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# Equidistribution of small points on abelian varieties

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## 1. Main theorem and consequences

Let $A$ be an abelian variety defined over a number field $K$. Fix an embedding $\sigma$: $K \to \mathbb{C}$ and let $\bar{\mathbb{Q}}$ be the algebraic closure of $K$ in $\mathbb{C}$. Then $A(\bar{\mathbb{Q}})$ is included in $A_\sigma(\mathbb{C})$ and we have an action on $A(\bar{\mathbb{Q}})$ by $\mathrm{Gal}(\bar{\mathbb{Q}}/K)$. For each $x \in A(\bar{\mathbb{Q}})$ let $O(x)$ denote the orbit $\mathrm{Gal}(\bar{\mathbb{Q}}/K)x$. Let $h$: $A(\bar{\mathbb{Q}}) \to \mathbb{R}$ be a Néron-Tate height with respect to a symmetric and ample line bundle on $A$.

We say a subvariety $Y$ is torsion if $Y = x + B$ is a translate of an abelian subvariety $B$ by a torsion point $x$. Also, a sequence $(x_n, n \in \mathbb{N})$ is strict if no (infinite) subsequence is contained in a proper torsion subvariety of $A$. A sequence $(x_n, n \in \mathbb{N})$ is generic if no subsequence is included in a proper subvariety of $A$. A sequence $(x_n, n \in \mathbb{N})$ is small if $h(x_n)$ approaches 0.

In this paper, using an idea of Ullmo [6], we will prove the following distribution theorem conjectured in [5] for small points on $A$ with respect to the height $h$:

THEOREM 1.1. *Let $(x_n, n \in \mathbb{N})$ be a small and strict sequence of points in $A(\bar{\mathbb{Q}})$. Then the sequence of orbits $O(x_n)$ is equidistributed with respect to the Haar measure $dx$ on $A_\sigma(\mathbb{C})$ with mass* 1.

Notice that the equidistribution of $O(x_n)$ means, for any continuous function $f$ on $A_\sigma(\mathbb{C})$, that the sequence

$$\int_{A_\sigma(\mathbb{C})} f\delta_{x_n} := \frac{1}{\#O(x_n)} \sum_{y\in O(x_n)} f(y)$$

converges to $\int_{A_\sigma(\mathbb{C})} f dx$.

*The idea of proving Bogomolov's conjecture by using arithmetic intersection theory is due to L. Szpiro. I would like to thank him for introducing me to this circle of problems. The idea of applying equidistribution theorem to a dominant but not smooth map is due to E. Ullmo, which is published in this issue. I would like to thank him for sending me his beautiful preprint. This research has been supported by NSF under the grant number DMS-9623053.

Letting $S$ be a proper closed subset of $A_\sigma(\mathbb{C})$, we can find a nontrivial function $f \geq 0$ whose support is disjoint with $S$. We say a torsion subvariety in $S$ is maximal if there is no other torsion subvariety in $S$ containing it. By the theorem we have:

COROLLARY 1. *Let $S$ be a closed subset of $A_\sigma(\mathbb{C})$. Then $S$ contains at most finitely many Galois orbits of maximal torsion subvarieties. Let $S'$ be the complement of the union of all Galois orbits of the maximal torsion subvarieties included in $S$. Then there is a positive number $\varepsilon$ such that $h(x) > \varepsilon$ for any $O(x) \subset S'$.*

Let $\mathbb{Q}^t$ be the maximal, totally real subfield of $\bar{\mathbb{Q}}$. Applying the corollary to the variety $B = \mathrm{Res}_{K/\mathbb{Q}}A$ and $S = B(\mathbb{R})$ we obtain the following corollary:

COROLLARY 2. *The following two assertions are true:*

1) *The set of torsion points in $A(K\mathbb{Q}^t)$ is finite.*

2) *There is a positive number $\varepsilon$ such that any nontorsion point $x$ in $A(K\mathbb{Q}^t)$ has height $\geq \varepsilon$.*

Notice that the assertion 1) more or less has been proved by Zarhin [7] by using Faltings' theorem on Tate's conjecture. Using our earlier paper [5], Ullmo and I were able to prove assertion 1) for abelian varieties and assertion 2) for elliptic curves.

When $S$ is a subvariety of $A$, if $x \in S(\bar{\mathbb{Q}})$ then $O(x) \subset S$. So we have the Bogomolov conjecture ([1], [9]).

COROLLARY 3 (Bogomolov's conjecture). *Let $X$ be a nontorsion subvariety of $A$. Then there is an $\varepsilon > 0$ such that the set*

$$\{x \in X(\bar{\mathbb{Q}}):\ h(x) \leq \varepsilon\}$$

*is not Zariski dense.*

One important case occurs when $X$ is a smooth curve of genus $> 1$, $A$ is the Jacobian variety of $X$ and the embedding $j_D$: $X \subset A$ is given by a divisor $D$ of degree 1: $j(x) = x - D$. Under these conditions, the conjecture is proved in [8] when $(2g-2)D - \Omega$ is not torsion in the Jacobian where $\Omega$ is a canonical divisor of $X$. Recently, E. Ullmo [6] gave a proof for the case that $(2g-2)D-\Omega$ is torsion. Previously, J.-F. Burnol [3] proved the same case as Ullmo with the additional assumption that $X$ has smooth reductions at all finite places and that $A$ has a complex multiplication.

Replacing $A$ by a subvariety, one may assume that $A$ is generated by $X - X$. Without restriction on the dimension of $X$ and the type of embeddings $X \subset A$, the generalized conjecture is proved for the case where the map $\mathrm{NS}(A)_\mathbb{Q} \to \mathrm{NS}(X)_\mathbb{Q}$ is not injective [9]. For example when $\dim X = 1$

this is equivalent to saying that the $\mathrm{End}(A)_{\mathbb{R}}$ is not a field. Notice also that the elementary method used in E. Bombieri and U. Zannier's work [2] on the Bogomolov conjecture for multiplicative groups may be used to prove the Bogomolov conjecture for CM-abelian varieties also.

Considering only the torsion points, we have Lang's conjecture as proved by Raynaud [4].

COROLLARY 4 (Lang's conjecture). *Let $X$ be a nontorsion subvariety of an abelian variety. Then the set of torsion points in $X$ is not Zariski dense in $X$.*

The proofs of the above results relies heavily on arithmetic intersection theory invented by Arakelov, and developed by Deligne, Faltings, Gillet, Soulé, Szpiro, and myself. We will first prove a generic equidistribution theorem (Theorem 1.1). This result for the case of a smooth variety was shown in our earlier paper [5]. Then for a subvariety $X$ of an abelian variety $A$ with no symmetries by translations, for $m$ sufficiently large, we will construct certain nonsmooth but birational morphisms $\alpha_m$ from $X^m$ to a subvariety of $A^{m-1}$ (Lemma 3.1). Then we will prove the Bogomolov conjecture (Corollary 3) by using an idea of Ullmo [6]: applying our generic distribution theorem to both $X^m$ and $\alpha_m(X^m)$. In Ullmo's paper, he applies the equidistribution theorem to $X^g$ and $\mathrm{Jac}(X)$. Finally we will use the Bogomolov conjecture and the generic equidistribution theorem to prove Theorem 1.1. We refer to [9] for a more general formulation of the Bogomolov conjecture and all results used in this paper.

## 2. The equidistribution theorem for generic small points

Let $X$ be a variety defined over a number field $K$ of dimension $d$. Let $\mathcal{L}$ be an ample line bundle on $X$ with a semipositive adelic metric $\|\cdot\| = \{\|\cdot\|_v\colon v \text{ places of } K\}$. Let $\sigma\colon K \to \mathbb{C}$ be an archimedean place. Assume that the height $h(X)$ of $X$ with respect to this adelic metrized line bundle is 0. Recall that for any irreducible subvariety $Z$, the height of $Z$ is defined by

$$h(Z) = \frac{\widehat{c}_1(\mathcal{L}|_Z, \|\cdot\|)^{\dim Z+1}}{(\dim Z+1)\deg\mathcal{L}}.$$

If $x$ is a point in $X(\bar{\mathbb{Q}})$ then we define $h(x)$ to be the height of the Zariski closure of $x$ in $X$. Then we have the following result proved in [9]:

THEOREM OF SUCCESSIVE MINIMA. *For $i = 1, \dots, d$, define numbers $\lambda_1 \leq \cdots \leq \lambda_d$ by*

$$\lambda_i = \sup_{Y\subset X,\, \dim Y = i}\ \inf_{p\in X(\bar{\mathbb{Q}})-Y(\bar{\mathbb{Q}})} h(p)$$

*where the $Y$ are subvarieties of $X$. Then*

$$\lambda_d \geq h(X) \geq \frac{1}{d}(\lambda_1 + \cdots + \lambda_d).$$

As in the introduction, for an abelian variety, we may define generic sequences and small sequences of points in $X(\bar{\mathbb{Q}})$ with respect to (the height of) the metrized line bundle. In this section we want to prove the following generic equidistribution theorem.

THEOREM 2.1. *Assume there is an embedding $i$: $X_\sigma(\mathbb{C}) \to Y$ from $X$ to a complex projective manifold $Y$ with an ample hermitian line bundle $(\mathcal{M}, \|\cdot\|_0)$ such that the curvature of $(\mathcal{M}, \|\cdot\|_0)$ is strictly positive and $(\mathcal{L}_\sigma, \|\cdot\|_\sigma)$ is isomorphic to the pullback of $(\mathcal{M}, \|\cdot\|_0)$ on $X$. Let $(x_n, n \in \mathbb{N})$ be a generic and small sequence of points on $X(\bar{\mathbb{Q}})$. Then the sequence of subsets $(O(x_n), n \in \mathbb{N})$ is equidistributed with respect to the measure*

$$dx: = c_1(\mathcal{L}_\sigma, \|\cdot\|_\sigma)^d / \deg(\mathcal{L}).$$

*Proof.* The proof is almost the same as in [5]. Let $f$ be a continuous function on $X_\sigma(\mathbb{C})$. We want to show that $\int_{X_\sigma(\mathbb{C})} f\delta_{x_n}$ converges to $\int_{X_\sigma(\mathbb{C})} f dx$. By the classical theorem of Stone-Weierstrass, for any $\varepsilon > 0$, there is a continuous function $g$ on $Y$ such that $|g(x) - f(x)| < \varepsilon$ for any $x \in X_\sigma(\mathbb{C})$. After approximation, we may now assume that $f$ is the restriction of a smooth function $g$ on $Y$. For any positive number $\lambda$ we let $\|\cdot\|_\lambda$ denote the norm $\|\cdot\|_0 \exp(-\lambda g)$ on $\mathcal{M}$. Assume $\lambda$ is small enough so that the curvature of $(\mathcal{M}, \|\cdot\|_\lambda)$ is positive. Let $\|\cdot\|'$ be the adelic metric on $\mathcal{L}$ with new metric $\|\cdot\|_\lambda$ at the place $\sigma$ and the same metrics at the remaining places as before. For any irreducible subvariety $Z$ of $X$, let $h'(Z)$ denote the height of $Z$ with respect to the line bundle $\mathcal{L}$ with this new adelic metric. Since the sequence $(x_n, n \in \mathbb{N})$ is generic, by the theorem of successive minima,

$$\liminf_n h'(x_n) \geq h'(X).$$

By definition, we have asymptotic expansions:

$$\begin{aligned} h'(x_n) &= h(x_n) + \lambda \int_{X_\sigma(\mathbb{C})} f\delta_{x_n}, \\ h'(X) &= h(X) + \lambda \int_{X_\sigma(\mathbb{C})} f dx + O(\lambda^2). \end{aligned}$$

As $\lim_{n\to\infty} h(x_n) = h(X) = 0$, the inequality implies

$$\liminf_n \int_{X_\sigma(\mathbb{C})} f\delta_{x_n} \geq \int_{X_\sigma(\mathbb{C})} f dx.$$

Replacing $f$ by $-f$ in this inequality, we have

$$\limsup_n \int_{X_\sigma(\mathbb{C})} f\delta_{x_n} \leq \int_{X_\sigma(\mathbb{C})} f dx.$$

It follows that the limit $\int f\delta_{x_n}$ exists and equals $\int f dx$. This finishes the proof of the theorem. □

## 3. A geometric lemma

Let $X$ be an integral subvariety of an abelian variety $A$ over an algebraically closed field. Assume that the algebraic group

$$G(X) = \{a \in A\colon\ a + X = X\}$$

is trivial. We want to prove the following lemma:

LEMMA 3.1. *For $m$ big enough, the map $\alpha_m$: $X^m \to A^{m-1}$ defined by*

$$\alpha_m(x_1,\ldots,x_m) = (x_1 - x_2, x_2 - x_3,\ldots,x_{m-1} - x_m)$$

*is a generic embedding.*

*Proof.* That $\alpha_m$ is a generic embedding means $\alpha_m$ is quasi-finite with generic degree 1. We need only to show that for $m$ big enough there is a fiber of $\alpha_m$ containing only one element. For any $(x_1,\ldots,x_m) \in X^m$, let $G(x_1,\ldots,x_m)$ denote the subvariety of $A$ of elements $a$ such that $a+x_1, a+x_2,\ldots,a+x_m \in X$. Then the fiber of $\alpha_m$ containing $(x_1,\ldots,x_m)$ is

$$\{(x_1 + a, x_2 + a,\ldots,x_m + a)\colon\ a \in G(x_1,\ldots,x_m)\}.$$

It is easy to see that the intersection of finitely many subvarieties of the form $G(x_1,\ldots,x_m)$ is still of the form $G(y_1,\ldots,y_n)$. Since the intersection of all subvarieties of the form $G(x_1,\ldots,x_m)$ is $G(X) = 0$, there is a point $(x_1,\ldots,x_{m_0}) \in X^{m_0}$ such that $G(x_1,\ldots,x_{m_0}) = 0$. Now for any $m \geq m_0$, the morphism $\alpha_m$ has one fiber containing only one element. □

## 4. Proof of Theorem 1.1

We are now ready to prove the results in the introduction. We begin with a lemma:

LEMMA 4.1. *Let $X$ be a variety defined over $\bar{\mathbb{Q}}$. Let $(x_n, n \in \mathbb{N})$ be a Zariski dense sequence in $X$. Then $(x_n, n \in \mathbb{N})$ has a generic subsequence.*

*Proof.* Since the set of proper subvarieties of $X$ is countable, we may list its subvarieties in a sequence $(Y_n, n \in \mathbb{N})$. For any $i$, let $(x_{a_n^i}, n \in \mathbb{N})$ be the complement of $\cup_{j=1}^{i} Y_j$ in $(x_n, n \in \mathbb{N})$. Then $(x_{a_1^i}, i \in \mathbb{N})$ will be a generic subsequence. □

Let us prove the generalized Bogomolov conjecture (Corollary 1.2) first.

*Proof.* Define $G = G(X)$ as in the last section; then $X' = X/G$ is a subvariety of the abelian variety $A' = A/G$. The assumption that $X$ is not torsion implies that $X'$ is of positive dimension. Let $A''$ be the connected component of $G$ containing 0. Then $A$ is isogenous to $A' \times A''$. Since the Bogomolov conjecture does not depend on the choice of symmetric ample line bundle $\mathcal{L}$ on $A$ we may assume that $\mathcal{L}$ is the pullback via some isogeny $A \to A' \times A''$ of the product of some symmetric ample line bundles $\mathcal{L}'$ and $\mathcal{L}''$ on $A'$ and $A''$ respectively. Now the Bogomolov conjecture for $X$ in $A$ is equivalent to the Bogomolov conjecture for subvariety $X'$ in $A'$. Replacing $X$ by $X'$ we may assume that $G(X) = 1$.

Now assume that $(x_n, n \in \mathbb{N})$ is a Zariski dense sequence of small points on $X$. Let $m$ be any positive integer. Fix any bijective map

$$\alpha\colon \mathbb{N} \to \mathbb{N}^m,\ \alpha(n) = [\alpha_1(n), \ldots, \alpha_m(n)].$$

Let

$$(x(n) := [x_{\alpha_1(n)}, \ldots, x_{\alpha_m(n)}],\ n \in \mathbb{N})$$

be the corresponding sequence on $X^m$. Then $(x(n), n \in \mathbb{N})$ is also Zariski dense in $X^m$. By Lemma 3.1, it has a generic subsequence $(x(n_i), i \in \mathbb{N})$.

Now assume that $m$ is big enough so that the map $\alpha_m$ is a generic embedding. Replacing $x(n_i)$ by a subsequence we may assume that $\alpha_m$ is smooth at all $x(n_i)$.

Let $\mathcal{L}$ be a symmetric and ample line bundle on $A$ and $\|\cdot\|_A$ be an admissible metric on $\mathcal{L}_A$. Let $(\mathcal{L}_X, \|\cdot\|_X)$ be the restriction of $(\mathcal{L}_A, \|\cdot\|_A)$ on $X$. Let $(\mathcal{L}_{X^m}, \|\cdot\|_{X^m})$ be the product of pullbacks of $(\mathcal{L}_X, \|\cdot\|_X)$ via the projections $p_i\colon X^m \to X$ to the single factors. Let $(\mathcal{L}_{A^{m-1}}, \|\cdot\|_{A^{m-1}})$ be the product of the pullbacks of $(\mathcal{L}_A, \|\cdot\|_A)$ via the projections $\pi_i\colon A^{m-1} \to A$ to the single factors. Since $(x_n, n \in \mathbb{N})$ is a small sequence, we have the following assertions:

1) The sequence $(x(n_i), i \in \mathbb{N})$ is a small sequence with respect to the metrized line bundle $(\mathcal{L}_{X^m}, \|\cdot\|_{X^m})$.

2) The sequence $(\alpha_m(x(n_i)), i \in \mathbb{N})$ is a small sequence with respect to the metrized bundle $(\mathcal{L}_{A^{m-1}}, \|\cdot\|_{A^{m-1}})$.

Applying the theorem of successive minima, we have the fact that the heights of $X^m$ and $\alpha_m(X^m)$ are 0 with these metrized line bundles respectively. Applying Theorem 2.1 to both $X^m$ and $\alpha_m(X^m)$, we obtain the following data:

1) The measures $\delta_{x(n_i)}$ converges to

$$dx_m := p_1^*(dx)p_2^*(dx)\cdots p_m^*(dx) \text{ where } dx = c_1(\mathcal{L}_{X,\sigma}, \|\cdot\|_{X,\sigma})^{\dim X}/\deg(\mathcal{L}_X).$$

2) The measures $\delta_{\alpha_m(x(n_i))}$ converges to the restriction on $\alpha_m(X^m)$ of

$$dx'_m = \left(\sum_{i=1}^{m-1} \pi_i^* c_1(\mathcal{L}_{A,\sigma}, \|\cdot\|_{A,\sigma})\right)^{m \dim X} / \deg(\alpha_m(X^m)).$$

It follows that $dx_m = \alpha_m^*(dx'_m)$. Let $x$ be any smooth point on $X_\sigma(\mathbb{C})$. Then the form $dx$ is nonzero at $x$ and therefore the form $dx_m$ is nonzero at $(x, x, \dots, x)$ in $X^m$. However the morphism $\alpha_m$ is singular at $(x, x, \dots, x)$ as $\alpha_m$ maps the diagonal to 0; $\alpha_m^* dx'_m$ is 0 at $(x, \dots, x)$ which is a contradiction. This finishes the proof of the generalized Bogomolov conjecture. □

Now let us prove Theorem 1.1.

*Proof.* Let $(x_n, n \in \mathbb{N})$ be a strict and small sequence on $A$. We claim that $(x_n, n \in \mathbb{N})$ is generic. Otherwise, there is a subsequence with Zariski closure a proper subvariety $X$ of $A$. By the Bogomolov conjecture, $X$ must be a torsion subvariety giving a contradiction. The claim is therefore proved.

By the theorem of successive minima, the height of $A$ is 0 with respect to any symmetric and ample line bundle on $A$. By Theorem 2.1, the measure $\delta_{x_n}$ on $A_\sigma(\mathbb{C})$ converges to the Haar measure. This finishes the proof of the theorem. □

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