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Linear forms, algebraic cycles, and derivatives of L-series

Dedicated to Professor Lo Yang on the Occasion of His 80th Birthday

Shouwu Zhang

Department of Mathematics, Princeton University, Princeton, NJ 08544, USA

Email: shouwu@math.princeton.edu

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Abstract In this note, we state some refinements of conjectures of Gan-Gross-Prasad and Kudla concerning the central derivatives of L-series and special cycles on Shimura varieties. The analogues of our formulation for special values of L-series are written in terms of invariant linear forms on automorphic representations defined by integrations of matrix coefficients.

Keywords linear forms, algebraic cycles, derivatives, L-series

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1 Introduction

This is the note for a talk given at the conference "Number Theory and Representation Theory" in honor of the 60th birthday of Benedict Gross held on June 2–June 5, 2010. In this note, we state some refinements of conjectures of Gan, Gross and Prasad (see [5–7]) and Kudla [17] concerning the central derivatives of L-series and special cycles on Shimura varieties. The analogues of our formulation for special values of L-series are written in terms of invariant linear forms on automorphic representations defined by integrations of matrix coefficients. These formulae look simpler than but are actually equivalent to the formulae in Ichino and Ikeda [14], Harris [9] and Rallis' inner product formula [11, 12, 24]. In the derivative situation, our formula will use another space of invariant forms defined by Beilinson-Bloch height pairings. Again our formulae look simpler than but equivalent to those given in Zhang [31] and Liu [22]. We will also state some recent work about this formulae by Yuan, Zhang and Zhang [28, 29] and Liu [23].

2 Modular L-functions as linear forms

Let k be a positive integer. There is an action of $\mathrm{GL}_2(\mathbb{R})_+$, the group of real matrix with positive determinants, on the space of holomorphic functions on the upper half plane $\mathcal{H}$ by the following formula:

$$f \mapsto f\,|_k\gamma, \quad f\,|_k\gamma(z) = f\left(\frac{az+b}{cz+d}\right)\frac{\det\gamma^{k/2}}{(cz+d)^k}, \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

A function f is called a modular form, if it is invariant under some congruence subgroup Γ and its fourier expansion vanishes at every cusp. Let S_k denote the space of cusp forms of weight k. Then S_k admits an action by $\mathrm{GL}_2(\mathbb{Q})_+$, the group of two-by-two rational invertible matrices with positive determinants. Such an action can be extended into an action by $\mathrm{GL}_2(\widehat{\mathbb{Q}})_+$, the group of matrices over $\widehat{\mathbb{Q}} = (\prod_p \mathbb{Z}_p) \otimes \mathbb{Q}$ with determinants in $\mathbb{Q}_{>0}^\times$.

For each $f \in S_k$, define its L-series by the Mellin transform

$$L(f, s) = \int_0^\infty f(\mathrm{i}y) y^s \frac{dy}{y}.$$

Then it is easy to see that this function is entire for $s \in \mathbb{C}$. For each diagonal element

$$\gamma = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix},$$

we have

$$L(f\,|_k\gamma, s) = \int_0^\infty f(\mathrm{i}ay/d)(a/d)^{k/2} y^s \frac{dy}{y} = (a/d)^{k/2-s} L(f, s).$$

Let A denote the subgroup of $\mathrm{GL}_2(\mathbb{Q})_+$ of diagonal elements. For each $t \in \mathbb{C}$, define a character χ_t by

$$\chi_t\left(\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}\right) = (a/d)^t.$$

We embed A into $A \times \mathrm{GL}_2(\mathbb{Q})_+$ diagonally, and then $f \mapsto L(f, s)$ defines an element in

$$\ell_s \in \mathrm{Hom}_{A(\widehat{\mathbb{Q}})}(S_k \otimes \chi_{s-k/2}, \mathbb{C}).$$

To understand this last space, we decompose S_k into a direct sum of irreducible representations π which is generated by a unique new form f_π. Write $L(s, \pi) = L(f_\pi, s)$. Let $\ell_{s,\pi}$ denote the restriction of ℓ_s on $\pi \otimes \chi_{s-k/2}$:

$$\ell_{s,\pi} \in \mathrm{Hom}_{A(\widehat{\mathbb{Q}})}(\pi \otimes \chi_{s-k/2}, \mathbb{C}).$$

On the other hand, it can be shown that the right-hand side is one-dimensional and is generated by one element e_s such that

$$\ell_s = L(\pi, s) e_s.$$

Thus we have just explained that the L-series for new forms are just factors of the Mellin transform over a base.

3 Gan-Gross-Prasad conjectures

3.1 Linear forms and pairing

Let F be a local field and (E, σ) be a semisimple algebra over F of degree less than or equal to 2 with an involution σ so that the fixed part of E is F, i.e., (E, σ) is either F with $\sigma = 1$, a quadratic field extension of F with σ the non-trivial element of $\mathrm{Gal}(E/F)$, or the direct sum $F \oplus F$ with σ switching two factors. Let V be a hermitian space over E of dimension n with respect to σ. Let $e \in V$ with nonzero norm and let W be the complement of e in V. Define algebraic groups over G and H as follows:

$$G = U(V) \times U(W), \quad H = U(W)$$

and embed H into G diagonally. For simplicity, we use G or H to denote the group of F-rational points. Fix a Haar measure on H. Notice that according to the three cases of (E, σ), $U(V)$ is either an orthogonal group, unitary group, or the general linear group. We consider the general linear case as the split unitary case.

Let π be an admissible representation of G. The space of H-invariant forms on π is define as

$$\mathcal{P}(\pi) = \mathrm{Hom}_H(\pi, \mathbb{C}).$$

Then it is known that $\mathcal{P}(\pi)$ is at most one-dimensional by recent work of Aizenbud, Gourevitch, Rallis and Schiffman [1] and Sun and Zhu [26]. When π is tempered, then one defines an element

$$I \in \mathcal{P}(\pi) \otimes \mathcal{P}(\widetilde{\pi})$$

by matrix coefficient integrals

$$I(v \otimes \widetilde{v}) = \int_H \langle \pi(h)v, \widetilde{v}\rangle dh.$$

Sakellaridis and Venkatesh [25] have recently shown that the non-vanishing of $\mathcal{P}(\pi)$ is equivalent to the non-vanishing of I. According to the computations of Ichino and Ikeda [14] in the orthogonal case and Harris [9] in the unitary case, we may normalize the I in the following way:

$$m := \frac{L(1, \pi, ad)}{L(M^\vee(1), 0)L(1/2, \pi)} I.$$

In this way, $m(v \otimes \widetilde{v}) = 1$ in the spherical case:

- π is unramified,
- G and H are unramified with standard measure,
- v and $\widetilde{v}$ are spherical elements such that $\langle v, \widetilde{v}\rangle = 1$.

Here, M is the Gross motive attached to $U(V)$ which has the L-series

$$L(M, 0) = \begin{cases} \zeta(2)\zeta(4)\cdots\zeta(n-1), & \text{if } n \text{ is odd},\\ \zeta(2)\zeta(4)\cdots\zeta(n-2)L(n/2, \chi_{\mathrm{disc}(V)}), & \text{if } n \text{ is even} \end{cases}$$

if $F = E$ and otherwise,

$$L(M, 0) = \begin{cases} L(1, \eta)\zeta(2)L(3, \eta)\cdots L(n-2, \eta)\zeta(n-1), & \text{if } n \text{ is odd},\\ L(1, \eta)\zeta(2)L(3, \eta)\cdots L(n-1, \eta), & \text{if } n \text{ is even}. \end{cases}$$

We assume that π is tempered and define a bilinear form

$$\langle \cdot, \cdot \rangle : \mathcal{P}(\pi) \otimes \mathcal{P}(\widetilde{\pi}) \longrightarrow \mathbb{C}$$

by

$$\langle \ell, \widetilde{\ell}\rangle = \frac{\ell \otimes \widetilde{\ell}}{m}, \quad \ell \otimes \widetilde{\ell} \in \mathcal{P}(\pi) \otimes \mathcal{P}(\widetilde{\pi}).$$

By the work of He [10], when π is unitary, this pairing is positive definite.

3.2 The global conjecture

Let F be a number field with adeles $\mathbb{A}$ and (E, σ) be a semisimple algebra over F of degree less than or equal to 2 such that $E^\sigma = F$. Let V be a hermitian space over (E, σ) and let W be the orthogonal complement for some element e with non-zero norm. Define analogously the group G and H as in the local case. Let $\pi = \bigotimes \pi_v$ be an irreducible cuspidal representation of $G(\mathbb{A}_F)$. We define

$$\mathcal{P}(\pi) = \bigotimes_v \mathcal{P}(\pi_v).$$

When it is not zero, then we define a bilinear form

$$\langle \cdot, \cdot \rangle : \mathcal{P}(\pi) \otimes \mathcal{P}(\widetilde{\pi}) \longrightarrow \mathbb{C}$$

by product of local ones.

Define an element $P_\pi \in \mathcal{P}(\pi)$ by

$$P_\pi(\phi) = \int_{H(F)\backslash H(\mathbb{A})} \phi(h)dh,$$

where the integration is taken using Tamagawa measure. Then we have the following conjecture of Gross and Prasad [6, 7] and Gan, Gross and Prasad [5], and the refinements by Ichino and Ikeda [14] and Harris [9].

Conjecture 3.1. Assume that $\mathcal{P}(\pi) \neq 0$. There is an integer β such that

$$\langle P_\pi, P_{\widetilde{\pi}} \rangle = \frac{L(M^\vee(1), 0)}{2^\beta L(1, \pi, ad)} L\bigg(\frac{1}{2}, \pi\bigg).$$

Moreover, if S is the centralizer in the dual group $\widehat{G}$ of the image of the Arthur parameter of π, then

$$2^\beta = |S|.$$

Remark 3.2. Decompose $L(s, \pi) = \prod L(s, M_i)$ into a product of self-dual L-series according to a decomposition of the Arthur parameter. Then $\mathcal{P}(\pi) = 0$ if one of the these L-series has odd sign of functional equation. If all of these L-series have even sign of the functional equation, then Gan, Gross and Prasad [5] have conjectured a unique pure inner form (H', G') of (H, G) such that $\mathcal{P}(\pi') \neq 0$ for a unique endoscopic transfer π' of π. In particular if $L(1/2, \pi) \neq 0$, such an inner form always exists.

Remark 3.3. The conjecture holds in the following cases:

(1) V is hermitian of dimension less than or equal to 2 or orthogonal of dimension less than or equal to 3, by Waldspurger [27];

(2) V is orthogonal of dimension 4 by Ichino [13].

Remark 3.4. For unitary group, Jacquet and Rallis [15] have formulated a relative trace formula approach to attack Gross-Prasasde conjecture. The corresponding fundamental lemma has just been proved by Yun [30].

3.3 The arithmetic Gan-Gross-Prasad conjecrture

Let F be a totally real number field and E either F or a quadratic CM-extension. Let $\mathbb{V}$ be a hermitian space over $\mathbb{A}_E$ which is incoherent in the sense that it is not isomorphic to $V \otimes_E \mathbb{A}_E$ for a Hermitian space over E, and let $\mathbb{W}$ be the orthogonal complement of an element e with the norm in $F^\times$. We define the group $\mathbb{G}$ and $\mathbb{H}$ for $U(\mathbb{V}) \times U(\mathbb{W})$ and $U(\mathbb{W})$ as algebraic groups over $\mathbb{A}$ and embed $\mathbb{H}$ into $\mathbb{G}$ diagonally.

We assume that $\mathbb{V}$ has signature $(n, 0)$ at all archimedean place of F. Then $\mathbb{G}$ defines a projective system X_K of Shimura varieties X_K over E parameterized by open compact subgroups K of $\mathbb{G}(\mathbb{A}_f)$. Let X denote the projective limit of X_K which is a scheme over E with an action by $\mathbb{G}(\mathbb{A}_f)$. We define $\mathrm{Ch}^*(X)$ to be the direct limit of $\mathrm{Ch}^*(X_K)$, the Chow group of cycles on X_K. For an irreducible representation π of $\mathbb{G}$ with trivial archimedean components, define

$$\mathrm{Ch}^*(\pi) := \mathrm{Hom}_{\mathbb{G}}(\pi, \mathrm{Ch}^*(X)_{\mathbb{C}}).$$

Then by standard conjectures on Chow groups of Shimura varieties, one expect that this group is non-vanishing only if π is automorphic in the sense that it has automorphic lifting to the quasi-split coherent group of $\mathbb{G}$ and that we have decomposition

$$\mathrm{Ch}^*(X)_{\mathbb{C}} = \bigoplus_{\pi \in \mathcal{A}(\mathbb{G})} \pi \otimes \mathrm{Ch}^*(\pi).$$

Here, $\mathcal{A}(\mathbb{G})$ denote the set of automorphic representations of $\mathbb{G}$. If π is cuspidal, we even conjecture that $\mathrm{Ch}^*(\pi)$ takes values in $\mathrm{Ch}^*(X)^{00}_{\mathbb{C}}$, the group of homologically trivial cycles. By conjectures of

Beilinson [2,3] and Bloch [4] $\mathrm{Ch}^*(\pi)$ is finite dimensional with dimension equal to $\mathrm{ord}_{s=1/2}L(s,\pi)$ and there is a positive definite height pairing

$$\langle\cdot,\cdot\rangle:\mathrm{Ch}^*(\pi)\otimes\mathrm{Ch}^*(\widetilde{\pi})\longrightarrow\mathbb{C}.$$

In duality, we may define $\mathrm{Ch}_*(X)_K=\mathrm{Ch}^{\dim X-*}(X_K)$, and $\mathrm{Ch}_*(X)$ to be the projective limit of $\mathrm{Ch}_*(X_K)$ under the projection maps as transition. Then we have a decomposition dual to the above:

$$\mathrm{Ch}_*(X)=\prod_{\pi\in\mathcal{A}(\mathbb{G})}\pi^\vee\otimes\mathrm{Ch}_*(\pi^\vee),$$

where $\pi^\vee=\mathrm{Hom}(\pi,\mathbb{C})$ as $\mathbb{G}$-module, and

$$\mathrm{Ch}_*(\pi^\vee)=\mathrm{Hom}_{\mathbb{G}}(\pi^\vee,\mathrm{Ch}_*(X)_{\mathbb{C}})=\mathrm{Ch}^{\dim X-*}(\widetilde{\pi}).$$

The group $\mathbb{H}$ defines a projective system Y_K of subvarieties of X_K with limit Y. The cycle Y has a class in $\mathrm{Ch}_*(X)$ and has component Y_π in $\pi^\vee\otimes\mathrm{Ch}_*(\pi^\vee)$. It is clear that Y is invariant in $\mathbb{H}$ and thus we may replace $\pi^\vee$ by $\mathcal{P}(\pi)=(\pi^\vee)^{\mathbb{H}}$. Thus we define the element

$$Y_\pi\in\mathcal{P}(\pi)\otimes\mathrm{Ch}^*(\widetilde{\pi}).$$

The arithmetical Gross-Prasad conjecture formulated in Gan, Gross and Prasad [5] and refined by Zhang [31] is as follows:

Conjecture 3.5. Assume that $\mathcal{P}(\pi)\neq 0$. Then

$$\langle Y_\pi,Y_{\widetilde{\pi}}\rangle=\frac{L(M^\vee(1),0)}{2^\beta L(1,\pi,ad)}L'(1/2,\pi).$$

Remark 3.6. Assume that the root number of π is -1 and S is a singlet. Then Gan, Gross and Prasad [5] have conjectured a unique incoherent pure inner form $(\mathbb{G}',\mathbb{H}')$ of $(\mathbb{G},\mathbb{H})$ such that $\mathcal{P}(\pi')\neq 0$ for a unique endoscopic transfer π' of π.

Remark 3.7. The conjecture holds in the following situations:

(1) $\mathbb{V}$ is unitary of dimension 2 by Yuan, Zhang and Zhang [28] as a generalization of the Gross-Zagier formula [8].

(2) $\mathbb{V}$ is orthogonal of dimension 4 by Yuan, Zhang and Zhang [29].

Remark 3.8. Inspired by the work of Jacquet and Rallis [15], Zhang [31] has formulated a relative trace formula approach to attack the arithmetic Gross-Prasad conjecture for unitary group. Zhang [32] also proved the corresponding fundamental lemma when $n=3$.

4 The Rallis inner product formula

4.1 Local theory

Let F be a local field. We consider a reductive pairing (G,H) of the following two types:

(1) $G=O(V)$ for V a quadratic space over F of odd dimension $2n+1$, $H=\mathrm{Sp}(F^{2n})$ with a standard symplectic form on F^{2n}.

(2) $G=U(V)$ for V a hermitian space over a quadratic field extension of even dimension $2n$, and $H=U(E^{2n})$ with a standard skew hermitian form on E^{2n}.

We embed $G\times H$ into the metaplectic group $\mathrm{Mp}(V^{2n})$ and let R be the Weil representation of $\mathrm{Mp}(V^{2n})$ realized on the space $\mathcal{S}(V_{\mathbb{A}}^n)$ of Schwartz functions on $V_{\mathbb{A}}^n$. For any irreducible representations π of H we consider the space of linear forms

$$\mathcal{P}(\pi):=\bigoplus_\sigma\mathrm{Hom}_{H\times G}(\pi\otimes R,\sigma),$$

where the sum is over a complete set of inequivalent irreducible representations σ of G. Then the Howe duality says that this space is at most one-dimensional. In particular there is at most one irreducible representation σ (called theta lifting of π) contributes to $\mathcal{P}(\pi)$. Moreover, if π is tempered, by recent work of Li, Harris and Sun [21], this space is non-vanishing if and only if the matrix coefficients integrations is non-vanishing: for some $f \in \pi \otimes R$, $\widetilde{f} \in \widetilde{\pi} \otimes \widetilde{R}$,

$$\int_H \langle hf, \widetilde{f} \rangle dh \neq 0.$$

Following the work of Li [20], we may normalize the matrix integral so that it takes value one in the spherical case by

$$m(f, \widetilde{f}) = \frac{L(1, M)}{L(\frac{1}{2}, \pi)} \int_H \langle hf, \widetilde{f} \rangle dh,$$

where

$$L(1, M) = \prod_{i=1}^{n} L(i, \eta_{E/F}^i)$$

in unitary case, and

$$L(1, M) = \zeta(1/2) \prod_{i=1}^{n} \zeta(2i)$$

in orthogonal case. It is easy to see

$$m \in \mathrm{Hom}_{H^2 \times \Delta(G)}(\pi \otimes \widetilde{\pi} \otimes R \otimes \widetilde{R}, \mathbb{C}).$$

Using m, we can define a bilinear form

$$\langle \cdot, \cdot \rangle, \mathcal{P}(\pi) \otimes \mathcal{P}(\widetilde{\pi}) \longrightarrow \mathbb{C}$$

by

$$\langle \ell, \widetilde{\ell} \rangle = \frac{\langle \ell(f), \widetilde{\ell}(\widetilde{f}) \rangle}{m(f, \widetilde{f})}$$

for any choice of f and $\widetilde{f}$ with nonzero m-value. When π is unitary and tempered, this pairing is positive definite.

4.2 The inner product formula

For cuspidal representations π and σ of H and G respectively, we define analogously the space $\mathcal{P}(\pi)$ and linear forms as product of local data.

One can define an element $P_\pi \in \mathcal{P}(\pi)$ as follows:

$$P_\pi(f \otimes \phi)(g) = \int_{[H]} \theta_\phi(g, h) f(h) dh.$$

Here, θ_ϕ is the theta series

$$\theta_\phi(g) = \sum_{x \in V^m} R(g)\phi(x), \quad g \in \mathrm{Mp}((V \otimes W)_\mathbb{A}).$$

The Rallis inner product formula is given as follows.

Theorem 4.1. *It holds that*

$$\langle P_\pi, P_{\widetilde{\pi}} \rangle = \frac{1}{2L(1, M)} \cdot L(1/2, \pi).$$

For the references, see Rallis [24], Kudla and Rallis [18], Li [20] and Ichino [11, 12].

4.3 The arithmetic inner product formula

Assume that F is totally real, and E/F is CM. Let $\mathbb{V}$ be an incoherent orthogonal space over $\mathbb{A}_F$ of rank $2n+1$ (resp. the hermitian space over $\mathbb{A}_E$ of rank $2n$) which is positive definite at all archimedean places. Let $\mathbb{G}$ denote $U(\mathbb{V})$ as an algebraic group over $\mathbb{A}$. Then we have a projective system of Shimura varieties X_K of dimension $2n-1$ parameterized by open compact group K of $\mathbb{G}_f$ as above. For each irreducible representation σ of $\mathbb{G}$ with trivial archimedean components, define

$$\mathrm{Ch}^*(\sigma) = \mathrm{Hom}_{\mathbb{G}}(\sigma, \mathrm{Ch}^*(X)).$$

Moreover if σ has cuspidal lifting to $H(\mathbb{A})$, then $\mathrm{Ch}^*(\sigma)$ is expected to have values in homologically trivial cycles. The Beilinson-Bloch conjecture says that this is a finite dimensional space with dimension equal to $\mathrm{ord}_{s=1/2}L(s,\pi)$ where π is the theta lifting of σ on $H(\mathbb{A})$, and that there is a positive definite pairing on $\mathrm{Ch}^*(\sigma)$.

The arithmetic Rallis inner product formula of Kudla [16] refined by Liu [22] is of the following form:

Conjecture 4.2. Assume that $\mathcal{P}(\pi) \neq 0$. Then the sign of the functional equation of $L(s,\pi)$ is -1 and there is an explicit

$$Y_\pi \in \mathcal{P}(\pi) \otimes \mathrm{Ch}(\sigma)$$

such that

$$\langle Y_\pi, Y_{\widetilde{\pi}} \rangle = \frac{1}{2L(1,M)} \cdot L'(1/2,\pi).$$

Here is a sketch of construction of Y_π. For each $x \in \mathbb{V}^n$ with the positive moment matrix, then the subspace $\langle x \rangle$ generated by components of x is coherent, and its orthogonal complement is incoherent and thus defines a Shimura variety in the same way as $\mathcal{V}$ and is embedded into X. We denote this subvariety by $Z(x)$. For any $\phi \in \mathcal{S}(V_{\mathbb{A}}^n)$, we define a Kudla generating series $Z_\phi \in \mathrm{Ch}^n(X)_{\mathbb{C}}$ formally by

$$Z_\phi := \int_{\mathbb{V}} \phi(x) Z(x) dx \in \mathrm{Ch}^*(X).$$

For $g \in H(\mathbb{A})$ define

$$Z_\phi(g) = Z_{R(g)\phi} \in \mathrm{Ch}^n(X).$$

The modularity conjecture says that $Z_\phi(g)$ as a function of $g \in H(\mathbb{A})$ is left invariant under $H(F)$ and defines an element

$$Z \in \mathrm{Hom}_{\mathbb{G}\times H(\mathbb{A})}(R, C^\infty(H(F)\backslash H(\mathbb{A})) \otimes \mathrm{Ch}^n(X)_{\mathbb{C}}).$$

Let π be an irreducible cuspidal representation of $H(\mathbb{A})$, the inner product with $C^\infty(H(F)\backslash H(\mathbb{A}))$ defines

$$\begin{aligned} Y_\pi &\in \mathrm{Hom}_{\mathbb{G}\times H(\mathbb{A})}(R\otimes\pi, \mathrm{Ch}^n(X)_{\mathbb{C}}) \\ &= \mathrm{Hom}_{\mathbb{G}\times H(\mathbb{A})}(R\otimes\pi, \sigma) \otimes \mathrm{Hom}_{\mathbb{G}}(\sigma, \mathrm{Ch}^n(X)_{\mathbb{C}}) \\ &= \mathcal{P}(\pi) \otimes \mathrm{Ch}^*(\sigma). \end{aligned}$$

Remark 4.3. The conjecture holds in the case $m = 1$ by Kudla, Rapoport and Yang [19] and Liu [23].

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