text{cuspidal}} \mathrm{Ch}^2(\rho) \otimes \rho.$$

If we normalize the height paring on $\mathrm{Ch}^{2,1}(X_U)$ by a factor $\mathrm{vol}(U_E)$ then we have a height pairing on $\mathrm{Ch}^{2,1}(X)$ which induces a bilinear pairing

$$\mathrm{Ch}^{2,1}(\rho) \otimes \mathrm{Ch}^{2,1}(\widetilde{\rho}) \longrightarrow \mathbb{C}.$$

Decomposition of homological group

The decomposition on the cohomological cycles induces a decomposition on the class group:

$$\mathrm{Ch}_1(X)_{\mathbb{C}} = \prod_{\rho} \mathrm{Hom}(\rho, \mathbb{C}) \otimes \mathrm{Ch}^2(\rho).$$

Let $\mathrm{Cl}_1(X)_{\mathbb{C}}^{00}$ denote sum over components where ρ is cuspidal.

The decomposition induces an inclusion

$$\mathrm{Ch}^2(X)_{\mathbb{C}} \subset \mathrm{Ch}_1(X)_{\mathbb{C}}$$

which is given concretely by assigning an element $\alpha \in \mathrm{Ch}^2(X_{U_E})$ to a unique element $\alpha^* \in \mathrm{Ch}_1(X)_{\mathbb{C}}$ with component $\alpha^*_{U'_E} := \mathrm{vol}(U'_E)\alpha$ if $U' \subset U$.

This decomposition induces a pairing between $\mathrm{Ch}^2(X)^{00}$ and $\mathrm{Ch}_1(X)_{\mathbb{C}}^{00}$ which is nothing than the one induced by the height pairing.

Gross–Schoen cycle

As in Introduction, let Δ be the image of the diagonal embedding of $Y \longrightarrow X$ considered as an element in $\mathrm{Ch}_1(X)$. Its projection to $\mathrm{Ch}_1(X)^{00}$ is called the *Gross–Schoen cycle* and denoted as Δ_ξ. For each cupidal represenation of $\mathbb{B}_E^\times$ of parallel weight 2, one has component $\Delta_\pi \in \mathscr{P}(\pi) \otimes \mathrm{Ch}^2(\widetilde{\pi})$.

For $f \otimes \widetilde{f} \in \pi \otimes \widetilde{\pi}$ we define

$$\mathrm{T}(f \otimes \widetilde{f})\Delta_\xi := \widetilde{f} \otimes \Delta_\pi(f) \in \widetilde{\pi} \otimes \mathrm{Ch}^2(\widetilde{\pi}) \subset \mathrm{Ch}^2(X)_{\mathbb{C}}^{00}.$$

We claim the following

$$(\Delta_\pi, \Delta_{\widetilde{\pi}})m(f \otimes \widetilde{f}) = (\mathrm{T}(f \otimes \widetilde{f})\Delta_\xi, \Delta_\xi).$$

Indeed, by definition, the left hand side is equal to

$$(\Delta_\pi(f), \Delta_{\widetilde{\pi}}(\widetilde{f})).$$

While the right hand side equals to

$$(\widetilde{f} \otimes \Delta_\pi(f), \Delta_{\widetilde{\pi}}).$$

These two are equal by definition. Thus we can rewrite our main theorem as

Conjecture 3.2.1.

$$\langle \mathrm{T}(f \otimes \widetilde{f})\Delta_\xi, \Delta_\xi \rangle = \frac{8\zeta_F(2)^2}{2L(1, \sigma, \mathrm{ad})} L'(1/2, \sigma) m(f \otimes \widetilde{f}).$$

3.3 Generating series of Hecke correspondences

Let $\mathbb{V}$ denote the orthogonal space $\mathbb{B}$ with quadratic form q. Recall that $\mathscr{S}(\mathbb{V})$ has an extended Weil representation on by

$$\mathscr{R} = \left\{(b_1, b_2, g) \in \mathbb{B}^\times \times \mathbb{B}^\times \times \mathrm{GL}_2(\mathbb{A}) : \quad q(b_1 b_2^{-1}) = \det g\right\}$$

by

$$r(h, g)\Phi(x) = |q(h)|^{-1} r(d(\det(g))^{-1} g)\Phi(h^{-1}x).$$

For $\alpha \in F_+^\times \backslash \mathbb{A}_f^\times$, let X_α denote the union

$$M_\alpha = \coprod_{a \in \pi_0(Y)} Y_a \times Y_{a\alpha}.$$

This is a Shimura subvariety of $Y \times Y$ stabilized by the subgroup $\mathrm{GSpin}(\mathbb{V})$ of $\mathbb{B}^\times \times \mathbb{B}^\times$ of elements with same norms. Define the group of cocycles:

$$\mathrm{Ch}^1(M_\alpha) := \varinjlim_{U_1} \mathrm{Ch}^1(M_{\alpha, U_1})$$

where U_1 runs through the open and compact subgroups of $\mathrm{GSpin}(\mathbb{V})$. For an $h \in \mathbb{B}^\times \times \mathbb{B}^\times$, the pull-back morphism $T(h)$ of right multiplication defines an isomorphism

$$\mathrm{T}(h) : \quad \mathrm{Ch}^1(M_\alpha) \longrightarrow \mathrm{Ch}^1(M_{\alpha\nu(h)^{-1}}).$$

Using Kudla's generating series and the modularity proved in [35], for each $\Phi \in \mathscr{S}(\mathbb{V})$ and $g \in \mathrm{GL}_2(F)_+ \backslash \mathrm{GL}_2(\mathbb{A})_+$, we will construct an element

$$Z(g, \Phi) \in \mathrm{Ch}^1(M_{\det g})$$

such that for any $(g', h') \in \mathscr{R}$,

$$Z(g, r(g', h')\Phi) = \mathrm{T}(h') Z(gg', \Phi).$$

Hecke correspondences

For any double coset UxU of $U\backslash\mathbb{B}_f^\times/U$, we have a Hecke correspondence

$$Z(x)_U \in Z^1(Y_U \times Y_U)$$

defined as the image of the morphism

$$(\pi_{U\cap xUx^{-1},U}, \pi_{U\cap x^{-1}Ux,U} \circ T_x): \qquad Z_{U\cap xUx^{-1}} \longrightarrow Y_U^2.$$

In terms of complex points at a place of F as above, the Hecke correspondence $Z(x)_U$ takes

$$(z, g) \longrightarrow \sum_i (z, gx_i)$$

for points on $X_{U,\tau}(\mathbb{C})$ represented by $(z, g) \in \mathscr{H}^\pm \times \mathbb{B}_f$ where x_i are representatives of UxU/U.

Notice that this cycle is supported on the component $M_{\nu(x)^{-1}}$ of $Y \times Y$.

Hodge class

On $Y \times Y$, one has a *Hodge bundle* $\mathscr{L}_K \in \mathrm{Pic}(Y \times Y) \otimes \mathbb{Q}$ defined as

$$\mathscr{L}_K = \frac{1}{2}(p_1^*\mathscr{L} + p_2^*\mathscr{L}).$$

Generating Function

Write $M = M_1$ which has an action by $\mathrm{GSpin}(\mathbb{V})$. For any $x \in \mathbb{V}$ and open and compact subgroup K of $\mathrm{GSpin}(\mathbb{V})$, let us define a cycle $Z(x)_K$ on M_K as follows. This cycle is non-vanishing only if $q(x) \in F^\times$ or $x = 0$. If $q(x) \in F^\times$, then we define $Z(x)_K$ to be the Hecke operator UxU defined in the last subsection. If $x = 0$, then we define $Z(x)_K$ to be the push-forward of the Hodge class on the subvariety M_α which is union of connected components $Y_a \times Y_a$ with $a \in \pi_0(Y)$. Let $\widetilde{K} = \mathrm{O}(F_\infty) \cdot K$ act on $\mathbb{V}$.

For $\Phi \in \mathscr{S}(\mathbb{V})^{\widetilde{K}}$, we can form a generating series

$$Z(\Phi) = \sum_{x \in \widetilde{K}\backslash\mathbb{V}} \Phi(x) Z(x)_K.$$

It is easy to see that this definition is compatible with pull-back maps in Chow groups in the projection $M_{K_1} \longrightarrow M_{K_2}$ with $K_1 \subset K_2$. Thus it defines an element in the direct limit $\mathrm{Ch}^1(M)_\mathbb{Q} := \lim_K \mathrm{Ch}^1(M_K)$ if it absolutely convergent. We extend this definition to $\mathscr{S}(\mathbb{V})$ by projection

$$\mathscr{S}(\mathbb{V}) \longrightarrow \mathscr{S}(\mathbb{V})^{\mathrm{O}(F_\infty)}, \qquad \Phi \longrightarrow \widetilde{\Phi} := \int_{\mathrm{O}(F_\infty)} r(g)\Phi dg$$

where dg is the Haar measure on $\mathrm{O}(F_\infty)$ with volume 1.

For $g \in \mathrm{SL}_2(\mathbb{A})$, define
$$Z(g,\Phi) = Z_{r(g)\Phi} \in \mathrm{Ch}^1(M).$$
By our previous paper [35], this series is absolutely convergent and is modular for $\mathrm{SL}_2(\mathbb{A})$:
$$Z(\gamma g,\Phi) := Z(g,\Phi), \qquad \gamma \in \mathrm{SL}_2(F) \tag{3.3.1}$$
Moreover, for any $h \in \mathbb{H}$,
$$Z(g, r(h)\Phi) = \mathrm{T}(h) Z(g,\Phi). \tag{3.3.2}$$
where $\mathrm{T}(h)$ denotes the pull-back morphism on $\mathrm{Ch}^1(M)$ by right translation of h_f.

Let $\mathrm{GL}_2(\mathbb{A})^+$ denote subgroup of $\mathrm{GL}_2(\mathbb{A})$ with totally positive determinant at archimedean places. For $g \in \mathrm{GL}_2(\mathbb{A}_F)^+$, define
$$Z(g,\Phi) = \mathrm{T}(h)^{-1} Z(r(g,h)\Phi) \in \mathrm{Ch}^1(M_{\det g}),$$
where h is an element in $\mathbb{B}^\times \times \mathbb{B}^\times$ with norm $\det g$. By (3.3.1), the definition here does not depend on the choice of h. It is easy to see that this cycle satisfies the property
$$Z(g, r(g_1,h_1)\Phi) = T(h_1) Z(gg_1,\Phi), \qquad (g,h) \in \mathscr{R}.$$
The following is the modularity of $Z(g,\Phi)$:

Proposition 3.3.1. *The cycle $Z(g,\Phi)$ is automorphic for $\mathrm{GL}_2(\mathbb{A})^+$: for any $\gamma \in \mathrm{GL}_2(F)^+$, $g \in \mathrm{GL}_2(\mathbb{A})$,*
$$Z(\gamma g,\Phi) = Z(g,\Phi).$$

Proof. Let $\gamma \in \mathrm{GL}_2(F)^+$ if suffices to show
$$\mathrm{T}(\alpha h)^{-1} Z(r(\gamma g, \alpha h)\Phi) = \mathrm{T}(h)^{-1} Z(r(g,h)\Phi)$$
where (γ,α) and (g,h) are both elements in $\mathscr{R}$. This is equivalent
$$\mathrm{T}(\alpha)^{-1} Z(r(\gamma g,\alpha h)\Phi) = Z(r(g,h)\Phi)$$
and then to
$$\mathrm{T}(\alpha)^{-1} Z(r(\gamma,\alpha)\Phi) = Z(\Phi)$$
with $r(g,h)\Phi$ replaced by Φ. Write $\gamma_1 = d(\gamma)^{-1}\gamma$. By definition, the left is equal to
$$\begin{aligned}
\mathrm{T}(\alpha)^{-1} Z(L(\alpha) r(\gamma_1)\Phi) &= \sum_{x \in \widetilde{K}\backslash \mathbb{V}} r(\gamma_1)\Phi(\alpha^{-1}x)\rho(\alpha)^{-1} Z(x)_K \\
&= \sum_{x \in K\cdot \mathrm{O}(F_\infty)\backslash \mathbb{V}} r(\gamma_1)\Phi(\alpha^{-1}x) Z(\alpha^{-1}x)_K \\
&= \sum_{x \in \widetilde{K}\backslash \mathbb{V}} r(\gamma_1)\Phi(x) Z(x)_K \\
&= Z(r(\gamma_1)\Phi) = Z(\Phi).
\end{aligned}$$
$\square$

Notice that the natural embedding $\mathrm{GL}_2(\mathbb{A}_F)^+ \longrightarrow \mathrm{GL}_2(\mathbb{A}_F)$ gives bijective map

$$\mathrm{GL}_2(F)^+\backslash \mathrm{GL}_2(\mathbb{A}_F)^+ \longrightarrow \mathrm{GL}_2(F)\backslash \mathrm{GL}_2(\mathbb{A}_F)$$

thus we can define $Z(g, \Phi)$ for $g \in \mathrm{GL}_2(\mathbb{A}_F)$ by

$$Z(g, \Phi) = Z(\gamma g, \Phi)$$

for some $\gamma \in \mathrm{GL}_2(F)$ such that $\gamma g \in \mathrm{GL}_2^+(\mathbb{A}_F)$. Then $Z(g, \Phi)$ is automorphic for $\mathrm{GL}_2(\mathbb{A})$.

3.4 Geometric theta lifting

Let σ be a cuspidal representation of $\mathrm{GL}_2(\mathbb{A})$ of parallel weight 2. For any $\varphi \in \sigma$, $\alpha \in F_+^\times\backslash \mathbb{A}_f^\times$, define

$$Z_\alpha(\Phi \otimes \varphi) := \int_{\mathrm{SL}_2(F)\backslash \mathrm{SL}_2(\mathbb{A})} Z(g_1 g, \Phi)\varphi(g_1 g) dg_1 \in \mathrm{Ch}^1(M_\alpha)$$

where $g \in \mathrm{GL}_2(\mathbb{A})$ with determinant equal to α. Then it is easy to see that By [36], Theorem 3.5.2, we have the following identity:

$$Z(\Phi \otimes \varphi) = \frac{L(1, \pi, \mathrm{ad})}{2\zeta_F(2)} \mathrm{T}(\theta(\Phi \otimes \varphi)). \tag{3.4.1}$$

The collection $(Z_\alpha(\Phi \otimes \varphi))$ defines an element

$$(Z_\alpha(\Phi \otimes \varphi)) \in \prod_\alpha \mathrm{Ch}^1(M_\alpha).$$

It is easy to see that this element is invariant under open compact subgroup $U \times U$ of $\mathbb{B}^\times \times \mathbb{B}^\times$. Thus is given by an element

$$Z(\Phi \otimes \varphi) \in \mathrm{Ch}^1(Y \times Y).$$

Kernel identity

Recall that the diagonal Y of $X = Y^3$ defines a holomogical cycle in $\mathrm{Ch}_1(X)$ whose projection to $\mathrm{Ch}_1(X)^{00}$ is denoted by P. For a $\Phi \in \mathscr{S}(\mathbb{V}_E)$, we can define a correspondence $Z(\Phi)$ on X by linear combination of product of correspondences on Y: if $\Phi = \Phi_1 \otimes \Phi_2 \otimes \Phi_3$ and $g = (g_1, g_2, g_3)$ with $\Phi_i \in \mathscr{S}(\mathbb{V})$ and $g_i \in \mathrm{GL}_2(\mathbb{A})$, then

$$Z(g, \Phi) = \pi_1^* Z(g_1, \Phi_1) \cdot \pi_2^* Z(g_2, \Phi_2) \cdot \pi_3^* Z(g_3 \Phi_3).$$

This correspondences maps homological cycles to cohomological cycles. In particular we have cycle $Z(\Phi)\Delta_\xi \in \mathrm{Ch}^2(X)^{00}$. Thus the number

$$Z(g, \Phi, \Delta_\xi) := \langle \Delta_\xi, Z(g, \Phi)\Delta_\xi \rangle$$

is well defined.

Our main theorem is reduced to the following identity of kernel functions:

Conjecture 3.4.1.
$$(-E'(\cdot, 0, \Phi), \varphi) = (Z(\cdot, \Phi, \Delta_\xi), \varphi).$$

Deduce conjecture 3.2.1 from the above kernel identity. By (2.3.7), the left hand side of the kernel identity is
$$\frac{L'(1/2, \sigma)}{\zeta_F(2)} m(\theta(\Phi \otimes \varphi).$$

By formula (3.4.1), the right hand side is
$$\frac{L(1, \pi, \mathrm{ad})}{8\zeta_F(2)^3} \langle \Delta_\xi, T(\theta(\Phi \otimes \varphi))\Delta_\xi \rangle.$$

As the image generates all $\pi \otimes \widetilde{\pi}$, the conjecture 3.2.1 follows. $\square$

By 3.1.3, we may write

$$Z(g, \Phi, \Delta_\xi) = \widehat{Z}^-(g_1, \Phi_1) \cdot \widehat{Z}^-(g_2, \Phi_2) \cdot \widehat{Z}^-(g_3, \Phi_3) \tag{3.4.2}$$

where the right hand side is the intersection of the admissible class extending the projection $Z(g_i, \Phi)^- \in \mathrm{Pic}^-(Y \times Y)$ of $Z(g_i, \Phi)$.

For the actually computation, we may replace $\widehat{Z}(g, \Phi)^-$ by arithmetic classes extending $Z(g, \Phi)$. In fact, since $Z(g, \Phi)$ will fix class $\mathrm{Pic}^\xi(Y)$, we see it is in the space
$$\pi_1^* \mathrm{Pic}^\xi(Y) \otimes \mathrm{Ch}^0(Y) + \mathrm{Ch}^0(Y) \otimes \pi_2^* \mathrm{Pic}^\xi(Y) + \mathrm{Pic}^-(Y \times Y).$$

Thus we have an decomposition
$$Z(g, \Phi) = Z_1^\xi(g, \Phi) + Z_2^\xi(g, \Phi) + Z^-(g, \Phi).$$

It is easy to see that both $Z_i^\xi(g, \Phi)$ are Eisenstein series with valued in Hodge cycles. Now for each $\alpha \in \mathrm{Pic}^\xi(Y)$, fix an arithmetic extension $\widehat{\alpha}$. Then the above decomposition defines an arithmetic extension $\widehat{Z}(g, \Phi)$. Now we define

$$Z(g, \Phi, \Delta) = \widehat{Z}(g_1, \Phi_1) \cdot \widehat{Z}(g_2, \Phi_2) \cdot \widehat{Z}(g_3, \Phi_3) \tag{3.4.3}$$

Then we have that the difference $Z(g, \Phi, P) - Z(g, \Phi)$ is sum of forms which is Eisenstein for at least one variable g_i. It follows that it has zero inner product with cusp forms. Thus we have the following equivalent form of the above theorem:

Conjecture 3.4.2 (Kernel identity)**.**
$$(-E'(\cdot, 0, \Phi), \varphi) = (Z(\cdot, \Phi, \Delta), \varphi).$$

Remark 3.4.1. Unlike the formalism $\Phi \mapsto Z(g, \Phi)$ which is equivariant under the action of $\mathbb{B}^\times \times \mathbb{B}^\times$, the formalism $\Phi \mapsto \widehat{Z}(g, \Phi)$ is not $\mathbb{B}^\times \times \mathbb{B}^\times$ equivariant in general.

3.5 Arithmetic Hodge class and Hecke operators

In this section, we want to introduce an arithmetic Hodge classes and then the arithmetic Hecke operators. The construction depends on the choice of integral models which in terms depends on a maximal order $\mathscr{O}_{\mathbb{B}}$ of $\mathbb{B}$ we fix here.

Moduli interprstation at an archimedean place

Let U be an open and compact subgroup of $\mathscr{O}_{\mathbb{B}}^{\times}$. Let τ be an archimedean place of F. Write B a quaternion algebra over F with ramification set $\Sigma \setminus \{\tau\}$. Fix an isomorphism $\mathbb{B}^{\tau} \simeq B \otimes \mathbb{A}^{\tau}$. Recall from §51 in our Asia journal paper, that the curve Y_U parameterizes the isomorphism classes of triples $(V, h, \bar{\kappa})$ where

1. V is a free B-module of rank 1;
2. h is an embedding $\mathbb{S} \longrightarrow \mathrm{GL}_B(V_{\mathbb{R}})$ which has has weight -1 at τ_1, and trivial component at τ_i for $i > 1$, where $\tau_1 := \tau, \tau_2, \cdots, \tau_g$ are all archimedean places of F;
3. $\bar{\kappa}$ is a $\mathrm{Isom}(\widehat{V}_0, \widehat{V})/U$, where $V_0 = B$ as a left B-module.

The Hodge structure h define a Hodge decomposition on $V_{\tau,\mathbb{C}}$:

$$V_{\tau,\mathbb{C}} = V^{-1,0} + V^{0,-1}.$$

By Hodge theory, the tangent space of Y at a point (V, h, κ) is given by

$$\mathscr{L}(V)_{\tau} = \mathrm{Hom}_B(V^{-1,0}, V_{\mathbb{C}}/V^{-1,0}) = \mathrm{Hom}_B(V^{-1,0}, V^{0,-1}).$$

Since the complex conjugation on $V_{\mathbb{C}}$ switches two factors $V^{-1,0}$ and $V^{0,-1}$, one has a canonical identification

$$\mathscr{L}(V)_{\tau} \otimes \overline{\mathscr{L}(V)_{\tau}} = \mathrm{Hom}_B(V^{-1,0}, V^{-1,0}) = \mathbb{C}.$$

This identification defines a Hermitian norm on $\mathscr{L}(V)_{\tau}$.

Lemma 3.5.1. *Let $\delta(V)$ denote the one dimensional vector space over F of reduced norms $\delta(v)$ for $v \in V$ with relation $\delta(bv) = \nu(b)\delta(v)$. Then we have a canonical isomorphism:*

$$\mathscr{L}(V) = \delta(V) \otimes_{F,\tau} \det(V_{\mathbb{C}}^{-1,0})^{\vee}.$$

Proof. Indeed, there is pairing $\psi : V \otimes V \longrightarrow \delta(V)$ define by

$$\psi(u, v) := \frac{1}{2}(\delta(u + v) - \delta(u) - \delta(v)).$$

Let $B^{\times}$ acts on this space by multiplication by $\nu : B^{\times} \longrightarrow F^{\times}$ then

$$\psi \in \mathrm{Hom}_{B^{\times}}(V \otimes V, \delta(V)).$$

This pairing is compatible with Hodge structures when $\delta(V)$ is equipped with action weight $(-1,-1)$. Thus on $V_{\tau,\mathbb{C}}$, the above pairing has isotropic spaces $V^{-1,0}$ and $V^{0,-1}$ and defines bilinaer $B_{\mathbb{C}}^{\times}$ equivariant pairing

$$V^{-1,0}\otimes V^{0,-1}\longrightarrow \delta(V)_{\mathbb{C}}.$$

On the other hand the wedge product defines a $B_{\mathbb{C}}^{\times}$ pairing

$$V^{-1,0}\otimes V^{0,-1}\longrightarrow \det(V^{-1,0})$$

when the later space is equipped with action $\nu : B^{\times}\longrightarrow F^{\times}$. The above two pairing define canonical identifications:

$$V^{0,-1}=\delta(V)_{\tau,\mathbb{C}}\otimes \mathrm{Hom}_{B^{\times}}(V^{-1,0},\mathbb{C})$$

$$V^{-1,0}=\det(V^{-1,0})\otimes \mathrm{Hom}_{B^{\times}}(V^{-1,0},\mathbb{C}).$$

Thus we have

$$\mathscr{L}(V)_{\tau}=\mathrm{Hom}(V^{-1,0},V^{0,-1})=\delta(V)_{\tau,\mathbb{C}}\otimes \det(V^{-1,0})^{\vee}.$$

$\square$

Modular interpretation at an finite place

Let v be a finite place. Recall from §5.3 in our Asia journal paper, the prime to v-part of $(\widehat{V}_U,\bar{\kappa})$ extends to an étale system over $\mathscr{Y}_U$, but the v-part extends to a system of special divisible $\mathscr{O}_{\mathbb{B}_v}$-module of dimension 2, height 4, with Drinfeld level structure:

$$(\mathscr{A},\bar{\alpha})$$

with an identification

$$\kappa_v(\mathscr{O}_v)\simeq \mathrm{T}_v(\mathscr{A}).$$

where $\mathrm{T}_v(\mathscr{A})$ is the Tate module of $\mathscr{A}$ for prime v.

The Lie algebra of the formal part $\mathscr{A}^0$ of $\mathscr{A}$ defines a two dimensional vector bundle $\mathrm{Lie}(\mathscr{A})$ on $\mathscr{Y}_U$. The tangent space of Y_U is canonically identified with $\mathscr{L}_v:=\delta(V)_{\mathscr{O}_F}\otimes \mathrm{Lie}(\mathscr{A})^{\vee}$. The level structure defines an integral structure on $\delta(V)$ at place v. Thus $\mathscr{L}_v$ has an integral structure by the tensor product.

If v is not split in $\mathbb{B}$, then $\mathscr{O}_{\mathbb{B}_v}$ is unique and then integral structure on $\mathscr{L}$ is unique. This can also been seen from the fact that the group $\mathscr{A}$ is totally formal and supersingular. Any isogeny $\varphi:\mathscr{A}_x\longrightarrow \mathscr{A}_y$ of two such $\mathscr{O}_{\mathbb{B}_v}$-modules representing two points x and y on $\mathscr{Y}_U$ smooth over $\mathscr{O}_v$ induces an isomorphism of $\mathscr{O}_v$-modules:

$$\mathscr{L}(\mathscr{A})\simeq \mathscr{L}(\mathscr{B}).$$

If v is split in $\mathbb{B}$, then we may choose an isomorphism $\mathscr{O}_{\mathbb{B}_v}=M_2(\mathscr{O}_v)$. Then the divisible module $\mathscr{A}$ is a direct sum $\mathscr{E}\oplus\mathscr{E}$ where $\mathscr{E}$ is a divisible $\mathscr{O}_F$-module of dimension 1 and height 2. Then we have an isomorphism

$$\mathscr{L}=\mathrm{Lie}(\mathscr{E})^{\otimes -2}\otimes \det \mathrm{T}_v(\mathscr{A}).$$

Let x be an ordinary $\mathscr{O}_v$-point of $\mathscr{Y}_U$ then we have an formal–etale decomposition

$$0 \longrightarrow \mathscr{E}_x^0 \longrightarrow \mathscr{E} \longrightarrow \mathscr{E}^{et} \longrightarrow 0.$$

This induces an isomorphism

$$\mathscr{L}_x = (\mathrm{Lie}(\mathscr{E})^\vee \otimes \mathrm{T}_v(\mathscr{E}^0))^{\otimes 2} \otimes (\mathrm{T}_v(\mathscr{E}_x^{et}) \otimes \mathrm{T}_v(\mathscr{E}_x^0)^\vee).$$

The first part does not depend on the level structure but the second part does. If $\varphi : \mathscr{E}_x \longrightarrow \mathscr{E}_y$ be an isogeny of orders a, b on the formal and etale part respectively, then it has order $b - a$ for the bundles $\mathscr{L}_x \longrightarrow \mathscr{L}_y$.

Admissible arithmetic classes

Combining the above, we have introduce an arithmetic structure $\widehat{\mathscr{L}}$ for $\mathscr{L}$. The roots of this defines an arithmetic structure on elements of Hodges classes $\mathrm{Pic}^\xi(Y)$. We denote the resulting the groups of arithmetic classes as as $\widehat{\mathrm{Pic}}^\xi(Y)$. Unlike $\mathrm{Pic}^\xi(Y)$, the group $\widehat{\mathrm{Pic}}^\xi(Y)$ is not invariant under the action of $\mathbb{B}^\times$ but invariant under $\mathscr{O}_{\mathbb{B}}^\times$. We normalize the the metric of $\widehat{\xi}$ at one archimedean place such that on each connected component of any Y_U

$$\widehat{\xi}^2 = 0.$$

Now for any class $\alpha \in \mathrm{Ch}^1(Y_{U,a} \times Y_{U,b})$ in some irreducible component of $Y \times Y$ in a finite level which fixes ξ by both push-forward and pull-back, we can attached a class $\widehat{\alpha}$ such that if $\alpha = \alpha^- + a\pi_1^*\xi_1 + b\pi_2^*\xi_2$ with $\alpha \in \mathrm{Pic}^-(Y \times Y)$ and $\xi_i \in \mathrm{Pic}^\xi(Y)$, then we

$$\widehat{\alpha} = \widehat{\alpha}^- + a\pi_1^*\widehat{\xi}_1 + b\pi_2\widehat{\xi}_2.$$

We call such a class $\widehat{\xi}$-*admissible.* Such a class can be characterized by the following properties:

1. for any point $(p_1, p_2) \in Y_{U,a} \times Y_{U,b}$, the induced class arithmetic classes $\widehat{\alpha}_1 := \widehat{\alpha}|_{p_1 \times U_b}$ and $\widehat{\alpha}_2 := \widehat{\alpha}_{Y_a \times p_2}$ on $Y_{U,a}$ or $Y_{U,b}$ is $\widehat{\xi}$-admissible in the sense that $\widehat{\alpha}_i - \deg \alpha_i \widehat{\xi}$ has curvature 0 at archimedean places and and zero intersection with vertical cycles.

2. $\widehat{\alpha} \cdot \pi_1^*\widehat{\xi}_1 \cdot \pi_2^*\widehat{\xi}_2 = 0.$

The class $\alpha \mapsto \widehat{\alpha}$ extends to whole group $\mathrm{Ch}^1(Y \times Y)$.

Arithmetic Hecke operators

Let Z be a Hecke operator as a divisor in $Y_U \times Y_U$. We want to define an adelic green's function $g = (g_v)$ such that the arithmetic cycle $\widehat{Z} = (Z, g)$ is $\widehat{\xi}$-admissible. Let p_1, p_2 be two projection of Z onto Y_U. Then p_i's have the same degree called d and there is an canonical isomorphism $p_1^*\xi \to p_2^*\xi$ of line bundles (with fractional power). This induces an isomorphism

$$\alpha : Z_*\xi_1 \simeq d\xi_2.$$

We want to construct a green function g for Z such that arithmetic class $\widehat{Z}$ satisfies the property 1 above and with property 2 replaced by the following refined one:

3. The isomorphisms α and β above induces isometry of adelic metrized line bundles:

$$\alpha : \widehat{Z}_*\widehat{\xi}_1 \simeq d\widehat{\xi}_2.$$

First of all, we define an adelic green's function $g^0 = (g^0_v)$ with the following two properties at each place v of F parallel to the properties as above,

4. (Z, g^0) has curvatures parallel to $c_1(\widehat{\xi}_i)_v$ at fibers $p_i^* y$ over two projections p_i to Y_U.

5. g^0_v has integral 0 against $c_1(\widehat{\xi}_1) \cdot c_1(\widehat{\xi}_2)$.

The class $\widehat{Z}^0 := (Z, g^0)$ will satisfies the property 1. Since the bundle $\widehat{Z}^0_*\widehat{\xi}_1$ will have same curvatures as $d\widehat{\xi}_2$, then we have constants $c = (c_v)$ such that α induces isometry

$$\widehat{Z}^0_*\widehat{\xi}_1 = d\widehat{\xi}_2 + c.$$

Define

$$\widehat{Z} := \widehat{Z}^0 - \frac{1}{d}c.$$

Then $\widehat{Z}$ will have required properties. Notice that by property 1 and 3, the classes is closed under composition.

Remark 3.5.2. The above class $\widehat{Z}$ may not give an isometry between $d\widehat{\xi}_1$ under $\widehat{Z}^*\widehat{\xi}_2$. In fact, we will give an expression of c in terms of bundles bundles $p_i^*\widehat{\xi}$. Notice that the difference

$$\widehat{Z}_*\widehat{\xi}_1 - d\widehat{\xi}_2 = \pi_{2*}(\widehat{Z} \cdot (\pi_1^*\xi_1 - \pi_2^*\xi_2)).$$

The class $\pi_1^*\xi_1 - \pi_2^*\xi_2$ is represented by a vertical divisor class $C = (C_v)$ and

$$c_v = \pi_{2*}(\widehat{Z} \cdot C_v).$$

We intersect this with $\widehat{\xi}_2$ then we have

$$c_v = \widehat{Z} \cdot C_v \cdot \pi_2^*\widehat{\xi}_2.$$

Since property 4, this las sum is equal to an intersection number on Z:

$$c_v = C_v|_Z \cdot p_2^*\widehat{\xi}_2.$$

If we redo the construction for Z^*, then we will obtain a class $\widehat{Z}' = \widehat{Z}^0 + c'$ with

$$c'_v = -C_v|_v \cdot \cdot p_1^*\xi_1.$$

Notice that sum of c_v and c'_v are both equal to $p_1^*\widehat{\xi}_1 \cdot p_2^*\widehat{\xi}_2$ since $\widehat{\xi}_i^2 = 0$. The sum $c_v + c'_v$ is equal to $-(C_v|_Z)^2$ which is nonnegative. This shows that $\widehat{Z}^0 \geq \widehat{Z}$.

First decomposition

With construction of cycles as above, we can decompose the intersection as follows

$$Z(g, \Phi, \Delta) := \widehat{Z}(g_1, \Phi_1) \cdot \widehat{Z}(g_2, \Phi_2) \cdot \widehat{Z}(g_3, \Phi_3).$$

First of this intersection is non-trivial only if all g_i have the same norm. In this case we have one $h \in \mathbb{B}^\times \times \mathbb{B}^\times$ such that

$$Z(g_i, \Phi_i) = \mathrm{T}(h) Z(r(g_i, h)\Phi_i).$$

Thus we have that

$$\widehat{Z}(g_1, \Phi_1) \cdot \widehat{Z}(g_2, \Phi_2) \cdot \widehat{Z}(g_3, \Phi_3) = \widehat{Z}(r(g_1, h)\Phi_1) \cdot \widehat{Z}(r(g_2, h)\Phi_2) \cdot \widehat{Z}(r(g_3, h)\Phi_3).$$

Assume that each $r(g_i, h_i)\Phi_i$ is invariant under K. In this case this intersection number is given by

$$Z(g, \Phi, \Delta) = \sum_{(x_1, x_2, x_3) \in (\widetilde{K} \backslash \mathbb{V})^3} r(g, h)\Phi(x_1, x_2, x_3) \widehat{Z}(x_1)_K \cdot \widehat{Z}(x_2)_K \cdot \widehat{Z}(x_3)_K.$$

We write $Z(g, \phi)_{sing}$ for the partial sum where $Z(x_i)$ has non-empty intersection at the generic fiber. Then the rest term can be decompose into local intersections. Thus we have a decomposition

$$Z(g, \Phi, \Delta) = Z(g, \Phi, \Delta)_{sing} + \sum_v Z(g, \Phi, \Delta)_v.$$

4 Local Whittaker integrals

As we have seen in section 2.5, we need to study the non-singular Fourier coefficients of the derivative of Eisenstein series for Schwartz function $\Phi \in \mathscr{S}(\mathbb{B}^3)$ on an incoherent (adelic) quaternion algebra $\mathbb{B}$ over the adeles $\mathbb{A}$ of a number field F. This is essentially reduced to the study of the derivative at the local Whittaker functions. In the case of unramified Siegel–Weil section section at a non-archimedean place, the computation is known. We will recall the results. Then we move to compute the archimedean Whittaker integrals.

4.1 Nonarchimedeanl local Whittaker integral

Now we recall some results about the local Whittaker integral and local density.

Let F be a nonarchimedean local field with integer ring $\mathscr{O}$ whose residue field is of *odd* characteristic p. We remark that all results in this subsection actually holds for $p = 2$. For simplicity of exposition, we only record the results for odd p. Let ϖ be a uniformizer and $q = |\mathscr{O}/(\varpi)|$ be the cardinality of the residue field. Assume further that the additive character ψ is unramified.

Let $V = B = M_2(F)$ with the quadratic form $q = det$. Let Φ_0 the characteristic function of $M_2(\mathscr{O})$. Let $T \in \mathrm{Sym}_3(\mathscr{O})^\vee$ (cf. "Notations"). It is a fact that $W_T(e, s, \Phi_0)$ is a polynomial of q^{-s}.

To describe the formula, we need several invariants of $T \in \mathrm{Sym}_3(\mathscr{O}_v)^\vee$. Suppose that $T \sim \mathrm{diag}[u_i \varpi^{a_i}]$ with $a_1 < a_2 < a_3 \in \mathbb{Z}$, $u_i \in \mathscr{O}^\times$. Then we define $\xi(T)$ to be the Hilbert symbol $\left(\frac{-u_1 u_2}{\varpi}\right) = (-u_1 u_2, \varpi)$ if $a_1 \equiv a_2 \,(mod\, 2)$ and $a_2 < a_3$, otherwise zero. Note that this doesnot depend on the choice of the uniformizer ϖ.

Firstly, we have a formula for the central value of Whittaker integral $W_{T,v}(e, 0, \Phi_0)$.

Proposition 4.1.1. *The Whittaker function at $s = 0$ is given by*

$$W_{T,v}(e, 0, \Phi_0) = \zeta_F(2)^{-2} \beta_v(T)$$

where

1. *When T is anisotropic, we have*
$$\beta_v = 0.$$

2. *When T is isotropic, we have three cases*

   (a) *If $a_1 \neq a_2 \, mod\, 2$, we have*

$$\beta_v(T) = 2\left(\sum_{i=0}^{a_1} (1+i) q^i + \sum_{i=a_1+1}^{(a_1+a_2-1)/2} (a_1+1) q^i\right).$$

(b) *If* $a_1 \equiv a_2 \, mod \, 2$ *and* $\xi = 1$, *we have*

$$\begin{aligned}\beta_v(T) =& 2(\sum_{i=0}^{a_1}(i+1)q^i + \sum_{i=a_1+1}^{(a_1+a_2-2)/2}(a_1+1)q^i) \\ &+ (a_1+1)(a_3-a_2+1)q^{(a_1+a_2)/2}.\end{aligned}$$

(c) *If* $a_1 \equiv a_2 \, mod \, 2$ *and* $\xi = -1$, *we have*

$$\begin{aligned}\beta_v(T) =& 2(\sum_{i=0}^{a_1}(i+1)q^i + \sum_{i=a_1+1}^{(a_1+a_2-2)/2}(a_1+1)q^i) \\ &+ (a_1+1)q^{(a_1+a_2)/2}.\end{aligned}$$

The second result we will need is a formula of the central derivative $W'_{T,v}(e, 0, \Phi_0)$.

Proposition 4.1.2. *We have*

$$W'_{T,v}(e, 0, \Phi_0) = \log q \cdot \zeta_F(2)^{-2}\nu(T),$$

where $\nu(T)$ *is given as follows: let* $T \sim \mathrm{diag}[t_i]$ *with* $a_i = \mathrm{ord}(t_i)$ *in the order* $a_1 \leq a_2 \leq a_3$, *then*

1. *If* $a_1 \not\equiv a_2 \, mod \, 2$, *we have*

$$\nu(T) = \sum_{i=0}^{a_1}(1+i)(3i - a_1 - a_2 - a_3)q^i + \sum_{i=a_1+1}^{(a_1+a_2-1)/2}(a_1+1)(4i - 2a - 1 - a_2 - a_3)q^i.$$

2. *If* $a_1 \equiv a_2 \, mod \, 2$, *we must have* $a_2 \not\equiv a_3 \, mod \, 2$. *In this case we have*

$$\begin{aligned}\nu(T) =& \sum_{i=0}^{a_1}(i+1)(3i - a_1 - a - 2 - a_3)q^i \\ &+ \sum_{i=a_1+1}^{(a_1+a_2-2)/2}(a_1+1)(4i - 2a - 1 - a_2 - a_3)q^i \\ &- \frac{a_1+1}{2}(a_3 - a_2 + 1)q^{(a_1+a_2)/2}.\end{aligned}$$

Proposition 4.1.3. *Let* Φ'_0 *be the characteristic function of* $\mathscr{O}_D^3$ *where* $\mathscr{O}_D$ *is the maximal order of the division quaternion algebra* D. *Then we have for all anisotropic* $T \in \mathrm{Sym}_3(\mathscr{O})^\vee$:

$$W_T(e, 0, \Phi'_0) = -2q^{-2}(1+q^{-1})^2.$$

For the proof of the three propositions above, we refer to [1, Chap. 15, 16] where a key ingredient is a result in [19] on the local representation density for Hermitian forms.

Proposition 4.1.4. *Let Φ_0' be the characteristic function of maximal order $\mathscr{O}_D$ of the division quaternion algebra D. Then we have for all anisotropic $T \in \mathrm{Sym}_3(\mathscr{O})^\vee$:*

$$I_T(e, \Phi_0') = \mathrm{vol}(SO(V')).$$

Proof. A prior we know that $I_T(e, \Phi_0')$ is a constant multiple of $W_T(e, 0, \Phi_0')$. Take any $x \in \mathscr{O}_D^3$ with moment T.Then it is easy to see that $h \cdot x$ is still in $\mathscr{O}_D^3$ for all $h \in SO(V')$. This completes the proof. □

4.2 Archimedean Whittaker integral

We want to compute the Whittaker integral $W_T(g, \Phi, g, s)$ when $F = \mathbb{R}$, $B = \mathbb{H}$ is the Hamiltonian quaternion algebra, and

$$\Phi_\infty(x) = \Phi(x) = e^{-2\pi tr(Q(x))}, \quad x \in B^3 = \mathbb{H}^3.$$

Recall that we have choose the additive character

$$\psi(x) = e^{2\pi i x}, \quad x \in \mathbb{R}.$$

Let K_∞ be the maximal compact subgroup of $\mathrm{Sp}_6(\mathbb{R})$:

$$K_\infty = \left\{ \begin{pmatrix} x & y \\ -y & x \end{pmatrix} \in \mathrm{Sp}_6(\mathbb{R}) \mid x + yi \in U(3). \right\}$$

Denote by χ_m the character of K_∞

$$\chi_m \begin{pmatrix} x & y \\ -y & x \end{pmatrix} = det(x + yi)^m.$$

Then the Siegel–Weil section attached to Φ transform by the character χ_2 under the action of K_∞ (cf. [11], [21]).

Lemma 4.2.1. *Let $g = n(b)m(a)k \in \mathrm{Sp}_6(\mathbb{R})$ be the Iwasawa decomposition. Then we have when* $\mathrm{Re}(s) \gg 0$*:*

$$W_T(g, s, \Phi) = \chi_2(k)\psi(Tb)\lambda_s(m({}^t a^{-1}))|\det(a)|^4 W_{{}^t aTa}(e, s, \Phi).$$

Proof. This follows the invariance under K_∞ and the property of Siegel–Weil section. □

Thus it suffices to consider only the identity element $g = e$ of $\mathrm{Sp}_6(\mathbb{R})$. It is easy to obtain a formula for $\lambda_s(wn(u))$ and we have

$$W_T(e, s, \Phi) = \int_{\mathrm{Sym}_3(\mathbb{R})} \psi(-Tu) \det(1 + u^2)^{-s} r(wn(u))\Phi(0) du.$$

Lemma 4.2.2. *When* $\mathrm{Re}(s) \gg 0$*, we have*

$$W_T(e, s, \Phi) = -\int_{\mathrm{Sym}_3(\mathbb{R})} \psi(-Tu) \det(1 + iu)^{-s} \det(1 - iu)^{-s-2} du,$$

where we have the usual convention $i = \sqrt{-1}$*.*

Proof. Let $u =^t kak$ be the Cartan decomposition where $a = diag(u_1, u_2, u_3)$ is diagonal and $k \in SO(3)$. Then it is easy to see that $n(u) = m(k)^{-1} n(a) m(k)$ and $wm(k)^{-1} = m(-k^{-1})w$. Note that $det(k) = 1$ and $\chi_2(m(k)) = 1$. We obtain by the previous lemma:

$$r(wn(u))\Phi(0) = r(wn(a))\Phi(0).$$

By definition we have

$$r(wn(a))\Phi(0) = \gamma(\mathbb{H}, \psi) \int_{\mathbb{H}^3} \psi(aQ(x))\Phi(x) dx,$$

where for our choice the Weil constant is

$$\gamma(\mathbb{H}, \psi) = -1.$$

Therefore we have

$$r(wn(a))\Phi(0) = -\prod_{j=1}^{3} \int_{\mathbb{H}} e^{\pi(iu_j - 1)q(x_j)} dx_j.$$

This is equal to a constant times

$$\prod_{j}^{3} \frac{1}{(1 - iu_j)^2} = \det(1 - iu)^{-2}.$$

To recover the constant, we let $u = 0$ and note that

$$r(w)\Phi(0) = \chi_2(w)\Phi(0) = -\Phi(0) = -1.$$

We thus obtain that

$$r(wn(u))\Phi(0) = r(wn(a))\Phi(0) = -\det(1 - iu)^{-2}.$$

Since $\det(1 + u^2) = \det(1 - iu)\det(1 + iu)$, the lemma now follows. $\square$

Following Shimura ([33, pp.274]), we introduce a function for $g, h \in \mathrm{Sym}_n(\mathbb{R})$ and $\alpha, \beta \in \mathbb{C}$

$$\eta(g, h; \alpha, \beta) = \int_{x > \pm h} e^{-gx} \det(x + h)^{\alpha - 2} \det(x - h)^{\beta - 2} dx$$

which is absolutely convergent when $g > 0$ and $\mathrm{Re}(\alpha), \mathrm{Re}(\beta) > \frac{n}{2}$. Here we use $x \pm h$ to mean that $x + h > 0$ and $x - h > 0$. Here we point out that the measure dx in [33] is the

Euclidean measure viewing $\mathrm{Sym}_n(\mathbb{R})$ as $\mathbb{R}^{n(n+1)/2}$ naturally. *This measure is not self-dual but only up to a constant* $2^{n(n-1)/4}$. In the following we always use the Euclidean measure as [33] does. For two elements $h_1, h_2 \in \mathrm{Sym}_n(\mathbb{R})$, by $h_1 \sim h_2$ we mean that $h_1 = kh_2k^{-1}$ for some $k \in O(n)$, the real orthogonal group for the standard positive definite quadratic form.

Before we proceed let us recall some well-known results. Let $z \in \mathrm{Sym}_n(\mathbb{C})$ with $\mathrm{Re}(z) > 0$, then we have for $s \in \mathbb{C}$ with $\mathrm{Re}(s) > \frac{n-1}{2}$,

$$\int_{\mathrm{Sym}_n(\mathbb{R})_+} e^{-tr(zx)} \det(x)^{s-\frac{n+1}{2}} dx = \Gamma_n(s) \det(z)^{-s}, \tag{4.2.1}$$

where the "higher" Gamma function is defined as

$$\Gamma_n(s) = \pi^{\frac{n(n-1)}{4}} \Gamma(s)\Gamma(s-\frac{1}{2})...\Gamma(s-\frac{n-1}{2}).$$

For instance, when $n = 1$, we have when $\mathrm{Re}(z) > 0$ and $\mathrm{Re}(s) > 0$

$$\int_{\mathbb{R}_+} e^{-zx} x^{s-1} dx = \Gamma(s) z^{-s}.$$

Consider

$$f(x) = \begin{cases} e^{-vx} \det(x)^{s-\frac{n+1}{2}} & x > 0, \\ 0 & \text{otherwise.} \end{cases}$$

Applying (4.2.1) to $z = v + 2\pi i u$ for $u, v \in \mathbb{R}$, we obtain when $\mathrm{Re}(s) > \frac{n-1}{2}$,

$$\widehat{f}(u) = \Gamma_n(s) \det(v + 2\pi i u)^{-s}.$$

Take the inverse Fourier transformation, we obtain

$$\int_{\mathrm{Sym}_n(\mathbb{R})} e^{2\pi i u x} \det(v + 2\pi i u)^{-s} du = \begin{cases} \frac{1}{2^{n(n-1)/2}\Gamma_n(s)} e^{-vx} \det(x)^{s-\frac{n+1}{2}} & x > 0, \\ 0 & \textit{otherwise.} \end{cases} \tag{4.2.2}$$

Lemma 4.2.3. *When* $\mathrm{Re}(s) > 1$, *we have*

$$W_T(e, s, \Phi) = \kappa(s)\Gamma_3(s+2)^{-1}\Gamma_3(s)^{-1}\eta(2\pi, T; s+2, s)$$

where

$$\kappa(s) = -2^{9/2}\pi^{6s+6}.$$

Proof. By (4.2.1) for $n = 3$, we may rewrite the Whittaker function in the previous lemma as

$$W_T(e, s, \Phi) = -\frac{\pi^{3s+6}}{\Gamma_3(s+2)} \int_{\mathrm{Sym}_3(\mathbb{R})} e^{-2\pi i T u} \det(1 + iu)^{-s} \int_{\mathrm{Sym}_3(\mathbb{R})_+} e^{-\pi(1-iu)x} \det(x)^s dx 2^{3/2} du.$$

Here du is changed to the Euclidean measure and the constant multiple $2^{3/2}$ comes from the ratio between the self-dual measure and the Euclidean one. Interchange the order of the two integrals

$$-2^{3/2}\frac{\pi^{3s+6}}{\Gamma_3(s+2)}\int_{\mathrm{Sym}_3(\mathbb{R})_+} e^{-\pi x}\det(x)^s\left(\int_{\mathrm{Sym}_3(\mathbb{R})} e^{2\pi iu(\frac{1}{2}x-T)}\det(1+iu)^{-s}du\right)dx.$$

By (4.2.2) for $n=3$, we obtain

$$\begin{aligned}&-2^{3/2}\frac{\pi^{3s+6}}{\Gamma_3(s+2)}\int_{x>0,x>2T} e^{-\pi x}\det(x)^s\frac{(2\pi)^6}{2^3\Gamma_3(s)}e^{-2\pi(\frac{x}{2}-T)}\det(2\pi(\frac{x}{2}-T))^{s-2}dx\\ =&-2^{9/2}\frac{\pi^{6s+6}}{\Gamma_3(s+2)\Gamma_3(s)}\int_{x>0,x>2T} e^{-2\pi(x-T)}\det(x)^s\det(x-2T)^{s-2}dx.\end{aligned}$$

Finally we may substitute $x\to T+x$ to complete the proof. $\square$

To compute the integral η in an inductive way, we recall the "higher" *confluent hypergeometric function* ([33, pp.280,(3.2)]). Let $\mathrm{Sym}_n(\mathbb{C})_+$ be the set of z with $\mathrm{Re}(z)>0$. Then for $z\in\mathrm{Sym}_n(\mathbb{C})_+$, we define

$$\zeta_n(z,\alpha,\beta)=\int_{\mathrm{Sym}_n(\mathbb{R})_+} e^{-zx}\det(x+1)^{\alpha-\frac{n+1}{2}}\det(x)^{\beta-\frac{n+1}{2}}dx. \tag{4.2.3}$$

It was first introduced by Koecher and its analytic continuation is settled by Shimura:

Lemma 4.2.4 (Shimura)**.** *For $z\in\mathrm{Sym}_n(\mathbb{C})$ with $\mathrm{Re}(z)>0$, the integral $\zeta_n(z;\alpha,\beta)$ is absolutely convergent for $\alpha\in\mathbb{C}$ and $\mathrm{Re}(\beta)>\frac{n-1}{2}$. And the function*

$$\omega(z,\alpha,\beta):=\Gamma_n(\beta)^{-1}\det(z)^{\beta}\zeta_n(z,\alpha,\beta)$$

can be extended to a holomorphic function of $(\alpha,\beta)\in\mathbb{C}^2$.

Proof. See [33, Thm. 3.1]. $\square$

The following proposition gives an inductive way to compute the Whittaker integral $W_T(e,s,\Phi)$, or equivalently $\eta(2\pi,T;s+2,s)$. From now on, to simplify notations, we use w' to denote the transpose of w if no confusion arises.

Proposition 4.2.5. *Assume that $sign(T)=(p,q)$ with $p+q=3$ so that we have $4\pi T\sim diag(a,-b)$ for $a\in\mathbb{R}^p_+, b\in\mathbb{R}^q_+$. Let $t=diag(a,b)$. Then we have*

$$\eta(2\pi,T;s+2,s)=2^{6s}e^{-t/2}|\det(T)|^{2s}\xi(T,s)$$

where

$$\begin{aligned}\xi(T,s)=&\int_M e^{-(aW+bW')}\det(1+W)^{2s}\zeta_p(ZaZ,s+2,s-\frac{3-p}{2})\\ &\times\zeta_q(Z'bZ',s,s+\frac{q+1}{2})dw.\end{aligned}$$

where $M=\mathbb{R}^p_q$, $W=w\cdot w'$, $W'=w'w$, $Z=(1+W)^{1/2}$ and $Z'=(1+W')^{1/2}$.

Proof. We may assume that $4\pi T = kt'k^{-1}$ where $k \in O(3)$ and $t' = diag(a, -b)$. Then it is easy to see that

$$\eta(2\pi, T; s+2, s) = \eta(2\pi, t'/(4\pi); s+2, s) = |\det(T)|^{2s}\eta(t/2, 1_{p,q}; s+2, s)$$

where $1_{p,q} = diag(1_p, -1_q)$.

By [33, pp.289, (4.16),(4.18),(4.24)], we have

$$\eta(2\pi, T; s+2, s) = 2^{6s}e^{-t/2}|\det(T)|^{2s}\xi(T, s).$$

□

Corollary 4.2.6. *Suppose that $sign(T) = (p, q)$ with $p + q = 3$. Then $W_T(e, s, \Phi)$ is holomorphic at $s = 0$ with vanishing order*

$$ord_{s=0}W_T(e, s, \Phi) \geq [\frac{q+1}{2}].$$

Proof. By Proposition 4.2.5, we know that

$$W_T(e, s, \Phi) \sim \frac{\Gamma_p(s - \frac{3-p}{2})\Gamma_q(s + \frac{q+1}{2})}{\Gamma_3(s+2)\Gamma_3(s)} \int_F e^{-(aW+bW^*)} \det(1+W)^{2s}$$
$$\times \frac{1}{\Gamma_p(s - \frac{3-p}{2})}\zeta_p(ZaZ; s+2, s - \frac{3-p}{2})\frac{1}{\Gamma_q(s + \frac{q+1}{2})}\zeta_q(Z'bZ'; s, s + \frac{q+1}{2})dw$$

where "$\sim$" means up to nowhere vanishing entire function. Lemma 4.2.4 implies that the latter two factors in the integral are entire functions. Thus we obtain that

$$ord_{s=0}W_T(e, s, \Phi) \geq ord_{s=0}\frac{\Gamma_p(s - \frac{3-p}{2})\Gamma_q(s + \frac{q+1}{2})}{\Gamma_3(s+2)\Gamma_3(s)} = [\frac{q+1}{2}].$$

□

Remark 4.2.1. 1. The same argument also applies to higher rank Whittaker integral. More precisely, let V be the $n+1$-dimensional positive definite quadratic space and Φ_0 be the standard Gaussian $e^{-2\pi tr(x,x)}$ on V^n. Then for T non-singular we have

$$order_{s=0}W_T(e, s, \Phi_0) \geq ord_{s=0}\frac{\Gamma_p(s - \frac{n-p}{2})\Gamma_q(s + \frac{q+1}{2})}{\Gamma_n(s + \frac{n+1}{2})\Gamma_n(s)} = [\frac{n-p+1}{2}] = [\frac{q+1}{2}].$$

And it is easy to see that when $T > 0$ (namely, represented by V), $W_T(e, 0, \Phi_0)$ is non-vanishing. One immediately consequence is that: $W_T(e, s, \Phi_0)$ vanishes with order precisely one at $s = 0$ only if the quadratic space with signature $(n-1, 2)$ represents T. We will see by concrete computation for $n = 3$ that the formula above actually gives the exact order of vanishing at $s = 0$. It should be true for general n but we have not tried to verify this.

Proposition 4.2.7. *When $T > 0$, we have*

$$W_T(e,0,\Phi) = \kappa(0)\Gamma_3(2)^{-1}e^{-2\pi T}.$$

Proof. Near $s = 0$, we have

$$\begin{aligned}
&\eta(2\pi, T; s+2, s)\\
=&e^{-2\pi T}\int_{x>0} e^{-2\pi x}\det(x+2T)^s\det(x)^{s-2}dx\\
=&e^{-2\pi T}\left(\int_{x>0} e^{-2\pi x}\det(2T)^s\det(x)^{s-2}dx + O(s)\right)\\
=&e^{-2\pi T}\left(\det(2T)^s(2\pi)^{-3s}\Gamma_3(s) + O(s)\right)
\end{aligned}$$

Note that

$$\Gamma_3(s) = \pi^{3/2}\Gamma(s)\Gamma(s-\frac{1}{2})\Gamma(s-1).$$

has a double pole at $s = 0$ and $\Gamma_3(s+2)$ is non-zero at $s = 0$. Thus when $s = 0$, we obtain

$$W_T(e,0,\Phi) = \kappa(0)\Gamma_3(2)^{-1}e^{-2\pi T}.$$

□

4.3 Indefinite Whittaker integrals

Now we consider a non-definite T. We will find certain nice integral representations of the central derivative of the Whittaker integral $W_T(e,s,\Phi)$ in the sequel when the sign of T is $(p,q) = (1,2)$ or $(2,1)$ respectively.

Case $(p,q) = (1,2)$

Proposition 4.3.1. *Suppose that $4\pi T \sim diag(a,-b)$, $b = diag(b_1,b_2)$. Then we have*

$$\begin{aligned}
W_T'(e,0,\Phi) = &-\frac{\kappa(0)}{2\pi^2\Gamma_3(2)}e^{t/2}\int_{\mathbb{R}^2} e^{-(a(1+w^2)+b_1(1+w_1^2)+b_2(1+w_2^2))}\zeta_2(diag(z_1,z_2),0,\frac{3}{2})\\
&\times\left(a(1+w^2)-1\right)dw_1dw_2,
\end{aligned}$$

where (z_1,z_2) are the two eigenvalues of $b(1+w'w)$ and $w^2 = w_1^2 + w_2^2$.

Proof. Recall by Prop. 4.2.5

$$W_T(e,s,\Phi) = \kappa(s)\Gamma_3(s+2)^{-1}\Gamma_3(s)^{-1}2^{6s}e^{-t/2}|\det(T)|^{2s}\xi(T,s), \tag{4.3.1}$$

where

$$\begin{aligned}
\xi(T,s) = \int_{\mathbb{R}^2} &e^{-(aW+bW^*)}\det(1+W)^{-2s}\zeta_p(ZaZ; s+2, s-\frac{3-p}{2})\\
&\times\zeta_q(Z'bZ'; s, s+\frac{q+1}{2})dw.
\end{aligned}$$

When $(p,q)=(1,2)$, $\zeta_1(ZaZ;s+2,s-1)$ has a simple pole at $s=0$. We here recall a fact that will be used frequently later, namely $\zeta_1(z;\alpha,\beta)$ satisfies a recursive relation ([33, pp. 282,(3.14)])

$$\beta\zeta_1(z,\alpha,\beta)=z\zeta_1(z,\alpha,\beta+1)-(\alpha-1)\zeta_1(z,\alpha-1,\beta+1). \tag{4.3.2}$$

Repeating this

$$(s-1)\zeta_1(z,s+2,s-1)=z\zeta_1(z,s+2,s)-(s+1)\zeta_1(z,s+1,s),$$

$$s\zeta_1(z,s+2,s)=z\zeta_1(z,s+2,s+1)-(s+1)\zeta_1(z,s+1,s+1),$$

$$s\zeta_1(z,s+1,s)=z\zeta_1(z,s+1,s+1)-s\zeta_1(z,s,s+1),$$

we obtain the residue at $s=0$

$$Res_{s=0}\zeta_1(z,s+2,s-1)=-(z^2\zeta_1(z,2,1)-2z\zeta_1(z,1,1)).$$

It is easy to see that

$$\zeta_1(z,1,1)=\int_{\mathbb{R}_+}e^{-zx}dx=\frac{1}{z}$$

and

$$\zeta_1(z,2,1)=\int_{\mathbb{R}_+}e^{-zx}(x+1)dx=\frac{1}{z}+\frac{1}{z^2}.$$

Thus we have

$$Res_{s=0}\zeta_1(z,s+2,s-1)=-z+1.$$

Suppose that $w=(w_1,w_2)$ and $b=(b_1,b_2)$. Note that $\Gamma_3(s)$ has a double pole at $s=0$ with leading term

$$\Gamma_3(s)=2\pi^{3/2}\Gamma(1/2)s^{-2}+...=2\pi^2s^{-2}+....$$

Since the trace $tr(t)=a+b_1+b_2$, we have:

$$\begin{aligned}W'_T(e,0,\Phi_\infty)=&-\frac{\kappa(0)}{2\pi^2\Gamma_3(2)}e^{t/2}\int_F e^{-(a(1+w^2)+b_1(1+w_1^2)+b_2(1+w_2^2))}\zeta_2(diag(z_1,z_2),0,\frac{3}{2})\\&\times(ZaZ-1)\,dw_1dw_2.\end{aligned}$$

Finally we note that $Z=(1+w^2)^{1/2}$. □

The next result involves the exponential integral Ei defined by

$$-Ei(-z)=\int_0^\infty\frac{e^{-z(1+t)}}{1+t}dt=e^{-z}\zeta_1(z,0,1),\quad z\in\mathbb{R}_+. \tag{4.3.3}$$

It satisfies

$$\frac{d}{dz}Ei(z) = \frac{e^z}{z}$$

and

$$Ei(z) = \gamma + log(-z) + \int_0^z \frac{e^t - 1}{t} dt$$

where γ is the Euler constant. Then it is easy to see that $Ei(z)$ has logarithmic singularity near 0.

Lemma 4.3.2. *For simplicity, we will denote*

(4.3.4)

$$F(w) = F(w_1, w_2) = e^{-(b_1(1+w_1^2)+b_2(1+w_2^2))}\zeta((z_1, z_2), 0, 3/2) = e^{-(z_1+z_2)}\zeta((z_1, z_2), 0, 3/2).$$

Then we have

$$W_T'(e, 0, \Phi) = \frac{\kappa(0)}{8\pi^2\Gamma_3(2)} e^{t/2-a} \left(\int_{\mathbb{R}^2} Ei(-aw^2)(2w_1F_1 + 2w_2F_2 + (1+w^2)\Delta F)dw - 4\pi F(0) \right)$$

where $F_i = \frac{\partial}{\partial w_i}F$ *and* $\Delta = \frac{\partial^2}{\partial w_1^2} + \frac{\partial^2}{\partial w_1^2}$ *is the Laplace operator.*

Proof. Note that

$$\Delta e^{-aw^2} = 4a(aw^2 - 1)e^{-aw^2}$$

$$\nabla Ei(-aw^2) = \frac{2e^{-aw^2}}{w^2}(w_1, w_2)$$

and

$$\Delta Ei(-aw^2) = -4ae^{-aw^2}.$$

We may thus rewrite our integral as

$$\begin{aligned}
&\int_{\mathbb{R}^2} e^{-a(1+w^2)}F(w)(a(1+w^2) - 1)dw \\
=&\int_{\mathbb{R}^2} e^{-a(1+w^2)}(aw^2 - 1)F(w)dw + \int_{\mathbb{R}^2} ae^{-a(1+w^2)}F(w)dw \\
=&1/(4a) \int \Delta e^{-a(1+w^2)}F(w)dw - (1/4)e^{-a} \int \Delta Ei(-aw^2)F(w)dw.
\end{aligned}$$

By Stokes theorem and noting that the function $Ei(z)$ has a logarithmic singularity near $z = 0$, the second term is equal to:

$$-(1/4)e^{-a} \left(\int Ei(-aw^2)\Delta F dw - \lim_{r\to 0} \int_{C_r} \nabla Ei(-aw^2)F(w)nds \right)$$

where C_r is the circle of radius r centered at the origin. It is not hard to simplify it as

$$-(1/4)e^{-a} \left(\int Ei(-aw^2)\Delta F dw - 4\pi F(0) \right).$$

Again by Stokes theorem the first term is equal to

$$-1/(4a)\int \nabla e^{-a(1+w^2)}\cdot\nabla F = 1/2\int e^{-a(1+w^2)}(w_1F_1+w_2F_2)dw.$$

Note that $\nabla Ei(-aw^2) = \frac{2e^{-aw^2}}{w^2}(w_1,w_2)$. This last term is equal to

$$(1/4)e^{-a}\int \nabla Ei(-aw^2)\cdot(w^2F_1, w^2F_2).$$

Apply Stokes again:

$$-(1/4)e^{-a}\int Ei(-aw^2)(2w_1F_1+2w_2F_2+w^2\Delta F)dw.$$

Together we have shown that

$$\begin{aligned}&\int_{\mathbb{R}^2} e^{-a(1+w^2)}F(w)(a(1+w^2)-1)dw\\ =&-\frac{1}{4}e^{-a}\left(\int Ei(-aw^2)(2w_1F_1+2w_2F_2+(1+w^2)\Delta F)dw-4\pi F(0)\right).\end{aligned}$$

□

In the following we want to find nice integral representations of $F(w)$ (4.3.4) and its various derivatives. First we deduce an integral expression of $\zeta_2(diag(z_1,z_2);0,\frac{3}{2})$ (recall (4.2.3)).

Lemma 4.3.3. *For $z=(z_1,z_2)\in \mathrm{Sym}_2(\mathbb{C})_+$, we have*

$$\zeta_2(diag(z_1,z_2);0,\frac{3}{2}) = 2\int_{x>0}\int_{y>0} e^{-z_1x-z_2y}(x+1)^{-1}(y+1)^{-1}\frac{\sqrt{xy}}{\sqrt{(x+y+1)}}dxdy.$$

Proof. By definition $\zeta_2(diag(z_1,z_2);0,\frac{3}{2})$ is given by

$$\int_{x>0}\int_{y>0} e^{-z_1x-z_2y}\int_{|t|<\sqrt{xy}}((x+1)(y+1)-t^2)^{-3/2}dtdxdy.$$

Substitute $t\to t(x+1)^{1/2}(y+1)^{1/2}$

$$\int_{x>0}\int_{y>0} e^{-z_1x-z_2y}(x+1)^{-1}(y+1)^{-1}\int_{|t|<\frac{\sqrt{xy}}{\sqrt{(x+1)(y+1)}}}(1-t^2)^{-3/2}dtdxdy.$$

It is easy to calculate the inner integral

$$2[t(1-t^2)^{-1/2}]|_0^{\frac{\sqrt{xy}}{\sqrt{(x+1)(y+1)}}} = 2\frac{\sqrt{xy}}{\sqrt{(x+y+1)}}.$$

□

Lemma 4.3.4. *We have*

$$\frac{\partial}{\partial w_1}F(w) = -4\Gamma(3/2)e^{-z_1-z_2}\int_{\mathbb{R}_+} e^{-x}\frac{\sqrt{x}}{(x+z_1)^{3/2}(x+z_2)^{3/2}}(\frac{w_1}{1+w^2}x + b_1w_1)dx \tag{4.3.5}$$

and

$$F_{11}(w) := \frac{\partial^2}{\partial w_1^2}F = -4\Gamma(3/2)e^{-z_1-z_2}\int_{\mathbb{R}_+} e^{-x}\frac{\sqrt{x}}{(x+z_1)^{3/2}(x+z_2)^{3/2}}A_{11}dx \tag{4.3.6}$$

where

$$\begin{aligned} A_{11} = &-2b_1w_1(\frac{w_1}{1+w^2}x + b_1w_1) + \frac{(1+w^2)-2w_1^2}{(1+w^2)^2}x + b_1 \\ &+ (-3/2)\frac{2b_1b_2w_1 + 2b_1w_1x}{(x+z_1)(x+z_2)}(\frac{w_1}{1+w^2}x + b_1w_1). \end{aligned}$$

Similar formula for w_2.

Proof. By Lemma 4.3.3, we have

$$\frac{\partial}{\partial w_1}F(w) = -2\frac{\partial z_1}{\partial w_1}\int e^{-z_1(1+x)-z_2(1+y)}\frac{\sqrt{xy}}{(y+1)\sqrt{1+x+y}}dxdy - 2\frac{\partial z_2}{\partial w_1}...$$

where we omit the similar term for z_2 and the integral is taken over $x, y \in \mathbb{R}_+$. All integrals in this proof below are taken over $\mathbb{R}_+$ which we hence omit.

Let us consider the integral right after $\frac{\partial z_1}{\partial w_1}$. Substitute $x \mapsto x(1+y)$:

$$\int e^{-z_1x-z_2y}\frac{\sqrt{xy}}{(y+1)\sqrt{1+x+y}}dxdy = \int e^{-z_1x(1+y)-z_2y}\frac{\sqrt{x(1+y)y}}{(y+1)\sqrt{1+x(1+y)+y}}(1+y)dxdy.$$

This can be simplified:

$$\int e^{-z_1x-y(z_1x+z_2)}\frac{\sqrt{xy}}{\sqrt{1+x}}dxdy.$$

Substitute $y \mapsto y(z_1x+z_2)^{-1}$ and separate variables:

$$\int e^{-z_1x}\frac{\sqrt{x}}{\sqrt{1+x}(z_1x+z_2)^{3/2}}dx\int e^{-y}y^{1/2}dy.$$

Substitute $x \mapsto xz_1^{-1}$:

$$\frac{1}{z_1}\Gamma(3/2)\int e^{-x}\frac{\sqrt{x}}{\sqrt{x+z_1}(x+z_2)^{3/2}}dx.$$

We have similar expression for the integral right after $\frac{\partial z_2}{\partial w_1}$. Thus we have

$$\frac{\partial}{\partial w_1}F(w) = -2\Gamma(3/2)e^{-z_1-z_2}\int e^{-x}\frac{\sqrt{x}}{(x+z_1)^{3/2}(x+z_2)^{3/2}}(\frac{\partial}{\partial w_1}\log(z_1z_2)x + \frac{\partial}{\partial w_1}(z_1+z_2))dx.$$

Note that $z_1z_2 = b_1b_2(1+w^2), z_1+z_2 = b_1(1+w_1^2)+b_2(1+w_2^2)$:

$$\frac{\partial}{\partial w_1}\log(z_1z_2) = \frac{2w_1}{1+w^2}, \quad \frac{\partial}{\partial w_1}(z_1+z_2) = 2b_1w_1.$$

From this we deduce further that

$$F_{11}(w) = -4\Gamma(3/2)e^{-z_1-z_2}\int e^{-x}\frac{\sqrt{x}}{(x+z_1)^{3/2}(x+z_2)^{3/2}}A_{11}dx,$$

where

$$A_{11} = -2b_1w_1(\frac{w_1}{1+w^2}x+b_1w_1)+\frac{(1+w^2)-2w_1^2}{(1+w^2)^2}x+b_1+(-\frac{3}{2})\frac{2b_1b_2w_1+2b_1w_1x}{(x+z_1)(x+z_2)}(\frac{w_1}{1+w^2}x+b_1w_1).$$

Similarly we have

$$F_{22}(w) = -4\Gamma(3/2)e^{-z_1-z_2}\int e^{-x}\frac{\sqrt{x}}{(x+z_1)^{3/2}(x+z_2)^{3/2}}A_{22}dx,$$

where

$$A_{22} = -2b_2w_2(\frac{w_2}{1+w^2}x+b_2w_2)+\frac{(1+w^2)-2w_2^2}{(1+w^2)^2}x+b_2+(-\frac{3}{2})\frac{2b_1b_2w_2+2b_2w_2x}{(x+z_1)(x+z_2)}(\frac{w_2}{1+w^2}x+b_2w_2).$$

□

Proposition 4.3.5. *We have*

$$W'_T(e,0,\Phi_\infty) = \frac{\kappa(0)}{8\pi^2\Gamma_3(2)}e^{t/2-a}\left(-4\pi e^{-b_1-b_2}\zeta_2((b_1,b_2),0,3/2)+\xi(T)\right)$$

where

$$\xi(T) = -4\Gamma(3/2)\int_{\mathbb{R}^2}Ei(-aw^2)e^{-z_1-z_2}(\int_{\mathbb{R}}e^{-u^2}\frac{-2(z_1+z_2-1-b_1-b_2)}{(u^2+z_1)^{1/2}(u^2+z_2)^{1/2}}du$$
$$+\int_{\mathbb{R}}e^{-u^2}\frac{(2z_1z_2-2b_1b_2-z_1-z_2)u^2+2z_1z_2(z_1+z_2-1-b_1-b_2)}{(u^2+z_1)^{1/2}(u^2+z_2)^{3/2}}du)dw.$$

Proof. Recall that we have

$$W'_T(e,0,\Phi) = \frac{\kappa(0)}{8\pi^2\Gamma_3(2)}e^{t/2-a}\left(\int_{\mathbb{R}^2}Ei(-aw^2)(2w_1F_1+2w_2F_2+(1+w^2)\Delta F)dw - 4\pi F(0)\right).$$

By Lemma 4.3.4, we obtain that

$$\Delta F(w) = -4\Gamma(3/2)e^{-z_1-z_2}\int_{\mathbb{R}_+}e^{-x}\frac{\sqrt{x}}{(x+z_1)^{3/2}(x+z_2)^{3/2}}Adx,$$

where

$$A=-2\frac{b_1w_1^2+b_2w_2^2}{1+w^2}x-2(b_1^2w_1^2+b_2^2w_2^2)+\frac{2}{(1+w^2)^2}x+b_1+b_2$$
$$+(-\frac{3}{2})\frac{2}{(x+z_1)(x+z_2)}(\frac{b_1w_1^2+b_2w_2^2}{1+w^2}x^2+\frac{b_1b_2w^2}{1+w^2}x+(b_1^2w_1^2+b_2^2w_2^2)x+b_1b_2(b_1w_1^2+b_2w_2^2)).$$

And we have

$$2(w_1F_1(w)+w_2F_2(w))=-4\Gamma(3/2)e^{-z_1-z_2}\int e^{-x}\frac{\sqrt{x}}{(x+z_1)^{3/2}(x+z_2)^{3/2}}Bdx,$$

where

$$B=\frac{2w^2x}{1+w^2}+2(b_1w_1^2+b_2w_2^2).$$

Together we obtain

$$(2(w_1F_1(w)+w_2F_2(w))+(1+w^2)\Delta F)(-4\Gamma(3/2))^{-1}=e^{-z_1-z_2}\int_{\mathbb{R}_+}e^{-x}\frac{\sqrt{x}}{(x+z_1)^{3/2}(x+z_2)^{3/2}}Cdx,$$

where C is given by

$$-2(b_1w_1^2+b_2w_2^2-1)x-(b_1-b_2)(w_1^2-w_2^2)+(b_1(1+w_1^2)+b_2(1+w_2^2))+2b_1b_2w^2-2(b_1-b_2)^2w_1^2w_2^2$$
$$-2(b_1w_1^2+b_2w_2^2)(b_1(1+w_1^2)+b_2(1+w_2^2))-3(b_1-b_2)^2w_1^2w_2^2\frac{x}{(x+z_1)(x+z_2)}.$$

We substitute $x\mapsto u^2$ and change the domain of integration from $x\in\mathbb{R}_+$ to $u\in\mathbb{R}$.

$$(2(w_1F_1(w)+w_2F_2(w))+(1+w^2)\Delta F)(-4\Gamma(3/2))^{-1}=e^{-z_1-z_2}\int_{\mathbb{R}}e^{-u^2}\frac{u^2}{(u^2+z_1)^{3/2}(u^2+z_2)^{3/2}}Cdu.$$

To finish the proof we need to compare the integral in the RHS of the above with:

$$\int_{\mathbb{R}}e^{-u^2}(\frac{-2(z_1+z_2-1-b_1-b_2)}{(u^2+z_1)^{1/2}(u^2+z_2)^{1/2}}+\frac{(2z_1z_2-2b_1b_2-z_1-z_2)u^2+2z_1z_2(z_1+z_2-1-b_1-b_2)}{(u^2+z_1)^{1/2}(u^2+z_2)^{3/2}})du,$$

which is also equal to

$$\int e^{-u^2}\frac{-2(b_1w_1^2+b_2w_2^2-1)u^4-2(b_1w_1^2+b_2w_2^2)(z_1+z_2)u^2+(2b_1b_2w^2+z_1+z_2)u^2}{(u^2+z_1)^{3/2}(u^2+z_2)^{3/2}}du.$$

Therefore it suffices to prove that the following integral vanishes:

$$(b_1-b_2)^2\int Ei(-aw^2)w_1^2w_2^2e^{-z_1-z_2}\int e^{-u^2}\frac{2u^2}{((u^2+z_1)(u^2+z_2))^{3/2}}+\frac{3u^4}{((u^2+z_1)(u^2+z_2))^{5/2}}dudw$$
$$+(b_1-b_2)\int Ei(-aw^2)(w_1^2-w_2^2)e^{-z_1-z_2}\int e^{-u^2}\frac{u^2}{((u^2+z_1)(u^2+z_2))^{3/2}}dudw.$$

By the definition $Ei(-aw^2) = -\int_1^\infty e^{-aw^2u}u^{-1}du$, it suffices to prove that the following integral vanishes

$$(b_1-b_2)\int e^{-aw^2-b_1w_1^2-b_2w_2^2}w_1^2w_2^2\int e^{-u^2}\frac{2u^2}{((u^2+z_1)(u^2+z_2))^{3/2}}+\frac{3u^4}{((u^2+z_1)(u^2+z_2))^{5/2}}dudw$$
$$+\int e^{-aw^2-b_1w_1^2-b_2w_2^2}(w_1^2-w_2^2)\int e^{-u^2}\frac{u^2}{((u^2+z_1)(u^2+z_2))^{3/2}}dudw.$$

We substitute $X=w_1^2+w_2^2$ and $Y=w_1^2-w_2^2$. Then we have

$$dXdY = 2w_1w_2dw_1dw_2 = \sqrt{X^2-Y^2}dw_1dw_2$$

and

$$\int e^{-aw^2-b_1w_1^2-b_2w_2^2}(w_1^2-w_2^2)\frac{u^2}{((u^2+z_1)(u^2+z_2))^{3/2}}dw$$
$$=\int_{X\geq 0}\int_{-X\leq Y\leq X}e^{-(a+b_1/2+b_2/2)X-(b_1-b_2)Y/2}Y\frac{u^2}{((u^2+z_1)(u^2+z_2))^{3/2}}\frac{dY}{\sqrt{X^2-Y^2}}dX.$$

We apply integration by parts to the inner integral

$$-\int_{X\geq 0}e^{-(a+b_1/2+b_2/2)X}\int_{-X\leq Y\leq X}e^{-(b_1-b_2)Y/2}\frac{u^2}{((u^2+z_1)(u^2+z_2))^{3/2}}d\sqrt{X^2-Y^2}dX$$
$$=\int_{X\geq 0}e^{-(a+b_1/2+b_2/2)X}\int_{-X\leq Y\leq X}\sqrt{X^2-Y^2}e^{-(b_1-b_2)Y/2}\frac{u^2}{((u^2+z_1)(u^2+z_2))^{3/2}}$$
$$(-\frac{b_1-b_2}{2}-\frac{3}{2}\frac{b_1-b_2}{2}\frac{u^2}{(u^2+z_1)(u^2+z_2)})dYdX.$$

We may simplify it and plug back

$$-(b_1-b_2)/2\int_{X\geq 0}e^{-(a+b_1/2+b_2/2)X}\int_{-X\leq Y\leq X}\sqrt{X^2-Y^2}e^{-(b_1-b_2)Y/2}\frac{u^2}{((u^2+z_1)(u^2+z_2))^{3/2}}$$
$$(2+\frac{3u^2}{(u^2+z_1)(u^2+z_2)})dYdX$$
$$=-(b_1-b_2)\int e^{-aw^2-b_1w_1^2-b_2w_2^2}w_1^2w_2^2(\frac{2u^2}{((u^2+z_1)(u^2+z_2))^{3/2}}+\frac{3u^4}{((u^2+z_1)(u^2+z_2))^{5/2}})dw.$$

This proves the desired vanishing result.

Finally note that $F(0)=e^{-b_1-b_2}\zeta_2((b_1,b_2),0,3/2)$ and we complete the proof.

□

Case $(p,q)=(2,1)$

Lemma 4.3.6. $\zeta_2([z_1,z_2];s+2,s-\frac{1}{2})$ *has a simple pole at* $s=0$ *with residue given by*

$$\frac{\sqrt{\pi}}{2}\left(\int_{\mathbb{R}}e^{-u^2}\frac{(4z_1z_2-(z_1+z_2))u^2+2z_1z_2(z_1+z_2-1)}{(u^4+u^2(z_1+z_2)+z_1z_2)^{3/2}}du+\int_{\mathbb{R}}e^{-u^2}\frac{4z_1z_2-2z_1-2z_2+2}{(u^4+u^2(z_1+z_2)+z_1z_2)^{1/2}}du\right).$$

Proof. By [33, pp. 283], we have an integral representation when $\mathrm{Re}(s) > 1$

$$\begin{aligned}&\zeta_2([z_1, z_2]; s+2, s-\frac{1}{2})\\ =&\int_{\mathbb{R}} e^{-z_2 w^2}(1+w^2)^{2s-1/2}\zeta_1(z_1+z_2w^2, s+2, s-1/2)\zeta_1(z_2(1+w^2), s+3/2, s-1)dw.\end{aligned}$$

By (4.3.2) we have

$$(s-1/2)\zeta_1(z, s+2, s-1/2) = z\zeta_1(z, s+2, s+1/2) + (-s-1)\zeta_1(z, s+1, s+1/2).$$

It is easy to obtain

$$\zeta_1(z, 2, 1/2) = \int_{\mathbb{R}_+} e^{-zx}(1+x)x^{-1/2}dx = z^{-1/2}\Gamma(1/2) + z^{-3/2}\Gamma(3/2)$$

and

$$\zeta_1(z, 1, 1/2) = \int_{\mathbb{R}_+} e^{-zx}x^{-1/2}dx = z^{-1/2}\Gamma(1/2).$$

Therefore we obtain

$$\zeta_1(z, 2, -1/2) = -\Gamma(1/2)z^{-1/2}(2z-1).$$

By (4.3.2) again we have

$$(s-1)\zeta_1(z, s+3/2, s-1) = z\zeta_1(z, s+3/2, s) + (-s-1/2)\zeta_1(z, s+1/2, s).$$

We may obtain

$$\begin{aligned}&Res_{s=0}\zeta_1(z, s+3/2, s-1)\\ =&-z(z\zeta_1(z, 3/2, 1) - \frac{1}{2}\zeta_1(z, 1/2, 1)) + \frac{1}{2}(z\zeta_1(z, 1/2, 1) + \frac{1}{2}\zeta_1(z, -1/2, 1))\\ =&-z^2\zeta_1(z, 3/2, 1) + z\zeta_1(z, 1/2, 1) + \frac{1}{4}\zeta_1(z, -1/2, 1).\end{aligned}$$

Applying integration by parts to the first and third integrals, we may evaluate the sum:

$$Res_{s=0}\zeta_1(z, s+3/2, s-1) = -z + \frac{1}{2}.$$

Therefore we obtain the residue of $\zeta_2([z_1, z_2]; s+2, s-\frac{1}{2})$ as an integral

$$2\Gamma(1/2)\int_{\mathbb{R}} e^{-z_2w^2}(1+w^2)^{-1/2}(z_1+z_2w^2)^{-1/2}(z_1+z_2w^2-\frac{1}{2})(z_2(1+w^2)-\frac{1}{2})dw.$$

Substitute $u = z_2 w^2$ to obtain

$$2\Gamma(1/2)\int_{\mathbb{R}_+} e^{-u}(z_1+u)^{-1/2}(z_2+u)^{-1/2}(u+z_1-1/2)(u+z_2-1/2)u^{-1/2}du.$$

Now let $A = z_1+z_2-1/2, B = -1/2$ so that $A+B = z_1+z_2-1$. Then we can split the integral into three pieces:

$$\int_{u\in\mathbb{R}} e^{-u^2}\frac{(u^2+z_1-1/2)(u^2+z_2-1/2)}{(z_1+u^2)^{1/2}(z_2+u^2)^{1/2}}du = I + II + III$$

where

$$I = \int_{\mathbb{R}} e^{-u^2}\frac{u^4+Au^2}{(u^4+u^2(z_1+z_2)+z_1z_2)^{1/2}}du$$

$$II = \int_{\mathbb{R}} e^{-u^2}\frac{Bu^2-1/4}{(u^4+u^2(z_1+z_2)+z_1z_2)^{1/2}}du$$

and

$$III = \int_{\mathbb{R}} e^{-u^2}\frac{1}{4}\frac{4z_1z_2-2z_1-2z_2+2}{(u^4+u^2(z_1+z_2)+z_1z_2)^{1/2}}du.$$

Now, we rewrite the first integral and apply integration by parts

$$\begin{aligned} I &= -\frac{1}{2}\int_{\mathbb{R}}\frac{u^3+Au}{(u^4+u^2(z_1+z_2)+z_1z_2)^{1/2}}de^{-u^2} \\ &= \frac{1}{2}\int_{\mathbb{R}} e^{-u^2}\left(\frac{3u^2+A}{(u^4+u^2(z_1+z_2)+z_1z_2)^{1/2}} + \frac{(u^3+Au)(-\frac{1}{2})(4u^3+2u^2(z_1+z_2))}{(u^4+u^2(z_1+z_2)+z_1z_2)^{3/2}}\right)du \end{aligned}$$

which can be simplified

$$\int_{\mathbb{R}} e^{-u^2}\frac{\frac{1}{2}u^2+\frac{1}{4}}{(u^4+u^2(z_1+z_2)+z_1z_2)^{1/2}}du + \frac{1}{2}\int_{\mathbb{R}} e^{-u^2}\frac{(A(z_1+z_2)-z_1^2-z_2^2)u^2+2z_1z_2A-z_1z_2(z_1+z_2)}{(u^4+u^2(z_1+z_2)+z_1z_2)^{3/2}}du.$$

Notice that the first term cancels II. Plugging $A = z_1+z_2-1/2$ into the above, we obtain

$$\begin{aligned} &\int_{u\in\mathbb{R}} e^{-u^2}\frac{(u^2+z_1-1/2)(u^2+z_2-1/2)}{(z_1+u^2)^{1/2}(z_2+u^2)^{1/2}}du \\ &= \frac{1}{4}\int_{\mathbb{R}} e^{-u^2}\frac{(4z_1z_2-(z_1+z_2))u^2+2z_1z_2(z_1+z_2-1)}{(u^4+u^2(z_1+z_2)+z_1z_2)^{3/2}}du + \frac{1}{4}\int_{\mathbb{R}} e^{-u^2}\frac{4z_1z_2-2z_1-2z_2+2}{(u^4+u^2(z_1+z_2)+z_1z_2)^{1/2}}du. \end{aligned}$$

$\square$

Proposition 4.3.7. *Suppose that $4\pi T \sim diag(a, -b)$, $a = diag(a_1, a_2)$. Then we have an integral representation*

$$W'_T(e, 0, \Phi_\infty) = -\frac{\kappa(0)}{4\pi^{3/2}\Gamma_3(2)} e^{t/2} \int_{\mathbb{R}^2} Ei(-b(1+w^2)) e^{-a_1(1+w_1^2)-a_2(1+w_2^2)} dw$$
$$\left(\int_{\mathbb{R}} e^{-u^2} \frac{(4z_1z_2 - (z_1+z_2))u^2 + 2z_1z_2(z_1+z_2-1)}{(u^4+u^2(z_1+z_2)+z_1z_2)^{3/2}} du + \int_{\mathbb{R}} e^{-u^2} \frac{4z_1z_2 - 2z_1 - 2z_2 + 2}{(u^4+u^2(z_1+z_2)+z_1z_2)^{1/2}} du \right)$$

where z_1, z_2 are the two eigenvalues of ZaZ.

Proof. Recall by Prop. 4.2.5

$$W_T(e, s, \Phi) = \kappa(s)\Gamma_3(s+2)^{-1}\Gamma_3(s)^{-1} 2^{6s} e^{t/2} |\det(T)|^{2s} \xi(T, s) \tag{4.3.7}$$

where

$$\xi(T, s) = \int_{\mathbb{R}^2} e^{-(aW+bW^*)} \det(1+W)^{-2s} \zeta_2(ZaZ; s+2, s-\frac{1}{2})$$
$$\times \zeta_1(Z'bZ'; s, s+1) dw.$$

Note that

$$\zeta_1(z; 0, 1) = \int_{\mathbb{R}_+} e^{-zx}(x+1)^{-1} dx = -e^z Ei(-z).$$

Now the statement follows from the previous Lemma and that $\Gamma_3(s) = 2\pi^2 s^{-2} + \ldots$. □

4.4 Holomorphic projection

In this section, we want to study holomorphic projection of $E'(g, 0, \Phi)$.

Firstly let us try to study holomorphic projection for a cusp form φ on $\mathrm{GL}_2(\mathbb{A})$. Fix a non-trivial additive character ψ of $F\backslash\mathbb{A}$, say $\psi = \psi_0 \circ \mathrm{tr}_{F/\mathbb{Q}}$ with ψ_0 the standard additive character on $\mathbb{Q}\backslash\mathbb{A}_{\mathbb{Q}}$, and let W be the corresponding Whittaker function:

$$W_\varphi(g) = \int_{F\backslash\mathbb{A}} \varphi(n(b)g)\psi(-b) db.$$

Then φ has a Fourier expansion

$$\varphi(g) = \sum_{a \in F^\times} W_\varphi\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} g\right).$$

We say that φ is *holomorphic of weight* 2 , if $W_\Phi = W_\infty \cdot W_f$ has a decomposition with W_∞ satisfying the following properties:

$$W_\infty(g) = \begin{cases} y e^{2\pi i(x+iy)} e^{2i\theta} & \text{if } y > 0 \\ 0 & \text{otherwise} \end{cases}$$

for the decomposition of $g \in \mathrm{GL}_2(\mathbb{R})$:

$$g = z \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}.$$

For any Whittaker function W of $\mathrm{GL}_2(\mathbb{A})$ which is holomorphic of weight 2 as above with $W_f(g_f)$ compactly supported modulo $Z(\mathbb{A}_f)N(\mathbb{A}_f)$, the Poinaré series is define as follows:

$$\varphi_W(g) := \lim_{t\to 0+} \sum_{\gamma \in Z(F)N(F)\backslash G(F)} W(\gamma g)\delta(\gamma g)^t,$$

where

$$\delta(g) = |a_\infty / d_\infty|, \qquad g = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} k, \qquad k \in K$$

where K is the standard maximal compact subgroup of $\mathrm{GL}_2(\mathbb{A})$. Let φ be a cusp form and assume that both W and φ have the same central character. Then we can compute their inner product as follows:

$$\begin{aligned}
(\varphi, \varphi_W) &= \int_{Z(\mathbb{A})\mathrm{GL}_2(F)\backslash \mathrm{GL}_2(\mathbb{A})} \varphi(g)\bar{\varphi}_W(g)dg \\
&= \lim_{t\longrightarrow 0} \int_{Z(\mathbb{A})N(F)\backslash \mathrm{GL}_2(\mathbb{A})} \varphi(g)\bar{W}(g)\delta(g)^t dg \\
&= \lim_{t\longrightarrow 0} \int_{Z(\mathbb{A})N(\mathbb{A})\backslash \mathrm{GL}_2(\mathbb{A})} W_\varphi(g)\bar{W}(g)\delta(g)^t dg. \qquad (4.4.1)
\end{aligned}$$

Let φ_0 be the holomorphic projection of φ in the space of holomorphic forms of weight 2. Then we may write

$$W_{\Phi_0}(g) = W_\infty(g_\infty)W_{\varphi_0}(g_f)$$

with W_∞ as above. Then (11.1) is a product of integrals over finite places and integrals at infinite places:

$$\int_{Z(\mathbb{R})N(\mathbb{R})\backslash \mathrm{GL}_2(\mathbb{R})} |W_\infty(g_\infty)|^2 dg = \int_0^\infty y^2 e^{-4\pi y} dy/y^2 = (4\pi)^{-1}.$$

In other words, we have

$$(\varphi, \varphi_W) = (4\pi)^{-g} \int_{Z(\mathbb{A}_f)N(\mathbb{A}_f)\backslash \mathrm{GL}_2(\mathbb{A}_f)} W_{\varphi_0}(g_f)\bar{W}(g_f)dg_f. \qquad (4.4.2)$$

As $\bar{W}$ can be any Whittaker function with compact support modulo $Z(\mathbb{A}_f)N(\mathbb{A}_f)$, the combination of (10.1) and (10.2) gives

Lemma 4.4.1. *Let φ be a cusp form with trivial central character at each infinite place. Then the holomorphic projection φ_0 of Φ has Wittacher function $W_\infty(g_\infty)W_{\varphi_0}(g_f)$ with $W_{\varphi_0}(g_f)$ given as follows:*

$$W_{\varphi_0}(g_f) = (4\pi)^g \lim_{t\longrightarrow 0+} \int_{Z(F_\infty)N(F_\infty)\backslash \mathrm{GL}_2(F_\infty)} W_\varphi(g_\infty g_f)\bar{W}_\infty(g_\infty)\delta(g_\infty)^t dg_\infty.$$

For more details, see [36, §6.4,6.5].

5 Local triple height pairings

In this section, we want to compute the local triple height pairings of Hecke operators at the unramified places and archimedean places.

For unramified places, we first study the modular interpretation of Hecke operators and reduce the question to the work of Gross–Keating on deforming endomorphisms of formal groups.

For archimedean places, we introduce Green functions for Hecke correspondences and compute their star product. The hard part is to relate the star product to the archimedean Whittaker function.

5.1 Modular interpretation of Hecke operators

In this section, we would like to study the reduction of Hecke operators. For an $x \in \mathbb{V}$ with positive norm in F, the cycle $Z(x)_K$ is the graph of the Hecke operator given by the coset UxU. Namely, $Z(x)_K$ is the correspondence defined by maps:

$$Z(x)_K \simeq Y_{U\cap xUx^{-1}} \longrightarrow Y_U \times Y_U.$$

Moduli interpretation at an archimedean place

First let us give some moduli interpretation of Hecke operators at an archimedean place τ. Let $B = B(\tau)$ be the nearby quaternion algebra. If we decompose $UxU = \coprod x_i U$, then $Z(x)_K$ as a correspondence sends one object $(V, h, \bar{\kappa})$ to sum of $(V, h, \overline{\kappa x_i})$. In other words, we may write abstractly,

$$Z(x)_K(V, h, \bar{\kappa}) = \sum_i (V_i, h_i, \bar{\kappa}_i), \tag{5.1.1}$$

where the sum is over the isomorphism class of $(V_i, h_i, \bar{\kappa}_i)$ such that there is an isomorphism $y_i : (V_i, h_i) \longrightarrow (V, h)$ such that the induced diagram is commutative:

$$\begin{tikzcd} \widehat{V}_0 \arrow[r, "\kappa_i"] \arrow[d, "x_i"] & \widehat{V}_i \arrow[d, "\widehat{y}_i"] \\ \widehat{V}_0 \arrow[r, "\kappa"] & \widehat{V} \end{tikzcd} . \tag{5.1.2}$$

Replacing κ and κ_i by equivalent classes, we may assume that $x_i = x$. Thus the subvariety $Z(x)_K$ of M_K parameterizes the triple:

$$(V_1, h_1, \bar{\kappa}_1), \qquad (V_2, h_2, \bar{\kappa}_2), \quad y,$$

where the first two are objects as described as above for $\bar{k}_1$ and $\bar{k}_2$ level structures modulo $U_1 := U \cap xUx^{-1}$ and $U_2 = U \cap x^{-1}Ux$ respectively, and $y : (V_2, h_2) \longrightarrow (V_1, h_1)$ such that the diagram

$$
\text{(5.1.3)} \qquad \begin{array}{ccc}
\widehat{V}_0 & \xrightarrow{\kappa_2} & \widehat{V}_2 \\
\downarrow x & & \downarrow \widehat{y} \\
\widehat{V}_0 & \xrightarrow{\kappa_1} & \widehat{V}_1
\end{array}
$$

is commutative.

Now we want to describe the above moduli interpretation with an integral Hodge structure with respect to a maximal open compact subgroup of the form $\widehat{\mathscr{O}}_B^\times = \mathscr{O}_{\mathbb{B}}^\times$ containing U, where $\mathscr{O}_B$ is a maximal order of B. Let $V_{0,\mathbb{Z}} = \mathscr{O}_B$ as an $\mathscr{O}_B$-lattice in V_0. Then for any triple $(V, h, \bar{\kappa})$ we obtain a triple $(V_{\mathbb{Z}}, h, \bar{\kappa})$ with $V_{\mathbb{Z}} = \kappa(V_{0\mathbb{Z}})$ which satisfies the analogous properties as above. In fact, M_U parameterizes such integral triples. The Hecke operator $Z(x)_K$ has the following expression:

$$Z(x)_K(V_{\mathbb{Z}}, h, \bar{\kappa}) = \sum_i (V_{i\mathbb{Z}}, h_i, \kappa_i)$$

where $V_{i\mathbb{Z}} = \kappa_i(V_{0\mathbb{Z}})$. We can't replace terms in the above diagram by integral lattices as y_i and x_i only define a quasi-isogeny:

$$y_i \in \mathrm{Hom}_{\mathscr{O}_B}(V_{i\mathbb{Z}}, V_{\mathbb{Z}}) \otimes F, \qquad x_i \in \widehat{B} = \mathrm{End}_{\mathscr{O}_B}(\widehat{V}_{0,\mathbb{Z}}) \otimes F.$$

When U is sufficiently small, we have universal objects $(V_U, h, \bar{k})$, $(V_{U,\mathbb{Z}}, h, \bar{\kappa})$. We will also consider the divisible $\mathscr{O}_B$-module $\widetilde{V}_U = \widehat{V}_U / \widehat{V}_{U,\mathbb{Z}}$. The subvariety $Z(x)_K$ also has a universal object $y : V_{U_2} \longrightarrow V_{U_1}$.

Let us return to curves Y_U over F. Though the rational structure V at a point on Y_U does not make sense, the local system $\widehat{V}$ and $\widehat{V}_{\mathbb{Z}}$ make sense as $\mathbb{B}_f$ and $\mathscr{O}_{\mathbb{B}_f}$ modules respectively. The Hecke operator parameterizes the morphism $\widehat{y} : \widehat{V}_{U_2} \longrightarrow \widehat{V}_{U_1}$.

Modular interpretation at a finite place

We would like to give a moduli interpretation for the Zariski closure $\mathscr{Z}(x)_K$ of $Z(x)_K$. The isogeny $y : \widehat{V}_{U_2} \longrightarrow \widehat{V}_{U_1}$ induces a quasi-isogeny on divisible $\mathscr{O}_{\mathbb{B}_f}$-modules. For prime to v-part, this is the same as over generic fiber. We need to describe the quasi-isogeny on formal modules. First lets us assume that $U_v = \mathscr{O}_{\mathbb{B}_v}^\times$ is maximal.

If v is not split in $\mathbb{B}$, then $U_{1v} = U_{2v} = U_v$. Thus the condition on y_v on the generic fiber is just required to have order equal to $\mathrm{ord}(\nu(x))$. Hence $\mathscr{Z}(x)_K$ parameterizes the quasi-isogeny of pairs whose order at v has order x. Recall from §5.3 in our Asia journal paper that the notion of quasi-isogeny as quasi-isogeny of divisible module which can be lifted to the generic fiber.

If v is split in $\mathbb{B}$, then we may choose an isomorphism $\mathscr{O}_{\mathbb{B}_v} = M_2(\mathscr{O}_v)$. Then the formal module $\mathscr{A}$ is a direct sum $\mathscr{E} \oplus \mathscr{E}$ where $\mathscr{E}$ is a divisible $\mathscr{O}_F$-module of dimension 1 and height 2. By replacing x by an element in $U_v x U_v$ we may assume that x_v is diagonal:

$x_v = \begin{pmatrix} \varpi^c & \\ & \varpi^d \end{pmatrix}$ with $c, d \in \mathbb{Z}$ and $c \le d$. It is clear that the condition on y on the generic fiber is a composition of a scalar multiplication by ϖ_v^c (as a quasi-isogeny) and an isogeny with kernel isomorphic to the cyclic module $\mathscr{O}_v/\varpi^{d-a}\mathscr{O}_v$. Thus the scheme $\mathscr{Z}(x)_K$ parameterizes quasi-isogenies f of geometric points of type (c, d) in the following sense:

1. the v-component $\varpi^{-c}y_v : \mathscr{E}_2 \longrightarrow \mathscr{E}_1$ is an isogeny;
2. the kernel of $\varpi^{-c}y_v$ is cyclic of order $d - c$ in the sense that it is the image of a homomorphism $\mathscr{O}_v/\varpi^{d-c} \longrightarrow \mathscr{E}_2$.

We also call such a quasi-isogeny of *type* (a, b). Notice that the number a, b can be defined without reference to U_v. Indeed, a is the minimal integer such that $\varpi^{-a}x_v$ is integral over $\mathscr{O}_v$ and that $a + b = \mathrm{ord}(\det x_v)$.

5.2 Supersingular points on Hecke correspondences

For a geometric point in M_K with formal object $\mathscr{E}_1, \mathscr{E}_2$, by Serre–Tate theory, the formal neighborhood $\mathscr{D}$ is the product of universal deformations $\mathscr{D}_i$ of $\mathscr{E}_i$. The divisor of $\mathscr{Z}(x)_K^{ss}$ in this neighborhood is defined as the sum of the universal deformation of quasi-isogenies. In the following, we want to study the behaviors of this divisor in a formal neighborhood of a pair of surpersingular points on M_K when $U = U_vU^v$ with U_v maximal.

Supersingular points on Y_U and M_K

Recall from §5.4 in our Asia journal paper, all supersingular points on Y_U are isogenous to each other. Fix one of the supersingular point P_0 representing the triple $(\mathscr{A}_0, \widetilde{V}_0^v, \bar{\kappa}_0^v)$. Let $B = \mathrm{End}^0(P_0)$ which is a quaternion algebra over F obtained from $\mathbb{B}$ by changing invariants at v. We may use κ_0 to identify $\widetilde{V}_0$ with $\widehat{V}_0/\widehat{V}_{0\mathbb{Z}}$. The action of $(B \otimes \mathbb{A}_f^v)^\times$ and $(\mathbb{B}_f^v)^\times$ both acts on $\widetilde{V}_0$. We may use κ_0 to identify them. In this way, the set $\mathscr{Y}_U^{ss}$ of supersingular point is identified with

$$\mathscr{Y}_{U,v}^{ss} = B_0 \backslash (B \otimes \mathbb{A}_f^v)^\times / U^v$$

so that the element $g \in (B \otimes \mathbb{A}_f^v)^\times$ represents the triple

$$(\mathscr{A}_0, \widehat{V}_0^v, gU^v),$$

where B_0 means the subgroup of $B^\times$ of elements with order 0 at v.

The supersingular points on M_K will be represented by a pairs of elements in $(B \otimes \mathbb{A}_f^v)^\times$ with the same norm. Thus we can describe the set of supersingular points on $\mathscr{M}_K$ using orthogonal space $V = (B, q)$ and the Spin similitudes:

$$H = \mathrm{GSpin}(V) = \{(g_1, g_2) \in B^\times, \quad \nu(g_1) = \nu(g_2)\},$$

which acts as

$$(g_1, g_2)x = g_1 x g_2^{-1}, \qquad g_i \in B^\times, x \in V.$$

We then have a bijection

$$\mathscr{M}_{K,v}^{ss} \simeq H(F)_0 \backslash H(\mathbb{A}_f^v)/K^v.$$

Supersingular points on $\mathscr{Z}(x)_K$

The set $\mathscr{Z}(x)_{K,v}^{ss}$ of supersingular points on the cycle $\mathscr{Z}(x)_K$ represents the isogeny $y : P_2 \longrightarrow P_1$ of two supersingular points of level $U_1 = U \cap xUx^{-1}$ and $U_2 = U \cap x^{-1}Ux$. In terms of triples as above, $\mathscr{Z}(x)_K^{ss}$ represents equivalent classes of the triples (g_1, g_2, y) of elements $g_i \in (B \otimes \mathbb{A}_f^v)^\times/U_i$ and $y \in B^\times$ with following properties

$$g_1^{-1} y^v g_2 = x^v, \qquad \mathrm{ord}_v(\det(x_v)) = \mathrm{ord}_v(q(y_v)). \tag{5.2.1}$$

Two triples (g_1, g_2, y) and (g_1', g_2', y') are equivalent if there is a $\gamma_i \in B_0^\times$ such that

$$\gamma_i g_i = g_i', \qquad \gamma_1 y \gamma_2^{-1} = y'. \tag{5.2.2}$$

By (5.2.1), the norms of g_1 and g_2 are in the same class modulo $F_+^\times$. Thus by (5.2.2) we may modify them so that they have the same norm. Thus in term of the group H, we may rewrite condition (5.2.1) as

$$x^v = g^{-1} y^v, \qquad g = (g_1, g_2) \in H(\mathbb{A}_f^v). \tag{5.2.3}$$

This equation is always solvable in g, y for given x. Indeed, since the norm of x is positive, we have an element $y \in B$ with the same norm as x. Then there is a $g \in H(\mathbb{A}_f^v)$ such that $x = g^{-1} y^v$ in $\widehat{V}^v$. In summary, we have shown the following description of $\mathscr{Z}(x)_{K,v}^{ss}$:

Lemma 5.2.1. *Let (y, g) be a solution to (5.2.3) and H_y be the stabilizer of y. Then we have*

$$\begin{aligned}\mathscr{Z}(x)_{K,v}^{ss} =& H(F)_0 \backslash H(F)_0 (H_y(\mathbb{A}_f^v) g) K^v / K^v \\ \simeq & H_y(F)_0 \backslash H_y(\mathbb{A}_f^v)/K_y,\end{aligned}$$

where $K_y := H_y(\mathbb{A}_f^v) \cap g K^v g^{-1}$.

Supersingular formal neighborhood on Hecke operators

Let $\mathscr{H}_v$ be the universal deformation of $\mathscr{A}_0$. Then the union of universal deformation of supersingular points is given by

$$\widehat{\mathscr{Y}_U^{ss}} := B_0 \backslash \mathscr{H}_v \times (B \otimes \mathbb{A}_f^v)^\times / U^v.$$

Notice that $\mathscr{H}_v$ is a formal scheme over $\mathscr{O}_v^{\mathrm{ur}}$. Thus the formal completion of $\mathscr{M}_K$ along its supersingular points is given by

$$\widehat{\mathscr{M}_K^{ss}} := H(F)_0 \backslash \mathscr{D}_v \times H(\mathbb{A}_f^v)/K^v.$$

where $\mathscr{D}_v = \mathscr{H}_v \otimes_{\mathscr{O}_v^{\mathrm{ur}}} \mathscr{H}_v$. Let $\mathscr{D}_y(c, d)$ be the divisor of $\mathscr{D}$ defined by universal deformation of y of type (c, d).

Lemma 5.2.2. *Let H_y be the stabilizer of y. Then for any $g \in H(\mathbb{A}_f^v)$, the formal neighborhood of $\mathscr{Z}(x)_{K,v}^{ss}$ is given by*

$$\begin{aligned}\widehat{\mathscr{Z}}(x)_K^{ss} =& H(F)_0 \backslash H(F)_0 (\mathscr{D}_y(c,d) \times H_y(\mathbb{A}_f^v) g) K^v / K^v \\ \simeq & H_y(F) \backslash \mathscr{D}_f(c,d) \times H_y(\mathbb{A}_f^v)/K_y,\end{aligned}$$

where $K_y = H_y(\mathbb{A}_f^v) \cap gK^v g^{-1}$.

5.3 Local intersection at unramified place

In this section, we want to study the local intersection at a finite place v which is split in $\mathbb{B}$.

We still work on $\mathbb{H} = \mathrm{GSpin}(\mathbb{V})$. Let x_1, x_2, x_3 be three vectors in $K \backslash \mathbb{V}_f$ such that the cycles $\mathscr{Z}(x_i)_K$ intersects properly in the integral model $\mathscr{M}_K$ of M_K. This means that there are no $k_i \in K$ such that the space

$$\sum F k_i x_i$$

is one or two dimensional with totally positive norms.

First let us consider the case where U_v is maximal. We want to compute the intersection index at a geometric point (P_1, P_2) in the spacial fiber over a finite prime v of F. The non-zero intersection of the three cycles will imply that there are three quasi-isogenies $y_i : P_2 \longrightarrow P_1$ with type determined by x_i's. Notice that P_1 is ordinary (resp. supersingular) if and only if P_2 is ordinary (resp. supersingular).

If they both are ordinary, then we have canonical liftings $\widetilde{P}_i$ to CM-points on the generic fiber. Since

$$\mathrm{Hom}(P_1, P_2) = \mathrm{Hom}(\widetilde{P}_1, \widetilde{P}_2),$$

all y_i can be also lifted to quasi-isogenies of $\widetilde{y}_i : \widetilde{P}_2 \longrightarrow \widetilde{P}_1$. This will contradict the assumption that the three cycles $Z(x_i)_K$ have no intersection. It follows that all P_i's are supersingular points.

Now lets us assume that all P_i's are supersingular. Then we have the nearby quaternion algebra B and quadratic space (V, q) as before. By Lemma (5.2.2), we know that $\mathscr{Z}(x_i)_K^{ss}$ has an extension

$$\widehat{\mathscr{Z}}(x_i)_K^{ss} = H_f(F) \backslash \mathscr{D}_{y_i}(c_i, d_i) \times H_{y_i}(\mathbb{A}_f^v)/K_{y_i}$$

on the formal neighborhood of supersingular points:

$$\widehat{\mathscr{M}}_K^{ss} = H(F)_0 \backslash \mathscr{D} \times H(\mathbb{A}_f^v)/K^v.$$

Here $c_i, d_i \in \mathbb{Z}$ such that $\begin{pmatrix} \varpi^{c_i} & \\ & \varpi^{d_i} \end{pmatrix} \in U_v x_{iv} U_v$, and $(y_i, g_i) \in B \times H(\mathbb{A}_f^v)$ such that $g_i^{-1}(y_i) = x_i^v$ in $\mathbb{V}_f^v$. If these three has nontrivial intersection at a supersingular point represented by $g \in H(F)_0 \backslash H(\mathbb{A}_f^v)/K^v$, then we can write $g_i = g k_i$ with some $k_i \in K^v$. The intersection scheme $\mathscr{Z}(k_1 x_1, k_2 x_2, k_3 x_3)_K$ is represented by

$$\mathscr{Z}(k_1 x_1, k_2 x_2, k_3 x_3)_K = [\mathscr{D}_{y_1}(c_1, d_1) \cdot \mathscr{D}_{y_2}(c_2, d_2) \cdot \mathscr{D}_{y_3}(c_3, d_3) \times g]$$

on $\mathscr{D}$, here $y = (y_i) \in V^3$ and $c = (c_i), d = (d_i) \in \mathbb{Z}^3$. As this intersection is proper, the space generated by y_i's is three dimensional and positive definite. Notice that $g \in H(\mathbb{A}_f^v)/K^v$ is completely determined by the condition $g^{-1}y_i \in K^v x_i^v$. Thus we have that the total intersection at supersingular points is given by

$$\mathscr{Z}(x_1)_K \cdot \mathscr{Z}(x_2)_K \cdot \mathscr{Z}(x_3)_K := \sum_{kx^v \in K^v \backslash (Kx_1^v, Kx_2^v, Kx_3^v)} \deg \mathscr{Z}(k_1x_1, k_2x_2, k_3x_3)_K,$$

where sum runs through cosets such that $k_i x_i^v$ generated a subspace of dimension 3.

In the following, we let us compute the intersection at v for cycles $\mathscr{Z}(\Phi_i)$ for $\Phi_i \in \mathscr{S}(\mathbb{V})$. Assume that $\Phi_i(x) = \Phi_i^v(x^v)\Phi_{iv}(x_v)$. By the above discussion, we see that the total supersingular intersection is given by

$$\begin{aligned}\mathscr{Z}(\Phi_1) \cdot \mathscr{Z}(\Phi_2) \cdot \mathscr{Z}(\Phi_3) =& \mathrm{vol}(\widetilde{K}) \prod_{i=1}^{3} \sum_{x_i \in \widetilde{K} \backslash \mathbb{V}} \Phi_i(x_i) \mathscr{Z}(x_i)_K \\ =& \mathrm{vol}(\widetilde{K}) \sum_{x^v \in \widetilde{K}^3 \backslash (\mathbb{V}^v)^3_+} \sum_{x_v \in K_v^3 \backslash (\mathbb{V}_v)^3_{x^v}} \Phi(x) \deg \mathscr{Z}(x)_K \\ =& \mathrm{vol}(\widetilde{K}) \sum_{x^v \in \widetilde{K}^v \backslash (\mathbb{V}^v)^3_+} \Phi^v(x^v) m(x^v, \Phi_v),\end{aligned}$$

where $(\widehat{V})^3_+$ denote the set of elements $x^v \in (\widehat{V}^v)^3$ such that the intersection matrix of x_i^v as a symmetric elements in $M_3(\mathbb{A}_f^v)$ takes entries in F_+, $(V_v)^3_{x^v}$ denote the set of elements (x_{i_v}) with norm equal to the norms of (x_i^v), and

$$m(x^v, \Phi_v) = \sum_{x_v \in K_v^3 \backslash (\mathbb{V}_v)^3_{x^v}} \Phi_v(x_v) \deg \mathscr{Z}(x^v, x_v)_K.$$

We note that the volume factor $\mathrm{vol}(\widetilde{K})$ is product of the volume of the image of K_v in $SO(V_v)$ with respect to the Tamagawa measure (cf. Notations). And by definition it also includes the archimedean factor $\mathrm{vol}(SO(\mathbb{B}_\infty))$.

In order to compare the above with theta series, let us rewrite the intersection in terms of the quadartic space $V = B$. Notice that every x^v can be written as $x^v = g^{-1}(y)$ with $y \in (V)^3_+$ of elements with non-degenerate moment matrix. Thus we have

$$\mathscr{Z}(\Phi_1) \cdot \mathscr{Z}(\Phi_2) \cdot \mathscr{Z}(\Phi_3) = \mathrm{vol}(\widetilde{K}) \sum_{y \in H(F) \backslash V_+^3} \sum_{g \in H(\mathbb{A}^v)/\widetilde{K}^v} \Phi^v(g^{-1}y) m(y, \Phi_v)_K,$$

where for $y \in (V_v)^3$

$$m(y, \Phi_v) = \sum_{x_v \in K_v^3 \backslash (\mathbb{V}_v)^3_{x^v}} \Phi_v(x_v) \deg \mathscr{Z}(y, x_v)_K.$$

This is a *pseudo-theta series* (cf. [36]) if $m(\cdot, \Phi_v)$ has no singularity over $y \in (V_v)^3$.

In the following we want to deduce a formula for the intersection using the work of Gross–Keating. For a element $y \in B_v$ with integral norm, let $\mathscr{T}_y$ denote the universal deformation divisor on $\mathscr{D}$ of the isogeny $y : \mathscr{A} \longrightarrow \mathscr{A}$. We extend this definition to arbitrary y by setting $\mathscr{T}_y = 0$ if y is not integral. Then we have the following relation:

$$\mathscr{D}_y(c, d) = \mathscr{T}_{\varpi^{-c}y} - \mathscr{T}_{\varpi^{-c-1}y}.$$

Indeed, for any $y \in \varpi\mathscr{O}_B$, there is an embedding from $\mathscr{T}_{y/\varpi}$ to $\mathscr{T}_y$ by taking any deformation $\varphi : \mathscr{E}_1 \longrightarrow \mathscr{E}_2$ to $\varpi\varphi$. The complement are exactly the deformation with cyclic kernel. It follows that $\deg \mathscr{Z}(x^v, x_v)$ is an alternative sum of intersection of Gross–Keating's cycles:

$$\deg \mathscr{Z}(x^v, x_v) = \sum_{\epsilon_i \in \{0,1\}} (-1)^{\epsilon_1+\epsilon_2+\epsilon_3} \mathscr{T}_{\varpi^{-c_1-\epsilon_1}y_1} \mathscr{T}_{\varpi^{-c_2-\epsilon_2}y_2} \mathscr{T}_{\varpi^{-c_3-\epsilon_3}y_3}$$

Theorem 5.3.1 (Gross–Keating, [10]). *Assume that Φ_v is the characteristic function of $\mathscr{O}_{B,v}^3$. Then for $y \in (V_v')^3$, the intersection number $m(y, \Phi_v)$ depends only on the moment $T = Q(y)$ and*

$$m(y, \Phi_v) = \nu(Q(y)),$$

where the ν-invariant is defined as in Prop. 4.1.2.

Corollary 5.3.2. *We have*

$$W'_{T,v}(g_v, 0, \Phi_v) = \zeta_v(2)^{-2} m_T(r(g_v)\Phi_v). \tag{5.3.1}$$

Proof. By Gross–Keating and Prop. 4.1.2, this is true when $g_v = e$ is the identity element. We will reduce the general g_v to this known case.

Suppose that

$$g_v = d(\nu)n(b)m(a)k$$

for b, a are both diagonal matrices and k in the standard maximal compact subgroup of $\mathbb{G}$. Then it is easy to see that the Whittaker function obeys the rule:

$$W'_{T,v}(g_v, 0, \Phi_v) = \psi(\nu T b)|\nu|^{-3}|\det(a)|^2 W'_{\nu a T a}(e, 0, \Phi_v).$$

On the intersection side, we have the similar formula:

$$\begin{aligned} m_T(r(g)\Phi_v) =& |\nu|^{-3} \sum_{x_v} r(g_1)\Phi_v(h_v x_v) \deg \mathscr{Z}_{\nu T}(x_v)_K \\ =& \psi_{\nu T}(b)|\nu|^{-3}|\det a|^2 \sum_{x_v} \Phi_v(x_v a) \deg \mathscr{Z}_{\nu T}(x_v)_K, \end{aligned}$$

where $h_v \in GO(V_v)$ with $\nu(h_v) = \nu^{-1}$ and the sum runs over all x_v with norm $\nu \cdot diag(T)$.

By our definition of cycles, for diagonal matrix a, we have

$$\mathscr{Z}_{\nu T}(x) = \mathscr{Z}_{\nu a T a}(xa).$$

It follows that

$$m_T(r(g)\Phi_v) = \psi_{\nu T}(b)|\nu|^{-3}|\det a|^2 m_{\nu a T a}(\Phi_v).$$

∎

Comparison

In this subsection we will relate the global v-Fourier coefficient of the analytic kernel function with the local intersection of triple Hecke correspondences when the Shimura curve has good reduction at v.

Recall that we have a decomposition of $E'(g,0,\Phi)$ according to the difference of Σ_T and Σ:

$$E'(g,0,\Phi)=\sum_v E'_v(g,0,\Phi)+E'_{sing}(g,0,\Phi), \tag{5.3.2}$$

where

$$E'_v(g,0,\Phi)=\sum_{\Sigma_T=\Sigma(v)} E'_T(g,0,\Phi), \tag{5.3.3}$$

and

$$E'_{sing}(g,0,\Phi)=\sum_{T,\det(T)=0} E'_T(g,0,\Phi).$$

On the height intersection part, we have analogous decomposition

$$Z(g,\Phi,\Delta)=Z(g,\Phi,\Delta)_{sing}+\sum_v Z(g,\Phi,\Delta)_v, \tag{5.3.4}$$

and each $Z(g,\Phi,\Delta)_v$ has a part $\mathscr{Z}(g,\Phi)_v$ of intersection of horizontal cycles.

Theorem 5.3.3. *Assume that $\Phi=\otimes_w\Phi_w$ with Φ_v is the characteristic function of $\mathscr{O}_{\mathbb{B}_v}^3$. And let S be the set of places outside which everything is unramified. Assume further that for $w\in S$, Φ_w is supported in $\mathbb{V}^3_{w,reg}$, the subspace of elements (x_i) such that the moment matrix $(x_i,x_j)\in M_3(F_w)$ is non-degenerate. Then for $g=(g_1,g_2,g_3)\in\mathbb{G}$ such that $g_{i,v}=1$ for $v\in S$, we have an equalities*

$$(\mathscr{Z}(g_1,\Phi_1)\cdot\mathscr{Z}(g_2,\Phi_2)\cdot\mathscr{Z}(g_3,\Phi_3))_v=-2E'_v(g,0,\Phi)$$

and

$$Z(g,\Phi,\Delta)_v=-2E'_v(g,0,\phi)+\sum_i c_v(g_i,\Phi_i)Z(g_j,\Phi_j)\cdot Z(g_k,\Phi_k)$$

where $c_v(g_i,\Phi_i)$ are some constants which vanish for almost all v, $\{j,k\}$ is the complement of i in $\{1,2,3\}$, and $Z(g_j,\Phi_j)\cdot Z(g_k,\Phi_k)$ is the intersection on $Y_U\times Y_U$.

Proof. Since Y_U has a smooth model $\mathscr{Y}_{U,v}$ over v, the restriction of $\widehat{Z}(g_i,\Phi_i)$ over v can be constructed from $\mathscr{Z}(g_i,\Phi)$ by adding some multiple of the special fiber V:

$$\widehat{Z}(g_i,\Phi_i)=\mathscr{Z}(g_i,\Phi_i)+c(g_i,\Phi_i)V$$

Here $c_v(g_i, \Phi_i)V$ is some constant which vanishes for almost all v. Since $V^2 = 0$ in $\mathscr{Y}_U^2$, one has This implies

$$Z(g, \Phi, \Delta) = \mathscr{Z}(g_1, \Phi_1) \cdot \mathscr{Z}(g_2, \Phi_2) \cdot \mathscr{Z}(g_3, \Phi_3) + \sum_i c(g_i, \Phi_i) Z(g_j, \Phi_j) \cdot Z(g_k, \Phi_k).$$

Thus we have the equality follows from the first equality.

By our choice of Φ, there is no self-intersection in $\mathscr{Z}(g_1, \Phi_1) \cdot \mathscr{Z}(g_2, \Phi_2) \cdot \mathscr{Z}(g_3, \Phi_3)_v$:

$$\begin{aligned}
&(\mathscr{Z}(g_1, \Phi_1) \cdot \mathscr{Z}(g_2, \Phi_2) \cdot \mathscr{Z}(g_3, \Phi_3))_v \\
=& \sum_{x^v \in (\widetilde{K}^v)^3 \backslash (\mathbb{V}^v)^3_+} r(g^v)\Phi^v(x^v) m(x^v, r(g_v)\Phi_v) \\
=& \sum_{\Sigma(T) = \Sigma(v)} \prod_{w \neq v} \int_{(\mathbb{B}_v^3)_T} r(g^v)\Phi_w(x_w) dx_w \cdot m_T(r(g_v)\Phi_v),
\end{aligned}$$

where

$$m_T(\Phi_v) = \sum_{x_v \in K_v^3 \backslash (\mathbb{B}_v)^3_{\mathrm{diag}(T)}} \Phi_v(x_v) \deg \mathscr{Z}_T(x_v)_K,$$

where the sum is over elements of $\mathbb{B}_v^3$ with norms equal to diagonal of T, and the cycle $\mathscr{Z}_T(x_v)$ is equal to $\mathscr{Z}(x^v, x_v)$ with $x^v \in (\mathbb{V}^v)$ with *non-singular* moment matrix T.

In summary, the intersection number is given by

$$\sum_T \mathrm{vol}(K_v) I_T(g^v, \Phi^v) m_T(r(g_v)\Phi_v). \tag{5.3.5}$$

We need to compare this with the derivative of Eisenstein series. We invoke the formula of Kudla ([21]):

$$E_T'(g, 0, \Phi) = \frac{W_T'(g, 0, \Phi_v)}{W_T(g, 0, \Phi_v')} E_T(g, 0, \Phi^v \otimes \Phi_v'). \tag{5.3.6}$$

Under our choice of measures, by Siegwl–Weil we have

$$E_T(g, 0, \Phi^v \otimes \Phi_v') = I_T(g, \Phi^v \otimes \Phi_v').$$

We therefore have

$$\begin{aligned}
E_T'(g, 0, \Phi) =& \frac{W_T'(g, 0, \Phi_v)}{W_T(g, 0, \Phi_v')} I_T(g, \Phi^v \otimes \Phi_v') \\
=& W_T'(g, 0, \Phi_v) \frac{I_{T,v}(g_v, \Phi_v')}{W_T(g, 0, \Phi_v')} I_T(g^v, \Phi^v).
\end{aligned}$$

Note that $\frac{I_{T,v}(g_v, \Phi_v')}{W_T(g, 0, \Phi_v')}$ is a constant independent of T, g, Φ_v'. By Corollary 5.3.2,

$$E_T'(g, 0, \Phi) = \zeta_v(2)^{-2} m_T(r(g_v)\Phi_v) \frac{I_{T,v}(e, \Phi_v')}{W_T(e, 0, \Phi_v')} I_T(g^v, \Phi^v).$$

It suffices to prove that

$$\zeta_v(2)^{-2}\frac{I_{T,v}(e,\Phi'_v)}{W_T(e,0,\Phi'_v)} = -\frac{1}{2}\mathrm{vol}(K_v).$$

Now the nearby quaternion B is non-split at v. And we have

$$I(e,\Phi'_v) = \mathrm{vol}(SO(B_v)).$$

So we need to show

$$\frac{\mathrm{vol}(SO(B_v))}{\mathrm{vol}(K_v)} = -\frac{1}{2}\zeta_v(2)^2 W_T(e,0,\Phi'_v).$$

It is easy to see that (cf. [1, chap. 16, §3.5]):

$$\frac{\mathrm{vol}(SO(B_v))}{\mathrm{vol}(K_v)} = \frac{1}{(q-1)^2}.$$

Indeed, we have an isomorphism (cf. Notations)

$$SO(B) \simeq B^\times/F^\times \times B^1.$$

We now may compute the ratio for a non-archimedean v:

$$\frac{\mathrm{vol}(GL_2(\mathscr{O}_v))}{\mathrm{vol}(\mathscr{O}_{B_v}^\times)} = \frac{\zeta_v(1)^{-1}\zeta_v(2)^{-1}}{\zeta_v(2)^{-1}} \cdot \frac{\mathrm{vol}(M_2(\mathscr{O}_v))}{\mathrm{vol}(\mathscr{O}_{B_v})} = (q-1).$$

Moreover we have

$$\frac{\mathrm{vol}(GL_2(\mathscr{O}_v))}{\mathrm{vol}(\mathscr{O}_{B_v}^\times)} = \frac{\mathrm{vol}(SL_2(\mathscr{O}_v))}{\mathrm{vol}(B_v^1)}.$$

From §4.1 we also have

$$\zeta_v(2)^2 W_T(e,0,\Phi'_v) = -\frac{2}{(q-1)^2}.$$

This completes the proof.

$\square$

5.4 Archimedean height

Let B be the Hamilton quaternion and let Φ be the standard Gaussian. Let $B' = M_{2,\mathbb{R}}$ be the matrix algebra. Let $x = (x_1, x_2, x_3) \in B'^3$ with non-singular moment matrix $Q(x)$ and let $g_i = g_{x_i}$ be a Green's function of D_{x_i}. Define the star product

$$\Lambda(x) = \int_{D_\pm} g_1 * g_2 * g_3, \tag{5.4.1}$$

where $D_\pm$ is the union of $\mathscr{H}_\pm^2$ and $\mathscr{H}_+ = \mathscr{H}$ (D_-, resp.) is the upper (lower, resp.) half plane.

Then $\Lambda(x)$ depends only on the moment $Q(x) \in \mathrm{Sym}_3(\mathbb{R})$ (with signature either $(1,2)$ or $(2,1)$ since B' has signature $(2,2)$). Hence we simply write it as $\Lambda(\frac{1}{4\pi}Q(x))$ (note that we need to shift it by a multiple 4π).

We will consider a Green's function of logarithmic singularity which we call *pre-Green function* since it does not give the admissible Green's function. Their difference will be discussed later.

Now we specify our choice of pre-Green functions. For $x \in B'$ consider a function $D_\pm = \mathscr{H}_\pm^2 \to \mathbb{R}_+$ defined by

$$s_x(z) := q(x_z) = 2\frac{(x,z)(x,\overline{z})}{(z,\overline{z})}.$$

In terms of coordinates $z = \begin{pmatrix} z_1 & -z_1 z_2 \\ 1 & -z_2 \end{pmatrix}$ and $x = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, we have

$$s_x(z) = \frac{(-az_2 + dz_1 - b + cz_1z_2)(-a\overline{z}_2 + d\overline{z}_1 - b + c\overline{z}_1\overline{z}_2)}{-(z_1 - \overline{z}_1)(z_2 - \overline{z}_2)}.$$

We will consider the pre-Green function of D_x on D given by

$$g_x(z) := \eta(s_x(z))$$

where we recall that

$$\eta(t) = Ei(-t) = -\int_1^\infty e^{-tu}\frac{du}{u}.$$

In the following we want to compute the star product for a non-singular moment $4\pi T = Q(x)$. Our strategy is close to that of [21], namely by steps: in the first step we will establish a $SO(3)$-invariance of $\Lambda(T)$ which simplifies the computation to the case T is diagonal; in the second step we compute $\Lambda(T)$ when T is diagonal and we compare the result with the derivative of the Whittaker integrals $W_T'(e,s,\Phi)$.

Step one: $SO(3)$-invariance

The following lemma is a special case of a more general result of Kudla-Millson. For convenience we give a proof here.

Lemma 5.4.1. *Let $\omega_x = \partial\overline{\partial}g_x$. For any $(x_1,x_2) \in V^2$, the $(2,2)$-form $\omega_{x_1} \wedge \omega_{x_2}$ on $\mathscr{H}^2$ is invariant under the action of $SO(2)$ on V^2.*

Proof. Let $k \in SO(2)$ be the matrix $\begin{pmatrix} cos\theta & sin\theta \\ -cos\theta & sin\theta \end{pmatrix}$ and for simplicity, we denote $c = cos\theta$ and $s = sin\theta$. Let $x = cx_1 + sx_2$ and $y = -sx_1 + cx_2$. Then, by the formula

$$\begin{aligned} e^{s_x(z)}\omega_x &= s_x(z)\partial log s_x(z)\overline{\partial} log s_x(z) - \partial\overline{\partial} log s_x(z) \\ &= \frac{(x,z)(x,\overline{z})}{(z,\overline{z})}\left(\frac{\partial(x,z)}{(x,z)} - \partial log(z,\overline{z})\right)\left(\frac{\overline{\partial}(x,\overline{z})}{(x,\overline{z})} - \overline{\partial} log(z,\overline{z})\right) - \partial\overline{\partial} log(z,\overline{z}) \end{aligned}$$

and similar formula for ω_y, we have that

$$e^{s_x(z)+s_y(z)}\omega_x \wedge \omega_y(z) = A + B \wedge \partial\overline{\partial} log(z,\overline{z}) + \partial\overline{\partial} log(z,\overline{z})\partial\overline{\partial} log(z,\overline{z})$$

where

$$\begin{aligned} A = & \frac{(x,z)(x,\overline{z})}{(z,\overline{z})}(\frac{\partial(x,z)}{(x,z)} - \partial log(z,\overline{z}))(\frac{\overline{\partial}(x,\overline{z})}{(x,\overline{z})} - \overline{\partial} log(z,\overline{z})) \\ \wedge & \frac{(y,z)(y,\overline{z})}{(z,\overline{z})}(\frac{\partial(y,z)}{(y,z)} - \partial log(z,\overline{z}))(\frac{\overline{\partial}(y,\overline{z})}{(y,\overline{z})} - \overline{\partial} log(z,\overline{z})) \end{aligned}$$

and

$$\begin{aligned} B = & \frac{(x,z)(x,\overline{z})}{(z,\overline{z})}(\frac{\partial(x,z)}{(x,z)} - \partial log(z,\overline{z}))(\frac{\overline{\partial}(x,\overline{z})}{(x,\overline{z})} - \overline{\partial} log(z,\overline{z})) \\ + & \frac{(y,z)(y,\overline{z})}{(z,\overline{z})}(\frac{\partial(y,z)}{(y,z)} - \partial log(z,\overline{z}))(\frac{\overline{\partial}(y,\overline{z})}{(y,\overline{z})} - \overline{\partial} log(z,\overline{z})). \end{aligned}$$

It is easy to see that

$$\begin{aligned} B = & (s_x(z)+s_y(z))\partial log(z,\overline{z})\overline{\partial} log(z,\overline{z}) - \partial((x,\overline{z})(x,z)+(y,\overline{z})(y,z))\overline{\partial} log(z,\overline{z})/(z,\overline{z}) \\ & - \overline{\partial}((x,\overline{z})(x,z)+(y,\overline{z})(y,z))\partial log(z,\overline{z})/(z,\overline{z}) + (\partial(x,z)\overline{\partial}(x,\overline{z}) + \partial(y,z)\overline{\partial}(y,\overline{z}))/(z,\overline{z}). \end{aligned}$$

Now it is easy to see that the above sum is invariant since the following two terms are respectively invariant

$$(x,\overline{z})(x,z)+(y,\overline{z})(y,z), \quad (\partial(x,z)\overline{\partial}(x,\overline{z}) + \partial(y,z)\overline{\partial}(y,\overline{z})).$$

Now we come to A:

$$\begin{aligned} (z,\overline{z})^2 A = & \partial(x,z)\overline{\partial}(x,\overline{z})\partial(y,z)\overline{\partial}(y,\overline{z}) - ((y,z)\partial(x,z) - (x,z)\partial(y,z))\overline{\partial}(x,\overline{z}))\overline{\partial}(y,\overline{z}))\partial log(z,\overline{z}) \\ & - ((y,\overline{z})\overline{\partial}(x,\overline{z}) - (x,\overline{z})\overline{\partial}(y,\overline{z}))\partial(x,z))\partial(y,z))\overline{\partial} log(z,\overline{z}) \\ & + (\partial(x,z)\overline{\partial}(x,\overline{z}) + \partial(y,z)\overline{\partial}(y,\overline{z}))\partial log(z,\overline{z})\overline{\partial} log(z,\overline{z}). \end{aligned}$$

This is invariant since the following four terms are respectively invariant

$$\begin{aligned} & \partial(x,z)\overline{\partial}(x,\overline{z}), \quad \partial(y,z)\overline{\partial}(y,\overline{z}), \\ & (y,z)\partial(x,z) - (x,z)\partial(y,z), \quad (y,\overline{z})\overline{\partial}(x,\overline{z}) - (x,\overline{z})\overline{\partial}(y,\overline{z}). \end{aligned}$$

This completes the proof. $\square$

Proposition 5.4.2 (Invariance under $SO(3)$)**.** *The local archimedean height pairing $\Lambda(T)$ is invariant under $SO(3)$, i.e.,*

$$\Lambda(T) = \Lambda(kTk^t), \quad k \in SO(3).$$

Proof. Note that the group $SO(3)$ is generated by matrices of the form $\begin{pmatrix} 1 & & \\ & cos\theta & sin\theta \\ & -cos\theta & sin\theta \end{pmatrix}$ and subgroup of even permutation of S_3, the symmetric group. Thus it suffices to prove that

$$\Lambda(x_1, x_2, x_3) = \Lambda(x, y, x_3)$$

for $x = cx_1 + sx_2$ and $y = -sx_1 + cx_2$ where $c = cos\theta, s = sin\theta$.

Further, since $g^*\omega_x = \omega_{g^{-1}x}$ for $g \in Aut(\mathscr{H}^2)$, we can assume that $x_3 = \sqrt{a}\begin{pmatrix} 1 & \\ & \pm 1 \end{pmatrix}$ depending on the sign of $\det(x_3)$. Then $Z_{x_3} = \Delta(\mathscr{H})$ is the diagonal embedding of $\mathscr{H}$ if $\det(x_3) > 0$, otherwise $Z_{x_3} = \emptyset$.

By definition,

$$\Lambda(x, y, x_3) = \int_{\mathscr{H}^2} g_{x_3}(z)\omega_x(z) \wedge \omega_y(z) + \int_{Z_{x_3}} g_x * g_y|_{Z_{x_3}}.$$

Now the first term is invariant by Lemma above and the second term is either zero (when $\det(x_3) < 0$) or has been treated in the work of Kudla ([21]) when x, y generates a plane of signature $(1, 1)$. The left case is when x, y generates a negative definite plane. In this case the proof of Kudla still applies. This completes the proof. □

Remark 5.4.1.

1. The proof of $SO(2)$-invariance in [21] is indeed very difficult though elementary.
2. Similarly, by induction we can prove invariance for $SO(n+1)$ for V of signature $(n, 2)$.

Step two: star product

It turns out that for the convenience of computation, it is better to consider the bounded domain $\mathbb{D}^2$ where $\mathbb{D} = \{z \in \mathbb{C} | |z| < 1\}$ is the unit disk. We have an explicit biholomorphic isomorphism from $\mathbb{D}^2 \to \mathscr{H}^2$ given by

$$(z_1, z_2) \mapsto (i\frac{1+z_1}{1-z_1}, i\frac{1+z_2}{1-z_2}).$$

Then, using the bounded model $\mathbb{D}^2$, we can express

$$s_x(z) = \frac{|(ai - b - c - di)z_1z_2 + (ai + b - c + di)z_1 + (-ai + b - c - di)z_2 + (-ai - b - c + di)|^2}{4(1 - |z_1|^2)(1 - |z_2|^2)}.$$

We first compute several differentials which will be used later on.

Lemma 5.4.3. *Let $a_i \in \mathbb{R}_+$ and $x_i \in B', i = 1, 2, 3, 4,$ be the following four elements*

$$x_1 = \sqrt{a_1}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad x_2 = \sqrt{a_2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$x_3 = \sqrt{a_3}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad x_4 = \sqrt{a_4}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

We will shorten $s_1(z) := s_{x_i}(z)$. *Then we have*

$$s_1(z) = a_1 \frac{|z_1 - z_2|^2}{(1-|z_1|^2)(1-|z_1|^2)}, \quad s_2(z) = a_2 \frac{|1 - z_1 z_2|^2}{(1-|z_1|^2)(1-|z_1|^2)}$$

$$s_3(z) = a_3 \frac{|1 + z_1 z_2|^2}{(1-|z_1|^2)(1-|z_1|^2)}, \quad s_4(z) = a_4 \frac{|z_1 + z_2|^2}{(1-|z_1|^2)(1-|z_1|^2)}.$$

Moreover, we have

$$e^{s_1(z)}\partial\overline{\partial}Ei(-s_1(z)) = \left(a_1 + a_1 \frac{|z_1 - z_2|^2}{(1-|z_1|^2)(1-|z_2|^2)} - 1\right)\frac{dz_1 \wedge d\overline{z}_1}{(1-|z_1|^2)^2} + ...$$
$$- a_1 \frac{(1-\overline{z}_1 z_2)^2}{(1-|z_1|^2)(1-|z_2|^2)} \frac{dz_1 \wedge d\overline{z}_2}{(1-|z_1|^2)(1-|z_2|^2)} + ...$$

where we the omitted terms can be easily recovered by the symmetry of z_1, z_2. *Similarly we have*

$$e^{s_2(z)}\partial\overline{\partial}Ei(-s_2(z)) = \left(a_2 \frac{|\overline{z}_1 - z_2|^2}{(1-|z_1|^2)(1-|z_2|^2)} - 1\right)\frac{dz_1 \wedge d\overline{z}_1}{(1-|z_1|^2)^2} + ...$$
$$- a_2 \frac{(\overline{z}_1 - z_2)^2}{(1-|z_1|^2)(1-|z_2|^2)} \frac{dz_1 \wedge d\overline{z}_2}{(1-|z_1|^2)(1-|z_2|^2)} + ...$$

$$e^{s_3(z)}\partial\overline{\partial}Ei(-s_3(z)) = \left(a_3 \frac{|\overline{z}_1 + z_2|^2}{(1-|z_1|^2)(1-|z_2|^2)} - 1\right)\frac{dz_1 \wedge d\overline{z}_1}{(1-|z_1|^2)^2} + ...$$
$$a_3 \frac{(\overline{z}_1 + z_2)^2}{(1-|z_1|^2)(1-|z_2|^2)} \frac{dz_1 \wedge d\overline{z}_2}{(1-|z_1|^2)(1-|z_2|^2)} + ...$$

$$e^{s_4(z)}\partial\overline{\partial}Ei(-s_4(z)) = \left(a_4 + a_4 \frac{|z_1 + z_2|^2}{(1-|z_1|^2)(1-|z_2|^2)} - 1\right)\frac{dz_1 \wedge d\overline{z}_1}{(1-|z_1|^2)^2} + ...$$
$$+ a_4 \frac{(1+\overline{z}_1 z_2)^2}{(1-|z_1|^2)(1-|z_2|^2)} \frac{dz_1 \wedge d\overline{z}_2}{(1-|z_1|^2)(1-|z_2|^2)} + ...$$

And moreover

$$\partial\overline{\partial}Ei(-s_1(z)) \wedge \partial\overline{\partial}Ei(-s_4(z)) = e^{-s_1(z)-s_4(z)}\frac{dz_1 \wedge d\overline{z}_1}{(1-|z_1|^2)^2} \wedge \frac{dz_2 \wedge d\overline{z}_2}{(1-|z_2|^2)^2}$$
$$\left(4a_1a_4 \frac{(1-|z_1z_2|^2)^2}{(1-|z_1|^2)^2(1-|z_2|^2)^2} - 2a_1 \frac{|1+\overline{z}_1 z_2|^2}{(1-|z_1|^2)(1-|z_2|^2)} - 2a_4 \frac{|1-\overline{z}_1 z_2|^2}{(1-|z_1|^2)(1-|z_2|^2)} + 2\right).$$

$$\partial\overline{\partial}Ei(-s_2(z))\wedge\partial\overline{\partial}Ei(-s_3(z)) = e^{-s_2(z)-s_3(z)}\frac{dz_1\wedge d\overline{z}_1}{(1-|z_1|^2)^2}\wedge\frac{dz_2\wedge d\overline{z}_2}{(1-|z_2|^2)^2}$$
$$\left(4a_2a_3\frac{(|z_1|^2-|z_2|^2)^2}{(1-|z_1|^2)^2(1-|z_2|^2)^2} - 2a_2\frac{|\overline{z}_1-z_2|^2}{(1-|z_1|^2)(1-|z_2|^2)} - 2a_3\frac{|\overline{z}_1+z_2|^2}{(1-|z_1|^2)(1-|z_2|^2)}+2\right).$$

Proof. Simple but tedious computation.

□

We also need

Lemma 5.4.4 (Change of variables)**.** *Define a diffeomorphism between* $\mathbb{D}^2$ *and* $\mathbb{C}^2 \simeq \mathbb{R}^4$ *by* $(z_1, z_2) \mapsto (w_1, w_2)$ *where* $w_i = u_i + \sqrt{-1}v_i$ *and*

$$u_i = \frac{x_i}{(1-|z_1|^2)^{1/2}(1-|z_2|^2)^{1/2}}, \quad v_i = \frac{y_i}{(1-|z_1|^2)^{1/2}(1-|z_2|^2)^{1/2}}.$$

Then the Jacobian is given by

$$\frac{\partial(u_1,v_1,u_2,v_2)}{\partial(x_1,y_1,x_2,y_2)} = -\frac{1-|z_1|^2|z_2|^2}{(1-|z_1|^2)^3(1-|z_2|^2)^3}.$$

Moreover we have

$$\frac{dx_1dy_1dx_2dy_2}{(1-|z_1|^2)^2(1-|z_2|^2)^2} = -\frac{du_1dv_1du_2dv_2}{\sqrt{(1+|w_1|^2+|w_2|^2)^2-4|w_1|^2|w_2|^2}}.$$

Proof. Let $\lambda = \frac{1}{(1-|z_1|^2)(1-|z_2|^2)}$. Note that

$$u_1^2+v_1^2 = \lambda - \frac{1}{1-|z_2|^2}$$

and similarly

$$u_2^2+v_2^2 = \lambda - \frac{1}{1-|z_1|^2}.$$

This shows that λ satisfies a quadratic equation

$$\lambda^2 - (1+|w_1|^2+|w_2|^2)\lambda + |w_1|^2|w_2|^2 = 0.$$

Denote its two roots by $\lambda_1 > \lambda_2$. Since $|z_i| < 1$, a careful check shows that $\lambda = \lambda_1$ is the larger one of its two roots. Moreover, we have

$$\frac{1-|z_1|^2|z_2|^2}{(1-|z_1|^2)(1-|z_2|^2)} = \lambda - \lambda^{-1}|w_1|^2|w_2|^2 = \lambda_1 - \lambda_2 = \sqrt{\Delta}$$

where Δ is the discriminant of the quadratic equation above.

□

Theorem 5.4.5. *We have for $T \in \mathrm{Sym}_3(\mathbb{R})$ with signature either $(1,2)$ or $(2,1)$,*

$$W'_{T,\infty}(e,0,\Phi) = \frac{\kappa(0)}{2\Gamma_3(2)} e^{-2\pi T} \Lambda(T).$$

In particular, everything depends only on the eigenvalues of T (presumedly not obvious).

Proof. By Proposition 5.4.2, we may assume that T is a diagonal matrix.

We first treat the case $(p,q) = (2,1)$ and let's assume that $4\pi T = (a_1, a_4, -b)$. And we may choose x_i as in Lemma 5.4.3 as long as we take $a_2 = b$. Then by the same lemma, $\Lambda(T)$ is given by the integral

$$\Lambda(T) = 2(-2\pi i)^{-2} \int_{\mathbb{D}^2} Ei(-b\frac{|1+z_1z_2|^2}{(1-|z_1|^2)(1-|z_2|^2)}) e^{-s_1(z)-s_4(z)} \frac{dz_1 \wedge d\overline{z}_1}{(1-|z_1|^2)^2} \wedge \frac{dz_2 \wedge d\overline{z}_2}{(1-|z_2|^2)^2}$$
$$(4a_1a_4\frac{(1-|z_1z_2|^2)^2}{(1-|z_1|^2)^2(1-|z_2|^2)^2} - 2a_1\frac{|1+\overline{z}_1z_2|^2}{(1-|z_1|^2)(1-|z_2|^2)} - 2a_4\frac{|1-\overline{z}_1z_2|^2}{(1-|z_1|^2)(1-|z_2|^2)} + 2)$$

Here the factor $\frac{1}{-2\pi i}$ is from the definition star product, and the factor 2 is due to the fact that $D_\pm$ has two copies.

Now let us make the substitution

$$u_1 = \frac{x_1+x_2}{(1-|z_1|^2)^{1/2}(1-|z_2|^2)^{1/2}}, \quad u_2 = \frac{x_1-x_2}{(1-|z_1|^2)^{1/2}(1-|z_2|^2)^{1/2}}$$
$$v_1 = \frac{y_1+y_2}{(1-|z_1|^2)^{1/2}(1-|z_2|^2)^{1/2}}, \quad v_2 = \frac{y_1-y_2}{(1-|z_1|^2)^{1/2}(1-|z_2|^2)^{1/2}}.$$

By Lemma 5.4.4 we may calculate the Jacobian of our substitutions to arrive at

$$\Lambda(T) = \frac{1}{2\pi^2}\int_{\mathbb{R}^4} Ei(-b(1+u_1^2+v_2^2)) e^{-a_1(u_2^2+v_2^2)-a_4(u_1^2+v_1^2)}$$
$$(4a_1a_4\Delta - 2a_1(1+u_1^2+v_1^2) - 2a_4(1+u_2^2+v_2^2)+2)\frac{du_1dv_1du_2dv_2}{\sqrt{\Delta}},$$

which we may rearrange as

$$\frac{1}{2\pi^2}\int_{\mathbb{R}^2} Ei(-b(1+u_1^2+v_2^2))e^{-a_4v_2^2-a_1u_1^2}dv_2du_1 \int_{\mathbb{R}^2} e^{-a_4u_2^2-a_1v_1^2}$$
$$(4a_1a_4\Delta - 2a_1(1+u_1^2+v_1^2) - 2a_4(1+u_2^2+v_2^2)+2)\frac{dv_1du_2}{\sqrt{\Delta}}.$$

Here

$$\Delta = 1+u_1^2+v_1^2+u_2^2+v_2^2+(u_1u_2+v_1v_2)^2. \tag{5.4.2}$$

Comparing with Proposition 4.3.7, it suffices to prove that the integral

$$\frac{1}{\sqrt{\pi}}\int_{\mathbb{R}^2} e^{-a_4y_2^2-a_1y_1^2}(4a_1a_4\Delta - 2a_1(1+x_1^2+y_1^2) - 2a_4(1+x_2^2+y_2^2)+2)\frac{dy_1dy_2}{\sqrt{\Delta}} \tag{5.4.3}$$

is equal to

(5.4.4)
$$\int_{\mathbb{R}} e^{-u^2}\frac{4AB-2A-2B+2}{((u^2+A)(u^2+B))^{1/2}}du+\int_{\mathbb{R}} e^{-u^2}\frac{(4AB-A-B)u^2+2AB(A+B-1)}{((u^2+A)(u^2+B))^{3/2}}du.$$

Here note that we rename the variables u_i, v_i to x_i, y_i and they should not be confused with the real/imaginary part of z_i (coordinates of the bounded domain $\mathbb{D}$). And A, B are the two eigenvalues (as the z_1, z_2 in Prop. 4.3.7) of 2×2 matrix $(1+xx')^{1/2}a(1+xx')^{1/2}$ for x be the column vector $(x_1, x_2)^t$and

$$\Delta = (1+x'x)(1+y'y)-(x'y)^2.$$

Now notice that the 2×2 matrix $(1+xx')^{-1} = 1-\frac{1}{1+x'x}xx'$. We have

$$1+y'(1+xx')^{-1}y = 1+y'y-\frac{y'xx'y}{1+x'x}=\frac{\Delta}{1+x'x}.$$

Substitute $y\mapsto (1+xx')^{1/2}y$, the integral (5.4.3) is reduced

$$\frac{1}{\sqrt{\pi}}\int_{\mathbb{R}^2} e^{-y'(1+xx')^{1/2}a(1+xx')^{1/2}y}(4AB(1+y'y)-2y'(1+xx')^{1/2}a(1+xx')^{1/2}y-2A-2B+2)\frac{dy_1dy_2}{\sqrt{1+y'y}}.$$

Now make another substitution $y\mapsto ky$ where $k\in SO(2)$ is such that

$$(1+xx')^{1/2}a(1+xx')^{1/2} = k'diag(A,B)k.$$

We obtain:

$$\frac{1}{\sqrt{\pi}}\int_{\mathbb{R}^2} e^{-y'diag(A,B)y}(4AB(1+y'y)-2y'diag(A,B)y-2A-2B+2)\frac{dy_1dy_2}{\sqrt{1+y'y}}. \tag{5.4.5}$$

Using the integral

$$\int_{x\in\mathbb{R}} e^{-Ax^2}dx = \frac{1}{\sqrt{A}}\Gamma(1/2)=\sqrt{\pi}\frac{1}{\sqrt{A}},$$

we may rewrite the integral (5.4.5) as

$$\int_{\mathbb{R}^3} e^{-w^2(1+y_1^2+y_2^2)-Ay_1^2-By_2^2}(4AB(1+y_1^2+y_2^2)-2(Ay_1^2+By_2^2)-2A-2B+2)dwdy_1dy_2.$$

We interchange the order of integrals

$$\int_{\mathbb{R}} e^{-w^2}\int_{\mathbb{R}^2} e^{-y_1^2(w^2+A)-y_2^2(w^2+B)}((4AB-2A)y_1^2+(4AB-2B)y_2^2-4AB-2A-2B+2)dy_1dy_2dw.$$

Now we can integrate against y_1, y_2 and it is easy to verify that we arrive at the integral (5.4.4). This finishes the proof when $(p,q)=(2,1)$.

We now treat the slightly harder case $(p, q) = (1, 2)$. Assume that $4\pi T = (a, -b_1, -b_2)$ and we may take $a_4 = a, b_1 = a_3, b_2 = a_2$ as in Lemma 5.4.3. Then the same substitution as before yields that $\Lambda(T)$ is the sum of two terms:

$$\frac{1}{2\pi^2}\int_{\mathbb{R}^4} Ei(-a(u_1^2+v_1^2))e^{-b_1(1+u_1^2+v_2^2)-b_2(1+u_2^2+v_1^2)}$$
$$\times\,(4b_1b_2(u_1u_2+v_1v_2)^2 - 2b_1(u_1^2+v_2^2) - 2b_2(u_2^2+v_1^2)+2)\frac{du_1du_2dv_1dv_2}{\sqrt{\Delta}},$$

and (note that $s_4(z)$ has zeros along the divisor defined $z_1 + z_2 = 0$ on $\mathbb{D}^2$)

$$-\frac{1}{2\pi i}\int_{\mathbb{D}} Ei(-s_2(z,-z))\partial\bar{\partial}Ei(-s_3(z,-z)).$$

By Proposition 4.3.5, the Whittaker integral also breaks into two pieces. It is easy to prove that the first one matches the second term above. Indeed this already appeared in the work [25, Thm. 5.2.7, (ii)]. It suffices to prove that the integral

$$\int_{\mathbb{R}^2} e^{-b_1v_2^2-b_2u_2^2}(4b_1b_2(u_1u_2+v_1v_2)^2 - 2b_1(u_1^2+v_2^2) - 2b_2(u_2^2+v_1^2)+2)\frac{du_2dv_2}{\sqrt{\Delta}} \tag{5.4.6}$$

is equal to

$$\sqrt{\pi}\int_{\mathbb{R}} e^{-u^2}\left(\frac{-2(A+B-1-b_1-b_2)}{(u^2+A)^{1/2}(u^2+B)^{1/2}} + \frac{(2AB-2b_1b_2-A-B)u^2+2AB(A+B-1-b_1-b_2)}{(u^2+A)^{1/2}(u^2+B)^{3/2}}\right)du. \tag{5.4.7}$$

Here A, B are the two eigenvalues of $(1+w'w)^{1/2}b(1+w'w)^{1/2}$ where $w = (u_1, v_1)$.

Similar to the previous case we may rewrite the integral (5.4.6) as

$$\frac{1}{\sqrt{\pi}}\int_{\mathbb{R}} e^{-x^2(1+u_1^2+v_1^2)}I(x)dx, \tag{5.4.8}$$

where

$$I(x) = \int_{\mathbb{R}^2} e^{-(b_1+x^2)v_2^2-(b_2+x^2)u_2^2-x^2(u_1u_2+v_1v_2)^2}(4b_1b_2(u_1u_2+v_1v_2)^2-2b_1(u_1^2+v_2^2)-2b_2(u_2^2+v_1^2)+2)du_2dv_2.$$

We now want to make the exponent in $I(x)$ as a linear combination of only square terms (we point out that the same idea also works for the previous case). This suggests to make the substitution

$$y_1 = u_1u_2 + v_1v_2, \quad y_2 = \sqrt{\frac{b_2+x^2}{b_1+x^2}}v_1u_2 - \sqrt{\frac{b_1+x^2}{b_2+x^2}}u_1v_2.$$

Then we have

$$u_2 = \eta^{-1}(-\sqrt{\frac{b_1+x^2}{b_2+x^2}}u_1y_1 - v_1y_2), \quad v_2 = \eta^{-1}(-\sqrt{\frac{b_2+x^2}{b_1+x^2}}v_1y_1 + u_1y_2),$$

where

$$\eta = \frac{(b_1+x^2)u_1^2 + (b_2+x^2)v_1^2}{\sqrt{(b_1+x^2)(b_2+x^2)}}.$$

Also note that

$$y_1^2 + y_2^2 = ((b_2+x^2)u_2^2 + (b_1+x^2)v_2^2)(\frac{u_1^2}{b_2+x^2} + \frac{v_1^2}{b_1+x^2}).$$

After a suitable substitution we obtain

$$I(x) = \int_{\mathbb{R}^2} e^{-(\frac{(b_1+x^2)(b_2+x^2)}{(b_1+x^2)u_1^2+(b_2+x^2)v_1^2}+x^2)y_1^2 - \frac{(b_1+x^2)(b_2+x^2)}{(b_1+x^2)u_1^2+(b_2+x^2)v_1^2}y_2^2}(Cy_1^2 + Dy_2^2 + E)\eta^{-1} dy_1 dy_2,$$

where

$$\begin{cases} C = 4b_1b_2 + \eta^{-2}(-2b_1\frac{b_2+x^2}{b_1+x^2}v_1^2 - 2b_2\frac{b_1+x^2}{b_2+x^2}u_1^2) \\ D = \eta^{-2}(-2b_1u_1^2 - 2b_2v_1^2) \\ E = -2b_1u_1^2 - 2b_2v_1^2 + 2. \end{cases}$$

Moreover let's denote

$$\begin{aligned} F =& (b_1+x^2)(b_2+x^2) + ((b_1+x^2)u_1^2 + (b_2+x^2)v_1^2)x^2 \\ =& (1+u_1^2+v_1^2)x^4 + (b_1(1+u_1^2) + b_2(1+v_1^2))x^2 + b_1b_2. \end{aligned}$$

Now we may fold the integrals against y_1, y_2 to obtain

$$I(x) = \pi F^{-1/2}\left(CF^{-1}((b_1+x^2)u_1^2 + (b_2+x^2)v_1^2)/2 + D\frac{(b_1+x^2)u_1^2 + (b_2+x^2)v_1^2}{(b_1+x^2)(b_2+x^2)}/2 + E\right),$$

which can be simplified as

$$\pi F^{-1/2}E + \pi F^{-3/2}\left(2b_1b_2(u_1^2+v_1^2) - b_1(1+u_1^2) - b_2(1+v_1^2))x^2 + 2b_1b_2(b_1u_1^2 + b_2v_1^2 - 1)\right).$$

Plug back to the integral (5.4.8) and make a substitution $u = x(1+u_1^2+v_1^2)$. Therefore, we have proved that the integral (5.4.6) is equal to

$$\begin{aligned} &\pi\int_{\mathbb{R}} e^{-u^2}\frac{-2b_1u_1^2 - 2b_2v_1^2 + 2}{(u^4 + (b_1(1+u_1^2) + b_2(1+v_1^2))u^2 + b_1b_2(1+u_1^2+v_1^2))^{1/2}}du \\ &+\pi\int_{\mathbb{R}} e^{-u^2}\frac{(2b_1b_2(u_1^2+v_1^2) - b_1(1+u_1^2) - b_2(1+v_1^2))u^2 + 2b_1b_2(1+u_1^2+v_1^2)(b_1u_1^2+b_2v_1^2-1)}{(u^4 + (b_1(1+u_1^2) + b_2(1+v_1^2))u^2 + b_1b_2(1+u_1^2+v_1^2))^{3/2}}du \end{aligned}$$

This is clearly equal to the integral (5.4.7). We then complete the proof. ∎

Comparison

Assume that $\tau|\infty$ and we want to treat the archimedean height at τ. Recall that the generating function is defined for $g \in \mathrm{GL}_2^+(\mathbb{A})$

$$Z(g,\Phi) = \sum_{x \in \widehat{V}/K} r(g_{1f})\Phi(x)Z(x)_K, W_{T(x)}(g_\infty)$$

where the sum runs over all admissible classes. And for our fixed embedding $\tau : F \hookrightarrow \mathbb{C}$ we have an isomorphism of $\mathbb{C}$-analytic varieties (as long as K is neat):

$$Y_{K,\tau}^{an} \simeq G(F)\backslash D \times G(\mathbb{A}_f)/K \cup \{\text{cusp}\}$$

where, for short, $G = G(\tau)$ is the nearby group.

For $x_i \in V, i = 1, 2, 3$, we define a Green function as follows: for $[z, h'] \in G(F)\backslash D \times G(\mathbb{A}_f)/K$

$$g_{x,hK}([z,h']) = \sum_{\gamma \in G(F)/G_x(F)} \gamma^*[\eta(s_x(z))1_{G_x(\widehat{F})hK}(h'))].$$

For an admissible class $x \in \widehat{V}$ we will denote by g_x its Green function. Note that this is *not* the right choice of Green function. We will get the right one when we come to the holomorphic projection of the analytic kernel function. Therefore we denote

$$(Z(x_1,h_1)_K \cdot Z(x_2,h_2)_K \cdot Z(x_3,h_3)_K)_{Ei,\infty} := g_{x_1,h_1K} * g_{x_2,h_2K} * g_{x_3,h_3K},$$

where Ei is indicate the current choice of Green functions.

Theorem 5.4.6. *Let $\tau|\infty$ and $g = (g_1, g_2, g_3) \in \mathbb{G}_{\mathbb{A}} = \mathrm{GL}_2^{+,3}(\mathbb{A})$. And assume that Φ_v is supported on non-singular locus at some finite place v. Then the archimedean contribution*

$$(Z(g_1,\Phi_1) \cdot Z(g_2,\Phi_2) \cdot Z(g_3,\Phi_3))_{Ei,\infty} = -2E'_v(g,0,\Phi),$$

where $E'_v(g,0,\Phi)$ is defined before Theorem 5.3.3.

Proof. First we consider $g = (g_1, g_2, g_3) \in \mathrm{SL}_2^3(\mathbb{A})$. Afterwards we extend this to $\mathrm{GL}_2^+(\mathbb{A})$.

By definition, $(Z(x_1,h_1)_K \cdot Z(x_2,h_2)_K \cdot Z(x_3,h_3)_K)_{Ei,\infty}$ is given by

$$Z(g,\Phi)_\infty = \mathrm{vol}(\widetilde{K}) \sum_{x=(x_i)\in(K\backslash\widehat{V})^3} \Phi(x)W_{T(x_\infty)}(g_\infty)\left(\int_{G(F)\backslash D_\pm \times G(\mathbb{A}_f)/K} *_{i=1}^3 g_{x_i}(z,h')d[z,h']\right),$$

where the sum is over all admissible classes.

Note that

$$\gamma^*[\eta(s_x(z)1_{G_x(\widehat{F})hK}(h'))] = \eta(s_{\gamma^{-1}x}(z)1_{G_{\gamma1x}(\widehat{F})\gamma^{-1}hK}(h')).$$

For a fixed triple (x_i), the integral is nonzero only if there exists a $\gamma \in G(F)$ such that

$$\gamma h' \in G_{\gamma_i^{-1}x_i}(\widehat{F})\gamma_i^{-1}h_iK \Leftrightarrow \gamma_i^{-1}h_i \in G_{\gamma_i^{-1}x_i}(\widehat{F})\gamma h'K.$$

Observe that the sum in the admissible classes can be written as $x_i \in G(F)\backslash V(F)$ and $h_i \in G_{x_i}(\widehat{F})\backslash G(\widehat{F})/K$. Here we denote for short $V = V(v)$ that is the nearby quadratic space ramified at $\Sigma(v)$. Thus we may combine the sum $x_i \in G(F)\backslash V(F)$ with $\gamma_i \in G(F)/G_{x_i}(F)$ and combine the sum over $\gamma \in G(F)$ with the quotient $G(F)\backslash D_\pm \times G(\mathbb{A}_f)/K$:

$$\mathrm{vol}(\widetilde{K}) \sum_{x\in G(F)\backslash V(F)^3} \left(\int_{h'\in G(\widehat{F})/K} \Phi(h'x)dh'\right)\left(\int_{D_\pm} *_{i=1}^3 g_{x_i}(z)dz\right).$$

Here we have used the fact that $G_x = \{1\}$ if $T(x)$ is non-singular and we are assuming that Φ_v is supported in the non-singular locus at some finite place v.

Therefore we have

$$Z(g,\Phi)_\infty = \sum_T \mathrm{vol}(SO(\mathbb{B}_\infty))e^{-2\pi T}\Lambda(T)I_T(g^\infty,\Phi^\infty), \tag{5.4.9}$$

where the sum is over all non-singular T with $\Sigma_T = \Sigma(\tau)$, namely those non-singular T represented by the nearby quaternion $B(\tau)$.

Similar to the unramified p-adic case, we compare this with the derivative of Eisenstein series for a regular T:

$$E'_T(g,0,\Phi) = \frac{W'_T(g_\infty,0,\Phi_\infty)}{W_T(g_\infty,0,\Phi'_\infty)}E_T(g,0,\Phi^\infty\otimes\Phi'_\infty), \tag{5.4.10}$$

where Φ'_∞ is any test function on $V_\infty'^3$ which makes v nonvanishing. We may also rewrite

$$Z(g,\Phi)_\infty = \sum_T \frac{\mathrm{vol}(SO(\mathbb{B}_\infty))e^{-2\pi T}\Lambda(T)}{I_T(g_\infty,\Phi'_\infty)}I_T(g,\Phi^\infty\otimes\Phi'_\infty). \tag{5.4.11}$$

Similar to the p-adic case, we may reduce the desired equality to the case $g = e$ which we assume now.

We need to evaluate the constant. Note that by local Siegel–Weil, the ratio

$$\frac{W_T(e,0,\Phi_v)}{I_T(\Phi_v)}$$

(whenever the denominator is non-zero) is independent of Φ_v, T ($det(T) \neq 0$) and depends only the measure on $SO(V_v)$ (and, of course, ψ_v). Let $c_{v,+}$ ($c_{v,+}$, resp.) be this ratio for the quaternion algebra over F_v that is split (division, resp.). We now use the Siegel–Weil formula of Kudla–Rallis to show that (under our choice of measures)

$$a_v := \frac{c_{v,+}}{c_{v,-}} = \pm 1.$$

Indeed, fix two distinct places v_1, v_2. Choose a global quaternion algebra B split at v_1, v_2. Let $B(v_1, v_2)$ be the quaternion algebra that differs from B only at v_1, v_2. Note that our choice of measures on the orthogonal groups associated to all quaternion algebras makes sure that we always get Tamagawa measures on the adelic points. Compare the Siegel–Weil (we may choose B anisotropic to apply) for B and $B(v_1, v_2)$:

$$a_{v_1} a_{v_2} = 1.$$

But v_1, v_2 are arbitrary, we conclude that a_v is independent of v and hence $a_v^2 = 1$.

From §4.2 Prop. 4.2.7, we have for $T > 0$

$$W_{T,\infty}(e, 0, \Phi_\infty) = \kappa(0)\Gamma_3(2)^{-1} e^{-2\pi T},$$

where $\kappa(0) < 0$. It is easy to see that

$$I_{T,\infty}(e, \Phi_\infty) = \mathrm{vol}(SO(\mathbb{B}_\infty)) e^{-2\pi T}.$$

Hence,

$$c_{\infty,-} = \frac{\kappa(0)\Gamma_3(2)^{-1}}{\mathrm{vol}(SO(\mathbb{B}_\infty))} < 0$$

On the other hand, it is not hard to see that $c_{\infty,+}$ is positive so we have

$$c_{\infty,+} = -c_{\infty,-} = -\frac{\kappa(0)\Gamma_3(2)^{-1}}{\mathrm{vol}(SO(\mathbb{B}_\infty))}.$$

Now note that $I_T(g, \Phi^\infty \otimes \Phi'_\infty) = E_T(g, 0, \Phi^\infty \otimes \Phi'_\infty)$, and by Theorem 5.4.5:

$$W'_T(g_\infty, 0, \Phi_\infty) = \frac{\kappa(0)}{2\Gamma_3(2)} e^{-2\pi T} \Lambda(T).$$

Hence the ratio of 5.4.10 over the T-th term of 5.4.11 is given by

$$\frac{\kappa(0)\Gamma_3(2)^{-1}}{2\mathrm{vol}(SO(\mathbb{B}_\infty))} \cdot \frac{1}{c_{\infty,+}} = -\frac{1}{2}.$$

This completes the proof.

$\square$

Holomorphic projection

Now we calculate the holomorphic projection of $E'(g, 0, \Phi)$ and come to the right choice of Green functions. By Lemma 4.4.1, we need to calculate the integral

$$\alpha_s(T) := \int_{\mathbb{R}_+^3} W'_T(\Phi, \begin{pmatrix} y^{1/2} & \\ & y^{-1/2} \end{pmatrix}, 0) \det(y)^{1+s} e^{-2\pi T y} \frac{dy}{\det(y)^2},$$

where $y = diag(y_1, y_2, y_3)$ and $T \in \mathrm{Sym}_3(\mathbb{R})$ with positive diagonal $diag(T) = t = diag(t_1, t_2, t_3)$.

Note that when $t > 1$ and $\mathrm{Re}(s) > -1$, we have an integral representation of the Legendre function of the second kind:

$$Q_s(t) = \int_{\mathbb{R}_+} \frac{du}{(t + \sqrt{t^2-1} cosh\, u)^{1+s}} = \frac{1}{2}\int_1^\infty \frac{(x-1)^s dx}{x^{1+s}(\frac{t-1}{2}x+1)^{1+s}}.$$

And the admissible pairing at archimedean place will be given by the constant term at $s = 0$ of (the regularized sum of, cf. [36, §8.1]) $Q_s(1 + 2s_{\gamma x}(z)/q(x))$.

Consider another closely related function for $t > 1, \mathrm{Re}(s) > -1$:

$$P_s(t) := \frac{1}{2}\int_1^\infty \frac{dx}{x(\frac{t-1}{2}x+1)^{1+s}}.$$

Then obviously we have

$$Q_0(t) = P_0(t).$$

One may use either of the three functions (i.e., Ei, Q_s and P_s) to construct Green's functions. As Theorem 5.4.5 shows that to match the analytic kernel function, the function Ei is the right choice; while the admissible pairing requires to use Q_s. The following proposition relates Ei to P_s and hence to Q_s by the coincidence $Q_0 = P_0$.

Proposition 5.4.7. *Let $x \in M^3_{2,\mathbb{R}}$ such that $T = T(x)$ is non-singular and has positive diagonal. Then we have*

$$\alpha_s(T) = \det(t)^{-1}\left(\frac{\Gamma(s+1)}{(4\pi)^{1+s}}\right)^3 \int_{D_\pm} \eta_s(x_1) * \eta_s(x_2) * \eta_s(x_3),$$

where

$$\eta_s(x,z) := P_s(1 + 2\frac{s_x(z)}{q(x)})$$

is a Green's function of D_x.

Proof. First by the definition we have

$$\alpha_s(T) = \int_{\mathbb{R}^3_+} \overset{2}{\det}(y^{1/2}) W'_{\sqrt{y}T\sqrt{y}}(\Phi, e, 0)\det(y)e^{-2\pi Ty}\det(y)^s\frac{dy}{y^2},$$

which is equal to

$$\int_{\mathbb{R}^3_+} W'_{\sqrt{y}T\sqrt{y}}(\Phi, e, 0)e^{-2\pi Ty}\det(y)^s dy.$$

If we modify $x \in M^3_{2,\mathbb{R}}$ with moment $T = T(x)$ to a new $x' = (x'_i)$ with $x'_i = x_i/q(x_i)^{1/2}$, we have $T(x') = t^{-\frac{1}{2}}Tt^{-\frac{1}{2}}$ (so that the diagonal are all 1). By Theorem 5.4.5 we have (after substitution $y \to yt$)

$$\alpha_s(T) = \det(t)^{-1-s}\int_{\mathbb{R}^3_+} \Lambda(y^{\frac{1}{2}}T(x')y^{\frac{1}{2}})e^{-4\pi y}\det(y)^s dy.$$

By the definition of $\Lambda(T)$, this is the same as

$$\det(t)^{-1-s}\int_{\mathbb{R}_+^3}\{*_{i=1}^3\eta(y_i^{\frac{1}{2}}x_i';z),1\}_D e^{-4\pi y}\det(y)^s dy$$

where

$$\{*_{i=1}^3\eta(y_i^{\frac{1}{2}}x_i';z),1\}_D=\int_D *_{i=1}^3\eta(y_i^{\frac{1}{2}}x_i';z).$$

We can interchange the star product and integral over y to obtain

$$\alpha_s(T)=\det(t)^{-1-s}\{*_{i=1}^3\int_{\mathbb{R}_+}\eta(y_i^{\frac{1}{2}}x_i;z)e^{-4\pi y}y_i^s dy_i,1\}_{D_\pm}.$$

Here we use $\langle\cdot,\cdot\rangle_{D_\pm}$ to denote the integration of the product over $D_\pm$. Now we compute the inner integral:

$$\begin{aligned}
&\int_{\mathbb{R}_+}\eta(y^{\frac{1}{2}}x;z)e^{-4\pi y}y^s dy\\
=&\int_{\mathbb{R}_+}Ei(-4\pi y s_x(z))e^{-4\pi y}y^s dy\\
=&\int_{\mathbb{R}_+}\int_1^\infty e^{-4\pi y s_x(z)u}\frac{1}{u}du e^{-4\pi y}y^s dy\\
=&\frac{\Gamma(s+1)}{(4\pi)^{1+s}}\int_1^\infty\frac{1}{u(1+s_x(z)u)^{1+s}}du\\
=&\frac{\Gamma(s+1)}{(4\pi)^{1+s}}P_s(1+2s_x(z)).
\end{aligned}$$

For more details, see [36, §8.1]. □

Based on the decomposition of $E'(g,0,\Phi)$ in §2.5, we can have a decomposition of its holomorphic projection, denoted by $E'(g,0,\Phi)_{hol}$:

$$E'(g,0,\Phi)_{hol}=\sum_v E'(g,0,\Phi)_{hol}, \tag{5.4.12}$$

and

$$E'(g,0,\Phi)_{hol}=\sum_{T,\Sigma(T)=\Sigma(v)}E'_T(g,0,\Phi)_{hol},$$

where the holomorphic projection only changes $E'_T(g,0,\Phi)$ only when $\Sigma(T)=\Sigma(v)$ for v is an archimedean place and in which case we give the formula only when $g_\infty=e$:

$$E'_T(g,0,\Phi)_{hol}=W_T(g_\infty)m_v(T)W_{T,f}(g_f,0,\Phi_f),$$

where $m(T)$ is the star product of $P_s(1+2s_x(z)/q(x))$ for x with moment T. For the general g_∞, it can be recovered by the transformation rule under Iwasawa decomposition Then all equalities above are valid for $g\in\mathbb{G}$ with $g_v=1$ when $v\in S$, the finite set of non-archimedean places outside which Φ_v is unramified.

Theorem 5.4.8. *Let τ be an archimedean place. Assume that for at least two non-archimedean v where $\Phi_v \in \mathscr{S}(V^3_{v,reg})$. Then for $g \in \mathbb{G}$ with $g_w = 1$ for $w \in S_f$, the set of finite place outside which Φ_v is unramified, Then we have*

$$Z(g, \Phi, \Delta)_\tau = -2E(g, 0, \Phi)_{\tau,hol}.$$

Proof. Under the assumption, all singular coefficients vanish on both sides. For the non-singular coefficients, the right choice of Green's function is the regularized limit of Q_s as $s \to 0$. Since $P_s - Q_s$ is holomorphic and equal to zero when $s = 0$, by the same argument of [36, §8.1], we may use P_s in the Green's function and then take the regularized limit. Then the result follows from Theorem 5.4.6 and the holomorphic projection above. ☐

6 Vanishing of singular Whittaker integerals

In this section, we study the vanishing property of the Fourier coefficients $E'_T(g, 0, \Phi)$ of the derivative of the Eisestein series for Siegel–Weil sections associated to an incoherent quaternion algebra $\mathbb{B}$. First of all we show that if for two places Φ is supported on elements in $\mathbb{B}^3$ whose components are linearly independent, then $E'_T(g, 0, \Phi) = 0$ when T is singular, see Proposition 6.1.1. Then we show that $E_T(e, s, \Phi) = 0$ if Φ is *exceptional of sufficiently high order*, see Proposition 6.2.3. Combination of these two facts implies that $E'(g, 0, \Phi)$ has only non-zero Fourier coefficients at non-singular T with $\Sigma(T) = \Sigma(v)$ for those unramified v if we choose Φ carefully enough, see formula (6.2.1). Finally, we conjecture that we can always make such a choice, see Conjecture 6.3.1. Meanwhile we can only prove this conjecture when π_v has at least two ramifield finite places and all of them are not split in $\mathbb{B}$, see Theorem 6.3.2.

6.1 Singular coefficients

In this subsection we deal with the singular part $E'_{sing}(g, 0, \Phi)$ of the Siegel-Eisenstein series.

Definition 6.1.1. *For a place v of F, we define the open subset $\mathbb{B}^3_{v,\mathrm{sub}}$ (resp. $\mathbb{B}^3_{v,\mathrm{reg}}$) of $\mathbb{B}^3_v$ to be all $x \in \mathbb{B}^3_v$ such that the components of x generates a dimension 3 subspace of $\mathbb{B}_v$ (resp. with non-degenerate restricted). We define the subspace $\mathscr{S}(\mathbb{B}^3_{v,\mathrm{sub}})$ (resp. $\mathscr{S}(\mathbb{B}^3_{v,\mathrm{reg}})$) of $\mathscr{S}(\mathbb{B}^3_v)$ to be the set of all Bruhat-Schwartz functions Φ with $\mathrm{supp}(\Phi) \subset \mathbb{B}^3_{v,\mathrm{sub}}$ (resp. $\mathrm{supp}(\Phi) \subset \mathbb{B}^3_{v,\mathrm{reg}}$).*

Note that $\mathscr{S}(\mathbb{B}^3_{v,\mathrm{sub}})$ is P_v-stable under the action defined by the Weil representation.

Proposition 6.1.2. *For an integer $k \geq 1$, fix non-archimedean (distinct) places $v_1, v_2, ..., v_k$. Let $\Phi = \otimes_v \Phi_v \in \mathscr{S}(\mathbb{B}^3)$ with $\mathrm{supp}(\Phi_{v_i}) \subset \mathbb{B}^n_{v_i,\mathrm{sub}}$ (i=1,2,...,k) . Then for T singular and $g \in G(\mathbb{A})$ with $g_{v_i} \in P_{v_i}, (i = 1, 2, ..., k)$, the vanishing order of the analytic function $ord_{s=0}E_T(g, s, \Phi)$ is at least $k - 1$. In particular, when T is singular, then $E_T(g, 0, f) = 0$ if $k \geq 1$; and $E'_T(g, 0, \Phi) = 0$ if $k \geq 2$, .*

Proof. We will use some results about Siegel-Weil formula and related representation theory. They should be well-known to experts and are proved mostly in series of papers by Kudla–Rallis ([23],[24]). We will sketch proofs of some of them but don't claim any originality and we are not sure if there are more straightforward ways.

Suppose $rank(T) = 3 - r$ with $r > 0$. Note that if $T = {}^t\gamma T'\gamma$, $T' = \begin{pmatrix} 0 & \\ & \beta \end{pmatrix}$ for some $\beta \in \mathrm{GL}_{3-r}$ and $\gamma \in \mathrm{GL}_3$, we have

$$E_T(g, s, \Phi) = E_{T'}(m(\gamma)g, s, \Phi).$$

Since $m(\gamma) \in P_{v_i}$, it suffices to prove the assertion for

$$T = \begin{pmatrix} 0 & \\ & \beta \end{pmatrix}$$

with β non-singular.

For $\mathrm{Re}(s) \gg 0$, we have

$$E_T(g, 0, \Phi) = \int_{[N]} \sum_{P(F)\backslash G(F)} f_{\Phi,s}(\gamma n g)\psi_{-T}(n)dn$$
$$= \int_{[N]} \sum_{i=0}^{3} \sum_{\gamma \in P\backslash P w_i P} f_{\Phi,s}(\gamma n g)\psi_{-T}(n)dn.$$

Here For $i = 1, 2, 3$,

$$w_i := \begin{pmatrix} 1_i & & & \\ & & & 1_{3-i} \\ & & 1_i & \\ & -1_{3-i} & & \end{pmatrix}.$$

Lemma 6.1.3. *For a place v, if a Siegel-Weil section $f_{\Phi,s} \in I(s)$ is associated to $\Phi \in \mathscr{S}_0(\mathbb{B}^3_{v,sub})$, then $f_{\Phi,s}$ is supported in the open cell Pw_0P for all s.*

Proof. By the definition $f_{\Phi,s}(g) = r(g)\Phi(0)\lambda_s(g)$. Thus it suffices to prove $supp(f_{\Phi,0}) \subset Pw_0P$. Note that by the Bruhat decomposition $G = \coprod_i Pw_iP$, it suffices to prove $r(pw_ip)\Phi(0) = 0$ for $i = 1, 2, 3$. Since $\mathscr{S}(\mathbb{B}^3_{v,sub})$ is P_v-stable, it suffices to prove $r(w_i)\Phi(0) = 0$ for $i = 1, 2, 3$. Since

$$r(w_i)\Phi(0) = \gamma \int_{\mathbb{B}^{n-i}} \Phi(0, ...0, x_{i+1}, ..., x_3)dx_{i+1}...dx_3$$

for certain eighth-root of unity γ, we compelete the proof since

$$\Phi(0, ..., 0, x_{i+1}, ..., x_3) \equiv 0$$

when $i \geq 1$. ∎

By Lemma 6.1.3, $f_{\Phi_v}(\gamma n_v g_v, s) \equiv 0$ for $\gamma \in Pw_iP, i > 0$, $v \in \{v_1, ..., v_k\}$ and $g_v \in P_v$. Thus for g as in the statement, only the open cell has nonzero contribution in the coefficients

$$E_T(g, s, \Phi) = \int_{N_{\mathbb{A}}} r(w_0 n g)\Phi(0)\psi_{-T}(n)dn.$$

This is exactly the Whittaker functional $W_T(g, s, \Phi) = W_T(e, s, r(g)\Phi)$.

Let $i : \mathrm{Sp}(3 - r) \to \mathrm{Sp}(3)$ be the standard embedding indicated by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} 1_r & & & \\ & a & & b \\ & & 1_r & \\ & c & & d \end{pmatrix}.$$

Then this induces a map by restriction: $i^* : I(s) \to I^{3-r}(s + \frac{r}{2})$ to the degenerate principal series on $\mathrm{Sp}(3 - r)$. We will frequently use upper/lower index $n - r$ to indicate the rank of the symplectic group we work on. Let $M(s)$ be the intertwining operator. We now simply denote by f the Siegel–Weil section f_Φ.

Lemma 6.1.4. *Let $E_\beta(g,s,i^*M(s)f)$ denote the β-Fourier coefficient of the Eisensetin series defined by section $i^*M(s)f$. Then*

$$W_T(e,s,f) = E_\beta(e,-s+\frac{r}{2}, i^*M(s)f)$$

Remark 6.1.1. Note that in general, besides $W_T(g,s,f)$ in $E_T(g,s,f)$ there are also other terms including $E_\beta(e,s+\frac{r}{2}, i^*r(g)f)$.

Proof. By [24] we have

$$\begin{aligned}
&W_T(e,s,f)\\
=&\int_N f_s(wng)\psi_{-T}(n)dn\\
=&\int_{N_r}\int_{N_{3-r,3}} f_s(wn_1n_2g)\psi_{-T}(n_1n_2)dn_1dn_2\\
=&\int_{N_r}\left(\int_{N_{3-r,3}} f_s(ww_{n-r}^{-1}w_{n-r}n_1(x,y)w_{n-r}^{-1}w_{n-r}n_2(z)g)dn_1\right)\psi_{-\beta}(n_2)dn_2\\
=&\int_{N_r}\left(\int_{U_{3-r,3}} f_s(w^{(r)}u(x,y)w_{n-r}n_2(z)g)du\right)\psi_{-\beta}(n_2)dn_2\\
=&E_\beta(e,s-\frac{r}{2}, i^*U(s)f),
\end{aligned}$$

where the matrices

$$u(x,y)=\begin{pmatrix} 1_r & y & x & \\ & 1_{3-r} & & \\ & & 1_r & \\ & & -{}^ty & 1_{3-r}\end{pmatrix},$$

$$n(x,y)=\begin{pmatrix} 1_r & & x & y\\ & 1_{3-r} & {}^ty & \\ & & 1_r & \\ & & & 1_{3-r}\end{pmatrix},$$

$$u(x,y)=w_{3-r}n(x,y)w_{3-r}^{-1},$$

and the operator

$$U_r(s)f=\int_{U_{3-r,3}} f_s(w^{(r)}ug)du, \quad w^{(r)}=\begin{pmatrix} & & 1_r & \\ & 1_{3-r} & & \\ -1_r & & & \\ & & & 1_{3-r}\end{pmatrix}.$$

Apply the functional equation to the Eisenstein series $E(g,s,i^*M(s)f)$,

$$W_T(e,s,f) = E_\beta(e,-s+\frac{r}{2}, M(s-\frac{r}{2})\circ i^*U(s)f).$$

By the relation ([24, page 37]),

$$M(s-\frac{r}{2})\circ i^*U(s)=i^*M(s),$$

we obtain

$$W_T(e,s,f)=E_\beta(e,-s+\frac{r}{2},i^*M(s)f).$$

□

Now we have an Euler product when $\mathrm{Re}(s)\gg 0$,

$$W_T(e,s,f)=\prod_v W_{\beta,v}(e,-s+\frac{r}{2},i^*M_v(s)f_v).$$

Note that by the standard Gindikin-Karpelevich type argument, for the spherical vector $f_v^0(s)$ at a non-archimedean v and when χ_v is unramified, we have

$$M_v(s)f_v^0(s)=\frac{a_v(s)}{b_v(s)}f_v^0(-s),$$

where

$$a_v(s)=L_v(s+\varrho_3-3,\chi_v)\zeta_v(2s-1),$$

and

$$b_v(s)=L_v(s+\varrho_3,\chi_v)\zeta_v(2s+2).$$

Thus, for a finite set outside which everything is unramified,

$$M(s)f(s)=\frac{a(s)}{b(s)}\left(\bigotimes_{v\in S}\frac{b_v(s)}{a_v(s)}M_v(s)f_v(s))\right)\otimes f_S^0(-s).$$

For a local Siegel-Weil section f_v for all v, $\frac{b_v(s)}{a_v(s)}M_v(s)f_v$ is holomorphic at $s=0$ and there is a non-zero constant independent of f such that

$$\frac{b_v(s)}{a_v(s)}M_v(s)f_v(s))|_{s=0}=\lambda_v f_v(0).$$

Thus we have

$$\begin{aligned}&W_T(e,s,f)\\ =&\prod_v W_{\beta,v}(e,-s+\frac{r}{2},i^*M_v(s)f_v)\\ =&\Lambda_{3-r}(-s+\frac{r}{2})\frac{a(s)}{b(s)}\prod_{v\in S'_\beta}\frac{1}{\Lambda_{3-r,v}(-s+\frac{r}{2})}W_{\beta,v}(e,-s+\frac{r}{2},i^*(\frac{b_v(s)}{a_v(s)}M_v(s)f_v))\\ =&\Lambda_{3-r}(-s+\frac{r}{2})\frac{a(s)}{b(s)}\prod_{v\in S'_\beta}A_{\beta,v}(s,f),\end{aligned}$$

where S_β is the set of all primes such that outside S_β, f_v is the spherical vector, ψ_v is unramifed and $\mathrm{ord}_v(\det(\beta)) = 0$.

Since $ord_{s=0}\Lambda_{3-r,v}(-s+\frac{r}{2}) = 0$, $\frac{b_v(s)}{a_v(s)}M_v(s)f_v$ is holomorphic and $W_\beta(e,s,f)$ extends to an entire function, we know that $A_{\beta,v}(s,f)$ is holomorphic at $s=0$. We have a formula

$$A_{\beta,v}(0,f) = \frac{\lambda_v}{\Lambda_{3-r,v}(0)} W^{3-r}_{\beta,v}(e, \frac{r}{2}, i^* f_v(0)).$$

Lemma 6.1.5. *Define a linear functional*

$$\begin{aligned} \iota : \mathscr{S}(\mathbb{B}^3_v) &\to \mathbb{C} \\ \Phi_v &\mapsto A_{\beta,v}(0, f_{\Phi_v}). \end{aligned}$$

Then, we have $\iota(r(n(b))\Phi_v) = \psi_{v,T}(b)\iota(\Phi_v)$, *i.e.,* $\iota \in \mathrm{Hom}_N(\mathscr{S}(\mathbb{B}^3_v), \psi_T)$.

Proof. Let $b = \begin{pmatrix} x & y \\ {}^t y & z \end{pmatrix} \in \mathrm{Sym}_3(F_v)$. Since M_n is Sp(3)-intertwining, we have

$$\begin{aligned}
&W_{\beta,v}(e, -s+\frac{r}{2}, i^*(M_v(s)r(n(b))f_v)) \\
=&W_{\beta,v}(e, -s+\frac{r}{2}, i^*(r(n(b))M_v(s)f_v)) \\
=&\int_{\mathrm{Sym}_{3-r}} (M_v(s)f_v)(w_{3-r}n\left(\begin{pmatrix} 0 & 0 \\ 0 & z' \end{pmatrix}\right)n(b))\psi_{-\beta}(z')dz' \\
=&\int_{\mathrm{Sym}_{3-r}} (M_v(s)f_v)(u(x,y)w_{3-r}n\left(\begin{pmatrix} 0 & 0 \\ 0 & z'+z \end{pmatrix}\right))dz' \\
=&\int_{\mathrm{Sym}_{3-r}} (M_v(s)f_v)(w_{3-r}n\left(\begin{pmatrix} 0 & 0 \\ 0 & z'+z \end{pmatrix}\right))\psi_{-\beta}(z')dz' \\
=&\psi_\beta(z)W_{\beta,v}(e, -s+\frac{r}{2}, i^*M_v(s)f_v) \\
=&\psi_T(b)W_{\beta,v}(e, -s+\frac{r}{2}, i^*M_v(s)f_v).
\end{aligned}$$

Thus, the linear functional $f_s \mapsto A_{\beta,v}(s,f)$ defines an element in $\mathrm{Hom}_N(I(s), \psi_T)$. In particular, when $s=0$, the composition ι of $A_{\beta,v}$ with the G-intertwining map $\mathscr{S}(\mathbb{B}^3_v) \to I(0)$ defines a linear functional in $\mathrm{Hom}_N(\mathscr{S}(\mathbb{B}^3_v), \psi_T)$. □

Then the map ι factors through the ψ_T-twisted Jacquet module $\mathscr{S}(\mathbb{B}^3_v)_{N,T}$ (i.e., the maximal quotient of $\mathscr{S}(\mathbb{B}^3)$ on which N acts by character ψ_T). Thus by the following result of Rallis, ι is trivial on $\mathscr{S}(\mathbb{B}^3_{v,\mathrm{sub}})$ when T is singular:

Lemma 6.1.6. *The map* $\mathscr{S}(\mathbb{B}^3) \to \mathscr{S}(\mathbb{B}^3)_{N,T}$ *can be realized as the restriction* $\mathscr{S}(\mathbb{B}^3) \to \mathscr{S}(\Omega_T)$.

Now since $\mathrm{ord}_{s=0}\frac{a(s)}{b(s)} = 0$, we can now conclude that

$$\mathrm{ord}_{s=0} W_T(e, s, f_\Phi) \geq k$$

if $\Phi_{v_i} \in \mathscr{S}(\mathbb{B}^3_{v_i,\mathrm{reg}})$ since the restriction to Ω_T is zero.

For a general $g \in G_{\mathbb{A}}$, we have

$$\begin{aligned} &W_T(g, s, f_\Phi) \\ =&W_T(e, s, r(g)\Phi) \\ =&\Lambda_{n-r}(-s + \frac{r}{2})\frac{a(s)}{b(s)} \prod_{v\in S'_\beta} A_{\beta,v}(s, r(g_{v_i})\Phi_v) \end{aligned}$$

where $S_{\beta,g}$ is a finite set of place that depends also on g.

Since $\mathscr{S}(\mathbb{B}^3_v) \to I(0)$ is G-equivariant, $A_{\beta,v}(0, r(g_v)f_v) = \iota(r(g_v)\Phi_v)$. Since $g_{v_i} \in P_{v_i}$, we have $r(g_{v_i})\Phi_{v_i} \in \mathscr{S}(\mathbb{B}^3_{v_i,sub})$ and by the same argument above $A_{\beta,v_i}(0, r(g_{v_i})f_{v_i}) = 0$. This completes the proof of Proposition 6.1.2. □

Remark 6.1.2. The proof would be much shorter if it were true that $W_{T,v}(g, s, f_v)$ extends to $\mathbb{C}$ and holomorphic at $s = 0$ for singular T.

Now it is easy to extend to the similitude group GSp_3. Recall that we have a decomposition of $E'(g, 0, \Phi)$ according to the difference of $\Sigma(T)$ and Σ:

$$E'(g, 0, \Phi) = \sum_v E'_v(g, 0, \Phi) + E'_{sing}(g, 0, \Phi), \tag{6.1.1}$$

where

$$E'_v(g, 0, \Phi) = \sum_{\Sigma(T)=\Sigma(v)} E'_T(g, 0, \Phi), \tag{6.1.2}$$

and

$$E'_{sing}(g, 0, \Phi) = \sum_{T,\, \det(T)=0} E'_T(g, 0, \Phi).$$

Corollary 6.1.7. *The same assumption as in Proposition 6.1.2, then we have for T singular and $g \in \mathrm{GSp}_3(\mathbb{A})$ with $g_{v_i} \in P_{v_i}, (i = 1, 2, ..., k)$, the vanishing order of the analytic function $\mathrm{ord}_{s=0} E_T(g, s, \Phi)$ is at least $k - 1$. In other words, for such g we have*

$$E'_{sing}(g, 0, \Phi) = 0.$$

Proof. For $g \in \mathrm{GSp}_3(\mathbb{A})$, we still have

$$W_T(g, s, \Phi) = \Lambda_{3-r}(-s + \frac{r}{2})\frac{a(s)}{b(s)} \prod_{v\in S'_\beta} A_{\beta,v}(s, r(g_{v_i})f_v)$$

for a finite set of places $S_{\beta,g}$. □

6.2 Functions with regular support

Let F be a non-archimedean field. Let B be a quaternion algebra over F. And we have the moment map

$$Q : B^3 \to \mathrm{Sym}_3(F).$$

Definition 6.2.1. *We call a function $\Phi \in \mathscr{S}(B^3_{reg})$ "exceptional of order k" if it satisfies the condition that $Q(\mathrm{supp}(\Phi)) + p^{-k}\mathrm{Sym}_n(\mathscr{O}) \subseteq Q(B^3_{\mathrm{reg}})$.*

Even though it looks that such functions are very special, they in fact generate $\mathscr{S}(B^3_{reg})$ under the action of a very small subgroup.

Lemma 6.2.2. *Let k be any fixed integer. Then $\mathscr{S}(B^3_{reg})$ is generated by all exceptional function of order k under the action of elements $m(aI_3) \in \mathrm{Sp}_3$ for all $a \in F^\times$.*

Proof. Without loss of generality, we can assume that k is even and that $\Phi = 1_U \in \mathscr{S}(B^3_{reg})$ is the characteristic function some open compact set $U \subseteq B^3$. Then $Q(U)$ is an compact open subset of $\mathrm{Sym}_3(F)_{\mathrm{reg}}$. Let $\mathbb{Z}^3_+ = \{(a_1, a_2, a_3) \in \mathbb{Z}^3 | a_1 \le a_2 \le a_3\}$. Then the "elementary divisors" defines a map $\delta : b \in \mathrm{Sym}_3(F) \to (a_1, a_2, a_3) \in \mathbb{Z}^3_+$. One can check that it is locally constant on $\mathrm{Sym}_3(F)_{reg}$. Hence the composition of this map and the moment map Q is also locally constant on B^3_{reg}. In particular, this gives a partition of U into disjoint union of finitely many open subsets. So we can assume that $\delta \circ Q$ is constant on U, say, $\delta \circ Q(U) = \{(a_1, a_2, a_3)\}$.

Consider $m(aI_3)\Phi$ which is certain multiple of 1_{aU}. Choose $a = p^{-A}$ for some integer $A > 1 + a_1 + (a_2 - a_1) + (a_3 - a_1)$. Then we are left to prove that that such $1_{p^{-A-k/2}U}$ is exceptional of order k. It suffices to prove that, for any $x \in U$ and $t \in \mathrm{Sym}_3(\mathscr{O})$, $Q(p^{-A-k/2}x) + p^{-k}t$ belongs to $Q(B^3_{\mathrm{reg}})$. Note that

$$Q(p^{-A-k/2}x) + p^{-k}t = p^{-k-2A+2[\frac{a_1-1}{2}]}(Q(p^{-[\frac{a_1-1}{2}]}x) + p^{2A-2[\frac{a_1-1}{2}]}t).$$

Now $Q(p^{-[\frac{a_1-1}{2}]}x) \in \mathrm{Sym}_3(\mathscr{O})$. It is well-known that for $T \in \mathrm{Sym}_3(\mathscr{O})_{\mathrm{reg}}$, T and $T + p^{2+\det(T)}T'$ for any $T' \in \mathrm{Sym}_3(\mathscr{O})$ defines isomorphic integral quadratic forms of rank n. Equivalently, $T + p^{2+\det(T)}T' = {}^t\gamma T\gamma$ for some $\gamma \in \mathrm{GL}_3(\mathscr{O})$. Now it is easy to see that $Q(p^{-A-k/2}x) + p^{-k}t \in Q(B^3_{\mathrm{reg}})$. $\square$

The nice property of an exceptional of high order is exhibited in the vanishing of the Whitatker function.

Proposition 6.2.3. *Suppose that $\Phi \in \mathscr{S}(B^3_{reg})$ is exceptional of sufficiently large order k depending on the conductor of the additive character ψ. Then we have*

$$W_T(\Phi, e, s) \equiv 0$$

for regular $T \notin Q(B^3_{\mathrm{reg}})$ and any $s \in \mathbb{C}$. In particular, $W_T(\Phi, e, 0) = W'_T(\Phi, e, 0) = 0$.

Proof. When $\mathrm{Re}(s) \gg 0$, we have

$$\begin{aligned}
&W_T(\Phi, e, s) \\
&= \gamma(V,\psi) \int_{\mathrm{Sym}_3(F)} \psi(-b(T - Q(x))) \int_{B^3} \Phi(x)\delta(wb)^s dx db \\
&= \gamma(V,\psi) c_v \int_{\mathrm{Sym}_3(F)} \psi(b(T' - T))\delta(wb)^s, I_{T'}(\Phi) db dT'
\end{aligned}$$

where c_v is a suitable non-zero constant and $I_{T'}(\Phi)$ is a certain orbital integral defined earlier. Then $T' \mapsto I_{T'}(\Phi)$ defines a function in $\mathscr{S}(\mathrm{Sym}_3(F)_{\mathrm{reg}})$ for our choice of Φ. Since as a function of $b \in \mathrm{Sym}_3(F)$, $\delta(wb)$ is invariant under the translation of $\mathrm{Sym}_3(\mathscr{O})$, we have

$$\begin{aligned}
&\int_{\mathrm{Sym}_3(F)} \psi(bt)\delta(wb)^s db \\
&= \left(\int_{\mathrm{Sym}_3(O)} \psi(xt) dx\right) \sum_{b \in \mathrm{Sym}_3(F)/\mathrm{Sym}_3(\mathscr{O})} \psi(bt)\delta(wb)^s,
\end{aligned}$$

which is zero unless $t \in p^{-k}\mathrm{Sym}^3(\mathscr{O})$ for some k depending on the conductor of the additive character ψ.

Therefore the nonzero contribution to the integral are from $T' - T \in p^{-k}\mathrm{Sym}^3(\mathscr{O})$ and $I_{T'}(\Phi) \neq 0$. The assumption in the proposition forces that T' is not in $Q(\mathrm{supp}(\Phi))$. But this in turn implies that $I_{T'}(\Phi) = 0$!

In conclusion, we proves that, if Φ is exceptional of order at least k and $\mathrm{Re}(s) \gg 0$, we have

$$W_T(\Phi, e, s) \equiv 0.$$

By analytic continuation, we still have $W_T(\Phi, e, s) \equiv 0$ for all $s \in \mathbb{C}$!

$\square$

From now on, we will choose Φ_v to be a test function "exceptional of sufficiently higher order" for $v \in S$ where S is a set of finite places with at least two elements such that any finite place outside S is spherical. And we always choose the standard Gaussian at all archimedean places. Then for $g \in \mathbb{G}(\mathbb{A}^S)$, we have

$$E'(g, 0, \Phi) = \sum_v \sum_{\Sigma(T) = \Sigma(v)} E'_T(g, 0, \Phi), \tag{6.2.1}$$

where the sum runs over v outside S and nonsingular T.

6.3 Local zeta integrals with regular support

Let $\sigma = \otimes_{i=1}^3 \sigma_i$ be unitary irreducible admissible representation of $\mathbb{G}$ with each σ_i of infinite dimensional and with $\Lambda(\sigma) < 1/2$. Recall that we let $\Lambda(\sigma_i)$ be zero if it is supercuspidal and $|\Lambda|$ if $\sigma = \mathrm{Ind}_B^G(\chi|\cdot|^{\lambda}|\cdot|^{-\lambda})$ for a unitary χ. Let $\Lambda(\sigma)$ be the sum of $\Lambda(\sigma_i)$. Note that if σ is local component of global automorphic cuspidal representation, we have $\Lambda(\sigma) < 1/2$ by work of Kim-Shahidi (Ramanujam conjecture predicts that $\Lambda(\sigma) = 0$).

Conjecture 6.3.1. *Assume that* $\mathrm{Hom}_{\mathbb{G}}(\mathscr{S}(V^3)\times\sigma,\mathbb{C})\neq 0$. *Then the local zeta integral* $Z(f,W)$ *is non-zero for some choice of* $W\in\mathscr{W}(\sigma,\psi)$ *and* $f\in I(B)$ *attached to* $\Phi\in\mathscr{S}(V^3_{reg})$.

Since that above space $\mathrm{Hom}_{\mathbb{G}}(\mathscr{S}(V^3)\times\sigma,\mathbb{C})$ is one dimensional and generated by zeta integral, thus the zeta integral defines an $\mathrm{SL}_2(F)^3$-equivariant map

$$\alpha:\mathscr{S}(V^3)\longrightarrow\widetilde{\sigma}.$$

We need only show that there is an element in $\mathscr{S}(B^3_{reg})$ with non-zero image.

In the following we want to show that the theorem is true in some special cases including the case when V is anisotropic by induction.

Theorem 6.3.2. *Assume that* V *is anisotropic. Let* $\alpha_i:\mathscr{S}(V)\longrightarrow\sigma_i$ $(i=1,\cdots m)$ *be some* $\mathrm{SL}_2(F)$*-surjective morphisms to irreducible and admissible* $\mathrm{SL}_2(F)$ *representations. Let* $\alpha=\alpha_1\otimes\alpha_2\otimes\cdots\alpha_m$ *be their product:*

$$\alpha:\quad\mathscr{S}(V^m)\longrightarrow\sigma:=\sigma_1\otimes\cdots\otimes\sigma_m.$$

Let W *be a non-degenerate subspace of* V *(with respect to the norm* $q|_W$*) such that*

$$\dim W+m\leq\dim V.$$

There is a function $\phi\in\mathscr{S}(V^m)$ *such that* $\alpha(\phi)\neq 0$ *and that the support* $\mathrm{supp}(\phi)$ *of* ϕ *contains only elements* $x=(x_1,\cdots,x_m)$ *such that*

$$W(x):=W+Fx_1+\cdots Fx_m$$

is non-degenerate of dimension $\dim W+m$.

Proof of the case $m=1$. Since W is non-degenerate, we have an orthogonal decomposition $V=W\oplus W'$, and an identification $\mathscr{S}(V)=\mathscr{S}(W)\otimes\mathscr{S}(W')$. The action of $\mathrm{SL}_2(F)$ is given by actions of the double cover $\widetilde{\mathrm{SL}}_2(F)$ on $\mathscr{S}(W)$ and $\mathscr{S}(W')$ respectively. Since V is anisotropic, W' is anisotropic. So the space $\mathscr{S}(W')$ is generated over $\mathrm{SL}_2(F)$ by the subspace $\mathscr{S}(W'_{q\neq 0})$ of function supported on nonzero elements. In fact, one has

$$\mathscr{S}(W')=\mathscr{S}(W'_{q\neq 0})+w\mathscr{S}(W'_{q\neq 0}).$$

Choose any ϕ such that $\alpha(\phi)\neq 0$ and that ϕ is a pure tensor:

$$\phi=f\otimes f',\qquad f\in\mathscr{S}(W),\quad f'\in\mathscr{S}(W').$$

Write

$$f'=f'_0+wf'_1,\qquad f'_i\in\mathscr{S}(W'_{q\neq 0})\quad g_i\in\widetilde{\mathrm{SL}}_2(F).$$

Then we have decomposition

$$\phi=\phi_0+w\phi_1,\qquad \phi_0=f\otimes f'_0,\quad \phi_1=w^{-1}f\otimes f'_1.$$

One of $\alpha(\phi_i)\neq 0$. Thus we may replace ϕ by this ϕ_i to conclude that the support of ϕ is contained in the set of $x=(w,w')$ with $W(x):=W\oplus Fw'$ non-degenerate. $\square$

Proof of the case $m > 1$. We prove by induction on m. We assume that we have a $\phi' \in \mathscr{S}(V^{m-1})$ with nonzero image in $\sigma_1 \otimes \cdots \otimes \sigma_{m-1}$ under

$$\alpha' = \alpha_1 \otimes \cdots \otimes \alpha_{m-1}$$

such that the support of ϕ is contained in the set of elements $x' = (x_1, \cdots, x_{m-1})$ with non-degenerate $W(x')$ of dimension $\dim W + m - 1$. For any $x \in \mathrm{supp}(\phi')$ by applying the proved case $m = 1$ to the subspace $W(x')$, we have a $\phi_{x'} \in \mathscr{S}(V)$ such that $\mathrm{supp}(\phi_{x'})$ contains only elements x_m with non-degenerate

$$W(x')(x_m) = W(x), \qquad x = (x_1, \cdots, x_m)$$

of dimension $\dim W + m$. By computing moment matrix of W'', we see that this last condition is open in x'. Thus there is an open subset $U(x')$ of x' such that above non-degenerate condition holds for all elements in $U(x')$.

As x' varies in $\mathrm{supp}(\phi')$, $U(x')$ covers $\mathrm{supp}(\phi')$. By the compactness of $\mathrm{supp}(\phi')$, we can find finitely many $U(x_i')$ to cover $\mathrm{supp}(\phi')$. Replacing $U(x_i')$ by sub-coverings of $U(x_i') \cap \mathrm{supp}(\phi')$, we may assume that ϕ_{x_i} takes constants c_i on every $U(x_i')$. Thus we have an decomposition

$$\phi' = \sum_i c_i 1_{U(x_i')}.$$

As $\alpha'(\phi') \neq 0$, for one of x_i, say y, $\alpha'(1_{U(y)}) \neq 0$. Now we define

$$\phi = 1_{U(y)} \otimes \phi_y \in \mathscr{S}(V^m).$$

Then ϕ satisfies the conditions in the theorem. $\square$

7 Local intersection at ramified places

In this subsection, we want to describe the local height pairing of Gross–Schoen cycles on the triple product of a Shimura curve at bad places. Our treatment is complete only if v is non-split in the corresponding quaternion algebra. Some further treatment needs to treat so called non surpersingular local intersection.

7.1 Analytic unformizations

In the following, we want to give an analytic description of Hecke correspondence when v is not split in $\mathbb{B}^\times$. Again, we let B denote the quaternion algebra over F such that

$$B \otimes \mathbb{A} \simeq M_2(F_v) \otimes \mathbb{B}^v.$$

Cerednik–Drinfeld uniformization

First we want recall the Cerednik–Drinfeld uniformization when $U_v = U_v^0 := \mathscr{O}_{B,v}^\times$ is maximal for the formal completion $\widehat{Y}_U$ along its special fiber over v. Let $\widehat{\Omega}$ denote Deligne's formal scheme over $\mathscr{O}_v$ obtained by blowing-up $\mathbb{P}^1$ along its rational points in the special fiber over the residue field k of $\mathscr{O}_v$ successively. So the generic fiber Ω of $\widehat{\Omega}$ is a rigid analytic space over F_v whose $\bar{F}_v$-points are given by $\mathbb{P}^1(\bar{F}_v) - \mathbb{P}^1(F_v)$. The group $\mathrm{GL}_2(F_v)$ has a natural action on $\widehat{\Omega}$. Let $\mathscr{H}_0 = \Omega \otimes F_v^{ur}$ be its base change to the maximal unramified extension of F_v. Then $\Sigma_0 := \mathrm{Res}_{F_v^{\mathrm{ur}}/F_v} \mathscr{H}_0$ when viewed as a formal scheme over F_v has a action by $B^\times \times \mathbb{B}_v^\times$ via action of $g_1 \in B_v^\times \in \mathrm{GL}_2(F_v)$ and the following action on $\mathscr{O}_v^{ur}$:

$$(g_1, g_2) \in B_v^\times \times \mathbb{B}_v^\times \longrightarrow g_1 \times \mathrm{Frob}^{-\mathrm{ord}_v(\nu(g_1)\nu(g_2))}.$$

The theorem of Cerednik–Drinfeld gives a natural isomorphism between two analytic spaces:

$$Y_U^{\mathrm{an}} \simeq B^\times \backslash \Sigma_0 \times (\mathbb{B}^v)^\times / U^v.$$

The projective system of these varieties when U^v various form a projective system with compatible action by $\mathbb{B}^\times / B_{0,v}^\times$. The analytic space Σ_0 over F_v^{ur} is geometrically connected but $\mathrm{Res}_{F_v^{\mathrm{ur}}} \Omega_0$ is not. In fact over F_v^{ur} is isomorphic to $\Omega_0 \times \mathbb{Z}$. Thus we have description over F_v^{ur}:

$$Y_{U, F_v^{\mathrm{ur}}}^{\mathrm{an}} \simeq B_0^\times \backslash \mathscr{H}_0 \times (\mathbb{B}^v)^\times / U^v$$

here $B_0^\times$ denote the subgroup of elements b such that $\mathrm{ord}_v \nu(b) = 0$. The action of $b \in \mathbb{B}_v^\times$ in this new description is given as follows:

$$[z, g] \mapsto [f^{-1}z, f^{-1}g]$$

for some elements $f \in B$ whose norm has the same order as b.

More generally, for any integer $n \geq 1$, let U_v^n denote the subgroup $1 + \pi^n \mathscr{O}_{\mathbb{B}_v}$. Then there is an etale covering $\mathscr{H}_n$ of $\mathscr{H}_0$ over F_v^{ur} with an compatible action of $\mathbb{B}_v^\times \times B_v^\times$ on $\Sigma_n := \mathrm{Res}_{F_v^{\mathrm{ur}}/F_v}(\mathscr{H}_n)$ over F such that

$$Y_{U_v^n \cdot U^v}^{\mathrm{an}} \simeq B^\times \backslash \Sigma_n \times (\mathbb{B}^v)^\times / U^v.$$

with compatible action by $\mathbb{B}_v^\times$. The analytic space $\mathscr{H}_n$ over F_v^{ur} has an action of by the subgroup of elements (γ, b) of $B_v^\times \times \mathbb{B}_v^\times$ with $\mathrm{ord}(\nu(\gamma)\nu(b)) = 0$. Thus we have an description over F_v^{ur} as

$$Y_{U_v^n \cdot U^v, F_v^{\mathrm{ur}}}^{\mathrm{an}} \simeq B_0^\times \backslash \mathscr{H}_n \times (\mathbb{B}^v)^\times / U^v.$$

The right action of an element $b \in \mathbb{B}_v^\times$ is given by

$$[z, g] \mapsto [(f^{-1}, b)z, f^{-1}g].$$

Here $f \in B$ is any element with the same norm as b.

Write $\mathscr{H}$ as projective limit of $\mathscr{H}_n$ which admit an action of $\mathscr{O}_{\mathbb{B}_v}^\times$ so that quotient by any compact subgroup U_v gives an rigid space $\mathscr{H}_{U_v}$ over F_v^{ur}. Write $\Sigma_n = \mathrm{Res}_{F_v^{\mathrm{ur}}/F_v}(\mathscr{H}_n)$ and $\Sigma = \mathrm{Res}_{F_v^{\mathrm{ur}}/F_v}(\mathscr{H})$. Then we have a unformization for general open compact subgroup U of $\mathbb{B}^\times$:

$$Y_U^{\mathrm{an}} = B^\times \backslash \Sigma_{U_v} \times (\mathbb{B}^v)^\times / U^v = B^\times \backslash \mathrm{Res}_{F_v^{\mathrm{ur}}/F_v} \Sigma \times (\mathbb{B}^v)^\times / U.$$

The projective system of these spaces when n and U^v various form a projective system with action by $\mathbb{B}^\times$. We have a similar description over F_v^{ur}:

$$Y_{U, F_v^{\mathrm{ur}}}^{\mathrm{an}} = B_0^\times \backslash \mathscr{H}_{U_v} \times (\mathbb{B}^v)^\times / U^v = B_0^\times \backslash \mathscr{H} \times (\mathbb{B}^v)^\times / U.$$

The analytic space at v associate to M_K can be described using orthogonal space $V = (B, q)$ and

$$H = \mathrm{GSpin}(V) = \{(g_1, g_2) \in B^\times, \quad \nu(g_1) = \nu(g_2)\},$$
$$(g_1, g_2)x = g_1 x g_2^{-1}, \qquad g_i \in B^\times, x \in V,$$
$$D_K := \mathscr{H}_{U_v} \times_{F_v^{\mathrm{ur}}} \mathscr{H}_{U_v}, \qquad D = \Sigma \times_{F_v^{\mathrm{ur}}} \Sigma.$$

In this case,

$$M_K^{\mathrm{an}} = H(F) \backslash \mathrm{Res}_{F_v^{\mathrm{ur}}/F_v} D_{K_v} \times H(\mathbb{A}_f^v)/K^v = H(F) \backslash \mathrm{Res}_{F_v^{\mathrm{ur}}/F_v} D \times H(\mathbb{A}_f^v)/K$$

$$M_{K, F_v^{\mathrm{ur}}}^{\mathrm{an}} = H(F)_0 \backslash D_{K_v} \times H(\mathbb{A}_f^v)/K^v = H(F)_0 \backslash D \times H(\mathbb{A}_f^v)/K$$

Uniformization of $Z(x)_K$

For $x \in \mathbb{B}^\times$, the Hecke correspondence $Z(x)_K$ represents the right action of x on Y_U if $q(x) \in F_+^\times$, then we can find $f \in B^\times$ with the same norm as x_v. Then $Z(x)_K$ over can be described in terms of

$$Y_{U, F_v^{\mathrm{ur}}}^{\mathrm{an}} = B_0^\times \backslash \mathscr{H} \times (\mathbb{B}^v)^\times / U :$$

$$[z, g] \mapsto [(f^{-1}, x_v)z, f^{-1}gx^v].$$

Let $D_f(x_v) \subset \mathscr{H}^2$ denote the graph of the action (f^{-1}, x_v) on $\mathscr{H}$. Notice that the equation $g_2 = f^{-1}g_1x$ in $g_i \in (\mathbb{B}^v)^\times$ is equivalent to

$$x = g^{-1}(f), \qquad g = (g_1, g_2) \in H(\mathbb{A}_f^v).$$

Thus we have the following description of $Z(x)_{K,F_v^{\mathrm{ur}}}$:

Lemma 7.1.1.

$$Z(x)_K^{\mathrm{an}} = H(F)_0 \backslash \bigcup_{g^{-1}f = x^v} (D_f(x_v) \times gK^v)/K^v$$

7.2 Local intersection at a non-split prime

In the following, we let us compute the local intersection over F_v^{ur} for cycles $\widehat{Z}(\Phi_i)$ for $\Phi_i \in \mathscr{S}(\mathbb{V})$. Assume that $\Phi_i(x) = \Phi_i^v(x^v)\Phi_{iv}(x_v)$ with Φ_{iv} invariant under $U_v \times U_v$ with $U_v = U_v^n$. We assume that the support of the function $\Phi := \Phi_1 \otimes \phi_2 \otimes \Phi_3$ on $\mathbb{V}^3$ is supported on set of $(\mathbb{V}^3)_{\mathrm{sub}}$ of points (x_1, x_2, x_3) whose components are linearly independent. This implies that the cycles $\widehat{Z}(\phi_i)$ have no intersection in the generic fiber.

Lemma 7.2.1. *Consider $Z(\phi) = Z(\phi_1) \times Z(\phi_2) \times Z(\phi_3)$ as a correspondence on Y_U^3. Then there is an arithmetic class $\widehat{\Delta}_v$ of the diagonal Δ in Y_U^3 at place v such that*

$$(\widehat{Z}(\phi_1) \cdot \widehat{Z}(\phi_2) \cdot \widehat{Z}(\phi_3))_v = \deg(Z(\phi)_* \widehat{\Delta}_v|_\Delta)$$

Proof. First of all, we recall the discuss in §3.5 that since v is not split in $\mathbb{B}$, the Hecke operators does not change the v-adic structure of Hodge classes. This implies that for each $x \in \mathbb{B}^\times$ the admissible class $\widehat{Z}(\phi_i)$ at place v can be written as

$$\widehat{Z}(x)_v = Z(x)_* \widehat{\Delta}_v := p_{2*}p_1^* \widehat{Z}(1)_v$$

where $\widehat{Z}(1)_v$ is the admissible class for the diagonal in $Y_U \times Y_U$, and p_i are two projections of $Z(x)$ onto Y_U. In this way we have

$$\widehat{Z}(\phi_i) = Z(\phi_i)_* \widehat{Z}(1).$$

Secondly, define the arithmetic class for $Z(\phi)$ on Y_U^6 as the intersection

$$\widehat{Z}(\phi) = p_{1,4}\widehat{Z}(\phi_1) \cdot p_{2,5}^* \widehat{Z}(\phi_2) \cdot p_{3,6}^* \widehat{Z}(\phi_3)$$

where for any subset I of $\{1, 2, \cdots, 6\}$, p_I denotes the projection to the project of factors $Y_U^{|I|}$ indexed by I. Then

$$\widehat{Z}(\phi_1) \cdot \widehat{Z}(\phi_2) \cdot \widehat{Z}(\phi_3) = \deg(\widehat{Z}(\phi)|_{\Delta \times \Delta}.$$

Write $\Delta \times \Delta$ as intersection $p_{123}^*\Delta \times p_{456}^*\Delta$ to obtain

$$\deg p_{456*}(\widehat{Z}(\phi)|_{p_{123}^*\Delta})_\Delta = \deg p_{456*}(\widehat{Z}(\phi)|_{p_{123}^*\Delta})|_\Delta = \deg p_{456*}(Z(\phi)_*\widehat{Z}(1)^3|_{p_{123}^*\Delta})|_\Delta.$$

It is easy to check that two pushfords p_{456*} and $Z(\phi)_*$ commute. Thus we have

$$Z(\Phi,\Delta)_v := (\widehat{Z}(\phi_1)\cdot\widehat{Z}(\phi_2)\cdot\widehat{Z}(\phi_3))_v = \deg(Z(\phi)_*\widehat{\Delta}_v|_\Delta)$$

where $\widehat{\Delta}_v$ is an arithmetic class of the diagonal in Y_U^3 defined by:

$$\widehat{\Delta}_v := p_{456*}(\widehat{Z}(1)_v^3|_\Delta)$$

$\square$

For each $g \in \mathbb{B}^\times$ denote the diagonal $\Delta(g)$ (resp. arithmetic class $\widehat{\Delta}(g)$ the diagonal) in component Y_U^3 indexed by (g,g,g) in the uniformization:

$$Y_U^{\mathrm{an},3} = (B_0^\times)^3\backslash\mathscr{H}_U^3 \times (\mathbb{B}^v)^{\times 3}/(U^v)^3.$$

Then we can rewrite the intersection by

$$Z(\Phi,\Delta) = \sum_{(g_1,g_2)\in(B_0^\times)^2\backslash(\mathbb{B}^v)^{\times 2}/U^2}\ \sum_{x\in K^3\backslash\mathbb{V}^3} \Phi(x)\deg(Z(x)_*\widehat{\Delta}(g_1)|_{\Delta(g_2)}).$$

Notice that Hecke operator does not change these component in Y_U^3. Thus we may assume that $\nu(g_1) = \nu(g_2)$. In other words, we may replace the index in the summer by group $H(F)_0\backslash H(\mathbb{A}^v)^\times/K^v$. Moreover, on the uniformization level, the Hecke operator $Z(x_i)$ is given by

$$(z,g) \longrightarrow ([f_i^{-1}, x_{iv}], f_i^{-1}gx_i^v)$$

where $f_i \in B$ has the same norm as x. If $\deg(Z(x)_*\widehat{\Delta}(g_1)|_{\Delta(g_2)} \neq 0$, we must have $g_2 = f_i^{-1}g_1x_i^v$. In terms of action of H on $V = B$, we have $x_i^v = g^{-1}f_i$. Thus we can rewrite the sum as

$$Z(\Phi,\Delta) = \sum_{(g_1,g_2)\in(B_0^\times)^2\backslash(\mathbb{B}^v)^{\times 2}/U^2}\ \sum_{f\in V^3} \Phi^v(g^{-1}f)m(f,\Phi_v)$$

where

$$m(f,\Phi_v) = \sum_{x_v\in K_v^3\backslash\mathbb{B}_v^3} \Phi_v(x_v)\deg(Z(x_v,f)_*\widehat{\Delta}(g_1)|_{\Delta(g_2)}).$$

Here $Z(x_v,f)$ is a correspondence from Y_{U,g_1} to Y_{U,g_2} by action by $[f^{-1},x_v]$ on the analytic space $\mathscr{H}_U^3$.

7.3 Compactness of local intersection

In the following, we want to show that following

Lemma 7.3.1. *For given Φ_v with compact support, the function $m(f, \Phi_v) \neq 0$ only if the moment matrix of f is supported in a compact subject of* $\mathrm{Sym}_3(F_v)$.

In the following, we want to study the horizontal local intersection at a finite place v which is split in $\mathbb{B}$. We can construct a regular integral model $\mathscr{D}$ for $D_n := \mathscr{H}^3_{U^n_v}$ over some base change of F_v^{ur} as follows. First all, over some base change of F_v^n of F_v^{ur}, the rigid space $\mathscr{H}_n$ has an equivariant semistable model $\widehat{\mathscr{H}}$ over $\mathscr{O}_{F_v^n}$. Then we blow up some closed subscheme in the special fiber of the triple fiber product $\widehat{\mathscr{H}_n^3}$ over $\mathscr{O}_{F_v^{\mathrm{ur}}}$ to get a regular model $\mathscr{D}$ of D, see Lemma 2.2.1 in [39]. In this way, we obtain integral models of $\mathscr{X}_U$ and $X_U = Y_U^3$ as follows:

$$\mathscr{X}_U^{\mathrm{an}} = (B_0^\times)^3 \backslash \mathscr{D}_n \times (\mathbb{A}_f^v)^{\times 3}/(U^v)^3.$$

In this way the arithmetic cycle $\widehat{\Delta}(x)_v$ has a decomposition

$$\widehat{\Delta}_v = \bar{\Delta}_v + V$$

where V is a vertical cycle. The intersection has decomposition

$$\deg(Z(x_v, f)\widehat{\Delta}(g_1)|_{\Delta(g_2)} = Z(x_v, f)\bar{\Delta}(g_1) \cdot \bar{\Delta}(g_2) + Z(x_v, f)V \cdot \bar{\Delta}(g_2).$$

We will prove the compactness by working on the horizontal and vertical separately.

Lemma 7.3.2. *Fix a x and g. The function* $\deg(Z(x_v, f)\bar{\Delta}(g_1)|_{\Delta(g_2)}) \neq 0$ *only if the moment metrix is supported in a compact subset of* $\mathrm{Sym}_3(F_v)$.

Proof. The cycle $Z(x_v, f)\bar{\Delta}$ has non-empty intersection with $\bar{\Delta}$ only if they have non-empty intersection in the minimal level, and only if any two of the graphs $\Gamma(f_i)$ of the isomorphisms $f_i : \Omega \longrightarrow \Omega$ have a non-empty intersection in the generic fiber $\mathbb{P}^1(\mathbb{C}_v) - \mathbb{P}^1(F_v)$. Or in the other words, the morphism $f_i f_j^{-1}$ does not have a fixed point in $\mathbb{P}^1(F_v)$. This will implies that $f_i \bar{f}_j = f_i f_j^{-1} q(f_j)$ is elliptic in the sense it generates a quadratic subfield E_{ij} in B_v over F_v. Recall that in a quadratic field, an element t is integral only if its norm is integral. If n is an integer such that $2n \geq -\mathrm{ord}(q(t))$, then $\varpi_v^n t$ has integral norm, thus $\mathrm{tr}(\varpi_v^n t)$ is integral. Take $n = -[\mathrm{ord}(q(t))/2]$, then we get for all $t \in E_{ij}$:

$$\mathrm{ordtr}(t) \geq -[\mathrm{ord}(q(t))/2].$$

Since $q(f_i) = q(x_i)$, we thus obtain that entries of $Q(f)$ has an estimate

$$\mathrm{ord}(\mathrm{tr}(f_i \bar{f}_j)) \geq -[\mathrm{ord}(x_i \bar{x}_j)/2].$$

This shows that $Q(f)$ is in a compact subset of $\mathrm{Sym}_3(F_v)$. $\square$

Now let us to compute vertical local intersection at v. We need only show that

Lemma 7.3.3. *For an irreducible vertical cycle S of $\mathscr{D}$, the support of the function $f \longrightarrow [f^{-1}, x_v]S \cdot \bar{\Delta} \neq 0$ has compact moments.*

Proof. Assume that S has image in Ω^3 included into $A_1 \times A_2 \times \times A_3$, where A_i's are irreducible components of special component of Ω. If $Z(x_v, f)S \cdot \bar{\Delta} \neq 0$, then $y_i^{-1}A_i$ has non-trivial intersection on $\bar{\Omega}$. This implies that for any $i \neq j$, $y_i y_j^{-1} A_j$ is adjacent to A_i.

Recall that after fixing an isomorphism $B_v \simeq M_2(F_v)$, the irreducible components in the special component of Ω are parameterized by homothety classes of lattices in F_v^2. Thus for each i, there is a finite set T_i of elements in $\mathrm{GL}_2(F_v)$ such that tA_i for $t_{i,j} \in T_i$ are all component with non-trivial intersection with A_i. Then we have $t^{-1} y_i y_j^{-1} A_i = A_i$ for some $t \in T$. Thus $y_i y_j^{-1} \in t_{i,j} F_v^{\times} \mathrm{GL}_2(\mathscr{O}_v)$. Since y_i has the same norm as x_i, this equation implies that $y_i \bar{y}_j$ is in a compact set which implies that the moment matrix is bounded.

□

By Lemma 7.3.1, we can replace $m(f, \Phi_v)$ by a Schwarts function Φ'_v on $\mathbb{B}_v^3$. Then we have shown that that the local triple product at v is given by integration over $[H]$ of the theta series attached to Φ'. By Siegel–Weil theorem, this integration is a coherent Eisenstein series:

Theorem 7.3.1. *Assume that $\Phi_v \in \mathscr{S}(V_v^3)$ is neat of sufficiently large order. Then there is an $\Phi' = \Phi^v \otimes \Phi'_v \in \mathscr{S}(V_{\mathbb{A}}^3)$ such that*

$$Z(g, \Phi, \Delta)_v = E(g, \Phi')$$

for $g \in \mathbb{G}$ such that g_v is in a small neighborhood of 1.

7.4 Proof of Main Theorem

In this section we will finish proving the main result 1.2.4 of this paper. Note that we need to prove conjecture 3.4.2 under the assumption of the theorem. Firstly we compile established facts. Note that the test functions are chosen as follows:

1. For $v \notin \Sigma$ we choose Φ_v to be the characteristic function of $\mathscr{O}_{B,v}^3$

2. For $v|\infty$, we have chosen Φ_v to the standard Gaussian.

3. For finite v in Σ, we choose Φ_v to be neat of sufficiently large order depending on ψ_v.

Now by the decomposition of $E'(\cdot, 0, \Phi)_{hol}$ (equation 5.4.12), we have for $g \in \mathbb{G}$ with $g_v = 1$ when $v \in S$:

$$E'(g, 0, \Phi)_{hol} = \sum_v E'_v(g, 0, \Phi)_{hol} \tag{7.4.1}$$

$$E'_v(g, 0, \Phi)_{hol} = \sum_{T, \Sigma(T) = \Sigma(v)} E'_T(g, 0, \Phi)_{hol}$$

where the sum runs only over non-singular T by the vanishing of singular coefficients for such g. And when $\Sigma(T) = \Sigma(v)$ for $v|\infty$, we have

$$E'_T(g, 0, \Phi)_{hol} = W_T(g_\infty) m_v(T) W_{T,f}(g_f, 0, \Phi_f)$$

where $m_v(T)$ is the star product of $P_s(1 + 2s_x(z)/q(x))$ for x with moment T.

On the intersection side, we also have a decomposition

$$Z(g, \Phi, \Delta) = \sum_v Z(g, \Phi, \Delta)_v \pmod{\mathscr{A}_{coh}(\mathbb{G})}$$

where $\mathscr{A}_{coh}(\mathbb{G})$ is the subspace of $\mathscr{A}(\mathbb{G})$ generated by restrictions of $E(\cdot, 0, \Phi)$ for $\Phi \in \mathscr{S}(V_{\mathbb{A}}^3)$ for all possible *coherent* $V_{\mathbb{A}}$, and forms on (g_1, g_2, g_3) which is Eisenstein for at least one variable g_i.

And we have proved the following comparison for $g \in \mathbb{G}$ with g_w in a small neighborhood U'_w when $w \in \Sigma_f$, subset of finite places in Σ:

1. For $v \notin \Sigma$, by Theorem 5.3.3,

$$Z(g, \Phi, \Delta)_v = E'_v(g, 0, \Phi) + \sum_i c_v(g_i, \Phi_i) Z(g_j, \Phi_j) \cdot Z(g_k, \Phi_k)$$

   where $c_v(g_i, \Phi_i)$ are some functions which are vanishes for almost all v, and $Z(g_j, \Phi_j) \cdot Z(g_k, \Phi_k)$ is the intersection of two divisors on $Y_U \times Y_U$.

2. When $v|\infty$, by Theorem 5.4.8,

$$Z(g, \Phi, \delta)_v = E'_v(g, 0, \Phi)_{hol}.$$

3. When $v \in S$, by Theorem 7.3.1,

$$Z(g, \Phi, \Delta)_v = E^{(v)}(g)$$

   for some $E^{(v)} \in \mathscr{A}_{coh}(\mathbb{G})$. And by Proposition 6.2.3, we have for g as above

$$E'_v(g, 0, \Phi) = 0.$$

To sum up, we have an automorphic form

$$\mathscr{F}(g) = Z(g, \Phi, \Delta) - E'(g, 0, \Phi)_{hol} - \sum_{v \in S} E^{(v)}(g) \in \mathscr{A}(\mathbb{G})$$

with the property that for all $g \in \prod_{v \in \Sigma_f} U'_v \mathbb{G}^{\Sigma_f}$:

$$\mathscr{F}(g) = \sum_i c(g_i, \Phi_i) Z(g_j, \Phi_j) \cdot Z(g_k, \Phi_k) \tag{7.4.2}$$

where $c(g_i, \Phi_i)$ are some functions of g_i. Since $F(g)$ and $Z(g_i, \Phi_i)$ are all automorphic, we have for any

$$\gamma \in \mathrm{SL}_2(F)^3 \cap \prod_{v \in \Sigma_f} \mathbb{G}^{\Sigma_f},$$

$$\sum_i (c(\gamma_i g_i, \Phi_i) - c(g_i, \Phi_i)) Z(g_j, \Phi_j) \cdot Z(g_k, \Phi_k) = 0$$

In particular if $Z(g_j, \Phi_j) \cdot Z(g_k, \Phi_k) \neq 0$, then $c(\gamma_i g_i, \Phi_i) = c(g_i, \Phi_i)$. Since

$$\mathrm{SL}_2(F^3) \cdot \prod_{v \in \Sigma_f} U'_v \mathbb{G}^{\Sigma_f} = \mathrm{SL}_2(\mathbb{A}) \prod_{v \in \Sigma_f} U'_v \mathbb{G}^{\Sigma_f}$$

There are unique functions $\lambda(g_i, \Phi_i)$ for $g \in \mathrm{SL}_2(\mathbb{A})^3 \prod_{v \in \Sigma_f} U'_v \mathbb{G}^{\Sigma_f}$ with the following properties:

1. $\lambda(\gamma_i g_i, \Phi_i) = \alpha(g_i, \Phi_i)$ if $\gamma_i \in \mathrm{SL}_2(F)$
2. $\lambda(g_i, \Phi_i) = c(g_i, \Phi_i)$ if there are g_j, g_k with the sam norm as g_i such that $Z(g_j, \Phi_j) \cdot Z(g_k, \Phi_k) \neq 0$
3. $\lambda(g_i, \Phi_i) = 0$ if there such that $Z(g_j, \Phi_j) \cdot Z(g_k, \Phi_k) \neq 0$ for all g_j, g_k with the same norm as g_i.

Now we have a new equation

$$\mathscr{F}(g) = \sum_i \lambda(g_i, \Phi_i) Z(g_j, \Phi_j) \cdot Z(g_k, \Phi_k), \qquad g \in \mathrm{SL}_2(\mathbb{A})^3 \prod_{v \in \Sigma_f} U'_v \mathbb{G}^{\Sigma_f}. \tag{7.4.3}$$

We want to show that $\mathscr{F}$ is perpendicular to any cusp form $\varphi \in \sigma$ in the Main theorem. By our definition of $\mathscr{F}$, we see that if

$$\int_{\mathbb{G}(F) \backslash \mathbb{G}(\mathbb{A})} \mathscr{F}(g) \varphi(g) dg = \ell(\theta(\Phi \otimes \varphi))$$

where $\ell \in \mathscr{P}(\Pi)$ is the difference of two linear forms in remark 1.5.1 following the Main Theorem 1.2.4 on the Shimizu lifting $\Pi = \pi \otimes \widetilde{\pi}$ of σ on $O(\mathbb{V})^3$, Choose a fundamental domain Ω of $F^\times \backslash \mathbb{A}^\times$ in $\prod_{v \in \Sigma_f} \det U'_v(\mathbb{A}^\times)^{\Sigma_f}$. Then we have a decomposition $\ell = \sum \ell_i$ with

$$\ell_i(\Phi \otimes \varphi) = \int_\Omega \lambda_\alpha(\Phi_i \otimes \varphi_i) Z_\alpha(\Phi_j \otimes \varphi_j) \cdot Z_\alpha(\Phi_k \otimes \varphi_k) d\alpha$$

where

$$\lambda_\alpha(\varphi_i) = \int_{\mathrm{SL}_2(F) \backslash \mathrm{SL}_2(\mathbb{A})} \lambda(g g_\alpha, \Phi_i) \varphi_i(g g_\alpha), \qquad Z_\alpha(\Phi_j \otimes \varphi_j) = \int_{\mathrm{SL}_2(F) \backslash \mathrm{SL}_2(\mathbb{A})} Z(g g_\alpha, \Phi) \varphi(g g_\alpha) dg.$$

Notice that for each α, j, and k the intersection number $Z_\alpha(\Phi_j \otimes \varphi_j) \cdot Z_\alpha(\Phi_k \otimes \varphi_k)$ is invariant under the orthogonal group $O(\mathbb{V})$. Thus this defines a form in

$$\mathrm{Hom}_{O(\mathbb{V})}(\Pi_j \otimes \Pi_k, \mathbb{C}).$$

It follows that if $\ell_i \neq 0$, then $\Pi_i = \widetilde{\Pi}_k$, and then $\ell_i = \mu_i \otimes \nu_{i,j}$ where μ_i is a functional on Π_i and ν_{ij} is natural contraction between Π_j and Π_k.

The invariance of ℓ under the diagonal action by $O(\mathbb{V})$ implies that for any $h \in O(\mathbb{V})$,

$$\sum_i \mu_i'(v_i)\nu_{jk}(v_j \otimes v_k) = 0, v_i \in \pi_i \otimes \widetilde{\pi}_i,$$

where $\mu_i'(v_i) = \mu_i(hv_i) - \mu_i(v_i))$. If all π_i are infinite dimensional, then we claim that that $\mathrm{Hom}(\Pi_i, \mathbb{C}) \otimes \nu_{ij}$ are independent in $\mathrm{Hom}(\Pi, \mathbb{C})$.

In fact, we need only show this independence when restricted to a finite dimensional spaces $V_i \subset \Pi_i$. We assume all $V_i \simeq \mathbb{C}^n$ with a basis of linear forms e_i. Assume that ν_{ij} is given by the diagonal form $\sum_m e_m \otimes e_m$, and μ_i' is given by $\sum a_{in} e_n$. Then ℓ_i is given by

$$\ell_1 = \sum_{m,n} a_n e_n \otimes e_m \otimes e_m, \qquad \ell_2 = \sum_{m,n} b_n e_m \otimes e_n \otimes e_m, \qquad \ell_3 = \sum_{m,n} c_n e_m \otimes e_m \otimes e_n.$$

If $\dim V_i > 1$, then the equation $\ell_1 + \ell_2 + \ell_3 = 0$ implies that all $\ell_i = 0$.

By our claim, $\mu_i' = 0$. In other words, μ_i is an $O(\mathbb{V})$- equivariant linear form. This is impossible since Π_i is irreducible of dimension > 1. In summary, we have shown that $\ell = 0$. Thus we have completed the proof of Main Theorem 1.2.4.

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