

The arithmetic Hodge index theorem for adelic line bundles I: number fields

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Contents

1	Introduction	1
1.1	Arithmetic Hodge index theorem	2
1.2	Calabi–Yau theorem	5
1.3	Algebraic dynamics	6
2	Arithmetic Hodge index theorem	9
2.1	Preliminaries on arithmetic positivity	9
2.2	Hodge index theorem on curves	12
2.3	Inequality in the general case	16
2.4	Equality in the general case	20
2.5	Calabi–Yau theorem	26
3	Algebraic dynamics	28
3.1	Admissible arithmetic classes	28
3.2	Sets of preperiodic points	36
3.3	Variants and questions	40
A	Lefschetz theorems for Normal Varieties	42

1 Introduction

The Hodge index theorem for divisors on arithmetic surfaces proved by Faltings [Fal] in 1984 and Hirijac [Hr] in 1985 is one of the fundamental results

in Arakelov theory. For example, it was used to prove the first case of Bogomolov’s conjecture for curves in tori by Zhang [Zh1]. In 1996, Moriwaki [Mo1] extended the Hodge index theorem for codimension one cycles on high-dimensional arithmetic varieties, and then confirmed the codimension one case of the arithmetic standard conjecture proposed by Gillet and Soulé in [GS3]. Despite its fundamental importance in number theory and arithmetic geometry, e.g, in Gross–Zagier type formula for cycles on Shimura varieties, the high-codimensional case of the Gillet–Soulé conjecture is still wide open.

The aim of this series of two papers is to prove an *adelic version* of the Hodge index theorem for (still) codimension one cycles on varieties over a finitely generated field K . Namely, these line bundles are equipped with metrics as limits of integral models of the structure morphisms. These matrices naturally appear in algebraic dynamical systems and moduli spaces of varieties. Here we will give two applications:

- (1) the uniqueness part of the Calabi–Yau theorem for metrized line bundles over non-archimedean analytic spaces,
- (2) a rigidity result of the sets of preperiodic points of polarizable endomorphisms of a projective variety over any field K .

The proof of our results uses Arakelov theory (cf. [Ar, GS1]) and Berkovich analytic spaces (cf. [Be]). In comparison with Moriwaki’s proof, one essential difficulty in adelic case is the lack of relative ampleness in projective systems of integral models. We have two new ideas to handle this difficulty: a new notion of $\bar{L}$ -boundedness and a variation method (inspired by Blocki’s work [Bl] in complex geometry).

In this paper, the first one of the series, we prove our Hodge index theorem and the application in (2) assuming K is a number field, and prove the application in (1) in the full generality. In [YZ], the second one of the series, we will treat our Hodge index theorem and the application in (2) in the full generality after introducing a theory of adelic line bundles on varieties over finitely generated fields. We separate the exposition into two papers due to the technicality of our theory over finitely generated fields. In the following, we state the main results of this paper.

1.1 Arithmetic Hodge index theorem

Let us first recall the arithmetic Hodge index theorem for Hermitian line bundles on arithmetic varieties.

Theorem 1.1 ([Fal, Hr, Mo1]). *Let K be a number field, and $\pi : \mathcal{X} \rightarrow \operatorname{Spec} O_K$ a regular arithmetic variety, geometrically connected of relative dimension $n \geq 1$. Let $\overline{\mathcal{L}}$ an arithmetically ample Hermitian line bundle on $\mathcal{X}$ in the following sense:*

- (1) *The curvature $c_1(\overline{\mathcal{L}})$ is strictly positive on $\mathcal{X}(\mathbb{C})$;*
- (2) *For any integral closed subscheme $\mathcal{Y}$ of $\mathcal{X}$, the top intersection number $\widehat{c}_1(\overline{\mathcal{L}})^{\dim \mathcal{Y}} \cdot \mathcal{Y} > 0$.*

Let $\overline{\mathcal{M}}$ be a Hermitian line bundle on $\mathcal{X}$ such that $\mathcal{M}_K \cdot \mathcal{L}_K^{n-1} = 0$ on the generic fiber $\mathcal{X}_K$. Then

$$\overline{\mathcal{M}}^2 \cdot \overline{\mathcal{L}}^{n-1} \leq 0,$$

and the equality holds if and only if $\overline{\mathcal{M}} = \pi^ \overline{\mathcal{M}}_0$ for some Hermitian line bundle $\overline{\mathcal{M}}_0$ on $\operatorname{Spec} O_K$.*

The result was due to Faltings [Fal] and Hriljac [Hr] for $n = 1$, and due to Moriwaki [Mo1] for general n .

The main result of this paper is a version of the above result for adelic line bundles. The importance of the Hodge index theorem in the adelic setting will be justified by our applications to the Calabi–Yau theorem and to algebraic dynamics.

To state our theorem, we start with some positivity notions. We refer to Zhang [Zh3] for basic definitions of adelic line bundles, and to Definition 2.1 for positivity of Hermitian line bundles on arithmetic varieties.

Definition 1.2. *Let K be a number field. Let X be a projective variety over K , and $\overline{L}, \overline{M}$ be adelic line bundles on X . We make the following definitions.*

- (1) *We say that $\overline{L}$ is nef if the adelic metric of L is a uniform limit of metrics induced by nef Hermitian line bundles on integral models of X .*
- (2) *We say that $\overline{L}$ is integrable if it is the difference of two nef adelic line bundles on X .*

- (3) We say that $\bar{L}$ is arithmetically positive if L is ample, $\bar{L}$ is nef, and $(\bar{L}|_Y)^{\dim Y+1} > 0$ for any closed subvariety Y of X .
- (4) We say that $\bar{M}$ is $\bar{L}$ -bounded if there is an integer $m > 0$ such that both $m\bar{L} + \bar{M}$ and $m\bar{L} - \bar{M}$ are nef.

The following is the main theorem of this paper.

Theorem 1.3. *Let K be a number field, and $\pi : X \rightarrow \operatorname{Spec} K$ be a normal and geometrically connected projective variety of dimension $n \geq 1$. Let $\bar{M}$ be an integrable adelic line bundle on X , and $\bar{L}_1, \dots, \bar{L}_{n-1}$, $n-1$ nef line bundles on X where each L_i is big on X . Assume $M \cdot L_1 \cdots L_{n-1} = 0$ on X . Then*

$$\bar{M}^2 \cdot \bar{L}_1 \cdots \bar{L}_{n-1} \leq 0.$$

Moreover, if $\bar{L}_i$ is arithmetically positive and $\bar{M}$ is $\bar{L}_i$ -bounded for each i , then the equality holds if and only if $r\bar{M} = \pi^\bar{M}_0$ for some adelic line bundle $\bar{M}_0$ on $\operatorname{Spec} K$ and some integer $r > 0$.*

If M is numerically trivial, we can strengthen the theorem by removing the condition “ L_i is big” in the inequality part, and by replacing the condition “ $\bar{L}_i$ is arithmetically positive” with “ L_i is ample” in the equality part of the theorem.

As in the classical case, the theorem explains the signature of the intersection pairing on certain space of adelic line bundles. Let W denote the subspace of $\operatorname{Pic}(X)_{\text{int}} \otimes_{\mathbb{Z}} \mathbb{Q}$ consisting of elements which are represented by $\mathbb{Q}$ -linear combinations of integrable adelic line bundles on X which are $\bar{L}_i$ -bounded for all i . Define a pairing on W by

$$\langle \bar{M}_1, \bar{M}_2 \rangle = \bar{M}_1 \cdot \bar{M}_2 \cdot \bar{L}_1 \cdots \bar{L}_{n-1}.$$

Denote $V = \pi^*\widehat{\operatorname{Pic}}(K) \otimes_{\mathbb{Z}} \mathbb{Q}$, viewed as a subspace of W . Then the theorem implies that the pairing on $V^\perp$ is negative semi-definite, that V is a maximal isotropic subspace of $V^\perp$, and thus that $V^\perp/V$ is negative definite.

The inequality part of Theorem 1.3 can be eventually reduced to Theorem 1.1 by taking limits (in the case all $\bar{L}_i$ are equal). However, the equality part of the theorem is more profound and significantly more difficult, since it is not a simple limit of its counterpart for Hermitian line bundles on integral models. The following are some new ingredients of our treatment regarding these difficulties:

- (1) We introduce the notion “ $\bar{L}$ -bounded” to overcome the difficulty that arithmetic ampleness of Hermitian line bundles is not preserved by pull-back by morphisms between integral models, and that arithmetic ampleness of adelic line bundles is not preserved by perturbations.
- (2) We use a variational method to show that, in the case that M is numerically trivial,

$$\bar{M}^2 \cdot \bar{L}_1 \cdots \bar{L}_{n-1} = 0 \implies \bar{M}^2 \cdot \bar{L}_1^0 \cdots \bar{L}_{n-1}^0 = 0$$

for any nef adelic line bundle $\bar{L}_i^0$ with the same underlying line bundle L_i . It gives much flexibility. The variational method is inspired by the idea of Blocki [Bl] on the complex Calabi–Yau theorem.

- (3) We only assume that X is normal and projective in order to cover all polarizable algebraic dynamical systems. For this purpose, we extend the classical Lefschetz theorems for line bundles to the normal case.

Remark 1.1. The assumption “ $\bar{M}$ is $\bar{L}_i$ -bounded for each i ” is necessary for the condition of the equality. For counterexample, assume $n = 2$ and $\bar{L}$ is induced from an arithmetically ample line bundle $\bar{\mathcal{L}}$ on an integral model $\mathcal{X}$ of X . Let $\alpha : \mathcal{X}' \rightarrow \mathcal{X}$ be the blowing-up of a closed point on the $\mathcal{X}$, and $\bar{\mathcal{M}}$ the (vertical) line bundle on $\mathcal{X}'$ associated to the exceptional divisor endowed with the trivial hermitian metric $\|1\| = 1$. Then $\mathcal{M}_K \cdot \mathcal{L}_K = 0$ and $\bar{\mathcal{M}}^2 \cdot \alpha^* \bar{\mathcal{L}} = 0$ by the projection formula. But $\bar{\mathcal{M}}$ is not coming from any line bundle on the base $\text{Spec } O_K$. The problem is that $\bar{\mathcal{M}}$ is not $\bar{\mathcal{L}}$ -bounded if we convert the objects to the adelic setting.

Remark 1.2. Definition 1.2 admits an obvious function field analog where we take $K = k(C)$ the function field $k(C)$ of algebraic curve C over a finite field. And the functional analog of Theorem 1.3 holds by the same proof.

1.2 Calabi–Yau theorem

By a non-archimedean field, we mean a field K endowed with a non-trivial non-archimedean absolute value $|\cdot|$, under which K is complete. We say that K has a *rational rank-one valuation* if the (multiplicative) value group $|K^\times|$ admits an injective homomorphism to the additive group $\mathbb{Q}$. For example, all discrete valuation fields and $\mathbb{C}_p$ for all primes p are non-archimedean fields with a rational rank-one valuation.

Let K be either $\mathbb{C}$ or a non-archimedean field. Let X be a geometrically connected projective variety over K , and L an ample line bundle on X endowed with a continuous semipositive K -metric $\|\cdot\|$. Then $(L, \|\cdot\|)$ induces a canonical semipositive measure $c_1(L, \|\cdot\|)^{\dim X}$ on the analytic space X^{an} . We explain it as follows.

If $K = \mathbb{C}$, then X^{an} is just the complex analytic space $X(\mathbb{C})$. The measure $c_1(L, \|\cdot\|)^{\dim X}$ is just the determinant of the Chern form $c_1(L, \|\cdot\|)$ which is locally defined by

$$c_1(L, \|\cdot\|) = \frac{\partial\bar{\partial}}{\pi i} \log \|\cdot\|$$

in complex analysis.

If K is non-archimedean, X^{an} is the Berkovich space associated to the variety X over K . It is a Hausdorff, compact and path-connected topological space. Furthermore, it naturally includes the set of closed points of X . We refer to [Be] for more details on the space. The metric $\|\cdot\|$ being semipositive, in the sense of [Zh3], means that it is a uniform limit of metrics induced by ample integral models of L . The canonical measure $c_1(L, \|\cdot\|)^{\dim X}$ is constructed by Chambert-Loir [Ch] in the case that K contains a countable and dense subfield, and by Gubler [Gu2] for the general case.

Theorem 1.4. *Let K be a non-archimedean field with a rational rank-one valuation, and X a projective variety over K . Let L be an ample line bundle on X , and $\|\cdot\|_1$ and $\|\cdot\|_2$ two semipositive metrics on L . Then*

$$c_1(L, \|\cdot\|_1)^{\dim X} = c_1(L, \|\cdot\|_2)^{\dim X}$$

if and only if $\frac{\|\cdot\|_1}{\|\cdot\|_2}$ is a constant.

Remark 1.3. Theorem 1.4 includes a non-archimedean analogue of the uniqueness part of the complex Calabi–Yau theorem which was formulated in terms of a Kahler form ω representing the first Chern class of a line bundle L on X , and the volume form Ω as follows.

In the 1950s, Calabi [Ca1, Ca2] made the following famous conjecture: *Let X be a compact complex manifold endowed with a Kahler form ω , and let Ω be a positive smooth volume form on X such that $\int_X \Omega = \int_X \omega^{\dim X}$. Then there exists a smooth real-valued function ϕ on X such that $(\omega + i\partial\bar{\partial}\phi)^{\dim X} = \Omega$.* Calabi proved that the function ϕ is unique up to scalars (if it exists). the continuous semipositive case is due to Kolodziej [Ko]. Afterwards Blocki

[Bl] provided a very simple proof of Kolodziej’s result. The existence of the function is much deeper theorem of S. T. Yau in his seminal paper [Ya] in 1977. Now the whole results are called the Calabi–Yau theorem.

1.3 Algebraic dynamics

Let X be a projective variety over a field K . A *polarizable algebraic dynamical system on X* is a morphism $f : X \rightarrow X$ such that there is an ample $\mathbb{Q}$ -line bundle $L \in \text{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ satisfying $f^*L = qL$ from some rational number $q > 1$. We call L a *polarization of f* , and call the triple (X, f, L) a *polarized algebraic dynamical system*. For such an f , let $\text{Prep}(f)$ denote the set of *preperiodic points*, i.e.,

$$\text{Prep}(f) := \{x \in X(\overline{K}) \mid f^m(x) = f^n(x) \text{ for some } m, n \in \mathbb{N}, m \neq n\}.$$

A well-known result of Fakhruddin [Fak] asserts that $\text{Prep}(f)$ is Zariski dense in X .

Denote by $\mathcal{DS}(X)$ the set of all polarizable algebraic dynamical systems f on X . Note that we do *not* require elements of $\mathcal{DS}(X)$ to be polarizable by the same ample line bundle or have the same dynamical degree q .

Theorem 1.5. *Let X be a projective variety over a number field K . For any $f, g \in \mathcal{DS}(X)$, the following are equivalent:*

- (1) $\text{Prep}(f) = \text{Prep}(g)$;
- (2) $g\text{Prep}(f) \subset \text{Prep}(f)$;
- (3) $\text{Prep}(f) \cap \text{Prep}(g)$ is Zariski dense in X .

The theorem will be proved over any field K in our second paper [YZ]. The proof of the number field case in this paper gives a rough idea of our approach for the general case.

Remark 1.4. In an early version of the series, we require f and g to be polarizable by the same ample line bundle. The proof combines the equidistribution theorem of Yuan [Yu] generalizing that of Szpiro–Ullmo–Zhang [SUZ] et al, and the Calabi–Yau theorem in Theorem 1.4. But we have removed this restriction in the current version by introducing the arithmetic Hodge index theorem, which is more powerful than the Calabi–Yau theorem.

Remark 1.5. The same proof will give a functional field analog of Theorem 1.5 where $K = k(C)$ is the function field of algebraic curve C over a finite field k .

Remark 1.6. When $X = \mathbb{P}^1$, the theorem is independently proved by M. Baker and L. DeMarco [BD] using an equidistribution argument similar to our early version of the paper.

Remark 1.7. The following are some works related to the above theorem (when K is a number field):

- (1) A. Mimar [Mi] proved the theorem in the case that $X = \mathbb{P}^1$.
- (2) If f is a Lattès map on $\mathbb{P}^1$ or a power map on $\mathbb{P}^d$ induced by $(\mathbb{G}_m)^d$, the theorem is implied by the explicit description of g by S. Kawaguchi and J. H. Silverman [KS].
- (3) In the case that $X = \mathbb{P}^1$, C. Petsche, L. Szpiro, and T. Tucker [PST] found a further equivalent statement in terms of heights and intersections.

Now we explain some ingredients in our proof of Theorem 1.5. The hard part is to show that (3) implies (1).

Let $f, g \in \mathcal{DS}(X)$ such that $\text{Prep}(f) \cap \text{Prep}(g)$ is Zariski dense. We first consider the case that f and g are polarizable by the same ample line bundle L . Then we have an f -invariant adelic line bundle $\bar{L}_f$ and a g -invariant adelic line bundle $\bar{L}_g$.

By the assumption, the essential minimum of $\bar{L}_f + \bar{L}_g$ is 0. By the successive minima, $(\bar{L}_f + \bar{L}_g)^{n+1} = 0$. By the nefness, $\bar{L}_f^i \cdot \bar{L}_g^{n+1-i} = 0$ for any i . Hence,

$$(\bar{L}_f - \bar{L}_g)^2 \cdot (\bar{L}_f + \bar{L}_g)^{n-1} = 0.$$

Then we apply our arithmetic Hodge index theorem to conclude that $\bar{L}_f = \bar{L}_g$. It follows that f and g have the same canonical height. Then they have the same set of preperiodic points.

The general case that f and g are not polarizable by the same ample line bundle is much more difficult to handle. We develop a theory of admissible adelic line bundles to overcome the difficulty. An f -admissible line bundle is defined to a sum of eigenvectors of f^* . For any ample class $\xi \in \text{NS}(X)_{\mathbb{Q}}$, we first prove that there is a unique “ f -admissible lifting” $L \in \text{Pic}(X)_{\mathbb{Q}}$ of ξ . Note that L does not necessarily polarize f . Then we construct an

f -admissible adelic line bundle $\overline{L}_f$ extending L . We prove that $\overline{L}_f$ is nef. Similarly, from ξ , we have a unique “ g -admissible lifting” $M \in \text{Pic}(X)_{\mathbb{Q}}$ of ξ , and a unique g -admissible adelic line bundle $\overline{M}_g$ extending M . Here we a priori do not have $L = M$. But we can still apply the arithmetic Hodge index theorem by the above argument to force $\overline{L}_f = \overline{M}_g$.

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2 Arithmetic Hodge index theorem

The goal of this section is to prove Theorem 1.3. After introducing some basic positivity notions, we prove the theorem by a few steps which are clear from the section titles.

2.1 Preliminaries on arithmetic positivity

Let K be a number field. By an *arithmetic variety* $\mathcal{X}$ over O_K , we mean an integral scheme $\mathcal{X}$, projective and flat over O_K . It is called an *arithmetic surface* if its absolute dimension is two.

Recall that a *Hermitian line bundle* on an arithmetic variety $\mathcal{X}$ is a pair $\overline{\mathcal{L}} = (\mathcal{L}, \|\cdot\|)$ consisting of a line bundle $\mathcal{L}$ on $\mathcal{X}$ and a metric $\|\cdot\|$ on $\mathcal{L}(\mathbb{C})$ invariant under the action of the complex conjugation. The metric is assumed to satisfy the regularity that, for any analytic map from any complex open ball B (of any dimension) to $\mathcal{X}(\mathbb{C})$, the pull-back of $\|\cdot\|$ gives a smooth Hermitian metric on the pull-back of $\mathcal{L}(\mathbb{C})$ to B . We say that the metric is *semipositive* if any such pull-back has a positive semi-definite Chern form.

Definition 2.1. (1) We say that a Hermitian line bundle $\overline{\mathcal{L}} = (\mathcal{L}, \|\cdot\|)$ on $\mathcal{X}$ is *nef* if the Hermitian metric $\|\cdot\|$ is semipositive, and the intersection $(\overline{\mathcal{L}}|_{\mathcal{Y}})^{\dim \mathcal{Y}} \geq 0$ for any closed subvariety $\mathcal{Y}$ of $\mathcal{X}$.

(2) We say that a Hermitian line bundle $\overline{\mathcal{L}} = (\mathcal{L}, \|\cdot\|)$ on $\mathcal{X}$ is *arithmetically positive* if the generic fiber $\mathcal{L}_K$ is ample on X , $\overline{\mathcal{L}}$ is nef, and the intersection $(\overline{\mathcal{L}}|_{\mathcal{Y}})^{\dim \mathcal{Y}} > 0$ for any horizontal closed subvariety $\mathcal{Y}$ of $\mathcal{X}$.

(3) We say that a Hermitian line bundle $\overline{\mathcal{L}} = (\mathcal{L}, \|\cdot\|)$ on $\mathcal{X}$ is *vertical* if the generic fiber $\mathcal{L}_K$ is trivial on X .

Denote by $\widehat{\text{Pic}}(\mathcal{X})$ the group of Hermitian bundles on $\mathcal{X}$. Denote by $\widehat{\text{Pic}}(\mathcal{X})_{\text{nef}}$ (resp. $\widehat{\text{Pic}}(\mathcal{X})_{\text{amp}}$, $\widehat{\text{Pic}}(\mathcal{X})_{\text{vert}}$) the set of nef (resp. arithmetically positive, vertical) Hermitian line bundles on $\mathcal{X}$.

Let X be a projective variety over K . By an *integral model* $\mathcal{X}$ of X over O_K , we mean an arithmetic variety $\mathcal{X}$ over O_K with a fixed isomorphism $\mathcal{X}_K = X$. Let L be a line bundle on X . If $\overline{\mathcal{L}}$ is a Hermitian line bundle on $\mathcal{X}$ with generic fiber $\mathcal{L}_K = L$, then we say that $\overline{\mathcal{L}}$ is an *integral model* of L on $\mathcal{X}$. We also say that $(\mathcal{X}, \overline{\mathcal{L}})$ is an *integral model* of (X, L) over O_K .

An *adelic line bundle* on X is a pair $\overline{L} = (L, \{\|\cdot\|_v\}_v)$ consisting of a line bundle L on X and a collection of K_v -metrics $\|\cdot\|_v$ of $L(\overline{K}_v)$ on $X(\overline{K}_v)$ over all places v of K . The K_v -metrics is assumed to be continuous and Galois invariant, and the collection is assumed to be coherent in that there is an integral model of (X, L) inducing a collection of K_v -metrics on $L(\overline{K}_v)$ which agrees with $\{\|\cdot\|_v\}_v$ for all but finitely many places v . We refer to [Zh2] for more details on the definition. The following is a copy of Definition 1.2, except that (5) and (6) is new.

Definition 2.2. Let $\overline{L}, \overline{M}$ be adelic line bundles on X . We make the following definitions.

(1) We say that $\overline{L}$ is *nef* if it the adelic metric of L is a uniform limit of metrics induced by nef Hermitian line bundles on integral models of X .

- (2) We say that $\overline{L}$ is integrable if it is the difference of two nef adelic line bundles on X .
- (3) We say that $\overline{L}$ is arithmetically positive if L is ample, $\overline{L}$ is nef, and $(\overline{L}|_Y)^{\dim Y+1} > 0$ for any closed subvariety Y of X .
- (4) We say that $\overline{M}$ is $\overline{L}$ -bounded if there is an integer $m > 0$ such that both $m\overline{L} + \overline{M}$ and $m\overline{L} - \overline{M}$ are nef.
- (5) We say that $\overline{L}$ is vertical if L is trivial on X .
- (6) We say that $\overline{L}$ is constant if L is isomorphic to the pull-back of an adelic line bundle on $\text{Spec } K$.

Denote by $\widehat{\text{Pic}}(X)$ the group of adelic line bundles on X . By the above definitions, $\widehat{\text{Pic}}(X)$ has subsets $\widehat{\text{Pic}}(X)_{\text{nef}}$, $\widehat{\text{Pic}}(X)_{\text{amp}}$, $\widehat{\text{Pic}}(X)_{\text{int}}$ and $\widehat{\text{Pic}}(X)_{\text{vert}}$.

All the positivity notions can be extended to the concepts of $\mathbb{Q}$ -line bundles and $\mathbb{R}$ -line bundles. For example, the space of integrable $\mathbb{Q}$ -line bundles form a vector space $\widehat{\text{Pic}}(X)_{\text{int}} \otimes_{\mathbb{Z}} \mathbb{Q}$.

In the end, we present a lemma which shows the strength of the arithmetic arithmetically positivity defined above.

Lemma 2.3. *Let $\overline{L}$ be an adelic line bundle on X . Then the following are equivalent:*

- (1) $\overline{L}$ is arithmetically positive, i.e., L is ample, $\overline{L}$ is nef, and $(\overline{L}|_Y)^{\dim Y+1} > 0$ for any closed subvariety Y of X .
- (2) L is ample, and $\overline{L} - \pi^*\overline{N}$ is nef for some adelic line bundle $\overline{N}$ on $\text{Spec } K$ with $\widehat{\deg}(\overline{N}) > 0$.

Proof. It is easy to see that (2) implies (1). The implication from (1) to (2) follows the idea of the successive minima of [Zh2]. Assume (1) in the following. We claim that the absolute minima

$$\lambda(\overline{L}) = \inf_{x \in X(\overline{K})} h_{\overline{L}}(x)$$

is strictly positive. Apply the arithmetic Hilbert–Samuel formula for adelic line bundles in [Zh3, Theorem 1.7], which extends the original formula of

Gillet–Soulé [GS2]. Combine it with the adelic Minkowski theorem. Replacing $\overline{L}$ by a multiple if necessary, we can find a nonzero section $s \in H^0(X, L)$ such that $\|s\|_{\mathbb{A}} < 1$. Here

$$\|s\|_{\mathbb{A}} = \prod_v \|s\|_{v, \sup}, \quad \|s\|_{v, \sup} = \sup_{x \in X(\overline{K}_v)} \|s(x)\|_v.$$

For any closed point $x \in X(\overline{K})$ such that $s(x) \neq 0$, use s to compute the height of x . It gives

$$\widehat{\deg}(\overline{L}|_x) \geq -\log \|s\|_{\mathbb{A}} > 0.$$

To show $\lambda > 0$, it suffices to consider algebraic points of the support of $\text{div}(s)$. It has a smaller dimension, and can be done by induction.

Once we have $\lambda > 0$, the proof is immediate. In fact, we simply have

$$\lambda(\overline{L} - \pi^*\overline{N}) = \lambda(\overline{L}) - \widehat{\deg}(\overline{N}) > 0$$

for any adelic line bundle $\overline{N}$ on $\text{Spec } K$ with $\widehat{\deg}(\overline{N}) < \lambda$. In this case, $\overline{L} - \pi^*\overline{N}$ is nef. $\square$

2.2 Hodge index theorem on curves

In this section, we prove Theorem 1.3 when $\dim X = 1$. Note that X is smooth and the line bundles $\overline{L}_i$ do not show up in this case. By the Hodge index theorem of Faltings and Hriljac, the problem is easily reduced to the vertical case that M is trivial on X . Then the rest of the section is for the vertical case.

Result of Faltings and Hriljac

Take any regular integral model $\mathcal{X}$ of X over O_K , which exists by the resolution of singularity by Lipman [Li].

By the results of Faltings [Fal] and Hriljac [Hr], we can find some $\overline{\mathcal{M}} \in \widehat{\text{Pic}}(\mathcal{X})_{\mathbb{Q}}$ extending M and perpendicular to $\widehat{\text{Pic}}(\mathcal{X})_{\text{vert}}$. The curvature form of $\overline{\mathcal{M}}$ is necessarily zero. Then their index theorem asserts that

$$\overline{\mathcal{M}}^2 = -2 \widehat{h}(M) \leq 0.$$

Here $\widehat{h} : \text{Pic}^0(X)_{\mathbb{Q}} \rightarrow \mathbb{R}$ is the Neron–Tate height. The equality is attained if and only if $\overline{\mathcal{M}}$ is torsion in $\widehat{\text{Pic}}(\mathcal{X})/\pi^*\widehat{\text{Pic}}(O_K)$.

To treat the adelic line bundle $\overline{M}$ in the theorem, denote by $\overline{M}_0 \in \widehat{\text{Pic}}(X)$ the adelic line bundle induced by $\overline{\mathcal{M}}$. Define a vertical adelic line bundle $\overline{N}$ by

$$\overline{M} = \overline{M}_0 + \overline{N}.$$

By definition, $\overline{M}_0 \cdot \overline{N} = 0$ since $\overline{M}_0$ is perpendicular to all vertical classes. It follows that

$$\overline{M}^2 = \overline{M}_0^2 + \overline{N}^2.$$

Note that $\overline{M}_0^2 = \overline{\mathcal{M}}^2 \leq 0$ is already discussed above, where the equality is attained if and only if $\overline{M}_0$ is torsion in $\widehat{\text{Pic}}(X)/\pi^*\widehat{\text{Pic}}(K)$. Assume Theorem 1.3 for vertical adelic line bundles on curves. Then $\overline{N}^2 \leq 0$, where the equality is attained if and only if $\overline{N} \in \pi^*\widehat{\text{Pic}}(K)$. It follows that

$$\overline{M}^2 = \overline{M}_0^2 + \overline{N}^2 \leq 0.$$

If the equality holds, then $\overline{M}$ is torsion in $\widehat{\text{Pic}}(X)/\pi^*\widehat{\text{Pic}}(K)$.

Hence, it suffices to prove Theorem 1.3 for vertical adelic line bundles. In the remaining part of this section, we assume that M is trivial on X .

The inequality

We first treat the model case, i.e., when $\overline{M}$ is induced by an integral model of M .

Proposition 2.4. *Let $\pi : \mathcal{X} \rightarrow \text{Spec } O_K$ be a regular and geometrically connected arithmetic surface. Let $\overline{\mathcal{M}}, \overline{\mathcal{M}}_1, \overline{\mathcal{M}}_2 \in \widehat{\text{Pic}}(\mathcal{X})_{\text{vert}}$ be vertical adelic line bundles with smooth metrics on $\mathcal{X}(\mathbb{C})$. Then the following are true:*

- (1) *The self-intersection $\overline{\mathcal{M}}^2 \leq 0$. The equality holds if and only if $\overline{\mathcal{M}} \in \pi^*\widehat{\text{Pic}}(O_K)$.*
- (2) *One has the Cauchy–Schwartz inequality $(\overline{\mathcal{M}}_1 \cdot \overline{\mathcal{M}}_2)^2 \leq (\overline{\mathcal{M}}_1^2)(\overline{\mathcal{M}}_2^2)$.*

Proof. The results are well-known. We include it here for convenience of readers.

Note that (2) is the Cauchy–Schwartz inequality induced by (1). In fact, let $x, y \in \mathbb{R}$ be variables. The quadratic form

$$(x\overline{\mathcal{M}}_1 + y\overline{\mathcal{M}}_2)^2 = x^2 \overline{\mathcal{M}}_1^2 + 2xy \overline{\mathcal{M}}_1 \cdot \overline{\mathcal{M}}_2 + y^2 \overline{\mathcal{M}}_2^2$$

is negative semi-definite. It follows that the discriminant is negative or zero, which gives the inequality.

To prove (1), let $\overline{D} = (D, g)$ be a vertical arithmetic divisor representing $\overline{\mathcal{M}}$. Here the Green's function g is a smooth function on $\mathcal{X}(\mathbb{C})$. By definition,

$$\overline{\mathcal{M}}^2 = \sum_{v \nmid \infty} D_v^2 + \int_{\mathcal{X}(\mathbb{C})} g \frac{\partial \overline{\partial}}{\pi i} g.$$

Here D_v denotes the part of D above v . The problem is reduced to prove each term on the right-hand side is non-positive and seek the condition for it to be zero.

By integration by parts, the integral becomes

$$-\frac{1}{\pi i} \int_{\mathcal{X}(\mathbb{C})} \partial g \wedge \overline{\partial} g \leq 0.$$

The equality holds if and only if g is constant.

As for $D_v^2 \leq 0$, enumerate the irreducible components of the special fiber $\mathcal{X}_v$ above v by $V_1, \dots, V_r$. We have $\mathcal{X}_v = \sum_{i=1}^r a_i V_i$ with multiplicity $a_i > 0$. For convenience, denote $E_i = a_i V_i$. Write $D_v = \sum_{i=1}^r b_i E_i$ with some $b_i \in \mathbb{R}$. We have

$$D_v^2 = \sum_{i,j=1}^r b_i b_j E_i \cdot E_j.$$

Note that

$$\sum_{j=1}^r b_i^2 E_i \cdot E_j = 0, \quad \forall i.$$

We obtain

$$\begin{aligned} D_v^2 &= -\frac{1}{2} \sum_{i,j=1}^r (b_i - b_j)^2 E_i \cdot E_j \\ &= -\frac{1}{2} \sum_{i \neq j} (b_i - b_j)^2 E_i \cdot E_j \\ &\leq 0. \end{aligned}$$

If the equality holds, then $b_i = b_j$ for any $i \neq j$ such that $V_i \cap V_j$ is nonempty, since $E_i \cdot E_j > 0$ for such pairs. It gives $b_1 = b_2 = \dots = b_r$ since the whole special fiber $\mathcal{X}_v$ is connected.

Summing all local contributions together, we have the result for $\overline{\mathcal{M}}^2$. $\square$

By taking limit, we have obtained the inequality $\overline{M}^2 \leq 0$ in Theorem 1.3. To get the inequality part, we first interpret $\overline{M}$ as the limit of its “push-forwards.”

Condition of equality

To reduce adelic metrics to model metrics, we first introduce the “push-forward” of adelic line bundles to integral models. The treatment relies on the resolution of singularities of arithmetic varieties, which is only known for arithmetic surfaces (by Lipman [Li]), so we restrict the exposition to the case when X is a curve.

Let $\overline{M}$ be an integrable adelic line bundle on a projective and smooth curve X with $M = \mathcal{O}_X$ trivial. For any place v of K , the function $\log \|1\|_v$ on $X(\overline{K}_v)$ coming from the metric of $\overline{M}$ extends to a continuous function on the analytic space $X_{K_v}^{\text{an}}$. By definition, $\log \|1\|_v = 0$ for all but finitely many places v .

Let $\mathcal{X}$ be a regular integral model of X over O_K . Define a vertical arithmetic $\mathbb{R}$ -divisor $\overline{D}_{\mathcal{X}}$ on $\mathcal{X}$ by

$$\overline{D}_{\mathcal{X}} := \left(- \sum_{(V,v)} \log \|1\|_v(\eta_V) V, - \log \|1\|_{\infty} \right).$$

Here the summation is over all pairs (V, v) , where v is a non-archimedean place of K , and V is an irreducible component of the fiber $\mathcal{X}_v$ of $\mathcal{X}$ above v . The point $\eta_V \in X_{K_v}^{\text{an}}$ denotes the Shilov point corresponding to V , which is the unique preimage of the generic point of V under the reduction map $X_{K_v}^{\text{an}} \rightarrow \mathcal{X}_v$.

To get a line bundle, we denote by $\overline{M}_{\mathcal{X}} \in \widehat{\text{Pic}}(\mathcal{X})_{\mathbb{R}}$ the Hermitian line bundle on $\mathcal{X}$ associated to $\overline{D}_{\mathcal{X}}$. We have the following basic result.

Lemma 2.5. (1) *For any Hermitian line bundle $\overline{N}$ on $\mathcal{X}$, we have*

$$\overline{M} \cdot \overline{N} = \overline{M}_{\mathcal{X}} \cdot \overline{N}.$$

(2) *There is a sequence $\{\mathcal{X}_m\}_m$ of regular integral models of X over O_K such that the adelic metric induced by the push-forward $\overline{M}_{\mathcal{X}_m}$ converges uniformly to the adelic metric of $\overline{M}$.*

Proof. By definition, there is a sequence of regular integral models $(\mathcal{X}_m, \overline{\mathcal{M}}_m)$ of (X, M) over O_K , which induces a sequence of adelic line bundles $\overline{M}_m$ convergent to $\overline{M}$. Here by abuse of terminology, convergence means uniform convergence of induced adelic metrics.

For (2), we prove that $\overline{M}_{\mathcal{X}_m}$ converges to $\overline{M}$. It suffices to show that $\overline{N}_m = \overline{M}_{\mathcal{X}_m} - \overline{M}_m$ converges to the trivial adelic line bundle. Let $\overline{E} = (E, g_E)$ be the vertical arithmetic divisor associated to the section 1 of $\overline{N}_m$. By definition, the function g_E on $X(\mathbb{C})$ and the multiplicities of irreducible components of special fibers of $\mathcal{X}$ in E converges uniformly to 0. It follows that $\overline{N}_m$ converges to the trivial adelic line bundle.

For (1), we can further assume that each $\mathcal{X}_m$ dominates $\mathcal{X}$ by a birational morphism $\alpha_m : \mathcal{X}_m \rightarrow \mathcal{X}$. By definition, $\overline{M}_{m, \mathcal{X}} = (\alpha_m)_* \overline{\mathcal{M}}_m$ converges to $\overline{M}_{\mathcal{X}}$, which can be understood as the convergence of the corresponding vertical divisors. Then (1) is just the limit of the projection formula

$$\overline{M}_m \cdot \overline{\mathcal{N}} = \overline{M}_{m, \mathcal{X}} \cdot \overline{\mathcal{N}}.$$

□

Now we are ready to prove the equality part of Theorem 1.3 in the vertical case. Let $\overline{M} \in \widehat{\text{Pic}}(X)_{\text{vert}}$ with $\overline{M}^2 = 0$. Assume that M is trivial. We need to prove $\overline{M} \in \pi^* \widehat{\text{Pic}}(K)$.

By the limit version of the Cauchy–Schwartz inequality in Proposition 2.4 (2), for any vertical adelic line bundle $\overline{N}$ on X , we have

$$(\overline{M} \cdot \overline{N})^2 \leq (\overline{M}^2)(\overline{N}^2) = 0.$$

It follows that

$$\overline{M} \cdot \overline{N} = 0.$$

It will give the result by varying $\overline{N}$.

In fact, we first treat the metric at archimedean places. Take $\overline{N}$ to have trivial metrics at all non-archimedean places, and to have metric $\|1\|_{\infty} = e^{-\phi}$ on $X(\mathbb{C})$. Then we have

$$\overline{M} \cdot \overline{N} = \int_{X(\mathbb{C})} \phi \, c_1(\overline{M}).$$

It is zero for any smooth function ϕ on $X(\mathbb{C})$. It follows that $c_1(\overline{M}) = 0$ as a current, so the metric of $\overline{M}$ is constant on every connected component of $X(\mathbb{C})$.

Now we treat non-archimedian places. Let $\mathcal{X}$ be a regular integral model of X . Let $\overline{M}_{\mathcal{X}}$ be the push-forward. Setting $\overline{N} = \overline{M}_{\mathcal{X}}$ in the above result, we have

$$\overline{M} \cdot \overline{M}_{\mathcal{X}} = 0.$$

By Lemma 2.5 (1), it becomes

$$\overline{M}_{\mathcal{X}} \cdot \overline{M}_{\mathcal{X}} = 0.$$

By Proposition 2.4 (1), $\overline{M}_{\mathcal{X}} \in \pi^* \widehat{\text{Pic}}(O_K)$ where $\pi : \mathcal{X} \rightarrow \text{Spec } O_K$ denotes the structure morphism by abuse of notation.

By Lemma 2.5 (2), the metric of $\overline{M}$ can be approximated uniformly by the metric induced by $\overline{M}_{\mathcal{X}}$ when varying $\mathcal{X}$. It follows that the limit $\overline{M}$ actually lies in $\pi^* \widehat{\text{Pic}}(K)$. It finishes the proof of Theorem 1.3 in this case.

2.3 Inequality in the general case

Now we prove the inequality of Theorem 1.3 in the general case by induction on $n = \dim X$. We have already treated the case $n = 1$, so we assume $n \geq 2$ in the following.

Reducing to the model case

Consider the inequality part of the theorem. Recall $\overline{M} \in \widehat{\text{Pic}}(X)_{\text{int}}$ and $\overline{L}_1, \dots, \overline{L}_{n-1} \in \widehat{\text{Pic}}(X)_{\text{nef}}$. The theorem assumes $M \cdot L_1 \cdots L_{n-1} = 0$ and that each L_i is big on X . We are going to prove

$$\overline{M}^2 \cdot \overline{L}_1 \cdots \overline{L}_{n-1} \leq 0.$$

By a resolution of singularities, we can assume that X is smooth. Furthermore, we claim that we can further assume that each $\overline{L}_i$ is arithmetically positive .

In fact, fix an arithmetically positive adelic line bundle $\overline{A}$ on X . Take a small rational number $\epsilon > 0$. Set $\overline{L}'_i = \overline{L}_i + \epsilon \overline{A}$ and $\overline{M}' = \overline{M} + \delta \overline{A}$. Here δ is a number such that

$$M' \cdot L'_1 \cdots L'_{n-1} = (M + \delta A) \cdot L'_1 \cdots L'_{n-1} = 0.$$

It determines

$$\delta = -\frac{M \cdot L'_1 \cdots L'_{n-1}}{A \cdot L'_1 \cdots L'_{n-1}}.$$

As $\epsilon \rightarrow 0$, we have $\delta \rightarrow 0$ since

$$M \cdot L'_1 \cdots L'_{n-1} \rightarrow M \cdot L_1 \cdots L_{n-1} = 0,$$

$$A \cdot L'_1 \cdots L'_{n-1} \rightarrow A \cdot L_1 \cdots L_{n-1} > 0.$$

The last inequality uses the assumption that L_i is big and nef for each i . Therefore, the inequality $\overline{M}^2 \cdot \overline{L}_1 \cdots \overline{L}_{n-1} \leq 0$ is the limit of the inequality $\overline{M}'^2 \cdot \overline{L}'_1 \cdots \overline{L}'_{n-1} \leq 0$. Here every $\overline{L}'_i$ is arithmetically positive. So the claim is achieved.

Go back to the inequality

$$\overline{M}^2 \cdot \overline{L}_1 \cdots \overline{L}_{n-1} \leq 0.$$

We have the condition $M \cdot L_1 \cdots L_{n-1} = 0$. We further assume that X is smooth, and $\overline{L}_i$ is arithmetically positive for $i = 1, \dots, n-1$. By approximation, it suffices to prove

$$\overline{\mathcal{M}}^2 \cdot \overline{\mathcal{L}}_1 \cdots \overline{\mathcal{L}}_{n-1} \leq 0$$

under the following assumptions:

- $\mathcal{X}$ is a normal integral model of X over O_K ;
- $\overline{\mathcal{M}}, \overline{\mathcal{L}}_1, \dots, \overline{\mathcal{L}}_{n-1}$ are Hermitian line bundles on $\mathcal{X}$ with smooth Hermitian metrics and with generic fiber $M, L_1, \dots, L_{n-1}$;
- $\overline{\mathcal{L}}_i$ arithmetically ample on $\mathcal{X}$ in the sense of Theorem 1.1.

It was proved by Moriwaki [Mol] in the case that all $\overline{\mathcal{L}}_i$ are equal. The current case is similar, but we still sketch it in the following.

The model case

Here we prove the inequality in the above model case.

First, by $M \cdot L_1 \cdots L_{n-1} = 0$, there is a smooth metric $\|\cdot\|_0$ on $\mathcal{M}$, unique up to scalars, such that the curvature form of $\overline{\mathcal{M}}' = (\mathcal{M}, \|\cdot\|_0)$ gives

$$c_1(\overline{\mathcal{M}}') c_1(\overline{\mathcal{L}}_1) \cdots c_1(\overline{\mathcal{L}}_{n-1}) = 0$$

pointwise on $X(\mathbb{C})$. It is given by an elliptic equation. The solution can be found in Gromov [Gr, Corollary 2.2 A2]. To convert our problem to Gromov's

result, write $\overline{\mathcal{M}} = (\mathcal{M}, \|\cdot\|)$ and $\|\cdot\|_0 = e^\phi \|\cdot\|$ for a real-valued C^∞ -function ϕ on $X(\mathbb{C})$. Then our equation becomes

$$\frac{1}{\pi i} \partial \bar{\partial} \phi \wedge c_1(\overline{\mathcal{L}}_1) \cdots c_1(\overline{\mathcal{L}}_{n-1}) = -c_1(\overline{\mathcal{M}}) c_1(\overline{\mathcal{L}}_1) \cdots c_1(\overline{\mathcal{L}}_{n-1}).$$

Set $\Omega = c_1(\overline{\mathcal{L}}_1) \cdots c_1(\overline{\mathcal{L}}_{n-1})$. Then the left-hand side is a scalar multiple of the Laplacian $\Delta \phi$ with respect to Ω . The right-hand side is an exact form on every connected component $X_\sigma(\mathbb{C})$ of $X(\mathbb{C})$. In fact, its cohomology class is represented by $-M \cdot L_1 \cdots L_{n-1}$, which is zero in $H^{2n}(X_\sigma(\mathbb{C}), \mathbb{C}) = \mathbb{C}$. Then the solution ϕ exists.

Go back to the intersection number $\overline{\mathcal{M}}^2 \cdot \overline{\mathcal{L}}_1 \cdots \overline{\mathcal{L}}_{n-1}$. As above, write the original metric $\|\cdot\|$ of $\overline{\mathcal{M}}$ as $e^{-\phi} \|\cdot\|_0$. Then we have

$$\overline{\mathcal{M}}^2 \cdot \overline{\mathcal{L}}_1 \cdots \overline{\mathcal{L}}_{n-1} = \overline{\mathcal{M}}'^2 \cdot \overline{\mathcal{L}}_1 \cdots \overline{\mathcal{L}}_{n-1} - \int_{X(\mathbb{C})} \phi \frac{1}{\pi i} \partial \bar{\partial} \phi \wedge c_1(\overline{\mathcal{L}}_1) \cdots c_1(\overline{\mathcal{L}}_{n-1}).$$

By integration by parts, the second term on the right

$$- \int_{X(\mathbb{C})} \phi \frac{1}{\pi i} \partial \bar{\partial} \phi \wedge c_1(\overline{\mathcal{L}}_1) \cdots c_1(\overline{\mathcal{L}}_{n-1}) = \int_{X(\mathbb{C})} \frac{1}{\pi i} \partial \phi \wedge \bar{\partial} \phi \wedge c_1(\overline{\mathcal{L}}_1) \cdots c_1(\overline{\mathcal{L}}_{n-1}).$$

It is non-positive since the volume from

$$\frac{1}{\pi i} \partial \phi \wedge \bar{\partial} \phi \wedge c_1(\overline{\mathcal{L}}_1) \cdots c_1(\overline{\mathcal{L}}_{n-1}) \leq 0.$$

Hence, it suffices to prove

$$\overline{\mathcal{M}}'^2 \cdot \overline{\mathcal{L}}_1 \cdots \overline{\mathcal{L}}_{n-1} \leq 0.$$

By Moriwaki's arithmetic Bertini theorem in [Mol], replacing $\overline{\mathcal{L}}_{n-1}$ by a tensor power if necessary, there is a nonzero section $s \in H^0(\mathcal{X}, \mathcal{L}_{n-1})$ satisfying the following conditions:

- The supremum norm $\|s\|_{\text{sup}} = \sup_{x \in X(\mathbb{C})} \|s(x)\| < 1$;
- The horizontal part of $\text{div}(s)$ on $\mathcal{X}$ is a generically smooth arithmetic variety $\mathcal{Y}$;
- The vertical part of $\text{div}(s)$ on $\mathcal{X}$ is a linear combination $\sum_{\wp} a_{\wp} \mathcal{X}_{\wp}$ of smooth fibers $\mathcal{X}_{\wp}$ of $\mathcal{X}$ above (good) prime ideals $\wp$ of O_K .

Then by the intersection formula,

$$\begin{aligned} & \overline{\mathcal{M}}'^2 \cdot \overline{\mathcal{L}}_1 \cdots \overline{\mathcal{L}}_{n-2} \cdot \overline{\mathcal{L}}_{n-1} \\ = & \overline{\mathcal{M}}'^2 \cdot \overline{\mathcal{L}}_1 \cdots \overline{\mathcal{L}}_{n-2} \cdot \mathcal{Y} + \sum_{\wp} a_{\wp} \overline{\mathcal{M}}'^2 \cdot \overline{\mathcal{L}}_1 \cdots \overline{\mathcal{L}}_{n-2} \cdot \mathcal{X}_{\wp} \\ & - \int_{X(\mathbb{C})} \log \|s\|_{c_1(\overline{\mathcal{M}}')}^2 c_1(\overline{\mathcal{L}}_1) \cdots c_1(\overline{\mathcal{L}}_{n-2}). \end{aligned}$$

It suffices to prove each term on the right-hand side is non-positive. The “main term”

$$\overline{\mathcal{M}}'^2 \cdot \overline{\mathcal{L}}_1 \cdots \overline{\mathcal{L}}_{n-2} \cdot \mathcal{Y} \leq 0$$

by the induction hypothesis. The vertical part

$$\overline{\mathcal{M}}'^2 \cdot \overline{\mathcal{L}}_1 \cdots \overline{\mathcal{L}}_{n-2} \cdot \mathcal{X}_{\wp} = M^2 \cdot L_1 \cdots L_{n-2} \log N_{\wp} \leq 0$$

follows from the geometric Hodge index theorem on the algebraic variety X (or $\mathcal{X}_{\wp}$). It remains to check

$$- \int_{X(\mathbb{C})} \log \|s\|_{c_1(\overline{\mathcal{M}}')}^2 c_1(\overline{\mathcal{L}}_1) \cdots c_1(\overline{\mathcal{L}}_{n-2}) \leq 0.$$

In fact, we have a pointwise inequality

$$c_1(\overline{\mathcal{M}}')^2 c_1(\overline{\mathcal{L}}_1) \cdots c_1(\overline{\mathcal{L}}_{n-2}) \leq 0.$$

Note that

$$c_1(\overline{\mathcal{M}}') c_1(\overline{\mathcal{L}}_1) \cdots c_1(\overline{\mathcal{L}}_{n-2}) c_1(\overline{\mathcal{L}}_{n-1}) = 0.$$

The pointwise inequality follows from Gromov [Gr, Lemma 2.1A] (by setting $\Omega = c_1(\overline{\mathcal{L}}_1) \cdots c_1(\overline{\mathcal{L}}_{n-2})$ and $\omega_0 = c_1(\overline{\mathcal{L}}_{n-1})$). It is called Aleksandrov’s lemma, since it was essentially due to Aleksandrov [Al].

Cauchy–Schwartz inequality

The following Cauchy–Schwartz inequality from the inequality part of Theorem 1.3 will be used repeatedly, so we list it here.

Corollary 2.6. *Let X be a projective variety of dimension $n \geq 1$ over a number field K . Let $\overline{M}$ and $\overline{M}'$ be two integrable adelic line bundles on X , and $\overline{L}_1, \dots, \overline{L}_{n-1}$ be $n - 1$ nef line bundles on X . Assume that M and M' are numerically trivial on X . Then*

$$(\overline{M} \cdot \overline{M}' \cdot \overline{L}_1 \cdots \overline{L}_{n-1})^2 \leq (\overline{M}^2 \cdot \overline{L}_1 \cdots \overline{L}_{n-1})(\overline{M}'^2 \cdot \overline{L}_1 \cdots \overline{L}_{n-1}).$$

2.4 Equality in the general case

Now we prove the second part of Theorem 1.3 in the general case. We have already treated the case $n = 1$, so we assume $n \geq 2$ in the following.

Argument on the generic fiber

Lemma 2.7. *Under the conditions in the equality part of Theorem 1.3, M is numerically trivial on X .*

Proof. By Lemma 2.3, $\overline{L}'_{n-1} = \overline{L}_{n-1} - \pi^* \overline{N}$ is nef for some $\overline{N} \in \widehat{\text{Pic}}(K)$ with $c = \widehat{\deg}(\overline{N}) > 0$. Then

$$\overline{M}^2 \cdot \overline{L}_1 \cdots \overline{L}_{n-2} \cdot \overline{L}_{n-1} = \overline{M}^2 \cdot \overline{L}_1 \cdots \overline{L}_{n-2} \cdot \overline{L}'_{n-1} + c M^2 \cdot L_1 \cdots L_{n-2}.$$

Applying the inequality of the theorem to $(\overline{M}, \overline{L}_1, \dots, \overline{L}_{n-2}, \overline{L}'_{n-1})$, we have

$$\overline{M}^2 \cdot \overline{L}_1 \cdots \overline{L}_{n-2} \cdot \overline{L}'_{n-1} \leq 0.$$

By the Hodge index theorem on X in the geometric case, we have

$$M^2 \cdot L_1 \cdots L_{n-2} \leq 0.$$

Hence,

$$\overline{M}^2 \cdot \overline{L}_1 \cdots \overline{L}_{n-2} \cdot \overline{L}'_{n-1} = M^2 \cdot L_1 \cdots L_{n-2} = 0.$$

On the variety X , we have

$$M \cdot L_1 \cdots L_{n-2} \cdot L_{n-1} = 0, \quad M^2 \cdot L_1 \cdots L_{n-2} = 0.$$

By the Hodge index theorem on algebraic varieties, we conclude that M is numerically trivial. See Theorem A.1 in the appendix. $\square$

In the following, we assume that M is numerically trivial.

Variational method

The key to obtain the equality part of Theorem 1.3 is the following result, which was inspired by [B1].

Lemma 2.8. *Let $\overline{M}, \overline{L}_1, \dots, \overline{L}_{n-1}$ be integrable adelic line bundles on X such that the following conditions hold:*

(1) $\overline{M}$ is numerically trivial on X ;

(2) $\overline{M}$ is $\overline{L}_i$ -bounded for every i ;

(3) $\overline{M}^2 \cdot \overline{L}_1 \cdots \overline{L}_{n-1} = 0$.

For any nef adelic line bundles $\overline{L}_i^0$ on X with underlying bundle L_i^0 numerically equivalent to L_i , and any integrable adelic line bundle $\overline{M}'$ with numerically trivial underlying line bundle M' , the following are true:

$$\overline{M} \cdot \overline{M}' \cdot \overline{L}_1^0 \cdots \overline{L}_{n-1}^0 = 0,$$

$$\overline{M}^2 \cdot \overline{M}' \cdot \overline{L}_1^0 \cdots \overline{L}_{n-2}^0 = 0.$$

Proof. The second equality follows from the first one by taking $\overline{L}_{n-1}^0 = \overline{L}_{n-1} \pm \epsilon \overline{M}$ with small ϵ . So it suffices to prove the first inequality.

Note condition (2) implies that every $\overline{L}_i$ is nef. Replacing $\overline{L}_i$ by a large multiple if necessary, we can assume that both $\overline{L}_i \pm \overline{M}$ are nef for each i . Denote $\overline{L}_i^\pm = \overline{L}_i \pm \overline{M}$ in the following.

First, we have the equality

$$\overline{M}^2 \cdot \overline{L}_1^{\epsilon(1)} \cdots \overline{L}_{n-1}^{\epsilon(n-1)} = 0$$

for any sign function $\epsilon : \{1, \dots, n-1\} \rightarrow \{+, -\}$. Moreover, we can find a constant $t > 0$ such that $L_i - tL_i^{\epsilon(i)}$ is nef for any i . Then the inequality part of Theorem 1.3 gives

$$0 = \overline{M}^2 \cdot \overline{L}_1 \cdots \overline{L}_{n-1} \leq \overline{M}^2 \cdot t\overline{L}_1^{\epsilon(1)} \cdots t\overline{L}_{n-1}^{\epsilon(n-1)} \leq 0.$$

It forces

$$\overline{M}^2 \cdot \overline{L}_1^{\epsilon(1)} \cdots \overline{L}_{n-1}^{\epsilon(n-1)} = 0.$$

Second, we claim that for any k and any sign function ϵ on $\{k, \dots, n-1\}$,

$$\overline{M}^2 \cdot \overline{L}_1^0 \cdots \overline{L}_{k-1}^0 \cdot \overline{L}_k^{\epsilon(k)} \cdots \overline{L}_{n-1}^{\epsilon(n-1)} = 0. \quad (1)$$

When $k = n$, it gives

$$\overline{M}^2 \cdot \overline{L}_1^0 \cdots \overline{L}_{n-1}^0 = 0.$$

By the Cauchy–Schwartz inequality in Corollary 2.6, we have

$$\overline{M} \cdot \overline{M}' \cdot \overline{L}_1^0 \cdots \overline{L}_{n-1}^0 = 0.$$

Third, we prove the claim by induction on k . We already have the case $k = 1$. Assume the equality (1) for a general k for all sign functions on $\{k, \dots, n-1\}$. Let ϵ be a sign function on $\{k+1, \dots, n-1\}$ which has two extensions ϵ^δ on $\{k, \dots, n-1\}$ ($\delta = \pm 1$) by $\epsilon^\delta(k) = \delta$. Applying Corollary 2.6 again to $\overline{M}$, $\overline{M}_k^\pm = \overline{L}_k^\pm - \overline{L}_k^0$, and net bundles $\overline{L}_1^0, \dots, \overline{L}_{k-1}^0, \overline{L}_k^{\epsilon^\pm(k)}, \dots, \overline{L}_{n-1}^{\epsilon(n-1)}$, we have for $\delta_1 = \pm, \delta_2 = \pm$ four equalities:

$$\overline{M} \cdot \overline{M}_k^{\delta_1} \cdot \overline{L}_1^0 \cdots \overline{L}_{k-1}^0 \cdot \overline{L}_k^{\delta_2} \cdots \overline{L}_{n-1}^{\epsilon(n-1)} = 0.$$

Take the difference for two equation with fixed δ_1 to obtain

$$\overline{M} \cdot \overline{M}_k^{\delta_1} \cdot \overline{L}_1^0 \cdots \overline{L}_{k-1}^0 \cdot \overline{M} \cdot \overline{L}_{k+1}^{\epsilon(k+1)} \cdots \overline{L}_{n-1}^{\epsilon(n-1)} = 0.$$

It is just

$$\overline{M}^2 \cdot \overline{L}_1^0 \cdots \overline{L}_{k-1}^0 \cdot (\overline{L}_k^{\delta_1} - \overline{L}_k^0) \cdot \overline{L}_{k+1}^{\epsilon(k+1)} \cdots \overline{L}_{n-1}^{\epsilon(n-1)} = 0.$$

The left-hand side splits to a difference of two terms. One term is zero by the induction assumption for k . It follows that the other term is also zero, which gives

$$\overline{M}^2 \cdot \overline{L}_1^0 \cdots \overline{L}_{k-1}^0 \cdot \overline{L}_k^0 \cdot \overline{L}_{k+1}^{\epsilon(k+1)} \cdots \overline{L}_{n-1}^{\epsilon(n-1)} = 0.$$

It is exactly the case $k+1$. The proof is complete. $\square$

Numerically trivial case

Resume the proof of the equality part of Theorem 1.3. Let $M, \overline{L}_1, \dots, \overline{L}_{n-1}$ be as in the equality part of the theorem. We have already proved that $\overline{M}$ is numerically trivial on X . The key is the following induction result.

Lemma 2.9. *For any closed subvariety Y of codimension one in X , we have*

$$\overline{M}^2 \cdot \overline{L}_1 \cdots \overline{L}_{n-2} \cdot Y = 0.$$

Proof. By Lemma 2.8, we can assume that $\overline{L}_{n-1}$ is induced by a (single) Hermitian line bundle $\overline{\mathcal{L}}_{n-1}$ on a normal integral model $\mathcal{X}$ of X . Replacing $\overline{L}_{n-1}$ by a power if necessary, we can assume that there is a global section s of L_{n-1} vanishing on Y . Viewing s as a rational section of $\mathcal{L}_{n-1}$ on $\mathcal{X}$, write

$$\operatorname{div}(s) = \sum_{i=1}^t a_i \mathcal{Y}_i + \sum_{j=1}^r b_j \mathcal{V}_j, \quad a_i, b_j > 0,$$

where each $\mathcal{Y}_i$ is a horizontal prime divisor of $\mathcal{X}$, and each $\mathcal{V}_j$ is a vertical prime divisor. Write $Y_i = \mathcal{Y}_{i,K}$ for the generic fiber, and assume $\mathcal{Y}_{1,K} = Y$. We will use this section to compute the intersection number $\overline{M}^2 \cdot \overline{L}_1 \cdots \overline{L}_{n-1}$.

First, we claim that for each j ,

$$\overline{M}^2 \cdot \overline{L}_1 \cdots \overline{L}_{n-2} \cdot \mathcal{V}_j = 0.$$

If $\mathcal{V}_j$ is a Cartier divisor, it follows from the second equality of Lemma 2.8. In general, we blow-up $\mathcal{X}$ along $\mathcal{V}_j$ and take the normalization to obtain a normal integral model $\mathcal{X}_j$ of X with a birational morphism $p_j : \mathcal{X}_j \rightarrow \mathcal{X}$. Then $\mathcal{V}'_j := p_j^* \mathcal{V}_j$ is a Cartier divisor. Now we have

$$\overline{M}^2 \cdot \overline{L}_1 \cdots \overline{L}_{n-2} \cdot \mathcal{V}_j = \overline{M}^2 \cdot \overline{L}_1 \cdots \overline{L}_{n-2} \cdot \mathcal{V}'_j = 0.$$

Second, we claim that the archimedean part

$$-\int_{X(\mathbb{C})} \log \|s\| \, c_1(\overline{M})^2 c_1(\overline{L}_1) \cdots c_1(\overline{L}_{n-2}) = 0.$$

It is still a consequence of Lemma 2.8. In fact, for any continuous real-valued function ϕ on $X(\mathbb{C})$, take $\overline{M}' = \mathcal{O}(\phi)$ to be the trivial line bundle $\mathcal{O}_X$, with the trivial metric at every non-archimedean place, and with the metric $e^{-\phi}$ at archimedean places. Then the lemma gives

$$\int_{X(\mathbb{C})} \phi \, c_1(\overline{M})^2 c_1(\overline{L}_1) \cdots c_1(\overline{L}_{n-2}) = 0.$$

It implies that the distribution $c_1(\overline{M})^2 c_1(\overline{L}_1) \cdots c_1(\overline{L}_{n-2}) = 0$. Hence the integration of $-\log \|s\|$ is also zero.

Therefore, the intersection formula gives

$$\overline{M}^2 \cdot \overline{L}_1 \cdots \overline{L}_{n-1} = \sum_{i=1}^t a_i \overline{M}^2 \cdot \overline{L}_1 \cdots \overline{L}_{n-2} \cdot Y_i.$$

Since either term on the right-hand side is non-positive, we obtain

$$\overline{M}^2 \cdot \overline{L}_1 \cdots \overline{L}_{n-2} \cdot Y = 0.$$

□

Remark 2.1. As Moriwaki points out, in the proof of Lemma 2.9, a Bertini-type result allows us to assume further that every vertical component $\mathcal{V}_j$ in $\text{div}(s)$ is the (irreducible) fiber $\mathcal{X}_v$ of $\mathcal{X}$ above a finite place v . Then $\overline{M}^2 \cdot \overline{L}_1 \cdots \overline{L}_{n-2} \cdot \mathcal{V}_j = 0$ since M is numerically trivial (without the variational method). In particular, the proof in the model case in the function field analogue does not need the variational method. However, in the number field case here, we still need it to treat the archimedean part, and to make possible the assumption “ $\overline{L}_{n-1}$ is induced by a (single) Hermitian line bundle $\overline{\mathcal{L}}_{n-1}$ on a normal integral model $\mathcal{X}$ of X . ”

Go back to the proof of Theorem 1.3. By the lemma,

$$\overline{M}^2 \cdot \overline{L}_1 \cdots \overline{L}_{n-2} \cdot Y = 0.$$

It holds for any prime divisor Y . If Y is normal, by induction, we can assume that some multiple of $\overline{M}|_Y$ is *constant* in the sense that it is the pull-back from a number field.

We only use the property that $M|_Y$ is torsion here. Choose Y to be a general hyperplane section of some very ample line bundle on X . Then Y is normal. By Theorem A.2, the natural map $\text{Pic}^0(X) \rightarrow \text{Pic}^0(Y)$ has a finite kernel. Hence, M is torsion on X .

By passing to a multiple, we can assume that M is trivial. It is the vertical case below.

Vertical case

It remains to treat the vertical case that M is trivial on X , the last case in our proof of Theorem 1.3.

Apply Lemma 2.9 again. By induction, for any closed subvariety Y of codimension one in X , the restriction $\overline{M}|_Y$ is constant on Y . Note that we do not need Y to be normal in the vertical case by the normalization procedure.

To conclude the proof, we re-interpret the property that $\overline{M}$ is constant. In fact, $\overline{M}$ is constant if and only if $\log \|1\|_v$ is a (Galois-invariant) constant function on $X(\overline{K}_v)$ for any place v of K , or equivalently $\log \|1\|_v$ gives a constant function on the Berkovich space $X_{K_v}^{\text{an}}$ for any place v of K . Here $\{\|\cdot\|_v\}_v$ denotes the adelic metric of $\overline{M}$, and 1 denotes the constant section of $M = \mathcal{O}_X$.

By what we have proved, for any closed subvariety Y of codimension one in X , geometrically connected over K , $\log \|1\|_v$ is constant on $Y(\overline{K}_v)$. This

is sufficient to conclude that $\log \|1\|_v$ is constant on $X(\overline{K}_v)$. In fact, it suffices to prove that $\log \|1\|_v$ is constant on $X(\overline{K})$ since $X(\overline{K})$ is dense in $X(\overline{K}_v)$ (under the v -adic topology). On the other hand, any two points of $X(\overline{K})$ are connected by a closed subvariety Y of codimension one in X , geometrically connected over K . The proof is complete.

2.5 Calabi–Yau theorem

The goal of this section is to treat Theorem 1.4 in the non-archimedean case. If (X, L) comes from a number field, it is essentially Theorem 1.3, the arithmetic Hodge index theorem for vertical adelic line bundles.

Let K be a non-archimedean field, X a projective variety over K , and L a line bundle on X . Recall that a K -metric on L is a continuous and $\text{Gal}(\overline{K}/K)$ -invariant collection of $\overline{K}$ -metric $\|\cdot\|$ on $L(x)$ indexed by $x \in X(\overline{K})$. A *metrized line bundle* $\overline{L}$ on X is a pair $(L, \|\cdot\|)$ consisting of a line bundle L on X and a K -metric $\|\cdot\|$ on L . Denote by $\widehat{\text{Pic}}(X)$ the group of all metrized line bundles on X .

The metric is said to be *semipositive* if it is a uniform limit of metrics induced by integral models $(\mathcal{X}_m, \mathcal{L}_m)$ of (X, L) over O_K where $\mathcal{L}_m \in \text{Pic}(\mathcal{X})_{\mathbb{Q}}$ is nef on fibers of $\mathcal{X}_m$ above O_K . The metric is said to be *integrable* if it is the quotient of two semipositive metrics. By abuse of notations, we also say that the corresponding metrized line bundle are semipositive or integrable if the metric is so. To be compatible with the global case, semipositive metrized line bundles are also called *nef metrized line bundles*. Finally, we also have the notion of $\overline{L}$ -bounded as in the global base.

Theorem 2.10 (local hodge index theorem). *Let K be a non-archimedean field with a rational rank-one valuation, and $\pi : X \rightarrow \text{Spec } K$ be a geometrically connected projective variety of dimension $n \geq 1$. Let $\overline{M}$ be an integrable metrized line bundle on X with M trivial, and $\overline{L}_1, \dots, \overline{L}_{n-1}$ be $n-1$ nef metrized line bundles on X . Then*

$$\overline{M}^2 \cdot \overline{L}_1 \cdots \overline{L}_{n-1} \leq 0.$$

Moreover, if L_i is ample and $\overline{M}$ is $\overline{L}_i$ -bounded for each i , then the equality holds if and only if $\overline{M} \in \pi^ \widehat{\text{Pic}}(K)$.*

Proof. The theorem is a local version of Theorem 1.3 in the vertical case. If K is the completion of a number field K_0 at some place, and (X, L) is the

base changes to K of a pair (X_0, L_0) over K_0 , then Theorem 2.10 is equivalent to Theorem 1.3.

The proof of Theorem 1.3 in the vertical case is easily translated to a proof of Theorem 2.10. The assumption that K has a rational rank-one valuation is used to reduce integral models over O_K to those over discrete valuation rings. In fact, by Lefschetz principle, any integral model of $(X, M, L_1, \dots, L_{n-1})$ over O_K is determined by finitely many polynomials. The coefficients of these polynomials generate a discrete valuation field K' in K . Then the integral model can be descended to the discrete valuation ring $O_{K'}$. The intersection theory works here with an appropriate normalizing factor when K' varies (as the integral model varies). The argument in previous section works here. $\square$

Remark 2.2. It is worth noting that, for a general non-archimedean field K , integral models of X over O_K are not Noetherian and cannot be descended to Noetherian schemes, so the usual intersection theory is not applicable. In that case, Gubler [Gu1] introduced an intersection theory using rigid-analytic geometry. One may try to generalize our theorem to the full case by his intersection theory.

Now we go back to Theorem 1.4. We will prove the following stronger result as in the case of model metrics.

Theorem 2.11. *Let K be a non-archimedean field with a rational rank-one valuation, and X a projective variety over K . Let L be an ample line bundle over X , and $\|\cdot\|_1$ and $\|\cdot\|_2$ two semipositive metrics on L . View $f = -\log(\|\cdot\|_1/\|\cdot\|_2)$ as a continuous function on X^{an} . Then*

$$\int_{X^{\text{an}}} f \, c_1(L, \|\cdot\|_1)^{\dim X} = \int_{X^{\text{an}}} f \, c_1(L, \|\cdot\|_2)^{\dim X}$$

if and only if f is a constant.

Proof. This theorem is just a simple consequence of Theorem 2.10. In fact, denote $\bar{L}_1 = (L, \|\cdot\|_1)$ and $\bar{L}_2 = (L, \|\cdot\|_2)$. Then the equality of the integrals is just

$$(\bar{L}_1 - \bar{L}_2) \cdot \bar{L}_1^n = (\bar{L}_1 - \bar{L}_2) \cdot \bar{L}_2^n.$$

Here $n = \dim X$. Equivalently,

$$\sum_{i=0}^{n-1} (\bar{L}_1 - \bar{L}_2)^2 \cdot \bar{L}_1^i \cdot \bar{L}_2^{n-1-i} = 0.$$

By Theorem 2.10, every term in the sum is non-positive. It forces

$$(\bar{L}_1 - \bar{L}_2)^2 \cdot \bar{L}_1^i \cdot \bar{L}_2^{n-1-i} = 0, \quad \forall i = 0, \dots, n-1.$$

It follows that

$$(\bar{L}_1 - \bar{L}_2)^2 \cdot (\bar{L}_1 + \bar{L}_2)^{n-1} = 0.$$

Note that $\bar{L}_1 - \bar{L}_2$ is vertical and $(\bar{L}_1 + \bar{L}_2)$ -bounded. Theorem 2.10 implies $\bar{L}_1 - \bar{L}_2 \in \pi^* \widehat{\text{Pic}}(K)$, which is equivalent to the statement that f is a constant. $\square$

3 Algebraic dynamics

In this section, we prove Theorem 1.5. We first introduce a theory of admissible adelic line bundles which will be needed in the proof. Then we prove the theorem using our arithmetic Hodge index theorem.

3.1 Admissible arithmetic classes

Let (X, f, L) be a *polarized dynamical system* over a number field K , i.e.,

- X is a projective variety over K ;
- $f : X \rightarrow X$ is a morphism over K ;
- $L \in \text{Pic}(X)_{\mathbb{Q}}$ is an ample line bundle such that $f^*L = qL$ from some $q > 1$.

By [Zh3], Tate's limiting argument gives an adelic $\mathbb{Q}$ -line bundle $\bar{L}_f \in \widehat{\text{Pic}}(X)_{\mathbb{Q}, \text{nef}}$ extending L and with $f^*\bar{L}_f = q\bar{L}_f$. In the following we generalize the definition to construct an admissible metric on any line bundle $M \in \text{Pic}(X)$.

Semisimplicity

By definition, f^* preserves the exact sequence

$$0 \longrightarrow \text{Pic}^0(X) \longrightarrow \text{Pic}(X) \longrightarrow \text{NS}(X) \longrightarrow 0.$$

It is known that $\text{NS}(X)$ is a finitely generated $\mathbb{Z}$ -module. Assume that X is normal. Then $\text{Pic}^0(X)$ is also a finitely generated $\mathbb{Z}$ -module. In fact,

the Picard functor $\underline{\text{Pic}}_{X/K}^0$ is represented by an abelian variety A . See [KL, Theorem 5.4] for example. Then $\text{Pic}^0(X) = A(K)$ is just the Mordell–Weil group. Alternatively, one can obtain the finiteness using only the Picard variety for a resolution of singularities.

Theorem 3.1. *Let (X, f, L) be a polarized dynamical system over a number field K . Assume that X is normal.*

- (1) *The operator f^* is semisimple on $\text{Pic}^0(X)_{\mathbb{C}}$ (resp. $\text{NS}(X)_{\mathbb{C}}$) with eigenvalues of absolute values $q^{1/2}$ (resp. q).*
- (2) *the operator f^* is semisimple on $\text{Pic}(X)_{\mathbb{C}}$ with eigenvalues of absolute values $q^{1/2}$ or q .*

Proof. The result can be proved using Hodge–Riemann bilinear relation (for Betti cohomology). See Serre [Ser] for the case when X is smooth. Here we present an algebraic proof which can be generalized to positive characteristics. It suffices to prove (1), since (2) is a consequence of (1).

We first consider $\text{NS}(X)_{\mathbb{C}}$. Write $n = \dim X$ as usual. Make the decomposition

$$\text{NS}(X)_{\mathbb{R}} := \mathbb{R}L \oplus P(X), \quad P(X) = \{\xi \in \text{NS}(X)_{\mathbb{R}} : \xi \cdot L^{n-1} = 0\}.$$

By Theorem A.1, the pairing

$$\langle \xi_1, \xi_2 \rangle = \xi_1 \cdot \xi_2 \cdot L^{n-2}$$

is a negative definite quadratic form on $P(X)$. The projection formula gives

$$\langle f^*\xi_1, f^*\xi_2 \rangle = q^2 \langle \xi_1, \xi_2 \rangle.$$

It follows that $q^{-1}f^*$ is an orthogonal transformation (with respect to the quadratic form). Then $q^{-1}f^*$ is diagonalizable with eigenvalues of absolute values 1.

Next we consider $\text{Pic}^0(X)_{\mathbb{C}}$. For $\xi_1, \xi_2 \in \text{Pic}^0(X)$, define a pairing

$$(\xi_1, \xi_2) = \bar{\xi}_1 \cdot \bar{\xi}_2 \cdot \bar{L}^{n-1}.$$

Here $\bar{\xi}_i$ is an adelic line bundle on X extending ξ_i and with zero intersection with any vertical classes on X for $i = 1, 2$, and $\bar{L}$ is any adelic line bundle

on X extending L . The intersection does not depend on the choice of the extension $\bar{L}$ since $\bar{\xi}_1$ is perpendicular to any vertical class.

The extension $\bar{\xi}_i$ always exists. In fact, we already know that $A = \underline{\text{Pic}}_{X/K}^0$ is an abelian variety. Let P be the universal bundle on $X \times A$. Then for any $\alpha \in A(\bar{K})$, the line bundle $P|_{X \times \alpha}$ on X is exactly the element of $\underline{\text{Pic}}_{X/K}^0(\bar{K})$ represented by α . Rigidify P by $P|_{x_0 \times A} = 0$ for some point $x_0 \in X(K)$. Here we assume x_0 exists by extending K if necessary. The multiplication $[2]_X : X \times A \rightarrow X \times A$ on the second component gives $[2]_X^* P = 2P$. By Tate's limiting argument, we obtain an adelic line bundle $\bar{P}$ extending P with $[2]_X^* \bar{P} = 2\bar{P}$. Then $\bar{P}|_{X \times \alpha_i}$ gives the desired extension of ξ_i , where $\alpha_i \in A(K)$ is the point representing ξ_i .

Go back to the pairing

$$(\xi_1, \xi_2) = \bar{\xi}_1 \cdot \bar{\xi}_2 \cdot \bar{L}^{n-1}.$$

It is extended to a pairing on $\text{Pic}^0(X)_{\mathbb{R}}$. We claim that the pairing is negative definite. In fact, let C be a closed non-singular curve in X representing L^{n-1} , which exists by Bertini's theorem. Then we have

$$(\xi_1, \xi_2) = \bar{\xi}_1|_C \cdot \bar{\xi}_2|_C = -2\langle \xi_1|_C, \xi_2|_C \rangle_{\text{NT}}.$$

Here we used the Hodge index theorem of [Fal, Hr]. By Theorem A.2, the map $\text{Pic}^0(X) \rightarrow \text{Pic}^0(C)$ has a finite kernel. It follows that the pairing is negative definite.

On the other hand, the projection formula gives

$$(f^* \xi_1, f^* \xi_2) = q(\xi_1, \xi_2).$$

It follows that $q^{-1/2} f^*$ is an orthogonal transformation (with respect to the pairing). Then $q^{-1/2} f^*$ is diagonalizable with eigenvalues of absolute values 1. The result is proved. □

By the theorem above, the exact sequence

$$0 \longrightarrow \text{Pic}^0(X)_{\mathbb{C}} \longrightarrow \text{Pic}(X)_{\mathbb{C}} \longrightarrow \text{NS}(X)_{\mathbb{C}} \longrightarrow 0.$$

has a splitting

$$\ell_f : \text{NS}(X)_{\mathbb{C}} \longrightarrow \text{Pic}(X)_{\mathbb{C}}$$

by identifying $\mathrm{NS}(X)_{\mathbb{C}}$ with the subspace of $\mathrm{Pic}(X)_{\mathbb{C}}$ generated by eigenvectors belonging to eigenvalues of absolute values q . It is easy to see that the splitting actually descends to

$$\ell_f : \mathrm{NS}(X)_{\mathbb{Q}} \longrightarrow \mathrm{Pic}(X)_{\mathbb{Q}}.$$

Definition 3.2. We say an element of $\mathrm{Pic}(X)_{\mathbb{C}}$ is *f-pure of weight 1* (resp. 2) if it lies in $\mathrm{Pic}^0(X)_{\mathbb{C}}$ (resp. $\ell_f(\mathrm{NS}(X)_{\mathbb{C}})$).

Admissible metrics

Now we introduce *f*-admissible metrics for line bundles. Note that the eigenvalues can be imaginary numbers. So we first extend the group of adelic line bundles.

Recall that $\widehat{\mathrm{Pic}}(X)$ is the group of adelic line bundles on X . Assume that $\pi : X \rightarrow \mathrm{Spec} K$ is geometrically connected. Write $\mathbf{F}$ for $\mathbb{Z}, \mathbb{Q}, \mathbb{R}$ or $\mathbb{C}$. Similar to [Mo4], we introduce

$$\widehat{\mathrm{Pic}}(X)_{[\mathbf{F}]} := \frac{\widehat{\mathrm{Pic}}(X) \otimes_{\mathbb{Z}} \mathbf{F}}{(\widehat{\mathrm{Pic}}(X)_{\mathrm{vert}} \otimes_{\mathbb{Z}} \mathbf{F})_0}.$$

where $(\widehat{\mathrm{Pic}}(X)_{\mathrm{vert}} \otimes_{\mathbb{Z}} \mathbf{F})_0$ is the subspace of elements $\overline{M} \in \widehat{\mathrm{Pic}}(X)_{\mathrm{vert}} \otimes_{\mathbb{Z}} \mathbf{F}$ represented by

$$\overline{M} = \sum_{i=1}^r a_i \otimes \overline{M}_i, \quad a_i \in \mathbf{F}, \quad \overline{M}_i \in \widehat{\mathrm{Pic}}(X),$$

which has an $\mathbf{F}$ -section s of $\overline{M}$ in the form

$$s = \sum_{i=1}^r a_i \otimes s_i, \quad s_i \in H^0(X, M) \setminus \{0\},$$

such that the function

$$-\log \|s\| := -\sum_{i=1}^r a_i \log \|s_i\| = 0$$

in the space

$$C(X, \mathbf{F}) = \oplus_v C(X_{K_v}^{\mathrm{an}}, \mathbf{F}),$$

where $C(X_{K_v}^{\text{an}}, \mathbf{F})$ denotes the space of continuous functions from $X_{K_v}^{\text{an}}$ to $\mathbf{F}$.

By definition, it is easy to have

$$\widehat{\text{Pic}}(X)_{[\mathbb{Z}]} = \widehat{\text{Pic}}(X), \quad \widehat{\text{Pic}}(X)_{[\mathbb{Q}]} = \widehat{\text{Pic}}(X)_{\mathbb{Q}}.$$

However, they are not true for $\mathbb{R}$ or $\mathbb{C}$. But we still have

$$\widehat{\text{Pic}}(X)_{[\mathbb{C}]} = \widehat{\text{Pic}}(X)_{[\mathbb{R}]} \otimes_{\mathbb{R}} \mathbb{C}.$$

Define $\widehat{\text{Pic}}(X)_{\text{int}, [\mathbf{F}]}$ to be the image of $\widehat{\text{Pic}}(X)_{\text{int}} \otimes_{\mathbb{Z}} \mathbf{F}$ in $\widehat{\text{Pic}}(X)_{[\mathbf{F}]}$. The intersection theory extends to $\widehat{\text{Pic}}(X)_{\text{int}, [\mathbf{F}]}$ by linearity. The positivity notions are extended to $\widehat{\text{Pic}}(X)_{\text{int}, [\mathbb{R}]}$.

The action $f^* : \widehat{\text{Pic}}(X) \rightarrow \widehat{\text{Pic}}(X)$ extends to $\widehat{\text{Pic}}(X)_{[\mathbf{F}]}$ naturally. The goal is to study the spectral theory of this action.

Definition 3.3. An element $\overline{M}$ of $\widehat{\text{Pic}}(X)_{[\mathbb{C}]}$ is called *f-admissible* if we can write $\overline{M} = \sum_{i=1}^m \overline{M}_i$ such that each $\overline{M}_i$ is an eigenvector of f^* in $\widehat{\text{Pic}}(X)_{[\mathbb{C}]}$.

The main result here asserts the existence of an admissible section of the forgetful map $\widehat{\text{Pic}}(X)_{[\mathbb{C}]} \rightarrow \text{Pic}(X)_{\mathbb{C}}$.

Theorem 3.4. *For any $M \in \text{Pic}(X)_{\mathbb{C}}$, there exists a unique f-admissible lifting $\overline{M}_f$ of M in $\widehat{\text{Pic}}(X)_{[\mathbb{C}]}$. Moreover, for $\mathbf{F} = \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$, if $M \in \text{Pic}(X)_{\mathbf{F}}$, then $\overline{M}_f \in \widehat{\text{Pic}}(X)_{[\mathbf{F}]}$.*

Proof. It suffices to assume that $M \in \text{Pic}(X)_{\mathbb{C}}$ is an eigenvector of f^* . Namely, $f^*M = \lambda M$ with $|\lambda| = q^{1/2}$ or q . We claim that Tate's limiting argument of [Zh3] still works here. Namely, let $\overline{M}$ be any lifting of M in $\widehat{\text{Pic}}(X)_{[\mathbb{C}]}$, then $(\lambda^{-1}f^*)^m \overline{M}$ converges *uniformly* to a unique element in $\widehat{\text{Pic}}(X)_{[\mathbb{C}]}$. This limit does not depend on the choice of $\overline{M}$, and we define $\overline{M}_f$ to be this limit. We only sketch an idea of the proof.

Let $\overline{M} \in \widehat{\text{Pic}}(X)_{[\mathbf{F}]}$ be an element represented by

$$\overline{M} = \sum_{i=1}^r a_i \otimes \overline{M}_i, \quad a_i \in \mathbf{F}, \quad \overline{M}_i \in \widehat{\text{Pic}}(X).$$

Take an $\mathbf{F}$ -section s of $\overline{M}$:

$$s = \sum_{i=1}^r a_i \otimes s_i, \quad s_i \in H^0(X, M) \setminus \{0\}.$$

It has *divisor*

$$\operatorname{div}(s) = \sum_{i=1}^r a_i \operatorname{div}(s_i) \in \operatorname{Div}(X)_{\mathbf{F}}.$$

Then the metrics turn to a function

$$-\log \|s\| = -\sum_{i=1}^r a_i \log \|s_i\|$$

in the space

$$S(X, \mathbf{F}) = \oplus_v S(X_{K_v}^{\text{an}}, \mathbf{F}),$$

where $S(X_{K_v}^{\text{an}}, \mathbf{F})$ denotes the space of functions from $X_{K_v}^{\text{an}}$ to $\mathbf{F}$ with logarithmic singularity along some $\mathbf{F}$ -divisor. Then we convert the uniform convergence of $\overline{M}$ to the uniform convergence of $-\log \|s\|$.

With these preparations, the original argument works here. We remark that we can further define the equivalence of two $\mathbf{F}$ -sections s_1 and s_2 by $\operatorname{div}(s_1) = \operatorname{div}(s_2)$. $\square$

Example 3.1. If $M \in \operatorname{Pic}^0(X)$, then $\overline{M}_f$ is represented by an arithmetic class which is perpendicular to all vertical arithmetic classes. The arithmetic class was used in the proof of Theorem 3.1.

Example 3.2. The admissible metrics for abelian varieties are well-known. Let X be an abelian variety and $f = [m]$ be the multiplication by m on X . Here m is an integer with $|m| > 1$. Any symmetric and ample line bundle L on X gives a polarization of (X, f) . We have $q = m^2$ in this case. Then $\ell_f(\operatorname{NS}(X)_{\mathbb{Q}})$ consists of exactly the rational multiples of symmetric line bundles, and $\operatorname{Pic}^0(X)$ is exactly the group of anti-symmetric line bundles. It is easy to see that f^* acts as m on $\operatorname{Pic}^0(X)$, and as m^2 on $\ell_f(\operatorname{NS}(X)_{\mathbb{Q}})$. The f -admissible classes can be obtained by the usual Tate's limiting argument (without get $\operatorname{Pic}(X)_{\mathbb{C}}$ involved).

Recall that we have a canonical section by Theorem 3.1:

$$\ell_f : \operatorname{NS}(X)_{\mathbb{Q}} \longrightarrow \operatorname{Pic}(X)_{\mathbb{Q}}$$

for the surjection $\operatorname{Pic}(X)_{\mathbb{Q}} \rightarrow \operatorname{NS}(X)_{\mathbb{Q}}$. By the f -admissible classes, we get a section

$$\widehat{\ell}_f : \operatorname{NS}(X)_{\mathbb{Q}} \longrightarrow \widehat{\operatorname{Pic}}(X)_{[\mathbb{Q}]}$$

for the surjection $\widehat{\operatorname{Pic}}(X)_{[\mathbb{Q}]} \rightarrow \operatorname{NS}(X)_{\mathbb{Q}}$.

Definition 3.5. For $\mathbf{F} = \mathbb{Q}, \mathbb{R}, \mathbb{C}$, define

$$\widehat{\ell}_f : \mathrm{NS}(X)_{\mathbf{F}} \longrightarrow \widehat{\mathrm{Pic}}(X)_{[\mathbf{F}]}$$

to be the map which sends $\xi \in \mathrm{NS}(X)_{\mathbf{F}}$ to the unique f -admissible class in $\widehat{\mathrm{Pic}}(X)_{[\mathbf{F}]}$ extending $\ell_f(\xi)$.

Positivity

The key result for our application is the following assertion.

Theorem 3.6. *Let (X, f, L) be a polarized dynamical system over a number field K with X normal. If $M \in \mathrm{Pic}(X)_{\mathbb{R}}$ is ample and f -pure of weight 2, then $\overline{M}_f$ is nef.*

Proof. By the action of the complex conjugation, we have $\overline{M}_f \in \widehat{\mathrm{Pic}}(X)_{[\mathbb{R}]}$.

Let $\overline{M}$ be any nef extension of M in $\widehat{\mathrm{Pic}}(X)_{\mathrm{nef}}$. Still consider the sequence $\overline{M}_m = (q^{-1}f^*)^m \overline{M}$. Here every term $\overline{M}_m$ is nef. We will pick a subsequence “convergent” to $\overline{M}_f$. Decompose

$$\overline{M} = \sum_{i=1}^r \overline{N}_i,$$

where $f^*N_i = \lambda_i N_i$ with $|\lambda_i| = q$ for any i . Then

$$\overline{M}_m = \sum_{i=1}^r (q^{-1}f^*)^m \overline{N}_i.$$

To compare with

$$\overline{M}_f = \sum_{i=1}^r \overline{N}_{i,f},$$

write

$$\overline{M}_m = \sum_{i=1}^r (q^{-1}\lambda_i)^m \cdot (\lambda_i^{-1}f^*)^m \overline{N}_i.$$

Here $(\lambda_i^{-1}f^*)^m \overline{N}_i$ converges to $\overline{N}_{i,f}$ uniformly.

Since $|q^{-1}\lambda_i| = 1$, we can find an infinite subsequence $\{m_k\}_k$ such that $(q^{-1}\lambda_i)^{m_k} \rightarrow 1$ for every i . Then $\overline{M}_{m_k}$ “converges” to $\overline{M}$ in the sense of

combining the uniform convergence of adelic metrics and the convergence of coefficients.

Then the theorem follows from the lemma below. Note that $\overline{N}_{i,f}$ and $(q^{-1}f^*)^m \overline{N}_i$ may lie in $\widehat{\text{Pic}}(X)_{[\mathbb{C}]}$ instead of in $\widehat{\text{Pic}}(X)_{[\mathbb{R}]}$. But $\overline{M}, \overline{M}_f$ and $\overline{M}_m$ are in $\widehat{\text{Pic}}(X)_{[\mathbb{R}]}$, and thus we can take the real parts of their decompositions above to get only elements of $\widehat{\text{Pic}}(X)_{[\mathbb{R}]}$ involved. $\square$

Lemma 3.7. *Suppose we are given*

$$\overline{M} = \sum_{i=1}^r a_i \overline{N}_i, \quad \overline{M}_m = \sum_{i=1}^r a_{i,m} \overline{N}_{i,m}, \quad a_i \in \mathbb{R}, \quad a_{i,m} \in \mathbb{R}.$$

Here every $\overline{N}_i$ and every $\overline{N}_{i,m}$ are adelic line bundles on a projective variety X over a number field. For any $i = 1, \dots, r$, assume the following convergence conditions:

- $a_{i,m} \rightarrow a_i$ as $m \rightarrow \infty$;
- $N_{i,m} = N_i$ for any m ;
- $\overline{N}_{i,m}$ converges uniformly to $\overline{N}_i$ as $m \rightarrow \infty$.

If M is ample and $\overline{M}_m$ is nef for any m , then $\overline{M}$ is nef.

Proof. Let $\overline{N}_i^0$ be any adelic line bundles extending M and N_i , with some conditions we will impose later. Denote

$$\overline{M}_m^0 = \sum_{i=1}^r a_{i,m} \overline{N}_i^0, \quad \overline{M}^0 = \sum_{i=1}^r a_i \overline{N}_i^0.$$

Let $\{\epsilon_m\}_m$ be a sequence in the interval $(0, 1)$ convergent to 0. Consider

$$\overline{M}_m' = (1 - \epsilon_m) \overline{M}_m - (1 - \epsilon_m) \overline{M}_m^0 + \overline{M}^0.$$

Note that

$$\overline{M}_m' - \overline{M} = (1 - \epsilon_m)(\overline{M}_m - \overline{M}_m^0) + (\overline{M}^0 - \overline{M}).$$

We see that the underlying line bundle of $\overline{M}_m'$ is exactly M , and $\overline{M}_m'$ converges to $\overline{M}$ uniformly. We are going to pick $\overline{M}^0, \overline{N}_i^0$ and ϵ_m so that $\overline{M}_m'$ is

nef, which will prove the lemma. The conditions assume that $\overline{M}_m$ is nef. It suffices to make

$$-(1 - \epsilon_m)\overline{M}_m^0 + \overline{M}^0 = \epsilon_m \left(\overline{M}^0 + (\epsilon_m^{-1} - 1)(\overline{M}^0 - \overline{M}_m^0) \right)$$

nef.

Let $\mathcal{X}$ be an integral model of X . Let $\overline{N}_i^0$ be induced by Hermitian line bundles $\overline{\mathcal{M}}$ and $\overline{\mathcal{N}}_i$ on $\mathcal{X}$. Denote

$$\overline{\mathcal{M}}_m = \sum_{i=1}^r a_{i,m} \overline{\mathcal{N}}_i, \quad \overline{\mathcal{M}} = \sum_{i=1}^r a_i \overline{\mathcal{N}}_i.$$

Assume $\overline{\mathcal{N}}_i$ and $\overline{\mathcal{M}}$ satisfy the following strong positivity conditions:

- There is an embedding of $\mathcal{X}(\mathbb{C})$ to a compact complex manifold Ω , such that $\overline{\mathcal{N}}_i(\mathbb{C})$ can be extended to line bundles $\overline{\mathcal{N}}'_i(\mathbb{C})$ on Ω with smooth metrics, and that the induced extension $\overline{\mathcal{M}}'(\mathbb{C})$ on Ω is ample with point-wise positive curvature.
- For any integral subvariety $\mathcal{Y}$ of $\mathcal{X}$,

$$(\overline{\mathcal{M}}|_{\mathcal{Y}})^{\dim \mathcal{Y}} > 0.$$

These assumptions make sure that, in the (finite-dimensional) real vector subspace V of $\widehat{\text{Pic}}(\mathcal{X})_{\mathbb{R}}$ generated by $\overline{\mathcal{N}}_1, \dots, \overline{\mathcal{N}}_r$ and $\overline{\mathcal{M}}$, the subset of ample $\mathbb{R}$ -line bundles in V form a neighborhood of $\overline{\mathcal{M}}$. Here V is endowed with the Euclidean topology. This is the open property of arithmetic ampleness and can be checked by the arithmetic Nakai–Moishezon criterion of Zhang [Zh2].

By definition, $\max_i(a_{im} - a_i)$ converges to 0 in V . Take

$$\epsilon_m = \max_i |a_{im} - a_i|^{1/2}.$$

Then for m sufficiently large,

$$\overline{\mathcal{M}} + (\epsilon_m^{-1} - 1)(\overline{\mathcal{M}} - \overline{\mathcal{M}}_m)$$

is ample on $\mathcal{X}$. The proof is complete. □

3.2 Sets of preperiodic points

The goal of this section is to prove Theorem 1.5.

Canonical heights

Let (X, f, L) be a polarized dynamical system over a number field K with X normal. Let M be any line bundle on X . The canonical height $h_{\overline{M}_f} : X(\overline{K}) \rightarrow \mathbb{R}$ associated to M is defined by

$$h_{\overline{M}_f}(x) = \frac{1}{\deg(x)} \widehat{\deg}(\overline{M}_f|_{\tilde{x}}), \quad x \in X(\overline{K}).$$

Here $\tilde{x}$ denotes the closed point of X representing x . The following are some basic properties:

- For any line bundle M , $h_{\overline{M}_f}(x) = 0$ if x is preperiodic.
- If M is ample, then $h_{\overline{M}_f}$ satisfies the Northcott property.
- If M is ample and f -pure of weight 2, then $h_{\overline{M}_f}(x) \geq 0$ for any $x \in X(\overline{K})$, since $\overline{M}_f$ is nef.
- For the polarization line bundle L , $h_{\overline{L}_f}(x) = 0$ if and only if x is preperiodic. It is an old result.

More generally, we define the canonical height of a closed subvariety Y of X associated to M by

$$h_{\overline{M}_f}(Y) := \frac{1}{(\dim Y + 1) \deg_M(Y)} (\overline{M}|_Y)^{\dim Y + 1}.$$

We still have the result that preperiodic subvarieties have canonical height 0. In fact, we have the following result.

Proposition 3.8. *Let Y be a preperiodic closed subvariety of X of dimension r . Let $M_1, M_2, \dots, M_{r+1}$ be line bundles in $\text{Pic}(X)_{\mathbb{C}}$ which are f -pure of weight two. Then we have*

$$\overline{M}_{1,f} \cdot \overline{M}_{2,f} \cdots \overline{M}_{r+1,f} \cdot Y = 0.$$

Proof. By the projection formula, we can assume that Y is periodic. Replacing f by a power if necessary, we can further assume that $f(Y) = Y$. By linearity, we can assume that M_i is an eigenvector of f^* for every i . Write $f^*M_i = \lambda_i M_i$. Then $f^*\overline{M}_{i,f} = \lambda_i \overline{M}_{i,f}$. Then the projection formula gives

$$f^*\overline{M}_{1,f} \cdot f^*\overline{M}_{2,f} \cdots f^*\overline{M}_{r+1,f} \cdot Y = q^r \overline{M}_{1,f} \cdot \overline{M}_{2,f} \cdots \overline{M}_{r+1,f} \cdot Y.$$

It follows that

$$(\lambda_1 \cdots \lambda_{r+1} - q^r) \overline{M}_{1,f} \cdot \overline{M}_{2,f} \cdots \overline{M}_{r+1,f} \cdot Y = 0.$$

The result follows since $\lambda_1 \cdots \lambda_{r+1} - q^r \neq 0$, as a consequence of the assumption $|\lambda_i| = q$. $\square$

Example 3.3. The result is not true without the assumption that the line bundles are f -pure of weight two. Take X to be an elliptic curve, and M_1, M_2 to be two line bundles of degree zero. Then they are f -pure of weight one if we take $f = [2]$. Then $\overline{M}_{1,f} \cdot \overline{M}_{2,f} = -2\langle M_1, M_2 \rangle_{\text{NT}}$ is often nonzero.

Preperiodic points

Now we are ready to prove the following refinement of Theorem 1.5 in the case of number fields. The condition of X being normal can be obtained by taking a normalization.

Theorem 3.9. *Let X be a normal projective variety over a number field K . For any $f, g \in \mathcal{DS}(X)$, the following are equivalent:*

- (1) $\text{Prep}(f) = \text{Prep}(g)$;
- (2) $g\text{Prep}(f) \subset \text{Prep}(f)$;
- (3) $\text{Prep}(f) \cap \text{Prep}(g)$ is Zariski dense in X ;
- (4) $\widehat{\ell}_f = \widehat{\ell}_g$ as maps from $\text{NS}(X)_{\mathbb{Q}}$ to $\widehat{\text{Pic}}(X)_{\mathbb{Q}}$.

We will prove $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4) \Rightarrow (1)$. We start the proof with some easy directions.

First, $(1) \Rightarrow (2)$ is trivial.

Second, $(2) \Rightarrow (3)$. For any integer $d > 0$, denote

$$\text{Prep}(f, d) := \{x \in \text{Prep}(f) \mid \deg(x) < d\}.$$

By Northcott's property, $\text{Prep}(f, d)$ is a finite set since its points have trivial canonical heights. Assuming (2), then g stabilizes the set $\text{Prep}(f)$. By definition,

$$\text{Prep}(f) = \bigcup_{d>0} \text{Prep}(f, d).$$

Since g is also defined over K , it stabilizes the set $\text{Prep}(f, d)$. By the finiteness, we obtain that

$$\text{Prep}(f, d) \subset \text{Prep}(g), \quad \forall d.$$

Hence,

$$\text{Prep}(f) \subset \text{Prep}(g).$$

Then (3) is true since $\text{Prep}(f)$ is Zariski dense in X by the result of Fakhruddin [Fak].

Third, we prove (4) $\Rightarrow$ (1). Let L be an ample line bundle on X polarizing f . By (4), $\overline{L}_f = \overline{L}_g$. For any $x \in \text{Prep}(g)$, we have $h_{\overline{L}_f}(x) = h_{\overline{L}_g}(x) = 0$. It follows that $x \in \text{Prep}(f)$. This proves $\text{Prep}(g) \subset \text{Prep}(f)$. By symmetry, we have the other direction and thus the equality.

From (3) to (4)

Assume that $\text{Prep}(f) \cap \text{Prep}(g)$ is Zariski dense in X . Write $n = \dim X$. We need to prove $\widehat{\ell}_f(\xi) = \widehat{\ell}_g(\xi)$ for any $\xi \in \text{NS}(X)_{\mathbb{Q}}$. By linearity, it suffices to assume that ξ is ample. Denote $L = \ell_f(\xi)$ and $M = \ell_g(\xi)$. They are ample $\mathbb{Q}$ -line bundles on X . Then $\overline{L}_f = \widehat{\ell}_f(\xi)$ and $\overline{M}_g = \widehat{\ell}_g(\xi)$ are nef by Theorem 3.6.

Consider the sum $\overline{N} = \overline{L}_f + \overline{M}_g$, which is still nef. By the successive minima of Zhang [Zh3],

$$\lambda_1(X, \overline{N}) \geq h_{\overline{N}}(X) \geq 0.$$

Here

$$h_{\overline{N}}(X) = \frac{1}{(n+1) \deg_N(X)} \overline{N}^{n+1}$$

and the essential minimum

$$\lambda_1(X, \overline{N}) = \sup_{U \subset X} \inf_{x \in U(\overline{K})} h_{\overline{N}}(x),$$

where the supremum is taken over all Zariski open subsets U of X . Note that $h_{\overline{N}}$ is zero on $\text{Prep}(f) \cap \text{Prep}(g)$, which is assumed to be Zariski dense in X . Hence, $\lambda_1(X, \overline{N}) = 0$. It forces $h_{\overline{N}}(X) = 0$.

Write in terms of intersections, we have

$$(\overline{L}_f + \overline{M}_g)^{n+1} = 0.$$

Expand by the binomial formula. Note that every term is non-negative. It follows that

$$\overline{L}_f^i \cdot \overline{M}_g^{n+1-i} = 0, \quad \forall i = 0, 1, \dots, n+1.$$

It particularly gives

$$(\overline{L}_f - \overline{M}_g)^2 \cdot (\overline{L}_f + \overline{M}_g)^{n-1} = 0.$$

Note that

$$(L - M) \cdot (L + M)^{n-1} = 0$$

since $L - M \in \text{Pic}^0(X)_{\mathbb{Q}}$ is numerically trivial. We are in the situation to apply Theorem 1.3 to

$$(\overline{L}_f - \overline{M}_g, \overline{L}_f + \overline{M}_g).$$

It is immediate that $(\overline{L}_f - \overline{M}_g)$ is $(\overline{L}_f + \overline{M}_g)$ -bounded. The only condition that does not match the theorem is that $(\overline{L}_f + \overline{M}_g)$ is not ample. However, since $L - M$ is numerically trivial, we can take any $\overline{C} \in \widehat{\text{Pic}}(K)$ with $\deg(\overline{C}) > 0$, and replace

$$(\overline{L}_f - \overline{M}_g, \overline{L}_f + \overline{M}_g)$$

by

$$(\overline{L}_f - \overline{M}_g, \overline{L}_f + \overline{M}_g + \pi^*\overline{C}).$$

Then all the conditions are satisfied. The theorem implies that

$$\overline{L}_f - \overline{M}_g \in \pi^*\widehat{\text{Pic}}(K).$$

By evaluating at any point x in $\text{Prep}(f) \cap \text{Prep}(g)$, we see that

$$\overline{L}_f - \overline{M}_g = 0$$

in $\widehat{\text{Pic}}(X)_{\mathbb{Q}}$. Here we have used the restriction $\overline{L}_f|_x = 0$, which is more delicate than $\widehat{\deg}(\overline{L}_f|_x) = 0$. It finishes the proof.

3.3 Variants and questions

Now we consider some variants, consequences and questions related to Theorem 1.5.

Variants

In a private communication, Barry Mazur points out that one direction of Theorem 1.5 can be generalized as follows:

Theorem 3.10. *Let X be a projective variety over a number field K . Let $f, g \in \mathcal{DS}(X)$, and denote by Y the Zariski closure of $\text{Prep}(f) \cap \text{Prep}(g)$ in X . Then*

$$\text{Prep}(f) \cap Y(\overline{K}) = \text{Prep}(g) \cap Y(\overline{K}).$$

The proof of Theorem 1.5 applies here. In fact, we can always restrict f -admissible (or g -admissible) adelic line bundles from X to Y . Then we apply Theorem 1.3 on Y .

One consequence of Theorem 3.9 is the following result.

Corollary 3.11. *Let X be a projective variety over a number field K , and $f, g \in \mathcal{DS}(X)$ be two polarizable algebraic dynamical systems. If $\text{Prep}(f) \cap \text{Prep}(g)$ is Zariski dense in X , then $d\mu_{f,v} = d\mu_{g,v}$ on $X_{K_v}^{\text{an}}$ for any place v of K .*

Here $d\mu_{f,v}$ denotes the equilibrium measure of (X, f) on the analytic space $X_{K_v}^{\text{an}}$. It can be obtained from any initial “smooth” measure on $X_{K_v}^{\text{an}}$ by Tate’s limiting argument. By a proper interpretation, it satisfies $f^*d\mu_{f,v} = q^{\dim X}d\mu_{f,v}$ and $f_*d\mu_{f,v} = d\mu_{f,v}$.

One can deduce the theorem by the equivalent condition $\widehat{\lambda}_f = \widehat{\lambda}_g$ in Theorem 3.9. Alternatively, one can apply the equidistribution theorem of Yuan [Yu] to any generic sequence in $\text{Prep}(f) \cap \text{Prep}(g)$ to obtain the result.

Semigroup

For any subset P of $X(\overline{K})$, denote

$$\mathcal{DS}(X, P) := \{g \in \mathcal{DS}(X) \mid \text{Prep}(g) = P\}.$$

We say that P is a *special set* of X if $\mathcal{DS}(P)$ is non-empty.

Question 3.4. Let P be a special set of X . Is the set $\mathcal{DS}(X, P)$ is a semigroup?

The question asks whether $g \circ h \in \mathcal{DS}(X, P)$ for any $g, h \in \mathcal{DS}(X, P)$. By Theorem 1.5, we can write:

$$\mathcal{DS}(X, P) = \{g \in \mathcal{DS}(X) \mid gP \subset P\}.$$

Then $P \subset \text{Prep}(g \circ h)$ by the simple argument proving (2) $\Rightarrow$ (3) of Theorem 1.5. Then we have $g \circ h \in \mathcal{DS}(X, P)$ if $g \circ h$ is polarizable.

The polarizability is automatically true if X is a projective space. Therefore, if $X = \mathbb{P}^n$, then $\mathcal{DS}(X, P)$ is a semigroup.

The results and the questions also apply to general fields by the treatment of [YZ].

Dynamical Manin–Mumford

The second author of this paper proposed in [Zh3] a dynamical analogue of the classical Manin–Mumford conjecture for abelian varieties. Namely, given a dynamical triple (X, L, f) over a field K of characteristic zero, a closed subvariety Y of X is preperiodic under f if and only if the set $Y(\overline{K}) \cap \text{Prep}(f)$ is Zariski dense in Y . Recently, Ghioca and Tucker found the following counter-example of this conjecture:

Proposition 3.12 (Ghioca and Tucker, [GTZ]). *Let E be an elliptic curve with complex multiplication by an order R in imaginary quadratic field K . Let f be an endomorphism on $E \times E$ defined by multiplications by two nonzero elements α and β in R with equal norm $N(\alpha) = N(\beta)$. Then f is polarized by any symmetric and ample line bundle*

$$\text{Prep}(f) = E_{\text{tor}} \times E_{\text{tor}}.$$

Moreover, the diagonal Δ_E in $E \times E$ is not preperiodic under f if α/β is not a root of unity,

Notice that the diagonal is preperiodic for multiplication by $(2, 2)$. Thus the proposition shows *an example that two endomorphisms of a projective variety, with the same set of preperiodic points, have different sets of preperiodic subvarieties.* We would like to propose the following revision of the dynamical Manin–Mumford conjecture:

Question 3.5. Let X be a projective variety over any field K , and P a special set of X . Let Y be a proper closed subvariety of X such that $Y(\overline{K}) \cap P$ is Zariski dense in Y . Do there exist two endomorphisms $f, g \in \mathcal{DS}(X, P)$, and a proper g -periodic closed subvariety Z such that $f(Y) \subset Z$?

If the answers to both questions are positive, then in the situation of the second question, we can find a finite sequence of subvarieties

$$X := X_0 \supset X_1 \supset X_2 \cdots \supset X_s$$

and endomorphisms $f_i, g_i \in \mathcal{DS}(X_i, P \cap X_i(\overline{K}))$ for $i = 0, 1, \dots, s$ such that

- (1) X_i is g_i -periodic, which implies that $P \cap X_i(\overline{K})$ is a special set of X_i ;
- (2) $f_i \circ f_{i-1} \cdots f_0(Y) \subset X_{i+1}$ for any $i = 0, 1, \dots, s-1$;
- (3) $f_{s-1} \circ f_{s-2} \cdots f_0(Y) = X_s$.

A Lefschetz theorems for Normal Varieties

We list some classical Lefschetz-type results applicable to normal projective varieties over any characteristic. Their counterparts for complex projective manifolds are even more classical, and we refer them to [La, Chapter 3].

Let X be a projective variety of dimension $n \geq 2$ over an algebraically closed field k . Consider the exact sequence

$$0 \longrightarrow \mathrm{Pic}^0(X) \longrightarrow \mathrm{Pic}(X) \longrightarrow \mathrm{NS}(X) \longrightarrow 0.$$

Here $\mathrm{Pic}^0(X)$ denotes the subgroup of algebraically trivial line bundles, and $\mathrm{NS}(X)$ denotes the quotient group.

Recall that a line bundle L on X is numerically trivial if $L \cdot C = 0$ for any closed curve C in X . It is well-known that a line bundle L is numerically trivial if and only if the multiple mL is algebraically trivial for some nonzero integer m . See [KL, Theorem 6.3].

Theorem A.1. *Let $L_1, \dots, L_{n-1}$ be ample line bundles on a projective variety X . For any $M \in \mathrm{Pic}(X)$ with $M \cdot L_1 \cdots L_{n-1} = 0$, one has*

$$M^2 \cdot L_1 \cdots L_{n-2} \leq 0.$$

Moreover, the equality holds if and only if M is numerically trivial.

Proof. If X is a smooth projective surface, the result is the classical Hodge index theorem. See [Ha, Theorem IV.1.9] for example. If X is a singular projective surface, the result is induced by a desingularization $X' \rightarrow X$. Note that the pull-back L'_1 of L_1 to X' is not ample, but $L'^2_1 = L^2_1 > 0$ is sufficient for the result.

In general, by Bertini's theorem, we can assume that $L_1 \cdot L_2 \cdots L_{n-2}$ is represented by an integral closed surface S in X . Then the inequality is proved by the Hodge index theorem on S .

For the condition of the equality, we need to prove that $M \cdot C = 0$ for any complete curve C in X . By Bertini's theorem, we can assume that $L_1 \cdot L_2 \cdots L_{n-2}$ is represented by an effective 2-cycle $\sum_{i=0}^r a_i S_i$ with $a_i > 0$ such that S_0 contains C . Then

$$\sum_{i=0}^r a_i S_i \cdot M^2 = 0$$

implies that $S_i \cdot M^2 = 0$ since each term is non-positive. It follows that $M|_{S_0}$ is numerically trivial on S_0 . Then $M \cdot C = 0$. $\square$

By a very ample linear system, we mean a subspace V of $H^0(X, L)$, for a very ample line bundle L on X , which gives an embedding $X \hookrightarrow \mathbb{P}(V)$. Denote by $|V| = \{\text{div}(s) : s \in V\}$ the space of hyperplane sections. By a general hyperplane section of V in X , we mean an element in a Zariski open subset of $|V|$. By the Bertini-type result of Seidenberg [Sei], if X is normal and projective, then a general element $Y \in |V|$ is also normal and projective. The following is the Lefschetz hyperplane theorem in the current setting.

Theorem A.2. *Let X be a normal projective variety of dimension n over k . Let Y be a general hyperplane section of a very ample linear system V on X .*

- (1) *The natural map $\text{Pic}^0(X) \rightarrow \text{Pic}^0(Y)$ has a finite kernel if $n \geq 2$.*
- (2) *The natural map $\text{NS}(X) \rightarrow \text{NS}(Y)$ has a finite kernel if $n \geq 3$.*
- (3) *The natural map $\text{Pic}(X) \rightarrow \text{Pic}(Y)$ has a finite kernel if $n \geq 3$.*

Proof. Part (3) is a consequence of (1) and (2). For (1), we refer to [KL, Remark 5.8] for a historical account. Part (2) is a consequence of Theorem A.1. In fact, assume that M lies in the kernel of $\text{NS}(X) \rightarrow \text{NS}(Y)$. In

Theorem A.1, set $L_1 = \cdots = L_{n-1} = \mathcal{O}(Y)$. We see that M is numerically trivial on X . Then some integer multiple of M lies in $\text{Pic}^0(X)$. Hence, the kernel of $\text{NS}(X) \rightarrow \text{NS}(Y)$ is a torsion subgroup. It must be finite since $\text{NS}(X)$ is a finitely generated abelian group. $\square$

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