Lectures on Algebraic Geometry

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Preface

This note is based on 12 lectures given by the second author at a summer school in "algebra and number theory" at the Chinese Academy of Sciences in 2019. It is conceived as the first course in abstract algebraic geometry for college student with some basic knowledge in commutative algebra and homological algebra. The main part of the book covers the basics of schemes, sheaves and their cohomology, and background materials for Borel–Serre's paper on the Grothendieck–Riemann–Roch theorem.

In Chapter 1 and 2, we will review the basic concepts in commutative algebra about rings and modules, then introduce the language of ringed spaces and sheaves. After these preparation, we will introduce in Chapter 3 the *affine schemes* which are the building blocks of the general *scheme* introduced in Chapter 4. Then in Chapter 5 and 6 we introduced the most useful objects in algebraic geometry, the *projective schemes* and most useful tool in algebraic geometry, *cohomology*. In Chapter 7, we specialize our study to the most classical object in algebraic geometry: *curves and Riemann–Roch*. In Chapter 8, we define *Chow groups* and *Chern class*, followed by the *Grothendieck-Riemann-Roch theorem*, which is our focal point of this book.

Because of our limitation of times and pages, we will not spend much time to review the classical algebraic geometry of varieties over complex numbers or general algebraic closed fields. We simply used them as examples of our more general spaces.

This book can be used as a textbook for a quick start tutorial in algebraic geometry, especially for understanding Grothendieck–Riemann–Roch theorem. The exercises are interspersed with the exposition in this book. Some of them are trivial, and some of them are tough as readers needs to study more background materials in appendix or elsewhere.

1 Rings and Modules

1.1 Rings

Let R be a ring. We have an object $(R, +, \cdot)$. This object contains a set R with 2 operations $+, \cdot$. The operations are maps from $R \times R \to R$. The pair $(R, +)$ is a commutative group, and the pair $(R, \cdot)$ is a semi-group or monoid. In the ring we require the 0 element in the abelian group satisfies the property $0x = x0 = 0$ for all $x \in R$. We also require an identity element 1, with the property that $1x = x1 = x$ for all x in R . The ring R also satisfies a distribution law with respect to the operations of addition and subtraction.

Examples of rings:

1. $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ (note $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ are fields);

2. $\mathbb{Z}[x]$, $\mathbb{Q}[x]$, $\mathbb{R}[x]$, $\mathbb{C}[x]$

Now we come to the category of commutative ring. The objects are commutative rings, and the arrows are homomorphisms of rings.

Definition 1.1.1. *If R is a commutative ring, then $R[x]$ is the ring of polynomials over R:*

$$R[x] = \left\{ \sum_{i=1}^{n} a_i x^i \mid a_i \in R, \text{ with operations } + \text{ and } \cdot \right\}$$

Definition 1.1.2. *Let R_1, R_2 be two rings. A homomorphism from R_1 to R_2 is a map*

$$\phi : R_1 \to R_2$$

which preserves the operations:

$$\phi(x + y) = \phi(x) + \phi(y), \qquad \phi(xy) = \phi(x)\phi(y)$$

Definition 1.1.3. *The kernel of a homomorphism $\phi : R_1 \to R_2$ is defined to be the preimage of 0 and it is denoted by $\operatorname{Ker}(\phi)$. It is and ideal of R_1*

$$\operatorname{Ker}(\phi) = \phi^{-1}(0) = \{ x \in M_1 \mid \phi(x) = 0 \}.$$

And if $\operatorname{Ker}(\phi) = 0$, then ϕ is injective

Definition 1.1.4. *Let $I \hookrightarrow R$ be a subset. We say I is an ideal if I satisfies two properties:*

1. *$x \in I, y \in I$ implies $x + y \in I$;*

2. *$x \in I$, $y \in R$ implies $xy \in I$.*

Example 1.1.5. The kernel of a homomorphism $\phi : R_1 \to R_2$ is an ideal in R_1.

Example 1.1.6. Given an ideal I in a ring R, the quotient R/I is a ring.

1.2 Modules

Definition 1.2.1. *Let R be a ring, an R-module is an abelian group endowed with a homomorphism $R \to \mathrm{End}(M)$, i.e. A map*

$$\begin{aligned} R \times M &\longrightarrow M \\ (a, m) &\longmapsto am \end{aligned}$$

where $a, b \in R$ saitisfying

$$(ab)m = a(bm), \quad a(m_1 + m_2) = am_1 + am_2,$$

$$(a+b)m = am + bm, \quad 1m = m.$$

Definition 1.2.2. *Let M_1, M_2 be two R-modules. A homomorphism from M_1 to M_2 is a map*

$$\phi : M_1 \to M_2$$

which preserves the operations:

$$\phi(m_1 + m_2) = \phi(m_1) + \phi(m_2), \qquad \phi(am) = a\phi(m),$$

where $a \in R; m, m_1, m_2 \in M_1$ Denote the set of R-homomorphisms from M to N by $\mathrm{Hom}_R(M, N)$. Then this set has a natural R-module structure: let $\varphi, \psi \in \mathrm{Hom}_R(M, N)$, and let $x \in M$

$$(\varphi + \psi)(x) = \varphi(x) + \psi(x); \tag{1.2.1}$$

$$a\varphi(x) = \varphi(ax), \qquad a \in R. \tag{1.2.2}$$

Similar to a subspace and a quotient space of a vector space, we can define a submodule and a quotient module of an R-module.

Definition 1.2.3. *Let $\phi : M_1 \longrightarrow M_2$ be a homomorphism.*

1. *The kernel of ϕ is defined to be the preimage of 0 and it is denoted by $\mathrm{Ker}(\phi)$. It is R-module*
$$\mathrm{Ker}(\phi) = \phi^{-1}(0) = \{\, x \in M_1 \mid \phi(x) = 0 \,\}.$$
And if $\mathrm{Ker}(\phi) = 0$, then ϕ is injective

2. *The image of ϕ is defined to be the set $\{\phi(m) | m \in M_1\} \subset M_2$, and it is denoted by $Im(\phi)$. It is a submodule of M_2. If $\mathrm{Im}(\phi) = M_2$, then we call ϕ is surjective.*

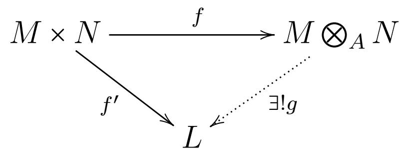
3. *The cockernel of ϕ is defined to be the quotient module $M_2/Im(\phi)$, and it is denoted by $\mathrm{coker}\phi$. It is also an R-modules*

Thus, all R-modules together form an abelian category $\mathcal{M}od_R$ with morphisms given by homomorphisms.

Example 1.2.4. Consider the commutative ring $\mathbb{Z}$. Then the category of $\mathbb{Z}$-modules is exactly the category $\mathcal{A}b$ of abelian groups.

Definition 1.2.5 (Tensor Products)**.** *Let A be a ring and let M, N be A-modules. Define $\otimes_A$ to be the functor from $\mathcal{M}od_A \times \mathcal{M}od_A$ to $\mathcal{M}od_A$ with $(M, N) \mapsto M \otimes_A N$.*

The tensor product of M and N, denoted by $M \otimes_A N$ is defined to be the A-module along with an A-bilinear map $f : M \times N \to M \otimes_A N$, with universal property, i.e given any A-bilinear map $f' : M \times N \to L$, there is a unique A-linear map $g : M \otimes_A N \to L$ such that $f' = g \circ f$.



Proposition 1.2.6. *The tensor product $M \otimes_A N$ is actually the free A-module generated by $M \times N$, quotiented by the submodule generated by*

$$(m_1 + m_2, n) - (m_1, n) - (m_2, n), \tag{1.2.3}$$
$$(m, n_1 + n_2) - (m, n_1) - (m, n_2), \tag{1.2.4}$$
$$a(m, n) - (m, an), a(m, n) - (am, n). \tag{1.2.5}$$

where $a \in A$, $m, m_1, m_2 \in M, n, n_1, n_2 \in N$.
Equivalently, the elements of the tensor product $M \otimes_A N$ are the finite A-linear combinations of symbols $m \otimes n (m \in M, n \in N)$, subject to relations

$$(m_1 + m_2) \bigotimes n = m_1 \bigotimes n + m_2 \bigotimes n \tag{1.2.6}$$
$$m \bigotimes (n_1 + n_2) = m \bigotimes n_1 + m \bigotimes n_2 \tag{1.2.7}$$
$$a(m \bigotimes n) = (am) \bigotimes n = m \bigotimes (an). \tag{1.2.8}$$

where $a \in A$, $m_1, m_2 \in M, n_1, n_2 \in N$. The image of (m, n) in this quotient is $m \otimes n$.

The following exercise showing that tensor functor can be recovered from Hom functor as a left adjoint:

Exercise 1.2.7. Let R be a commutative ring and L, M, N three R-modules. Show that there is a canonical isomorphism of R-modules

$$\mathrm{Hom}_R(L, \mathrm{Hom}_R(M, N)) \xrightarrow{\sim} \mathrm{Hom}_R(L \otimes_R M, N).$$

Example 1.2.8. $\mathbb{Q}/\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z} = 0$.

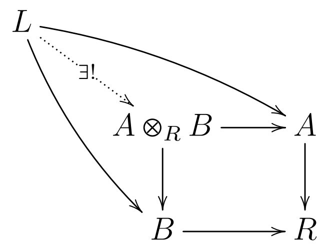
Definition 1.2.9 (Tensor product of algebras)**.** *Let R be a commutative ring. If A and B are R-algebras, the tensor product $A \otimes_R B$ is an R-algebra by putting*

$$(a_1 \otimes b_1) \cdot (a_2 \otimes b_2) = (a_1 \cdot a_2) \otimes (b_1 \cdot b_2).$$

where $a_1, a_2 \in A$, $b_1, b_2 \in B$. For example, if $f(x) \in R[x]$ be a polynomial, then

$$R[x]/f(x) \otimes_R B \xrightarrow{\sim} B[x]/f(x).$$

Let L be any R-algebra. Note that $A \otimes_R B$ also have the universal property:



Exercise 1.2.10. Let R be a ring and let M, N be two R-modules. If M and N are finitely generated then $\mathrm{Hom}_R(M, N)$ is finitely generated.

Restriction of scalar and base change

Let $\varphi : A \to B$ be a ring homomorphism. Then B is an A-algebra and also an A-module. Let N be a B-module. Then N can be equiped with an A-module structure via φ, denoted by N_A.

Conversely, given an A-module, how to make a B-module? There are two possible ways:

1. "base change" $M_B := B \otimes_A M$, where the tensor product $B \otimes_A M$ is defined by viewing B as an A-module. The multiplication on B defines a B-module structure.

2. "induced module": $M^B := \mathrm{Hom}_A(B, M)$, with $b\phi(x) = \phi(xb)$ where $x, b \in B$, $\phi \in M^B$.

The functor $\cdot \otimes_A B$ and $\mathrm{Hom}_A(B, \cdot)$ are functors from $\mathcal{M}od_A$ to $\mathcal{M}od_B$.

tensor-hom *Exercise* 1.2.11. Prove that there are natural bijection of A-module M and B-module N:

$$\mathrm{Hom}_A(M, N_A) \cong \mathrm{Hom}_B(M \otimes_A B, N),$$
$$\mathrm{Hom}_A(N_A, M) \cong \mathrm{Hom}_B(N, \mathrm{Hom}_A(B, M)),$$

Consequently, $M \otimes_A B$ and $\mathrm{Hom}_A(B, M)$ are uniquely determined by these property.

- $M \mapsto \mathrm{Hom}_A(B, M)$ is the right-adjoint of $N \mapsto N_A$;

- $M \mapsto M \otimes_A B$ is the left-adjoint of $N \mapsto N_A$.

Example 1.2.12 (Analogue of vector spaces). Let $\mathcal{C}_1$ be the category of $\mathbb{R}$-vector spaces and $\mathcal{C}_2$ be the category of $\mathbb{C}$-vector spaces. If $V \in \mathcal{C}_1$, then $V \otimes_{\mathbb{R}} \mathbb{C} = V \oplus iV$ is a vector space over $\mathbb{C}$. Also, $Hom_{\mathbb{R}}(\mathbb{C}, V) = V \oplus iV$ is the same vector space over $\mathbb{C}$

Remark 1.2.13. If B/A is free of finite rank, then $B \otimes_A M = \mathrm{Hom}_A(B, M)$. In general their dimention are different.

Example 1.2.14. Let $A = \mathbb{R}$, $B = \mathbb{R}[x]$, Then as A modules,

$$A \otimes_A B \simeq \bigoplus_{i=1}^{\infty} A, \qquad \mathrm{Hom}_A(B, A) \simeq \prod_{i=1}^{\infty} A.$$

1.3 Short exact sequence

Let R be a commutative ring.

Definition 1.3.1. *Let* $V_1 \xrightarrow{\alpha} V_2 \xrightarrow{\beta} V_3$ *be a sequence of homomorphisms of* R*-modules. We say that this sequence is* exact at V_2 *if* $\operatorname{Im}(\alpha) = \ker(\beta)$*.*

A short exact sequence *of* R*-modules is a sequence of* R*-modules of following form*

$$0 \to V_1 \xrightarrow{\alpha} V_2 \xrightarrow{\beta} V_3 \to 0$$

such that it's exact at V_1*,* V_2 *and* V_3*, i.e* α *is injective,* β *is surjective, and* $\operatorname{Im}(\alpha) = \ker(\beta)$*.*

Example 1.3.2 (Split sequence). Let V_1, V_2 be R-modules, then there is a short exact sequence:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & V_1 & \longrightarrow & V_1 \oplus V_2 & \longrightarrow & V_2 & \longrightarrow & 0 \\ & & v_1 & \longmapsto & (v_1, 0) & & & & \\ & & & & (v_1, v_2) & \longmapsto & v_2 & & \end{array} \tag{1.3.1}$$

Such a sequence is called split sequence.

In the category of R-modules, denote the isomorphism object class of R-module V as $[V]$. In some sense, short exact sequence gives a relation $[V_1] \oplus [V_3] = [V_2]$. If R is a field, then the isomophism class is uniquely determined by the dimension of the vector spaces. As a consequence, the short exact sequences is always split. But this is not true if R is a general ring.

Definition 1.3.3. *An additive functor* $F : \mathcal{M}od_A \longrightarrow \mathcal{A}b$ *is called exact if it preserves the short exact sequence. i.e. For any short exact sequence of* A*-modules*

$$0 \to M_1 \xrightarrow{\alpha} M_2 \xrightarrow{\beta} M_3 \to 0, \tag{1.3.2}$$

the sequence

$$0 \to F(M_1) \xrightarrow{\alpha} F(M_2) \xrightarrow{\beta} F(M_3) \to 0 \tag{1.3.3}$$

is also exact;

A functor is called left exact if

$$0 \to F(M_1) \xrightarrow{\alpha} F(M_2) \xrightarrow{\beta} F(M_3)$$

is exact. A functor is called right exact if

$$F(M_1) \xrightarrow{\alpha} F(M_2) \xrightarrow{\beta} F(M_3) \to 0$$

is exact.

Proposition 1.3.4. *The functor* $\operatorname{Hom}_A(M, \cdot)$ *is left exact.*

Exercise 1.3.5. Prove the above proposition.

Proposition 1.3.6. *Let* A *be a ring. A sequence of* A*-modules* $M_1 \xrightarrow{\phi} M_2 \xrightarrow{\psi} M_3 \to 0$ *is exact if*

$$0 \to Hom_A(M_3, P) \xrightarrow{\psi^*} Hom_A(M_2, P) \xrightarrow{\phi^*} Hom_A(M_1, P).$$

is exact for all A*-modules.*

Questions on exactness.

Let A be a ring and let N, M_1, M_2, M_3 be A-modules. Given short exact sequence

$$0 \to M_1 \xrightarrow{\alpha} M_2 \xrightarrow{\beta} M_3 \to 0 \tag{1.3.4}$$

we get a sequence

$$0 \to M_1 \otimes N \to M_2 \otimes N \to M_3 \otimes N \to 0. \tag{1.3.5}$$

In general, it is not exact. For example, consider the sequence $0 \to \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0$, the functor $\otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}$ is not exact. Since the morphism $\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \otimes \mathbb{Z}/2\mathbb{Z}$ is clearly zero morphism
But one can show that

Proposition 1.3.7. *For any short exact sequence of A-modules*

$$0 \to M_1 \xrightarrow{\alpha} M_2 \xrightarrow{\beta} M_3 \to 0,$$

the sequence

$$M_1 \otimes N \xrightarrow{\alpha \otimes \mathrm{Id}} M_2 \otimes N \xrightarrow{\beta \otimes \mathrm{Id}} M_3 \otimes N \to 0$$

is exact.

Proof. By applying $Hom_A(\cdot, P)$ to 1 1.3.4 we get the left exact sequence

$$0 \to \mathrm{Hom}_A(M_3, P) \to \mathrm{Hom}_A(M_2, P) \to \mathrm{Hom}_A(M_1, P), \tag{1.3.6}$$

and by applying $\mathrm{Hom}_A(N, \cdot)$ we get the left exact sequence

$$0 \to \mathrm{Hom}_A(N, \mathrm{Hom}_A(M_3, P)) \to \mathrm{Hom}_A(N, \mathrm{Hom}_A(M_2, P)) \to \mathrm{Hom}_A(N, \mathrm{Hom}_A(M_1, P))$$

then by applying the natural isomorphism in Exercise tensor-hom 1.2.11

$$\mathrm{Hom}_A(M \otimes N, P) \cong \mathrm{Hom}_A(M, \mathrm{Hom}_A(N, P)),$$

we get the left exact sequence

$$0 \to \mathrm{Hom}_A(M_3 \otimes N, P) \to \mathrm{Hom}_A(M_2 \otimes N, P) \to \mathrm{Hom}_A(M_1 \otimes N, P). \tag{1.3.7}$$

Then we use the fact about the above lemma, then

$$M_1 \otimes N \to M_2 \otimes N \to M_3 \otimes N \to 0 \tag{1.3.8}$$

is exact ∎

Definition 1.3.8. *Let N be an R-module.*

1. *N is called a* projective *R-module if the functor* $\mathrm{Hom}_R(N,-)$ *is exact.*

2. *N is called a* injective *R-module if the functor* $\mathrm{Hom}_R(-,N)$ *is exact.*

3. *N is called a* flat *R-module if the functor* $-\otimes_R N$ *is exact.*

Theorem 1.3.9. *Let N be an R-module. Then N projective if and only it's a direct summand of a free R-module, i.e. there exists an R-module N' and a free R-module F such that $N \oplus N' \cong F$.*

Sketch of proof. "$\Leftarrow$": it's easy to see that a free R-module is projective. From this it's easy to see that a direct summand of a free R-module is also projective. □

Exercise 1.3.10. Prove the "$\Rightarrow$" part.

(Hint: for any R-module N, we can always find a free R-module F with a surjective R-module homomorphism $F \twoheadrightarrow N$.)

Exercise 1.3.11. The N is an injective R-module if and only if for any ideal I of R, the natural map $N = \mathrm{Hom}_R(R,N) \to \mathrm{Hom}_A(I,N)$ is surjective. (This is called Baer's criterion.)

Exercise 1.3.12. The N is a flat R-module if and only if for any ideal I of R, the natural map $I \otimes_R N \to N$ is injective (equivalently, $I \otimes_A N = I \cdot N \subset N$).

Exercise 1.3.13. Classify these modules when R is a PID.

1.4 Noetherian condition

Definition 1.4.1. *Let R be a ring, and let M be an R-module. We say M is noetherian, if for any increasing chain of submodules $0 \hookrightarrow M_1 \hookrightarrow M_2 \hookrightarrow \ldots$, there is a maximal M_0 such that $M_n = M_0$ for $n \gg 0$. We say R is noetherian, if R is noetherian as R-module(submodule becomes ideals of R).*

Example 1.4.2. The polynomial ring $\mathbb{R}[x]$ is noetherian. Since the ideal in $\mathbb{R}[x]$ is of the form $(f(x))$, where $(f(x))$ is the polynomial of finite degree in $\mathbb{R}[x]$. Then for any increasing chain $(f_1(x)) \hookrightarrow (f_2(x)) \hookrightarrow (f_3(x) \hookrightarrow \ldots)$, we have $\ldots|f_3|f_2|f_1$. Thus the chain is stable.

Theorem 1.4.3 (Hilbert base theorem)**.** *If a ring R is noetherian, then $R[x]$ is also noetherian.*

Proof. Let I be an ideal of $R[x]$. We want to prove that I is finitely generated. Let $J \hookrightarrow R$ be an ideal of leading coefficients of polynomials in I. Since R is Noetherian, J is finitely generated. Let $f_1, f_2, \ldots, f_n \in I$ be elements whose leading coefficients generate J. We write $f_i = c_i x^{n_i} + \ldots$ with n_i the degree of f_i and a_i the leading coefficient of f_i.

Let $f \in I$. We can write $f = a_n x^n + \ldots + a_0$ with a_n non-zero. As $a_n \in J$, we can write $a_n = \sum b_i \cdot c_i$. If $n \geqq n_i$ for every i we can write $g = f - \sum b_i f_i x^{n-n_i}$ where the n-th coefficient of g is zero. By induction on n, we have proved that every polynomial $f \in I$ can be written as $f = \sum_{i=1}^{n} g_i f_i + f'$ where $\deg(f') \leqq \max(n_i)$.

Let $m = \max(n_i)$ and

$$I' = \{f \in I \text{ where } \deg(f) \leqq m\} \tag{1.4.1}$$

Then I' is not an $R[x]$-module but it is a R -submodule of $\sum_{i=1}^{n} R_i x^i \cong R^m$. Since R is Noetherian, I' is finitely generated. There are elements $f_1', f_2', \ldots, f_\ell'$ in I' generating I' as an R -module. Thus $f_1, f_2, \ldots, f_l, f_1', f_2', \ldots, f_\ell'$ generating I. □

Theorem 1.4.4. *Let R be a ring, then the following conditions are equivalent.*

1. *R is a noetherian ring;*
2. *Any ideal of R is finitely generated as R-module;*
3. *For any finitely generated R-module M, every submodule N of M is finitely generated.*

Proof. (1) $\Longrightarrow$ (2): Let I be an ideal of R, and let Σ be the set of all finitely generated ideals contained in I. Then Σ is not empty(since $0 \in \Sigma$) and thus by (1) and Zorn's lemma it has a maximal element, say I_0. If $I_0 \neq I$, consider the ideal $I_0 + Rx$ where $x \in I - I_0$; this is finitely generated and strictly contains I_0, contradiction! Hence $I_0 = I$ and therefore I is finitely generated.

(2) $\Longrightarrow$ (1): let $I_1 \subset I_2 \subset \ldots$ be an ascending chain of ideals of R. Then $I = \bigcup_{n=1}^{\infty} I_n$ is an ideal of R, hence is finitely generated. Suppose the generators are $x_1, ..., x_r$. If $x_i \in I_{n_i}$, let n be the maximal number among n_i. Then each each $x_i \in I_n$. Thus $I = I_n$ therefore the chain is stable.

(1) $\Longrightarrow$ (3): Write M as the quotient of R^n for some n, then submodules of M are quotients of submodules of R^n. Thus it suffices to prove the assertion for $M = R^n$. Take an exact sequence $0 \longrightarrow R^{n-1} \longrightarrow R^n \longrightarrow R \longrightarrow 0$ by projection onto the first fact. Then for any submodule $N \in R^n$, we have an exact sequence $0 \longrightarrow N_1 \longrightarrow N \longrightarrow N_2 \longrightarrow 0$ with $N_1 \subset R$ and $N_2 \subset R^{n-1}$. By induction, N_1 and N_2 are finitely generated. Now it is easy to see that N is finitely generated.

(3) $\Longrightarrow$ (2): It is clear when replacing M by R. □

Exercise 1.4.5. Let k be a field. Prove that the rational function field $k(x) = \left\{ \frac{f(x)}{g(x)} \middle| \, f, g \in k[x] \right\}$ is *not* a finitely generated k-algebra.

Exercise 1.4.6. Let R be a Noetherian ring, M be a finitely generated R-module. Let $\varphi : M \to M$ be a surjective R-module homomorphism. Prove that φ is an isomorphism.

iski lemma

Lemma 1.4.7 (Zariski's lemma)**.** *Let $A \hookrightarrow B \hookrightarrow C$ be injective ring homomorphisms of Noetherian rings. Assume that*

1. *C is finitely generated A-algebra,*
2. *C is finite B-algebra (i.e. C is finitely generated as a B-module).*

Then B is finitely generated A-algebra.

Proof. Strategy: let us construct $B' \subset B$ such that

1) B'/A is finitely generated as an A -algebra.

2) C/B' is finitely generated as a B' -module.

This will imply

3) B' is Noetherian.

4) B/B' is finitely generated as a B' -module (consider B as a B' submodule of C).

Then 1) and 4) imply B/A is finitely generated as an A -algebra. Write

$$C = A[x_1, \dots, x_n], \qquad C = \sum_{j=1}^{m} By_i$$

with x_i, y_j in C. Let b_{ijk} 's be elements of B such that

$$x_i y_j = \sum_{k=0}^{m} b_{ijk} y_k \tag{1.4.2}$$

where $y_0 = 1$.

Claim: Let $B' = A[b_{ijk}]$. Then B' satisfies 1) and 2).

Obviously, B' satisfies 1). For 2) we need only show that C is generated by y_j 's as a B'-module. Let C' denote this B' -module: $C' = \sum_{k=0}^{m} B' y_k$. Then

$$x_i C' = \sum_{k=0}^{m} B' x_i y_k \subset C' \tag{1.4.3}$$

In other words, C' is closed under multiplication by x_i 's. As C is generated by x_i 's, C' is closed under multiplication by C. Since C contains $y_0 = 1$, it follows that $C \subset C'$. This concludes the proof of Lemma.

□

1.5 The Grothendieck group over rings

Let $\mathcal{C}$ be an additive category. If X is an object in $\mathcal{C}$, then denote the isomorphic class of X by $[X]$. Let $F(\mathcal{C})$ be the free abelian group on the group of isomorphism classes, i.e. an element in $F(\mathcal{C})$ is a finite formal sum of the isomorphism classes:

$$\sum n_X [X] \tag{1.5.1}$$

where n_X is an integer and is zero for almost all $[X]$.

Definition 1.5.1. *Define $E(\mathcal{C})$ to be the subgroup generated by all expressions*

$$[A_2] - [A_1] - [A_3] \tag{1.5.2}$$

whenever there is an exact sequence

$$0 \to A_1 \to A_2 \to A_3 \to 0 \tag{1.5.3}$$

in $\mathcal{C}$.

Remark 1.5.2. If $A = A_1 \oplus A_2$, it induces an exact sequence

$$0 \to A_1 \to A \to A_2 \to 0. \tag{1.5.4}$$

Thus we have $[A] = [A_1] + [A_2]$.

Then the Grothendieck group of $\mathcal{C}$ is defined by the quotient $F(\mathcal{C})/E(\mathcal{C})$.

Let R be a noetherian ring. Now Let $\mathcal{C}_1$ be the category of finitely generated R-modules and $\mathcal{C}_2$ be the category of projective R-modules of finite type.

Definition 1.5.3. *Define*

$$K_0(R) := F(\mathcal{C}_1)/E(\mathcal{C}_1), \qquad K^0(R) := F(\mathcal{C}_2)/E(\mathcal{C}_2) \tag{1.5.5}$$

Remark 1.5.4. $(K^0(R), +, \cdot)$ is a ring with multiplication given by $[E] \cdot [F] = [E \otimes F]$ and $(K_0(R), +)$ has $K^0(R)$-module structure given by tensor product.

Question. How to describe $K(R)$ and $K'(R)$?

Example 1.5.5. if $R = k$ is a field, then $K(R) = K'(R) \cong \mathbb{Z}$ given by $[V] \mapsto \dim_k V$.

Example 1.5.6. If R is PID, then we can describe the structure of finitely generated R-module. Let M be a finitely generated R-module, then $M \cong R^r \oplus \oplus_{i=1}^s R/a_i$ for some integer r and some non-zero elements $a_i \in R$, we define the rank of M to be r, denoted by $\mathrm{rank}_R M$. Thus in $K'(R)$, we have

$$[M] = r[R] + \sum_{i=1}^{t} [R/a_i R] = 0$$

where the second equality follows from the following exact sequence:

$$0 \longrightarrow R \xrightarrow{\cdot a_i} R \longrightarrow R/a_i \longrightarrow 0.$$

Thus we have shown that $K^0(R) = K_0(R) \simeq \mathbb{Z}$ by map $[M] \mapsto \mathrm{rank} M$.

The next example shows that the Grothendieck group of category of all R-modules is trivial.

Example 1.5.7. Let $\mathcal{C}$ be the category of R-modules; it is a very large category; in this category, $M \oplus_{n\in\mathbb{Z}} M \cong \oplus_{n\in\mathbb{Z}} M$, thus $[M] = [\oplus_{n\in\mathbb{Z}} M] - [\oplus_{n\in\mathbb{Z}} M] = 0$ in $K'(\mathcal{C})$.

In general, the map $K^0(R) \longrightarrow K_0(R)$ is not isomorphism. In fact, we will prove this is an isomorphism when R is Noetherian and regular. Here is one example that this map is not an isomorphism when R is not regular.

Example 1.5.8. Let k be a field, $A = k[x,y]/(xy)$ and $m = (x,y) \subset A$. Then consider the infinite resolution of free A-modules for $k = A/m$

$$\ldots \xrightarrow{g} A^2 \xrightarrow{h} A^2 \xrightarrow{g} A^2 \xrightarrow{h} A^2 \xrightarrow{g} A^2 \xrightarrow{f} A \to k \to 0 \tag{1.5.6}$$

with

$$f : (s,t) \mapsto sx + ty, \quad g : (s,t) \mapsto (sy, tx) \text{ and } h : (s,t) \mapsto (sx, ty). \tag{1.5.7}$$

This fact can be used to show that the element $[k] \in K_0(A)$ is not in the image of $K^0(A)$.

Exercise 1.5.9. If $A \to B$ is a ring homomorphism between Noetherian rings, prove that there is a well-defined natural map $K(A) \to K(B)$ given by $[M] \mapsto [M \otimes_A B]$. If moreover B is a finite A-algebra, prove that there is a well-defined natural map $K'(B) \to K'(A)$ given by $[N] \mapsto [N_A]$.

2 Sheaves and ringed spaces

2.1 Sheaves

Definition 2.1.1 (Presheaves). *Let X be a topological space. A preshef $\mathcal{F}$ of abelian groups is a pair of assignments $U \mapsto \mathcal{F}(U)$ to each open subset an abelian groups, and $\gamma_{UV} \in \operatorname{Hom}(\mathcal{F}(U), \mathcal{F}(V))$ to each inclusion $V \subset U$ of open sets an homomorphism such that following conditions are satisfied:*

1. *$\mathcal{F}(\varnothing) = 0$, where $\varnothing$ is the empty set,*
2. *γ_{UU} is the identity map $\mathcal{F}(U) \to \mathcal{F}(U)$, and*
3. *if $W \subseteq V \subseteq U$ are three open subsets, then $\gamma_{UW} = \gamma_{VW} \circ \gamma_{UV}$.*

We will also use notation $\Gamma(U, \mathcal{F})$ for $\mathcal{F}(U)$.

In other words, a presheaf on X is a contravariant functor from the category of open sets whose morphisms are embedings to the category $\mathcal{A}b$ of abelian groups. The collection of presheaves on X for a category $\operatorname{PreSh}(X)$ whose morphism are natural transform of functors. Many notions about abelian groups can be extended to presheaves. So we can define the notion of kernel, image, cokernel for morphisms of presheaves. In particular, the category of presheaves is an abelian categories. We can also define the notion of sheaves of rings, and modules, the tensor product, etc.

For each $x \in X$, we can define a functor $\operatorname{PreSh}(X) \longrightarrow \mathcal{A}b$ by sending a presheaf $\mathcal{F}$ on X to

$$\mathcal{F}_x := \varinjlim_{x \in U} \mathcal{F}(U).$$

Thus for each open subset U of X, we have a map

$$\mathcal{F}(U) \longrightarrow \prod_{x \in U} \mathcal{F}_x : \qquad s \mapsto (s_x)_{x \in U}.$$

Definition 2.1.2 (Sheaves). *Let X be a topological space. A sheaf $\mathcal{F}$ of abelian groups is a presheaf which satisfies following glueing conditions: the sequence*

$$0 \to \mathcal{F}(U) \xrightarrow{d_0} \prod_i \mathcal{F}(U_i) \xrightarrow{d_1} \prod_{i,j} \mathcal{F}(U_{ij}) \tag{2.1.1}$$

defined by $d_0 : s \mapsto (s|_{U_i})_i$ and $d_1 : (s_i)_i \mapsto \left(s_i|_{U_{ij}} - s_j|_{U_{ij}} \right)_{i,j}$ is exact.

Definition 2.1.3 (Morphisms of Sheaves). *Let $\mathcal{F}$, $\mathcal{G}$ be presheaves on topological space X, a morphism $\varphi : \mathcal{F} \to \mathcal{G}$ is defined to be the natural transformation.*

Proposition 2.1.4. *Let X be a topological space, let $\mathcal{U} = \{U_i\}$ be an open cover of X. And suppose we are given for each i a sheaf $\mathcal{F}_i$ on U_i, and for each i, j an isomorphism*

$$\varphi_{ij} : \mathcal{F}_i\big|_{U_i \cap U_j} \to \mathcal{F}_j\big|_{U_i \cap U_j} \tag{2.1.2}$$

such that

(1) for each $i, \varphi_u = \mathrm{id}$;

(2) for each $i, j, k, \varphi_{ik} = \varphi_{jk} \circ \varphi_{ij}$ on $U_i \cap U_j \cap U_k$.

Then there exists a unique sheaf $\mathcal{F}$ on X, together with the isomorphisms $\psi_i : \mathcal{F}|_{U_i} \Rightarrow \mathcal{F}_i$ such that for each i, j, $\psi_j = \varphi_{ij} \circ \psi_i$ on $U_i \cap U_j$. We say loosely that $\mathcal{F}$ is obtained by glueing the sheaves $\mathcal{F}$ via the isomorphisms φ_{ij}.

All sheaves on X and their morphisms form an additive category $\mathrm{Sh}(X)$. tIn the next section, we want to show that this is actually an abelian category.

Proposition 2.1.5. *Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves on a topological space X. Then φ is an isomorphism if and only if the induced map on stalk $\varphi_p : \mathcal{F}_p \to \mathcal{G}_p$ is an isomorphism for every $P \in X$.*

Remark 2.1.6. The condition 'Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves on a topological space X' in the above proposition is very important, and the map on stalks must be induced from φ. For example, you can find two sheaves such that there exists an isomorphism between their stalks, but there is no isomorphism between themselves. Let $\mathcal{F} = (\bigoplus_p \mathbb{Z}/p\mathbb{Z}) \oplus \mathbb{Q}$ and $\mathcal{G} = \mathbb{Z}$, then $\mathcal{F}_p \xrightarrow{\sim} \mathcal{G}_p$ on every prime p, but $\mathcal{F}$ and $\mathcal{G}$ are not isomorphic.

Remark 2.1.7. The surjectivity of $\varphi : \mathcal{F} \to \mathcal{G}$ doesn't imply the surjectivity of $\varphi : \mathcal{F}(U) \to \mathcal{G}(U)$. We will give an example to explain this and give a criterian of surjectivity of sheaves.

Example 2.1.8. Let $X = \mathbb{C}^\times$ and let $\mathcal{F}$ be the functor from the category of the open subsets in X to the category of holomorphic functions. i.e for any open subset $U \subset X$, $\mathcal{F}(U)$ is the set of holomorphic functions on U. Suppose $\mathcal{G}$ is another sheaf such that $\mathcal{G}(U)$ is the set of invertible holomorphic functions on U. Then the exponential map $exp : \mathcal{F} \to \mathcal{G}$ is surjective. But $\mathcal{F}(X) \to \mathcal{G}(X)$ is not surjective.

for sheaf

Lemma 2.1.9. *Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves on X. Then φ is surjective if and only if the following condition holds: for every open subset $U \subset X$, and for every $s \in \mathcal{G}(U)$, there is a covering $\{U_i\}$ of U, and there are elements $t_i \in \mathcal{F}(U_i)$ such that $\varphi(t_i) = s|_{U_i}$ for all i.*

Proof. exercise ∎

2.2 Sheafication

Given a topological space X, we have a natural embedding functor $\text{Sh}(X) \longrightarrow \text{PreSh}(X)$ from the category of sheaves on X to the category of presheaves. This functor has a left adjoint called a sheafication defined as follows. For any presheaf $\mathcal{F}$ on X, its sheafication $\mathcal{F}^+$ is defined as follows: for any open subset U of X, the space $\mathcal{F}^+(U)$ of sections of $\mathcal{F}^+(U)$ is defined as elements $s \in \prod_{x \in U} \mathcal{F}_x$ such that there is a open covering $U = \cup_{i \in I} U_i$ by opens U_i, and sections $s_i \in \mathcal{F}(U_i)$, such that for any $x \in U_i$,

$$s_x = (s_i)_x.$$

Thus we have a natural morphism $\mathcal{F} \longrightarrow \mathcal{F}^+$ which induces isomorphism on stacks:

$$\mathcal{F}_x = \mathcal{F}_x^+$$

for any $x \in X$.

The functor sheafication allows us to extend notions from abelian groups to sheaves, including subsheaves, quotient sheaves, kernel, image, cokernel, sheaves of rings, modules, tenso products, etc.

Definition 2.2.1. *A subsheaf of a sheaf $\mathcal{F}$ is a sheaf $\mathcal{F}'$ such that for every open set $U \subset X$, $\mathcal{F}'(U)$ is a subgroup of $\mathcal{F}(U)$, and the restriction maps of the sheaf $\mathcal{F}'$ are induced by those of $\mathcal{F}$. It follows that for any point P, the stalk $\mathcal{F}'_P$ is a subsheaf of $\mathcal{F}_P$.*

Example 2.2.2 (Kernel and Image). Given a morphism $\varphi : \mathcal{F} \to \mathcal{G}$ of sheaves, we define the kernel of φ, denoted $\ker \varphi$, to be the presheaf kernel of φ(which is a sheaf). Thus $\ker \varphi$ is a subsheaf of $\mathcal{F}$.

Example 2.2.3 (Quotient Sheaf). Let $\mathcal{F}'$ be a subsheaf of a sheaf $\mathcal{F}$. We define the quotient sheaf $\mathcal{F}/\mathcal{F}'$ to be the sheaf associated to the presheaf $U \to \mathcal{F}(U)/\mathcal{F}'(U)$. It follows that for any point P, the stalk $(\mathcal{F}/\mathcal{F}')_P$ is the quotient $\mathcal{F}_P/\mathcal{F}'_P$.

Definition 2.2.4 (Exact Sequence of Sheaves)**.** *We say that a sequence*

$$\cdots \to \mathcal{F}^{i-1} \overset{\varphi^{i-1}}{\to} \mathcal{F}^i \overset{\varphi^i}{\to} \mathcal{F}^{i+1} \to \cdots$$

of sheaves and morphisms is exact if for every i, $\ker \varphi^i = \operatorname{im} \varphi^{i-1}$. Thus a sequence $0 \to \mathcal{F} \overset{\varphi}{\to} \mathcal{G}$ is exact if and only if φ is injective, and $\mathcal{F} \overset{\varphi}{\to} \mathcal{G} \to 0$ is exact if and only if φ is surjective.

Lemma 2.2.5. *A sequence $\cdots \to \mathcal{F}^{i-1} \overset{\varphi^{i-1}}{\to} \mathcal{F}^i \overset{\varphi^i}{\to} \mathcal{F}^{i+1} \to \cdots$ of sheaves and morphisms is eact if and only if for each $P \in X$ the corresponding sequence of stalks is exact as a sequence of abelian groups.*

2.3 Push forward and pull back

Definition 2.3.1. *Let $f : X \to Y$ be a continuous map of topological spaces. Then given an open subset $V \subset Y$, $f^{-1}(V)$ is an open subset in X. Then we can define the push forward functor f_* from the category of sheaves on X to the category of sheaves on Y as follows.*

For any sheaf $\mathcal{F}$ on X, and any open subset $V \subset Y$, define $f_*\mathcal{F}(V) = \mathcal{F}(f^{-1}(V))$. And we can define the functor f^{-1} from the category of sheaves on Y to the category of sheaves on X as follows:

For any sheaf $\mathcal{G}$ on Y, and any open subset $U \subset X$, define the sheaf $f^{-1}\mathcal{G}$ on X to be the sheaf associated to the presheaf $U \mapsto \lim\limits_{V \supseteq f(U)} \mathcal{G}(V)$, and the limit is taken over all open subsets $V \subset Y$ containing $f(U)$.

It is easy to prove that the functor f_* is left exact and f^{-1} is right exact.

op for f_* **Proposition 2.3.2** (Adjoint property)**.** *Let $f : X \to Y$ be a continuous map of topological spaces. Then for any sheaf $\mathcal{F}$ on X and sheaf $\mathcal{G}$ on Y, there is a canonical isomorphism*

$$\mathrm{Hom}_X(f^{-1}\mathcal{G}, \mathcal{F}) \xrightarrow{\sim} \mathrm{Hom}_Y(\mathcal{G}, f_*\mathcal{F}).$$

This means that the functor f_* is a right adjoint of f^{-1} and f^{-1} is a left adjoint of f_*.

Proof. Let V be open subset of Y and U be open subset of X. Since $f^{-1}f_*\mathcal{F}$ is the sheaf associated to the presheaf

$$U \mapsto \varinjlim_{U \subset f^{-1}(V)} \mathcal{F}(f^{-1}(V)).$$

Then as a presheaf, it has a natural morphism to $\mathcal{F}$ induced by direct limit. After sheafication we get a morphism $\alpha : f^{-1}f_*\mathcal{F} \to \mathcal{F}$. Since we have $f(f^{-1}(V)) \subset V$, then for any $V \subset Y$, we have

$$\mathcal{G}(V) \to \varinjlim_{f(f^{-1}(V)) \subset V'} \mathcal{G}(V') \to f^{-1}\mathcal{G}(f^{-1}(V)) = f_*f^{-1}\mathcal{G}(V)$$

which defines a morphism $\beta : \mathcal{G} \to f_*f^{-1}\mathcal{G}$.

Thus we can define

$$\begin{aligned} \phi :\ \mathrm{Hom}_X(f^{-1}\mathcal{G}, \mathcal{F}) &\longrightarrow \mathrm{Hom}_Y(\mathcal{G}, f_*\mathcal{F}) \\ (\varphi : f^{-1}\mathcal{G} \to \mathcal{F}) &\longmapsto (f_*\varphi\beta : \mathcal{G} \to f_*f^{-1}\mathcal{G} \to f_*\mathcal{F}) \end{aligned}$$

and

$$\begin{aligned} \psi :\ \mathrm{Hom}_Y(\mathcal{G}, f_*\mathcal{F}) &\longrightarrow \mathrm{Hom}_X(f^{-1}\mathcal{G}, \mathcal{F}) \\ (\psi : \mathcal{G} \to f_*\mathcal{F}) &\longmapsto (\alpha f^{-1}\psi : f^{-1}\mathcal{G} \to f^{-1}f_*\mathcal{F} \to \mathcal{F}) \end{aligned}.$$

Clearly $\phi \circ \psi = \mathrm{Id}$ and $\psi \circ \phi = \mathrm{Id}$. □

2.4 Some special sheaves

Example 2.4.1 (Constant Sheaves). Let A be an abelian group. We define the *constant sheaf* $\mathcal{A}$ on a topological space X determined by A to be the sheaf associated to the presheaf $U \mapsto A$ for any nonempty open subset $U \subset X$, with restriction maps the identity.

Example 2.4.2. For any open subset $U \subset X$, the functor $\Gamma(U, \cdot)$ is a left exact functor from the category of sheaves to the category of abelian groups.

Example 2.4.3. Let X be a topological space and let $\mathcal{F}$ and $\mathcal{G}$ be sheaves on X. For any open set $U \subset X$, then the set $\mathrm{Hom}(\mathcal{F}|_U, \mathcal{G}|_U)$ has a nature sturcture of abelian group. The functor $U \mapsto \mathrm{Hom}(\mathcal{F}|_U, \mathcal{G}|_U)$ denoted by $\mathcal{H}om(\mathcal{F}, \mathcal{G})$ is a sheaf.

Example 2.4.4 (Direct Limit.). Let $\{\mathcal{F}_i\}$ be a direct system of sheaves and morphisms on topological space X. For any open subset $U \subset X$, we have direct limit of abelian groups $\varinjlim \mathcal{F}_i(U)$. Let $\varinjlim \mathcal{F}$ denote the sheaf associated to the presheaf $U \mapsto \varinjlim \mathcal{F}_i(U)$. Then $\varinjlim \mathcal{F}$ is the *direct limit* of $\{\mathcal{F}_i\}$ in the category of sheaves on X.

Note that if X is a noetherian topological space, then the presheaf $U \mapsto \varinjlim \mathcal{F}_i(U)$ is already a sheaf.

Example 2.4.5 (Inverse limit.). Let $\{\mathcal{F}_i\}$ be a inverse system of sheaves and morphisms on topological space X. Then the presheaf $U \mapsto \varprojlim \mathcal{F}_i(U)$ is already a sheaf. Denote this sheaf by $\varprojlim \mathcal{F}_i$, then it is the *inverse limit* of $\{\mathcal{F}_i\}$ in the category of sheaves on X.

Example 2.4.6 (Skyscraper Sheaves.). Let X be a topological space and $P \in X$ is a point. Let A be an abelian group and $\mathcal{A}$ the constant sheaf on the closed subspace $\{P\}^-$ determined by A. Consider the embedding $i : \{P\}^- \longrightarrow X$, then the sheaf $i_*\mathcal{A}$ has the following property: for any open subset $U \subset X$,

$$i_*\mathcal{A}(U) = \begin{cases} A & \text{if } P \in U & (2.4.1) \\ 0 & \text{otherwise.} & (2.4.2) \end{cases}$$

$$(i_*\mathcal{A})_Q = \begin{cases} A & \text{if } Q \in \{P\}^- & (2.4.3) \\ 0 & \text{otherwise.} & (2.4.4) \end{cases}$$

This sheaf is called the *skyscraper sheaf* with value A at P.

2.5 Ringed space

In the following we want to generalize the notion of functions on a topological space. We will use the language of sheaves.

Definition 2.5.1. *A* ringed space *is a pair* $(X, \mathcal{O}_X)$ *of a topological space* X *with a sheaf* $\mathcal{O}_X$ *of rings such that for each* $x \in X$*, the stack* $\mathcal{O}_{X,x}$ *is a local ring. Let* m_x *be the maximal ideal of* $\mathcal{O}_{X,x}$*. The field* $\mathcal{O}_{X,x}/m_x$ *is called the*residue field *of* x *which is denoted by* $k(x)$*.*

A morphism *of ringed spaces* $f = (f, f^\#) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ *consists of a continuous map* $f : X \to Y$ *and a morphism of sheaves* $f^\# : \mathcal{O}_Y \to f_*\mathcal{O}_X$*, here* $f_*\mathcal{O}_X$ *is a sheaf on* Y*, called the* direct image *of* $\mathcal{O}_X$*, and is defined by* $(f_*\mathcal{O}_X)(V) := \mathcal{O}_X(f^{-1}(V))$ *for any open subset* V *of* X*.*

The rigned spaces with morphisms form the category $\mathcal{RS}$ of ringed space

Intuitively, a ringed space is a topological space equipped with the ring of functions $\mathcal{O}_X(U)$ for each open subset U, and for each open subset U. At each point x, the local ring $\mathcal{O}_{X,x}$ is the ring of functions defined in a neighborhood of x; the map $\mathcal{O}_{X,x} \longrightarrow k(x)$ is the valuation of these functions at x. The difference with the usual definition of functions is that we don't require that the "values" of these functions are in a single field.

Remark 2.5.2. The functions in $\mathcal{O}_{X,x}$ are not determined by values at points. In other words, if a section $f \in \mathcal{O}_X(U)$ has zero value at every point $x \in U$, then it is possible that $f \neq 0$. For example, let $X = \mathrm{Spec}\mathbb{Z}/4\mathbb{Z}$. Then X has only one point $x = (2)$ and $2 \in (2)$, thus the section 2 has zero value in the residue field of x. But $2 \neq 0$.

Example 2.5.3. Let $X = \mathbb{R}$, U is an open subset in X,

$$\mathcal{O}_X(U) = \begin{cases} C(U) & \text{continuous functions on U} \\ C'(U) & \text{the differenciatable functions on U} \\ C^\infty(U) & \text{holomorphic functions on U} \end{cases}$$

Push forward and pull back

By same way, we define the abelian category of $\mathcal{O}_X$-modules on a ringed space $(X, \mathcal{O}_X)$. In this category, we have operators $\otimes_{\mathcal{O}_X}$ and $\mathcal{H}om_{\mathcal{O}_X}$ using sheafication and they satisfies the usual adjoint property:

$$\mathrm{Hom}(\mathcal{E}, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})) \simeq \mathrm{Hom}(\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{F}, \mathcal{G}).$$

Let $f : X \to Y$ be a morphism of ringed spaces, and $\mathcal{E}$ an $\mathcal{O}_X$-module over X, and $\mathcal{F}$ an $\mathcal{O}_Y$-module over Y. Then $f_*(\mathcal{E})$ is an $f_*(\mathcal{O}_X)$-module, and thus an $\mathcal{O}_Y$-module through homomorphism $\mathcal{O}_Y \longrightarrow f_*\mathcal{O}_X$. But $f^{-1}\mathcal{F}$ is only an $f^{-1}\mathcal{O}_Y$-module. We define an $\mathcal{O}_X$-module by

$$f^*\mathcal{F} := \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{F}.$$

on formula **Lemma 2.5.4** (Projection formula). *Let $f : X \longrightarrow Y$ be a morphism of a ringed spaces and $\mathcal{E}$ and $\mathcal{F}$ are sheaves of $\mathcal{O}_X$ and $\mathcal{O}_Y$ respectively. Assume that $\mathcal{F}$ is locally free. Then there is a canonical isomorphism*

$$f_*(\mathcal{E}) \otimes_{\mathcal{O}_Y} \mathcal{F} \xrightarrow{\sim} f_*(\mathcal{E} \otimes_{\mathcal{O}_X} f^*(\mathcal{F})).$$

Proof. By definition of tensor sheaves, the left hand side is the sheafication of the presheaf $\mathcal{G}$ on Y defined by

$$\mathcal{G}(U) := f_*(\mathcal{E})(U) \otimes_{\mathcal{O}_Y(U)} \mathcal{F}(U).$$

Using defintion of f^*, we may write the last term as

$$\begin{aligned} \mathcal{E}(f^{-1}U) \otimes_{\mathcal{O}_Y(U)} \mathcal{F}(U) =& \mathcal{E}(f^{-1}(U)) \otimes_{\mathcal{O}_X(f^{-1}U)} (\mathcal{O}_X(f^{-1}U) \otimes_{\mathcal{O}_Y(U)} \mathcal{F}(U)) \\ =& \mathcal{E}(f^{-1}(U)) \otimes_{\mathcal{O}_X(f^{-1}U)} (\mathcal{O}_X(f^{-1}U) \otimes_{f^{-1}\mathcal{O}_Y(f^{-1}U)} f^{-1}\mathcal{F}(f^{-1}U)). \end{aligned}$$

This term has a natural map to

$$\mathcal{E}(f^{-1}(U)) \otimes_{\mathcal{O}_X(f^{-1}U)} f^*\mathcal{F}(f^{-1}(U)),$$

and thus to

$$(\mathcal{E} \otimes_{\mathcal{O}_X} f^*(\mathcal{F}))(f^{-1}U) = f_*(\mathcal{E} \otimes_{\mathcal{O}_X} f^*(\mathcal{F}))(U).$$

By university of sheafication, we have a morphism of sheaves

$$f_*(\mathcal{E}) \otimes_{\mathcal{O}_Y} \mathcal{F} \longrightarrow f_*(\mathcal{E} \otimes_{\mathcal{O}_X} f^*(\mathcal{F})).$$

To show that this is actually an isomorphism, we need only check in level of stacks. In this case, we may assume that $\mathcal{F}$ is free: $\mathcal{F} = \mathcal{O}_Y^{\oplus I}$. It then reduce to the case $\mathcal{F} = \mathcal{O}_Y$. In this case, the assertion is clear. □

3 Affine schemes

3.1 Spectrum

The term "Spectrum" comes from the use in operator theory. Given a linear operator T on a finite-dimensional vector space V, one can consider the vector space with operator as a module over the polynomial ring in one varieble $R = K[T]$. Then the spectrum of $K[T]$(as a ring) equals the spectrum of T(as an operator). The theory of Jordan form follows from in the structure theorem for finitely generated modules over a PID.

Definition 3.1.1. *Let R be a commutative ring. The spectrum the spectrum of R to defined to be the set*

$$\mathrm{Spec} R = \left\{ \mathfrak{p} \subsetneqq R \middle| \ \mathfrak{p} \ \textit{is prime ideal} \right\}. \tag{3.1.1}$$

Remark 3.1.2. For connection with classical algebraic geometry, let K be a algebraic closed field, and $A \hookrightarrow K^n$ a set of zeros of some polynomial functions $f_1, ..., f_m$, one considers the commutative ring $R = K[x_1, ..., x_n]/(f_1...f_m)$ of all polynomial functions $A \to K$. Then the maximal ideals of R correspond to the ponits of A and the prime ideals of R correspond to the subvarieties of A. In the above definition of spectrum, we include all prime ideals.

Given $f : R \to K$ a homomorphism from R to a field K, we can show that $\ker f$ is a prime ideal in R, thus $\mathrm{Im} f$ is an integral ring.

Exercise 3.1.3. An ideal p is a prime ideal in ring $R \Leftrightarrow p$ is kernel of a homomorphism from R to a field.

Exercise 3.1.4. A ring R is integral$\Leftrightarrow$ R is the subring of a field.

3.2 Topology

Now let R be commutative ring, we will define a topology of on $X = \mathrm{Spec}(R)$ with closed subsets $V(I)$ defined as follows:

Definition 3.2.1. *For any ideal $I \subset R$, define*

$$V(I) = \{p \in \mathrm{Spec}R \,|\, p \supseteq I\} \tag{3.2.1}$$

For an ideal I of R, we define the radical the ideal of elements a which has a positive power a^n belonged to I and denote it by $\sqrt{I}$.

Lemma 3.2.2. *Let R be a commutative ring.*

1. *If I_1 and I_2 are two ideals of R, then $V(I_1 I_2) = V(I_1) \bigcup V(I_2)$.*
2. *If I_s $(s \in S\ )$ be any set of ideals of R, then $V(\sum_{s \in S} I_s) = \bigcap V(I_s)$.*
3. *If I_1 and I_2 are two ideals, $V(I_1) \subset V(I_2)$ if and only if $\sqrt{I_2} \subset \sqrt{I_1}$*
4. *$V(I) = \bigcap_{f \in I} V(f)$*

Proof.

1. Clearly if a prime ideal $p \supseteq I_i$, $i = 1, 2$, then $p \supseteq I_1 I_2$. Conversely if $p \supseteq I_1 I_2$. Suppose $p \not\supseteq I_1$, then there is an $a \in I_1$ such that $a \notin p$. Since for any $b \in I_2$, $ab \in p$. Thus $b \in p$, therefore $I_2 \subseteq p$.
2. Since $\sum I_n$ is the smallest ideal containing all I_n, thus $p \supseteq \sum I_n$ if and only if $p \supseteq I_n$ for any n.
3. Since $\sqrt{I_1} = \cap_{\supseteq I_1}$, so $V(I_1) \subset V(I_2)$ if and only if $\sqrt{I_2} \subset \sqrt{I_1}$

□

By (1) and (2) of Lemma, we arrive to a topology on $\mathrm{Spec}R$:

Definition 3.2.3. *The* Zariski topology *of* $\mathrm{Spec}R$ *is a topology on* $\mathrm{Spec}R$ *with closed subsets $V(I)$ indexed by ideals I of R. For each $f \in R$, the complement of $V(fR)$ is called a principle open set and denoted by $D(f)$. Thus naturally we have* $\mathrm{Spec}R = V(0)$, $\varnothing = V(R)$.

Remark 3.2.4. For connection with classical algebraic geometry for the algebraic set A with ring R of algebraic functions, one can view the topological space $\mathrm{Spec}R$ as an erichment of the topological space A: for every subvariety of A, one additional non-closed point has been introduced, and this point "keeps track" of the corresponding subvariety. One thinks of this point as the generic point for the subvariety. Furthermore, the sheaf on $\mathrm{Spec}R$ and the sheaf of polynomial functions on A are essentially identical. By studying spectra of polynomial rings instead of algebraic sets with Zariski topology, one can generalize the concepts of algebraic geometry to non-algebraically closed fields and beyond, eventually arriving at the language of schemes.

Proposition 3.2.5. *If $\phi : A \to B$ is a homomorphism of rings, then ϕ induces a natural morphism of ringed spaces*

$$\phi_* : \mathrm{Spec}B \to \mathrm{Spec}A, \tag{3.2.2}$$

and this morphism is continuous.

Proof. Let q be a prime ideal of B, then ϕ induces a inclusion $A/\phi^{-1}q \hookrightarrow B/q$. Since B/q is integral, then $A/\phi^{-1}q$ is integral. Thus $\phi^{-1}q$ is prime ideal in A. Let I be an ideal in B. Given a closed subset $V(I) \subseteq \mathrm{Spec}B$,

$$\phi_*^{-1}(V(I)) = \left\{\phi^{-1}(q) | I \subseteq q\right\} = V(\phi^{-1}I).$$

Thus ϕ_* is continuous. □

Example 3.2.6. Let A be a ring, given an ideal $I \subseteq A$, the ring homomorhism $\phi : A \to A/I$ is surjective. Since any prime ideal $q \hookrightarrow A/I$ is the prime ideal in A that contains I. then $\phi^* : \mathrm{Spec}A/I \to \mathrm{Spec}A$ is injective. And $\mathrm{Im}\phi^* = V(I)$.

Definition 3.2.7. *Let A be a ring, and S a semi-multiplicated subgroup of A. Define the* localization *A_S of A on S to be the quotiont $A[x_s, s \in S]/(x_s s - 1)$, denoted by $A[\frac{1}{S}]$.*

It is equally to say that $A[\frac{1}{S}]$ is a ring generated by symbols $\frac{a}{s}$ module the following relation for $a, b \in A, s, t \in S$:

$$\frac{a}{s} = \frac{b}{t} \Leftrightarrow uta - usb = 0$$

for some $u \in S$.

If $S = \{f^n, n \in \mathbb{N}\}$ is generated by one element, we also denote $A_S = A_f = A[\frac{1}{f}]$.

Then $A[\frac{1}{S}] \neq 0$ if and only if $0 \notin S$.

Lemma 3.2.8. *let A be a ring, then the ring homomorphism $\phi : A \to A_f$ induces a morphism between topological spaces $\phi_* : \mathrm{Spec}A_f \to \mathrm{Spec}A$ and $\mathrm{Im}\phi_* = D(f)$*

Proof. By definition, $D(f) = \{p \in \mathrm{Spec}A | f \notin p\}$.

Let $q \in \mathrm{Spec}A_f$ be a prime ideal. Since ϕ is a ring homomorphism then $p = \phi^{-1}(q)$ is a prime ideal in A. If $f \in \phi^{-1}(q)$, it follows that $\phi(f) \in q$. Then $f \in q \subset A_f = A[x]/(xf - 1)$ is invertible, thus $q = A$, contradiction. Therefore $\mathrm{Im}\phi_* \subset D(f)$.

Conversely, for each $p \in D(f)$, we have an ideal $q := pA_f$ of A_f. It is each to see that this is a prime ideal, and that the induced map $D(f) \longrightarrow \mathrm{Spec}A)_f$ is continuous and inverse to the map $\phi_* : \mathrm{Spec}A_f \longrightarrow D(f)$. □

Let A be a ring and $f_1, f_2 \in A$, then $D(f_1) \cap D(f_2) = D(f_1 f_2)$. More general let $S \hookrightarrow A$ be nonzero semi-group, then we have a morphism

$$\phi_* : \mathrm{Spec}A[\frac{1}{S}] \to \mathrm{Spec}A. \tag{3.2.3}$$

It is easy to see that $\mathrm{Im}\phi_* = \cap_{f\in S} D(f)$.

If $p \in \mathrm{Spec}A$, take $S = A - p$. Then the localization of A at S is defined by $A_p := A[\frac{1}{S}]$. We also have a morphism of topological spaces

$$\phi_* : \mathrm{Spec}A_p \to \mathrm{Spec}A \tag{3.2.4}$$

The image is the intersection of all open neighborhoods of p.

Definition 3.2.9. *Let A be a ring. For any $p \in \mathrm{Spec}A$, let A_p be the local ring at p, and pA_p its maximal ideal. We define the residue field of p on $\mathrm{Spec}A$ to be the field $k(p) = A_p/pA_p$. And it is isomorphis to the fraction field of A/p.*

Naturally we have a sequence of homomorphism of rings

$$A \to A/p \to k(p) \tag{3.2.5}$$

which induces a sequence of morphisms of topological spaces

$$\mathrm{Spec}k(p) \to \mathrm{Spec}A/p \to \mathrm{Spec}A. \tag{3.2.6}$$

$$\begin{array}{ccccc} \mathrm{Spec}k(p) & \longrightarrow & \mathrm{Spec}A/p & \longrightarrow & \mathrm{Spec}A \\ x & \longmapsto & \text{generic point} & \longmapsto & p \end{array} \tag{3.2.7}$$

where x is the only point in $\mathrm{Spec}k(p)$.

Exercise 3.2.10. Show that $\psi : \mathrm{Spec}A/p \to \mathrm{Spec}A$ is injective and $\mathrm{Im}\psi$= the closure of $\{p\}$.

Remark 3.2.11. As a topological space, $\mathrm{Spec}A$ is T_0. But the good thing is only $\mathrm{Spec}A$ is compact. Which means if $\cap_{\lambda\in\Lambda} V(I_\lambda) = \varnothing$, then there exists a finite subset $\Lambda' \hookrightarrow \Lambda$ such that $\cap_{\lambda\in\Lambda} V(I_\lambda) = \varnothing$.

Proof. $\cap_{\lambda\in\Lambda} V(I_\lambda) = V(\sum_{\lambda\in\Lambda} I_\lambda) = \varnothing$, let $I = \sum_{\lambda\in\Lambda} I_\lambda$. Then for any prime ideal p, $p \not\supseteq I$. By Zorn's lemma, $I = A$. □

Example 3.2.12. A prime ideal $p \subset A$ is maximal ideal if only if the set $\{p\}$ is closed.

Definition 3.2.13. *Let R be a ring, S a subset of R. We say S is a multiplicative subset of R if 1$\in S$ and S is closed under multiplication, i.e., $s, s' \in S \Rightarrow ss' \in S$.*

Proposition 3.2.14. *The local ring $A[\frac{1}{S}]$ is flat as an A-module.*

Proof. It is enough to prove it preserves short exact sequences: $0 \to M \to N \to P \to 0$. As the tensor product is right-exact, and $S^{-1}M \simeq M \otimes_A S^{-1}A$, it is even enough to prove it preserves injectivity:
Consider an injective morphism $\varphi : M \to N$ and suppose $S^{-1}\varphi\left(\frac{m}{s}\right) = 0$ in $S^{-1}N$. This means there exists $t \in S$ such that $t\varphi(m) = \varphi(tm) = 0$. But then

$$\frac{m}{s} = \frac{tm}{ts} = \frac{0}{ts} = 0 \tag{3.2.8}$$

which shows $S^{-1}\varphi$ is injective. □

3.3 Structure sheaves

Let $X = \mathrm{Spec}A$. We will define structure sheaf on X so that $\mathcal{O}(D(f)) = A_f$ for all $f \in A$. Thus for any open subset U with an open covering $U = \cup_{f\in\Lambda} D(f)$ we have the following short exact sequence

$$\mathcal{O}_X(U) \to \Pi_{f\in\Lambda} A_f \xrightarrow{\alpha} \Pi_{f,g\in\Lambda} A_{fg} \tag{3.3.1}$$

We may use this exact sequence to define $\mathcal{O}(U)$. But then we have to check that the definition doesn't depend on the choice of any $U = \cup_{f\in\Lambda} D(f)$. We will give another definition and then prove above as a theorem.

Definition 3.3.1. *Given a ring A, and Let $X = \mathrm{Spec}A$, U be an open subset in X. Define the stalk of $\mathcal{O}_X$ at the point p to be the direct limit:*

$$\mathcal{O}_{X,p} = \varinjlim_{f\notin p} A_f \tag{3.3.2}$$

$$= \varinjlim_{f\notin p} \mathcal{O}_X(D(f)) \tag{3.3.3}$$

$$= A_p \tag{3.3.4}$$

The elements of the stalk $\mathcal{O}_{X,p}$ are called the germs of sections of $\mathcal{O}_X$ at the point p.

Now we come to the following definition:

$$\mathcal{O}_X(U) := \left\{ (s_p)_p \in \Pi_{p\in U} A_p \;\middle|\; \begin{array}{l} \exists \text{ a covering } U = \bigcup_{i\in I} D(a_i) \\ \text{such that on} D(a_i), \exists s_i : A_{a_i} \to A_p, s_p \text{ is the image of} f_i \end{array} \right\} \tag{3.3.5}$$

It is clear that $\mathcal{O}_X$ is a sheaf of rings on $\mathrm{Spec}A$.

Remark 3.3.2. Let X be a topological space, Y be any set, let $x \in X$. Given two maps $f, g : X \to Y$. Then f and g define the same germ at x if there is a neighborhood U of x such that the restriction of f and g on U are equal. In this way, A_p is the analogue of germ of functions at p.

Example 3.3.3. Let D be a domain in $\mathbb{C}$. Let $C(D)$ denote the set of continuous functions on D. Define

$$C^{holo}(D) = \{f \in C(D) | \forall p \in D, \text{f is analytic at p}\} \tag{3.3.6}$$

to be the set of holomorphic functions on D. i.e. $f \in C^{holo}(D)$, $f = \sum_{n=0}^{\infty} a_n (z - z_p)^n$, where $a_n, z \in \mathbb{C}$.
$\forall p \in D$, define

$$\mathcal{O}_{D,p} = \{f \in C^{holo} | f = \sum_{n=0}^{\infty} a_n (z - z_p)^n \text{ convergent in neighborhoods of } p\}$$

to be the set of germs of analytic functions on D.

m **Theorem 3.3.4** (Main theorem). *Let A be a ring, and $\mathcal{O}$ is the sheaf of rings on* Spec$A$ *defined as above.*

(a) For any $p \in \text{Spec}A$, the stalk $\mathcal{O}_p$ of the sheaf $\mathcal{O}$ is isomorphic to the local ring A_p

(b) For any element $f \in A$,the ring $\mathcal{O}(D(f))$ is isomorphic to the localized ring A_f.

(c) In particular, $A \xrightarrow{\sim} \Gamma(\text{Spec}A, \mathcal{O})$.

Proof. (a) First we define a homomorphism

$$\varphi : \mathcal{O}_p \longrightarrow A_p \qquad (3.3.7)$$
$$s \longmapsto s(p)$$

where s is a local section in a neighborhood U of p, and by definition of $\mathcal{O}$, $s(p)$ is the image of p by $s : U \to \prod A_p$.

The map is surjective: any element of A_p can be represented as a quotient a/f, where $a, f \in A$ and $f \notin p$. Then $D(f)$ will be an open neighborhood of p, then we get a section $s : D(f) \to \prod_{p \in D(f)} A_p$ with $p \mapsto a/f$, $f \notin p$.

To show the map is injective, let U be an open neighborhood of p. We need to show that given a local sections$s : U \to \prod A_p$, if $s(p) = 0$, then $s(q) = 0$ for every $q \in U$. By shrinking U if necessary, we may assume that $s = a/f$ on U, where $a, f \in A$ and $f \notin p$. Since the image of a/f in A_p is zero, then there exists an $h \notin p$ such that $ha = 0$ in A. Therefore $a/f = 0$ in every local ring A_q such that $f, h \notin q$. Hence $s = 0$ in a whole neighborhood of p, so they have the same stalk at p. So φ is an isomorphism.

b. We define a homomorphism

$$\psi : A_f \longrightarrow \mathcal{O}(D(f))$$
$$\frac{a}{f^n} \longmapsto s$$

where s is the section which assigns each p the image of $\frac{a}{f^n}$ in A_p.

First we show that ψ is injective. If $\psi(a/f^n) = \psi(b/f^m)$, then for every $p \in D(f)$, here is an element $h \notin p$ such that $h(f^m a = f^n b) = 0$ in A. Let α be the annihilator of $f^m a = f^n b$. Then $h \in \alpha$, and $h \notin p$, so $\alpha \nsubseteq p$. This holds for any $p \in D(f)$, so we conclude that $V(\alpha) \cap D(f) = \varnothing$. Therefore $f \in \sqrt{\alpha}$, so $f^l \in \alpha$ for some integer l, then $f^l(f^m a - f^n b) = 0$, which shows that $a/f^n = b/f^m$ in A_f. Hence ψ is injective.

Second we need to show that ψ is surjective. So let $s \in \mathcal{O}(D(f))$. Then by definition of $\mathcal{O}$, we can assume that $D(f)$ is covered by opensubsets $D(h_i)$ and s is represented by a_i/h_i on each $D(h_i)$

Observe that the cover $\{D(h_i)\}$ is finite. Indeed, $D(f) \subset \bigcup D(h_i)$ if and only if $V((f)) \supseteq \bigcap V((h_i)) = V(\sum(h_i))$. This is equivalent to say that $f \in \sqrt{\sum(h_i)}$, then $f^n \in \sum(h_i)$ for some integer n. This means that f^n can be expressed as a finite sum $f^n = \sum b_i h_i$, where $b_i \in A$. Hence a finite subset of h_i will do. So we suppose that $D(f) = \bigcup_{i=1}^r D(h_i)$.

Note that $D(h_i) \bigcap D(h_j) = D(h_i h_j)$. By the property of sheaves, the restriction of s on $D(h_i)$ and $D(h_j)$, namely $a_i/h_i, a_j/h_j$ have the same image on $D(h_i h_j)$. Then from the injectivity of ψ we know that $a_i/h_i = a_j/h_j$ in $D(h_i h_j)$. Hence there exists an integer n such that

$$(h_i h_j)^n (h_j a_i - h_i a_j) = 0. \tag{3.3.8}$$

Since there are only finitely many $D(h_i)$, we may pick n sufficiently large such that it works for all i, j. Rewrite this equation as

$$h_j^{n+1}(h_i^n a_i) - h_i^{n+1}(h_j^n a_j) = 0. \tag{3.3.9}$$

Replace each h_i, h_j by h_i^{n+1}, h_i^{n+1} and a_i, a_j by $h_i^n a_i, h_j^n a_j$ respectively. Then $h_j a_i = h_i a_j$ for all i, j, a_i/h_i represent s.

Since $D(f) = \bigcup_{i=1}^r D(h_i)$. Now write $f^n = \sum b_i h_i$ as above, let $a = \sum b_i a_i$. Then for each j we have

$$h_j a = \sum_{i=1}^r b_i a_i h_j = \sum_{i=1}^r b_i a_j h_i = f^n a_j. \tag{3.3.10}$$

This means that $a/f^n = a_j h_j$ on $D(h_j)$. So $\psi(a/f^n) = s$ everywhere, which shows that ψ is surjective, hence an isomorphism.

c. Let $f = 1$ in 1, $D(f)$ is the whole space.

□

Proposition 3.3.5. *Let A be a ring and let $X = \mathrm{Spec} A$. Then $f \in A$ is nilpotent if and only if $D(f)$ is empty.*

Proof. Since the nilradical of A is the intersection of all prime ideals of A, say $\mathrm{Nil}(A) = \bigcap_{\mathfrak{p}} \mathfrak{p}$. Then $f \in \mathrm{Nil}(A)$ if and only if $f \in \mathfrak{p}$ for all prime ideals $\mathfrak{p} \subset A$ if and only if $D(f) = \varnothing$. □

3.4 Affine Schemes

In the last section we defined a functor from the opposite category of rings to ringed spaces:

$$\begin{aligned} \mathrm{Spec}: \ \mathit{Ring}^{opp} &\longrightarrow \text{Ringed spaces} \\ A &\longmapsto (\mathrm{Spec} A, \mathcal{O}_{\mathrm{Spec} A}) \end{aligned} \tag{3.4.1}$$

Definition 3.4.1. *A covariant functor $F : \mathcal{C} \to \mathcal{D}$ is called faithful if for all $C_1, C_2 \in \mathcal{C}$, the map $Hom_{\mathcal{C}}(C_1, C_2) \to Hom_{\mathcal{D}}(F(C_1), F(C_2))$ is injective, and full if it is surjective. A functor that is full and faithful is called fully faithful.*

Definition 3.4.2. *Let A be a ring. The local ringed space* $(\mathrm{Spec} A, \mathcal{O}_{\mathrm{Spec} A})$ *is called an affine scheme.*

Proposition 3.4.3. *The functor* Spec *is fully faithful.*

Proof. Given two rings A, B, and let $X = \text{Spec}A, Y = \text{Spec}B$. Given a homomorphism $\varphi : A \to B$, in last chapter we have defined the induced map $f : \text{Spec}B \to \text{Spec}A$ by $f(p) = \varphi^{-1}(p)$ for any $p \in \text{Spec}B$. And we have proven that f is continuous. Since $\mathcal{O}(D(f))$=A_f for any open subset $D(f)$, and $D(f)$ form a basis of Spec$A$, and for any $p \in \text{Spec}B$ φ induces a local homomorphism $\varphi_p : A_{\varphi^{-1}(p)} \to B_p$. Considering the stalks on each p, then for any open subset V of X we obtani a homomorphism of rings $f^\sharp : \mathcal{O}_X(V) \to \mathcal{O}_Y(f^{-1}(V))$. Then we have diagram

$$\begin{array}{ccc} \mathcal{O}_X(V) & \xrightarrow{f^\sharp} & \mathcal{O}_Y(f^{-1}(V)) \\ \downarrow & & \downarrow \\ \mathcal{O}_{X,p} & \xrightarrow{\varphi_p} & \mathcal{O}_{Y,p} \end{array}$$

This gives the morphism of sheaves $f^\sharp : \mathcal{O}_X \to f_*(\mathcal{O}_Y)$.
Conversely, Given a morphism $f : Y \to X$, taking global sections it induces a homomorphism of rings $\varphi : \mathcal{O}_X(X) \to \mathcal{O}_Y(Y)$. But we know that $\mathcal{O}_X(X) = A$ and $\mathcal{O}_Y(Y) = B$, so we have $\varphi : A \to B$. For any p $\in$ Spec B, we have an induced local homomorphism on the stalks, $\mathcal{O}_{X,f(p)} \to \mathcal{O}_{Y,p}$. Then there is a commutative diagram:

$$\begin{array}{ccc} A & \xrightarrow{\varphi} & B \\ \downarrow & & \downarrow \\ A_{f(p)} & \xrightarrow{f_p^\sharp} & B_p \end{array}$$

since f'' is a local homomorphism, it follows that $\varphi^{-1}(p) = f(p)$, which shows that f coincides with the map B pec $B \to$ Spec A induced by φ . Now it is immediate that f^* also is induced by φ, so that the morphism (f, f^*) of locally ringed spaces does indeed come from the homomorphism of rings φ. □

Lemma 3.4.4. *Let $\varphi : A \to B$ be a homomorphism of rings, and let $f : Y = \text{Spec}B \to X = \text{Spec}A$ be the induced morphism of affine schemes. Then*

(1) φ is injective iff $f^\sharp : \mathcal{O}_X \to f_*\mathcal{O}_Y$ is injective. And in this case f is dominant.

(2) If φ is surjective, then f is a homeomorphism of Y onto a closed subset of X and $f^\sharp$ is surjective

Proof. (1) Let $\mathfrak{p} \in \text{Spec}A$ and $a \in A$, for any open subset $D(a) \hookrightarrow X$, since

$$(f_*\mathcal{O}_Y)_\mathfrak{p} = \varinjlim_{\mathfrak{p} \in D(a)} \mathcal{O}_Y(f^{-1}(D(a))) = \varinjlim_{a \notin \mathfrak{p}} \mathcal{O}_Y(D(\varphi(a))) \cong B_{\varphi(a)}.$$

Then consider the map on stalks:

$$f_\mathfrak{p}^\sharp : \mathcal{O}_\mathfrak{p} = A_\mathfrak{p} \to B_{\varphi(a)}.$$

Note that $B_{\varphi(a)} = B \otimes A_\mathfrak{p} = S^{-1}B$, where $S = A/\mathfrak{p}$. Then it is easy to check that φ is injective iff $f_\mathfrak{p}^\sharp$ is injective. To prove that f is dominant, we only need to prove that for any nonempty principle open subset $D(a) \subset X$, $f(Y) \bigcap D(a) \neq \varnothing$, or equivalently, $D(\varphi(a)) = f^{-1}(D(a)) \neq 0$. But it is clear as $\varphi(a)$ is not nilpotent if φ is injective.

(2) Since φ is surjective then $A/\ker\varphi \cong B$. Denote $\ker\varphi$ by I. Then $Y = \mathrm{Spec}A/I = V(I) \subset X$. The map on stalks

$$f^\sharp_\mathfrak{p} : \mathcal{O}_\mathfrak{p} = A_\mathfrak{p} \to A/I \otimes_A A_\mathfrak{p}$$

is clearly surjective. □

Example 3.4.5. Let k be a field. Then Spec$k$ is an affine scheme whose topological space contains only one point. And whose structure sheaf consists of the field k.

Example 3.4.6. Spec$\mathbb{Z} = \{(p)|$ p is prime number$\}$. Since $\mathbb{Z}$ is PID, and an ideal $(a) \subset (p)$ if and only if $p|a$. Thus a closed subset of Spec$\mathbb{Z}$ is a finite set.

Example 3.4.7. Let k be a field, then Spec$k[x] = \{(0), (f)|$ f is monic irreducible polynomial$\}$. Similarly, since $k[x]$ is PID. Then let p be a point in Spec$k$, suppose $p = (f)$, f is an polynomial. An ideal $(g) \subset p$ if and only if $f|g$. Thus the closed subset of Spec$k[x]$ is a finite set.

If k is algebraic closed, then Spec$k[x] = k \bigcup \{\zeta\}$, where ζ is generic point. The correspondence $k \to$ Spec$k[x]$ is given by $a \mapsto (x-a)k[x]$

Spec$k[x]$ is compact. Every open cover has a finite subcover.

Example 3.4.8. Let k be an algebraic field. $\mathbb{A}^n_k =$ Spec$k[x_1...x_n]$ is affine scheme of dimension n.

Theorem 3.4.9 (Hilbert-Nullstellensatz)**.** *Let k be an algebraically closed field. Then the set of closed points in $\mathbb{A}^n_k$ is isormorphic to k^n*

Proof. Let $A = k[t_1, ..., t_n]$, I an ideal of A and V the common zeros of I in k^n. Clearly, $\sqrt{I} \subset I(V)$. Let $f \notin \sqrt{I}$. Then $f \notin p$ for some prime ideal $p \supseteq I$ in A. Let $R = (A/\mathfrak{p})[f^{-1}]$ and m a maximal ideal in R. Apply Zariskı's lemma zariski lemma 1.4.7 on $k \hookrightarrow B \hookrightarrow k/m$, where B is $k(x_1, ..., x_r)$ with $\{x_1, .., x_r\}$ is a transcendental basis of k/m over k. $R/\mathfrak{m}$ is a finite extension of k; thus, is k since k is algebraically closed. Let x_i be the images of t_i under the natural map $A \to k$. It follows that $x = (x_1, \dots, x_n) \in V$ and $f(x) \neq 0$. □

Exercise 3.4.10. Let $f : X \to Y$ be a morphism of schemes, and suppose that Y can be covered by open subsets U_i, such that for each i, the induced map $f^{-1}(U_i) \to U_i$ is an isomorphism. Then f is an isomorphism.

X_f **Lemma 3.4.11.** *Let X be a scheme, let $f \in \Gamma(X, \mathcal{O}_X)$, and define $X_f = \{x \in X|\ f_x \notin m_x\mathcal{O}_x\}$, then*

(a) X_f is an open subset of X;

(b) assume that X is quasi-compact. Let $A = \Gamma(X, \mathcal{O}_X)$, and let $a \in A$ be an element whose restriction to X_f is 0. Then $f^n a = 0$ for some integer n;

(c) *Now assume that X has a finite affine open covering $\{U_i\}$ such that $U_i \bigcap U_j = U_{ij}$ is compact, $\forall i,j$, and a section b of $\mathcal{O}_{X_f}$ can be extend to a global section. i.e. let $b \in \Gamma(X_f, \mathcal{O}_{X_f})$. There is an integer n such that $f^n b$ is the restriction of an element of A;*

(d) $\Gamma(X_f, \mathcal{O}_{X_f}) \cong A_f$.

Proof. (a) Let $U = \mathrm{Spec}B$ be an affine open subset of X and let $\overline{f} \in B$ be the restriction of f. Then $x \in D(\overline{f})$ if and only if the stalk $\overline{f}_x \notin m_x \mathcal{O}_{U,x}$. Then $U \bigcap X_f = D(\overline{f})$, thus X_f is an open subset.

(b) Let $\{U_i = \mathrm{Spec}A_i\}$ be an open affine cover of X. Since X is quasi-compact, assume that $X = \bigcup_{i=1}^m U_i$. $\overline{f_i} \in A_i$ be the restriction of f on U_i. Since the restrictoin of a on X_f is 0, then $a_i = a|_{U_i}$ restricts on $D(\overline{f_i}) = \mathrm{Spec}(A_i)_{\overline{f_i}}$ is 0. Then for each i, there is an integer n_i such that $\overline{f_i}^{n_i} a_i = 0$. Denote $n = max\{n_i\}$ we have $\overline{f_i}^n a_i = 0$ on each U_i. By the property of sheaf, we get $f^n a = 0$.

(c) On every U_i, since $b|_{U_i \bigcap X_f} = b|_{D(\overline{f_i})}$, there exists integers n_i such that $\overline{f_i}^{n_i} b|_{D(\overline{f_i})}$ can be extended into U_i. Denote $n = max\{n_i\}$ we get $\overline{f_i}^n b|_{D(\overline{f_i})}$ can be extended into U_i as some b_i. On every U_{ij}, we may assume $U_{ij} = \bigcup_{k \in K_{ij}} U_{ijk}$ for some finite index set K_{ij}. Since on every U_{ijk}, $b_i|_{U_{ijk}} - b_j|_{U_{ijk}} = 0$, there exists some $m_i jk$ such that $(f|_{U_{ijk}})^{m_i jk} b_i|_{U_{ijk}} - b_j|_{U_{ijk}} = 0$. Then take $m = max\{m_{ijk}\}$ we get $f^m b_i|_{U_{ijk}} - b_j|_{U_{ijk}} = 0$. So $f^{n+m} b$ can be extended to some global section by property of sheaf.

(d) Consider morphism $A_f \to \Gamma(X_f, \mathcal{O}_{x_f})$ defined by $\frac{a}{f^n} \mapsto a|_{X_f}$. (b) means injection and (c) means surjection.

□

affineness **Theorem 3.4.12** (A Criterion for Affineness)**.** *A scheme X is affine if and only if there is a finite set of elements $f_1, ..., f_r \in A = \Gamma(X, \mathcal{O}_X)$ such that the open subsets X_{f_i} are affine, and $f_1, ..., f_r$ generate the unit ideal in A.*

Proof. The identity map $A \to \mathcal{O}_X(X)$ induces a morphism $\varphi : X \to \mathrm{Spec}A$. Since $f_1, ..., f_r$ generate the unit ideal in A, $\{D(f_i)\}$ forms an open covering of $\mathrm{Spec}A$. Since clearly $\varphi^{-1}(D(f_i)) = X_f$, we have $\mathcal{O}_X(X_{f_i}) \cong A_{f_i}$. Since φ is ismorphic on $\varphi^{-1}(D(f_i))$, by the exercise, φ is an isomorphism. □

3.5 Dimension Theory

We have our usual notion of dimension:

- a line is 1 -dimensional
- a plane is 2-dimensional
- a space is 3 -dimensional.

The dimension measures how much freedom do you have, or equivalntly, how many constrains you may have. If a space X has dimension n, then for generic $n+1$ functions $f_1, ..., f_{n+1}$ the system $f_1 = f_2 = ... = 0$ should not have solutions, but any n of them will have solutions. Note that $\mathbb{A}_k^n = \mathrm{Spec} k[x_1...x_n]$ has dimension n. And we have a chain of closed subsets

$$\mathbb{A}^n \supsetneq V(x_1) \supsetneq V(x_2) \supsetneq ... \supsetneq V(x_1...x_n).$$

Definition 3.5.1. *A topological space X is called noetherian if any decreasing chain of closed subsets $Y_0 \supsetneq Y_1 \supsetneq .... \supsetneq ...$ is stable. i.e there is an integer r such that $Y_r = Y_{r+1} = ...$*

Definition 3.5.2. *A topological space X is called quasi-compact if any open cover of X has a finite subcover.*

Exercise 3.5.3. If a topological space X is noetherian then it is quasi-compact.

Definition 3.5.4. *Let A be a ring, and let I be an ideal in A. The radical of I is defined as*

$$\sqrt{I} = \{f \in A | \ f^r \in I for\ some\ r > 0\}$$

Example 3.5.5. Let A, and $X = \mathrm{Spec}A$. If $\{I_i\}$ is an family of ideals in A, $Y_i = V(I_i)$. We say I_i is radical if $\sqrt{I_i} = I_i$. Then $Y_0 \supsetneq Y_1 \supsetneq .... \supsetneq ... \Longrightarrow I_0 \subsetneq I_1 \subsetneq ... \subsetneq ...$

Exercise 3.5.6. If a ring A is noetherian, then $\mathrm{Spec}A$ is noetherian as a topological space. But conversely it is not true. Give an example such that $\mathrm{Spec}A$ is noetherian but A is not noetherian.

Definition 3.5.7. *A nonempty subset Y of a topological space X is irreducible if Y cannot be expressed as the union $Y = Y_1 \bigcup Y_2$ of two proper subsets, each one of which is closed in Y.*

Theorem 3.5.8. *In a noetherian topological space X, any nonempty closed subset $Y \subset X$ can be expressd as a finite union of irreducible closed subsets Y_i. If $Y_i \not\supseteq Y_j$ for any i, j, then Y_i are uniquely determined. They are called the irreducible component of Y*

Proof. Suppose that $\mathcal{C}$ is the set of nonempty closed subsets of X which cannot be expressed as a finite union of irreducible closed subsets. If $\mathcal{C}$ is not empty, since X is noetherian, then it has a minimal element, say Y. Then Y is not irreducible, thus we can write $Y = Y' \bigcup Y''$, where Y' and Y'' are proper closed subsets in Y. Since Y is minimal, Y' and Y'' are not contained in $\mathcal{C}$. Then each of Y' and Y'' can be expressed as a finite union of closed irreducible subsets, hence Y also, contradiction. Now suppose $Y = Y_1 \bigcup ... \bigcup Y_r$, where Y_i are closed irreducible subsets.

Assume $Y_i \not\supseteq Y_j$ for any i, j and $Y = Y_1' \bigcup ... \bigcup Y_r'$ is another such expression. Then $Y_1' \subset Y = Y_1 \bigcup ... Y_r$. so $Y_1' = \bigcup (Y_1' \cap Y_i)$. But Y_1' is irreducible, so $Y_1' \subset Y_i$ for some i, say $i = 1$. Similarly, $Y_1 \subset Y_j'$ for some j. Then $Y_1' \subset Y_j'$, so $j = 1$. Thus $Y_1 = Y_1'$. Now let $Z = (Y - Y_1)^-$. Then $Z = Y_2 \bigcup ... \bigcup Y_r$ and also $Z = Y_2' \bigcup ... \bigcup Y_s'$. So preceeding by induction on r, we obtain the uniqueness of the Y_i. □

Definition 3.5.9. *Let X be a topological space. We define the dimension of X, denoted by $dimX$ to be the maximal length of the chain $Y_0 \subset Y_1 \subset ... \subset Y_n$ of distinct irreducible closed subsets of X. We define the dimension of an affine or quasi-affine variety to be its dimension as a topological space.*

Proposition 3.5.10. *Let X = Spec A where A is Noetherian. Then $\dim X$ is equal to the maximal length of chains $\mathfrak{p}_1 \supseteq \mathfrak{p}_2 \supseteq \ldots \supseteq \mathfrak{p}_m$, where $\mathfrak{p}_i$ is a non-maximal prime ideal of A.*

Proof. The proof is left to the reader as an exercise. □

Example 3.5.11. We have dim Spec $\mathbb{Z} = 1$, $\dim \operatorname{Spec} \mathbb{Z}[x_1, \ldots, x_{n-1}] = n$.

Example 3.5.12. Let k be a field and $k[x]$ be its polynomial ring. Then $\dim k = 0$, $\dim k[x] = 1$, $\dim \operatorname{Spec} k[x_1, \ldots, x_n] = n$.

Corollary 3.5.13. *Let A be a ring, I be an radical ideal in A. In* Spec$A$*, $V(I)$ is irreducible if and only if I is an prime ideal. Then the dimension of A equals the maximal length of chain of prime ideals. Furthermore, if A is noetherian then A has finitely many minimal prime ideal p_i, $\bigcap_i p_i = \sqrt{0}$*

Proof. Let I be a prime ideal in A. If $V(I) = Y_1 \bigcup Y_2$ for some proper closed subsets Y_i, $i = 1, 2$. Then there are ideals I_1, I_2 such that $Y_i = V(I_i)$. Then $V(I) = V(I_1) \bigcup V(I_2) = V(I_1 I_2)$, which implies $I \supseteq I_1 I_2$. Since I is prime, either $I \supseteq I_1$ or $I \supseteq I_2$. Then $V(I_1) \subset V(I)$ or $V(I_2) \subset V(I)$, contradiction.

Conversely if $V(I)$ is irreducible. Suppose that there are elements $a, b \in A$ such that $a, b \notin I$ but $ab \in I$. $V(I) = V(a) \bigcup V(b)$. Thus $V(I) \subset (V(I) \bigcap V(a)) \bigcup (V(I) \bigcap Z(b)$. Since $V(I)$ is irreducible, then $V(I) = V(I) \bigcap V(a)$or $V(I) = V(I) \bigcap V(b)$, which implies $V(I) \subset V(a)$ or $V(I) \subset V(b)$. Assume that $V(I) \subset V(a)$. Thus $a \in \sqrt{I} = I$ □

3.6 Some Classical Algebraic Geometry

Definition 3.6.1. *Let L/K be an extension of fields. Let $x \in L$. We say x is algebraic in K if x saitisfies an equation $a_n x^n + \ldots + a_0 = 0$ with $a_n \neq 0$, $a_0, \ldots, a_n \in K$, otherwise we say x is transcendental over K.*

Definition 3.6.2. *Let L/K be a field extension. A subset S of L is a transcendence basis of L/K if it is algebraically independent over K and $L/K(S)$ is algebraic. The dimension of the subset S is the transcendence degree of L over K, denoted by* $\operatorname{trdeg}_K(L)$

Recall that if an element $\alpha \in L$ is algebraic over K iff there is an intermediate field $K \subset F \subset L$ such that $\alpha \in F$ and $[F : K] < \infty$.

Lemma 3.6.3. *If $\alpha_1, ..., \alpha_n \in L$ are algebraic over K then*

$$[K(\alpha_1, ..., \alpha_n) : K] < \infty \tag{3.6.1}$$

Definition 3.6.4. *The transcendental dimension of* $X = \mathrm{Spec}(A)$ *relative to* k*, denoted by* $\mathrm{trdim}_k X$ *is the transcendental degree of* K/k *which is the cardinality* r *of any subset* $\{x_1, \cdots, x_r\}$ *of* K *with the following properties:*

1. $x_1, \ldots, x_r$ *are algebraically independent.*

2. *every element* $x \in K$ *is algebraic over* $k(x_1, \cdots, x_r)$*.*

Such a set $\{x_1, \cdots, x_r\}$ *is called a transcendental base of* K/k*.*

This definition makes sense because r does not depend on the choice of $x_1, \ldots, x_r$.

Theorem 3.6.5. *Let* k *be a field, and let* A *be an integral domain which is finitely generated* k*-algebra. Then:*

1. *The dimension of* A *is equal to the transcendence degree of the quotient field* $K(A)$ *of* A *over* k*.*

2. *For any prime ideal* p *in* A*, we have*

$$\mathrm{height} p + \dim A/p = \dim A \tag{3.6.2}$$

Proof of part 1. .

Step 1: $\dim A \leq \mathrm{tr}_k \dim X$

If we have a maximal sequence of prime ideals

$$0 \subseteq \mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \mathfrak{p}_2 \subsetneq \ldots \subsetneq \mathfrak{p}_n. \tag{3.6.3}$$

We have a sequence of surjections $A \to A/\mathfrak{p}_0 \to A/\mathfrak{p}_1 \to \ldots$. It is easy to see tr deg $A_i \geq \mathrm{tr}\deg A_{i+1}$, where $A_i = A/\mathfrak{p}_i$. We need only to show that tr deg $A_i \neq$ tr deg A_{i+1}. So the first step is reduced to the following

Lemma 3.6.6. *Let* A *be an integral* k*-algebra of finite type. Let* $\mathfrak{p} \subseteq A$ *be a non-zero prime ideal. Then* $\mathrm{trdeg} A \neq \mathrm{trdeg} A/\mathrm{p}$*.*

Proof. Assume tr deg $A = \mathrm{tr}\deg A/\mathfrak{p} = r$. Write $A = k[x_1, \ldots, x_n]$ with $x_1, \ldots, x_r$ algebraically independent over k. Let $S = k[x_1, \ldots, x_r] - (0)$. The assumption implies $S \cap \mathrm{p} = \varnothing$. Therefore

$$S^{-1}A \overset{\neq}{\to} S^{-1}(A/\mathfrak{p}), \quad S^{-1}A = k(x_1, \ldots, x_r)[x_{r+1}, \ldots, x_n] \tag{3.6.4}$$

Claim: $S^{-1}A$ is a field.

Lemma 3.6.7. *Let* L *be a field with* A/L *an integral algebra which is finite dimensional as an* L*-vector space. Then* A *is a field.*

Proof. Let $x \in A - \{0\}$. The set $\{1, x, x^2, \ldots, x^n, \ldots\}$ must be linearly dependent so $\sum_{n=0}^{m} a_n x^n = 0$ for some $m > 0$ and $a_n \in L$ which are not all 0. By eliminating minimal power of x we may assume $a_0 \neq 0$. We have

$$a_0 + x\left(a_1 + a_2 x + \ldots\right) = 0 \quad \text{or} \quad \frac{x\left(a_1 + a_2 x + \ldots\right)}{-a_0} = 1 \tag{3.6.5}$$

Therefore x is invertible so A is a field. Now since $S^{-1}A$ is a field, the homomorphism to $S^{-1}(A/p)$ must be injective as it is nonzero, so must be bijective as it is already surjective. This is a contradiction! □

Exercise 3.6.8. Let S be a multiplicative system of R. Let $\mathfrak{p}$ be a prime ideal of R such that $S \cap \mathfrak{p} = 0$. Prove that

(a) $S^{-1}(A/\mathfrak{p}) = S^{-1}A/S^{-1}\mathfrak{p}$

(b) $S^{-1}\mathfrak{p}$ is a prime in $S^{-1}A$

(c) $S^{-1}\mathfrak{p} \cap A = \mathfrak{p}$

Moreover, every prime in $S^{-1}A$ has the form $S^{-1}\mathfrak{p}$ as above.

Step 2: $\dim X \geq \operatorname{tr}_k \dim X$

We will do this by induction on $\operatorname{tr}\dim X$. If $\dim X = 0$ then we are done. Assume $\operatorname{trdim} X = n > 0$ and $\operatorname{tr}\dim X \leq \dim X$ is true for varieties with dimension less than n. Write $A = k\left[x_1, \ldots, x_m\right]$ and assume x_1 is transcendental over k. Let $S = k\left[x_1\right] - (0)$, $k' = k\left(x_1\right)$, and

$$B = S^{-1}A = k'\left[x_2, \ldots, x_m\right] \tag{3.6.6}$$

then

$$\operatorname{tr}_{k'} \dim B = \operatorname{tr}_k \dim A - 1 = n - 1 \tag{3.6.7}$$

By the induction hypothesis there is a chain of primes in B of length $n - 1$

$$0 \subsetneq Q_1 \subsetneq Q_2 \subsetneq \ldots \subsetneq Q_{n-1} \tag{3.6.8}$$

Intersect this chain with A:

$$0 \subsetneq P_1 \subsetneq P_2 \subsetneq \ldots \subsetneq P_{n-1}, \quad P_i = Q_i \cap A \tag{3.6.9}$$

Exercise 3.6.9. Show that $P_i \neq P_{i+1}, S^{-1}P_i = Q_i, P_i \cap S = \varnothing$

Claim: P_{n-1} is not maximal (This will imply $\dim X \geq n$).

Otherwise, A/P_{n-1} is a field then by the Hilbert Nullstellensatz $A/P_{n-1} = k$. By exercise, $P_{n-1} \cap S = \varnothing$, thus the composition map

$$k[x] \longrightarrow A \longrightarrow A/\mathfrak{p} = k \tag{3.6.10}$$

is injective. This is impossible as x is transendental over k. The second step is proved □

□

Proof of part 2. Take a transcendental base $\{x_1, .., x_r\}$ of A as in part 1 with $r = \dim A$, and put $p' = p \bigcap k[y_1, ..., y_r]$. Then

$$\dim(A/p) = \dim(k[x_1, ..., x_r]/p'), \qquad \text{height} p = \text{height}\, p'.$$

As $k[x_1, ..., x_r]$ is isomorphic to the polynomial ring in r variables, we have

$$\text{height}\, p' + \dim(k[x_1, ..., x_r]/p') = r$$

by the theorem. □

Exercise 3.6.10. Let B be noetherian and integral ring. Let A be a finitely generated integral algebra over B. Then A is a field$\Longrightarrow$ B is a field

Exercise 3.6.11. If A is integral of finite type over $\mathbb{Z}$, $0 \neq f \in A$ is a non-invertible element, then $\dim A/fA = \dim A - 1$

3.7 Sheavs of Modules

Definition 3.7.1. *Let $(X, \mathcal{O}_X)$ be a ringed space. An $\mathcal{O}_X$-module over X is a sheaf $\mathcal{F}$ on X together with an $\mathcal{O}_X(U)$-module structure on $\mathcal{F}(U)$ for each open subset U on X such that for each inclusion of open sets $V \subset U$, the restriction homomorphism $\mathcal{F}(U) \to \mathcal{F}(V)$ is compatible with the module structures via the ring homomorphism $\mathcal{O}_X(U) \to \mathcal{O}_X(V)$.*

Let A be a ring and $X = \text{Spec}A$. Consider the affine scheme $(X, \mathcal{O}_X)$.

We define a functor ~ from the category of A-modules(denoted by $\mathcal{M}_A$) to the category of sheaf of $\mathcal{O}_X$-modules(denoted by $\mathcal{M}_{\mathcal{O}_X}$):

$$\begin{aligned} \sim\colon \mathcal{M}_A &\longrightarrow \mathcal{M}_{\mathcal{O}_X} \\ M &\longmapsto \tilde{M} \end{aligned} \tag{3.7.1}$$

Definition 3.7.2. *We define $\tilde{M}$ to be the sheaf associated to M on* Spec$A$.

For each prime ideal $p \subset A$, define the localization of M at p to be the tensor product $M \otimes A_p$, denoted by M_p. Since localization is exact, it is easy to see that:

$$M_p = \left\{ \frac{m}{g} \middle| \ g \notin p \right\} / \left\{ \frac{m_1}{g_1} = \frac{m_2}{g_2} \Leftrightarrow (g_2 m_1 - g_1 m_2) g_3 = 0 \text{ for some } g_3 \notin p \right\}, \tag{3.7.2}$$

where $m \in M$, $g_i \notin p$, $i = 1, 2, 3$.

If $f \in A$, we can also define the localization of M at f, denoted by $M[\frac{1}{f}]$ or , to be $M \otimes A[\frac{1}{f}]$.

Similar as the definition of $\mathcal{O}_X$, we can define the global sections of $\tilde{M}$ to be the function s from Spec$A$ to $\prod_{p \in \text{Spec}A} M_p$, gluing by M_f, $f \in A$.

Exercise 3.7.3. Write the definition of $\tilde{M}$.

odule prop **Proposition 3.7.4.** *Let $\varphi : A \to B$ be a ring homomorphism and let $f : \mathrm{Spec}B \to \mathrm{Spec}A$ be the correponding morphism of spectra. Then:*

(a) If M and N are two A-modules, then $(M \otimes_A N)^{\sim} \cong \tilde{M} \otimes_{\mathcal{O}_X} \tilde{N}$;

(b) if $\{M_i\}$ is any family of A-modules, then $(\oplus M_i)^{\sim} \cong \oplus \tilde{M}_i$;

(c) for any B-module N we have $f_*(\tilde{N}) \xrightarrow{\sim} \widetilde{N_A}$, where $\widetilde{N_A}$ means N considered as an A-module.

Proof. The functor $\sim$ commutes with tensor product and direct sum, since these commute with localization.

To prove (c), we cover $\mathrm{Spec}A$ with principle open subsets $\{D(f)\}_{f \in A}$. Then $f^{-1}(D(f)) = D(\varphi(f))$. Since $(N_A)_f \xrightarrow{\sim} N_{\varphi(f)}$ and $(N_A)_f \cong \widetilde{N_A}(D(f))$ then we have an isomorphism $\widetilde{N_A}(D(f)) \xrightarrow{\sim} \tilde{N}(D(\varphi(f)))$. By construction of $\tilde{N}$ we can glue $D(f)$ so $f_*(\tilde{N}) \xrightarrow{\sim} \widetilde{N_A}$. □

gamma tilde **Proposition 3.7.5.** *Let A be a ring, let M be an A-module, and let $\tilde{M}$ be the sheaf on $X = \mathrm{Spec}A$ associated to M. Then:*

(a) $\tilde{M}$ is an $\mathcal{O}_X$-module;

(b) for each $p \in X$, the stalk $(\tilde{M})_p$ of the sheaf $\tilde{M}$ at p is isomorphic to the localized module M_p;

(c) for any $f \in A$, the A_f-module $\tilde{M}(D(f))$ is isomorphic to the localized module M_f;

(d) in particular, $M \xrightarrow{\sim} \Gamma(X, \tilde{M})$.

Proof. Recalling the construction of the structure sheaf $\mathcal{O}_X$ from 3.3.1, it is clear that $\tilde{M}$ is an $\mathcal{O}_X$-module. The proofs of $(b), (c), (d)$ are identical to the proofs of the main theorem 3.3.4, replacing A by M at appropriate places. □

By part (b), we have the following:

Corollary 3.7.6. *The functor $\sim$ is exact: the exactness of sequence of A-modules*

$$0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow M_3 \longrightarrow 0 \tag{3.7.3}$$

implies the exactness of the sequence of O_X-modules:

$$0 \longrightarrow \widetilde{M_1} \longrightarrow \widetilde{M_2} \longrightarrow \widetilde{M_3} \longrightarrow 0 \tag{3.7.4}$$

Definition 3.7.7. *Let $(X, \mathcal{O}_X)$ be an affine scheme. A Sheaf $\mathcal{F}$ on X is called* quasi-coherent *if there is an affine covering $\{U_i = \mathrm{Spec}A_i\}$ of X such that for every i, $\mathcal{F}|_{U_i} \cong \tilde{M}_i$ for some A_i-module M_i. We say that $\mathcal{F}$ is* coherent *if A is noetherian and furthermore each M_i is finitely generated.*

Question 3.7.8. *Given any quasi-coherent sheaf $\mathcal{F}$ on a scheme X, can we construct an A-module corrsponding to $\mathcal{F}$?*

Proposition 3.7.9. *Let $\mathcal{F}$ be a quasi-coherent sheaf on a scheme X. Let $U = \mathrm{Spec} A$ be any open affine subscheme. Then $\mathcal{F}|_{\mathrm{Spec} A} = \widetilde{M}$ for some A-module M.*

It suffices to show the following

equi def qc **Proposition 3.7.10.** *Let $X = \mathrm{Spec} A$ be an affine scheme. Let $\mathcal{F}$ be a quasi-coherent sheaf on X. Then $F \simeq \widetilde{M}$ for some A-module M.*

(This means the functor $\sim: \mathcal{M}_A \to \mathcal{M}_{O_X}$ is fully faithful. And the image is the subcategory of quasi-coherent category.

Proof. Let $M = \Gamma(X, \mathcal{F})$. Then there is a morphism $\widetilde{M} \to \mathcal{F}$. It suffices to show that for any $f \in A$, the homomorphism $Q: M \otimes A_f \to \Gamma(X_f, \mathcal{F})$ of A_f-modules is an isomorphism. This follows from the following lemma. □

lem affine **Lemma 3.7.11.** *Let $X = \mathrm{Spec} A$ and let $\mathcal{F}$ be a quasi-coherent sheaf, $X_f \in \mathrm{Spec} A_f \hookrightarrow X$.*

1) If $s \in \Gamma(X, \mathcal{F})$, $s|_{X_f} = 0$ then $f^n s = 0$ for some n.

2) If $s \in \Gamma(X_f, \mathcal{F})$ is a section then there is a section $t \in \Gamma(X, \mathcal{F})$ such that $t|_{X_f} = f^n s$ for some n.

Proof. Let $M = \Gamma(X, \mathcal{F})$. For part 1, suppose $s \in M$ maps to $0 \in \Gamma(X_f, \mathcal{F})$. We want $s = 0$ in $M \otimes A_f$. We know $s = m/f^n$, $m \in M$. Cover X by affines X_{g_i}, $i = 1, \dots, n$, where $g_1, \dots, g_n \in A$ such that $F|_{X_{g_i}} = \widetilde{M_i}$, M_i is A_{g_i}-module.

$$\begin{array}{ccc} X & \longleftarrow & X_f \\ \uparrow & & \uparrow \\ \coprod X_{g_i} & \longleftarrow & \coprod X_{g_i} \cap X_f = \coprod X_{g_i \cdot f} \end{array}$$

Let $s_i = s|_{X_{g_i}}$ then $s_i \in M_i \otimes A_{fg_i}$ then $s_i = 0$ in $M_i \otimes A_{fg_i}$,

$$s|_{X_{g_i}} = \frac{m_i}{f^n}.$$

Thus some multiple of $f^m \cdot m_i = 0$, so $f^N \cdot s = 0$ in every X_{g_i}. Therefore $f^N \cdot s = 0$ on X.

For part 2, let $s_i = s|_{X_{g_i f}}$, $\mathcal{F}|_{X_{g_i}} = \widetilde{M_i}$. We want to lift s_i to some section in $\mathcal{F}|_{X_{g_i}} = \widetilde{M}$. But this is obvious:

$$\mathcal{F}|_{X_{g_i f}} = \widetilde{M_i \otimes A_f}.$$

Thus we have sections $t_i \in \Gamma(X_{g_i}, \mathcal{F})$ such that $t_i|_{X_{g_i f}} = f^n s|_{X_{g_i f}}$. Can we glue t_i to obtain a section of $\mathcal{F}$ on X? We want to check $(t_i - t_j)|_{X_{g_i g_j}} = 0$. But

$$(t_i - t_j)|_{X_{g_i g_j f}} = \left(f^n s - f^n s\right)|_{X_{g_i g_j f}} = 0.$$

Therefore by the first part of the lemma $f^m(t_i - t_j) = 0$ on $X_{g_i g_j}$ for some $m \geq 0$. This yields that we can glue $f^m t_i$ and we are done. □

The proposition 3.7.5 and 3.7.10 gives the following theorem:

Theorem 3.7.12. *Let A be a ring, and Let $X = \operatorname{Spec} A$, then the functor $\sim: \mathcal{M}_A \to \mathcal{M}_{\mathcal{O}_X}$ induces an equivalence of categories. The converse of $\sim$ is given by $\mathcal{F} \mapsto \Gamma(X, \mathcal{F})$, where $\mathcal{F}$ is any quasi-coherent sheaf on X.*

Lemma 3.7.13. *Let X be a scheme. The kernel, cokernel, and image of morphism of quasi-coherent sheaves are quasi-coherent.*

Proof. This question is local, so we may assume $X = \operatorname{Spec} A$ is affine. Since the kernel, cokernel, and image of morphism of A-modules are A-modules, by the theorem 3.7.12 we get this lemma. □

Remark 3.7.14. By this theorem, the exactness of a sequence $0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{G} \to 0$ of quasi-coherent sheaves on X is equivalent to the exactness of the sequence of global sections

$$0 \to \Gamma(X, \mathcal{E}) \to \Gamma(X, \mathcal{F}) \to \Gamma(X, \mathcal{G}) \to 0.$$

In fact, we can use 3.7.11, we can strength this statement as follows.

Proposition 3.7.15. *Let X be an affine scheme and let $0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{G} \to 0$ be an exact sequence of $\mathcal{O}_X$-modules, and assume that $\mathcal{E}$ is quasi-coherent. Then the sequence*

$$0 \to \Gamma(X, \mathcal{E}) \to \Gamma(X, \mathcal{F}) \xrightarrow{\varphi} \Gamma(X, \mathcal{G}) \to 0 \tag{3.7.5}$$

is exact.

Proof. We know already that Γ is a left-exact functor so we only need to show that the last map is surjective. Let $s \in \Gamma(X, \mathcal{G})$ be a global section of $\mathcal{G}$. Since the map of sheaves $\mathcal{F} \to \mathcal{G}$ is surjective, by 2.1.9 we know that for any $x \in X$ there is an open neighborhood $D(f)$ of x such that $s|_{D(f)}$ lifts to a section $t \in \mathcal{F}(D(f))$.

Claim: for some $n > 0$, $f^n s$ lifts to a global section of $\mathcal{F}$.

Indeed, we can cover X with a finite number of open subsets $D(g_i)$ such that for each i, $s|_{D(g_i)}$ lifts to a section $t_i \in \mathcal{F}(D(f_i))$. On $D(f) \cap D(g_i) = D(fg_i)$, we have two sections $t, t_i \in \mathcal{F}(D(fg_i))$ both lifting s. Therefore $t - t_i \in \mathcal{E}(D(fg_i))$. Since $\mathcal{E}$ is quasi-coherent, by 3.7.11 for some $n > 0$, $f^n(t - t_i)$ extends to a section $u_i \in \mathcal{E}(D(g_i))$. As usual, pick n sufficiently large to work for all i. Let $t_i' = f^n t_i + u_i$. Now on $D(g_i g_j)$ we have two sections t_i' and t_j' are equal on $D(fg_ig_j)$, so by 3.7.11 we have $f^m(t_i' - t_j') = 0$ for some $m > 0$ and independent i, j. Now the sections $f^m t_i'$ of $\mathcal{F}$ glue to give a global section t'' of $\mathcal{F}$ over X, which lifts $f^{n+m}s$. This proves the claim.

Now cover X by a finite number of open sets $D(f_i)$, $i = 1, 2, \ldots, r$ such that $s|_{D(f)}$ lifts to a section of $\mathcal{F}$ over $D(f_i)$ for each i .Then by the claim, we can find an integer $n \gg 0$ and global section s $t_i \in \Gamma(X, \mathcal{F})$ such that t_i is a lifting of $f_i^n s$. Since $X = \bigcup_{i=1}^r D(f_i)$, so the ideal $(f_1^n, \ldots, f_r^n)$ unit ideal of A. Then we can write $1 = \sum_{i=1}^r a_i f_i^n$ with $a_i \in A$. Let $t = \sum a_i t_i$. Then t is a global section of $\mathcal{F}$ whose image in $\Gamma(X, \mathcal{G})$ is $\sum a_i f_i^n s = s$. This complete the proof.

□

Exercise 3.7.16. Let A be a noetherian ring and let $\mathcal{F}$ be quasi-coherent sheaf. Then $\mathcal{F}$ is coherent if and only if for any $f \in A$, $\mathcal{F}(D(f))$ is finitely generated over A_f.

Remark 3.7.17. Let $\mathcal{F}$ be a quasi-coherent sheaf on a topological space X. If for any $p \in X$, the stalk $\mathcal{F}_p$ is finitely generated, we can not conclude that $\mathcal{F}$ is finitely generated. Here is an example: $\oplus_p \mathbb{Z}/p\mathbb{Z}$ is finitely generated on every local places, but itself is not finitely generated.

Exercise 3.7.18. Let A be a ring and let M be a finitely generated A-module. Then M is projective if and only if M is locally free. In particular, if A is local ring, then every projective module of finite rank is free.

l q-c sheaf **Proposition 3.7.19.** *Let $f : X \to Y$ be a morphism of schemes. Assume that X is noetherian, then if $\mathcal{F}$ is a quasi-coherent sheaf of $\mathcal{O}_X$-module, $f_*\mathcal{F}$ is a quasi-coherent sheaf of $\mathcal{O}_Y$-module.*

Proof. Since X is noetherian, then we can cover X by a finite number of open affine subsets U_i. Since an open subset of a noetherian topological space is noetherian, then we can cover $U_{ij} = U_i \cap U_j$ by finite number of open affine subsets U_{ijk}. Now given any open subset V of Y, $\mathcal{F}|_{f^{-1}(V)}$ saitisfies gluing condition with respect to U_i and U_{ijk}. Thus there is an exact sequence of sheaves:

$$0 \to f_*\mathcal{F} \to \bigoplus_i f_*(\mathcal{F}|_{U_i}) \to \bigoplus_{i,j} f_*(\mathcal{F}|_{U_{ijk}}), \tag{3.7.6}$$

now $f_*(\mathcal{F}|_{U_i})$ and $f_*(\mathcal{F}|_{U_{ijk}})$ are quasi-coherent by module prop 3.7.4. Thus $f_*\mathcal{F}$ is also quasi-coherent as a kernel of morphism quasi-coherent sheaves on affine schemes. □

Remark 3.7.20. We sometimes use words 'vector bundles' and 'locally free sheaves' interchangeably, if no confusion seems likely to result.

3.8 The Grothendieck group on affine schemes

Definition 3.8.1. *Let A be a ring, let $\mathcal{C}$ be the category of coherent sheaves on* Spec$A$ *and let $\mathcal{D}$ be the category of locally free sheaves. With the definition of Grothendieck group in chapter 1, we can define the Grothendieck groups on affine scheme is by quotient:*

$$K_0(\mathrm{Spec}A) = F(\mathcal{C})/E(\mathcal{C}) \qquad K^0(\mathrm{Spec}A) = F(\mathcal{D})/E(\mathcal{D}). \tag{3.8.1}$$

Exercise 3.8.2. Let A be a ring, prove that $K^0(\mathrm{Spec}A) = K^0(A)$, and $K_0(\mathrm{Spec}A) = K_0(A)$

Remark 3.8.3. The above result shows the close connection among finitely generated projective modules on a ring A and locally free sheaves on Spec$A$ and vector bundles on Spec$A$. This can be made precise by stating that the following map is an equivalence of categories:

$$\left\{\begin{matrix}\text{finitely generated projective}\\ \text{modules over } A\end{matrix}\right\} \xrightarrow{\sim} \left\{\begin{matrix}\text{locally free sheaves}\\ \text{on Spec}A\end{matrix}\right\} \tag{3.8.2}$$

4 Schemes

4.1 Schemes and quai-coherent sheaves

Affine schemes are building blocks of schemes.

Definition 4.1.1. *A scheme X is a* locally ringed space $(X, \mathcal{O}_X)$ *in which every point has an open neighborhood U such that the topological space U, together with the restricted sheaf $\mathcal{O}_X|_U$, is an affine scheme. All schemes together form a category* Sch *with morphisms defined as morphisms of ringed spaces.*

On each scheme X, we have a category $\mathrm{QCoh}(X)$ of quasi-coherent sheaves $\mathcal{F}$ which are defines to be $\mathcal{O}_X$-modules so that on each affine open subset $U = \mathrm{Spec}A$ of X, $\mathcal{F}|_U = \widetilde{M}$ for some A-modules. If $f : X \longrightarrow Y$ be a morphism of schemes, then we have pushforward and pullback functors

$$f_* : \mathrm{QCoh}(X) \longrightarrow \mathrm{QCoh}(Y), \qquad f^* : \mathrm{QCoh}(Y) \longrightarrow \mathrm{QCoh}(X).$$

They are adjoin to each other in the following sense: for any $\mathcal{F} \in \mathrm{QCoh}(X)$, $\mathcal{G} \in \mathrm{QCoh}(Y)$, there is a canonical isomorphism

$$\mathrm{Hom}(f^*\mathcal{G}, \mathcal{F}) \xrightarrow{\sim} \mathrm{Hom}(\mathcal{G}, f_*\mathcal{F}).$$

Moreover f_* is left exact, and f^* is right adjoint.

The category $\mathrm{QCoh}(X)$ is closed under operator $\otimes$, and $\mathcal{H}om$ with adjoint formula and adjunction formula.

Definition 4.1.2. *A scheme X is called noetherian if and only if X is covered by finitely many affine open subset $U = \mathrm{Spec}A$ with A a noetherian ring.*

Definition 4.1.3. *Let X be a Noetherian scheme. A Sheaf $\mathcal{F}$ on X is called coherent if there is an affine covering $\{U_i = \mathrm{Spec}A_i\}$ of X such that for every i, $\mathcal{F}|_{U_i} \cong \tilde{M}_i$ for some finitely generated A_i-module M_i. All coherent sheaves on X form an abelian category* $\mathrm{Coh}(X)$.

All results for quasi-coherent sheaves hold for coherent sheaves, except push-forward morphism. We will show that f_* can be defined for projective morphism in future section.

Here is a standard way to construct a quasi-coherent sheaves:

si-coherent **Theorem 4.1.4.** *Let X be a scheme. For each open affine subset $U = \mathrm{Spec}A$ of X, let $M(U)$ be an A-module such that for each $f \in A$, $M(U_f) = M_f$. Then there is a quasi-coherent sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules with isomorphism $\alpha_U : \widetilde{M(U)} \xrightarrow{\sim} \mathcal{F}_U$ of $\mathcal{O}_X(U)$-modules. Moreover the collection $(\mathcal{F}, \alpha_U)$ is unique up to isomorphisms.*

Sketch of proof. For each point $x \in X$, we define the stack M_x as the direct limit of M_U for affine neighborhood of x. For each open subset U of X, we define a module $\mathcal{F}(U)$ as the subset of elements $m \in \prod_{x \in U} M_x$ so that there is an affine covering $U = \bigcup_i V_i$ and elements

$m_i \in M_{V_i}$ such that for $x \in V_i$, $m_x = m_{ix}$. It is clear that $\mathcal{F}$ is a sheave of $\mathcal{O}_X$-modules on X. and that there is homomorphism $\alpha_U : \widetilde{M(U)} \longrightarrow \mathcal{F}_U$ of $\mathcal{O}_X(U)$-modules for each affine subset U of X.

We want to prove that α_U is an isomorphism. Thus we are reduce to the affine case $X = \mathrm{Spec}A$. Write $M = M_X$. Since ops, ens $D(f)$ form a fundamental system of open subsets of X, we may replace affine subsets of X by $D(f)$ in the construction of M_x and $\mathcal{F}$. Thus we obtain that $\mathcal{F} = \widetilde{M}$. Thus α_U is an isomorphism.

The uniqueness of $(\mathcal{F}, \alpha_U)$ is clear. □

Exercise 4.1.5. Let U be an open subset of a scheme X, denote $i : U \hookrightarrow X$ the natural inclusion. Let $\mathcal{F}$ be a coherent sheaf on U. Then there exists a coherent sheaf $\mathcal{G}$ on X such that $i^*\mathcal{G} = \mathcal{F}$.

4.2 Open immersion and closed immersion.

Definition 4.2.1. *Let $X \to Y$ be a morphism of schemes. We say f is an* open immersion *if*

1) f is a homomorphism from X_{top} to an open subset of Y_{top}.

2) The induced morphism of schemes

$$\big(f(X_{top}), \mathcal{O}|_{f(X_{top})} f(X_{top})\big) \longrightarrow (X_{top}, \mathcal{O}_X)$$

is an isomorphism. In other words the induced morphism of schemes is isomorphic.

We say f is a closed immersion *if the image of X_{top} is a closed subset of Y and the induced morphism $f^\sharp : f^{-1}(\mathcal{O}_y) \to \mathcal{O}_x$ is surjective.*

If $X \to Y$ is a closed morphism then we call $X \to Y$ is a closed subscheme (embedding).

Example 4.2.2. Let $X = \mathrm{Spec}A$, $Y = \mathrm{Spec}B$ and let $f : X \to Y$ be given by $B \to A$.

1) $A = B_g$, where $g \in A$. In this case $X \xrightarrow{\sim} D(f) \hookrightarrow X$ is an open morphism.

2) $A = B/I$. Here $X \to Z(I) \hookrightarrow Y$ is a closed immersion. $B \to B/I$.

Remark 4.2.3. Let X be a scheme and Y be a closed subspace of the topological space X. Then there is at least one closed subscheme structure on Y. This means $\mathcal{O}_X|_Y$ has a quotient sheaf $\mathcal{O}_Y$ such that $(Y, \mathcal{O}_Y)$ is a scheme.

Example 4.2.4. Let k be an algebraically closed field. Let $X = \mathrm{Spec}k[x,y] = \mathbb{A}^2$ and let ideal $I = (y - x^2 - x)$. Then suppose Y is the subspace defined by the zeros of the function $y = x^2 + x$, i.e.

$$Y = \{(x,y) \in k^2 | y = x^2 + x\}.$$

Thus for any positive integer n, Y has a scheme structure $Y = Spec k[x,y]/I^n$, with $\mathcal{O}_Y \cong (k[x,y]/I^n)^\sim$.

Definition 4.2.5. *Let X be a scheme and let $Y \subset X$ be a closed subscheme. Let $i : Y \to X$ be the inclusion morphism. We define the* ideal sheaf *of Y, denoted by $\mathcal{I}_Y$, to be the kernel of the morphism $i^\sharp : \mathcal{O}_X \to i_*\mathcal{O}_Y$.*

sheaf prop **Proposition 4.2.6.** *If Y is a closed subscheme of an affine scheme $X = \mathrm{Spec}A$, then the ideal sheaf corresponding to Y is a quasi-coherent sheaf of ideals of X. Moreover, there is an ideal $I \subset A$ such that $Y \xrightarrow{\sim} \mathrm{Spec}A/I$*

Proof. Let $i : Y \hookrightarrow X$ be the closed embedding, and let $\mathcal{I}$ be the kernel of corresponding morphism of structure sheaves $i^\sharp : \mathcal{O}_X \to i_*\mathcal{O}_Y$. Then we have an exact sequence of sheaves:

$$0 \to \mathcal{I} \to \mathcal{O}_X \to i_*\mathcal{O}_Y \to 0. \tag{4.2.1}$$

Since $\mathcal{I}$ is the kernel of morphism of two quasi-coherent sheaves, thus $\mathcal{I}$ is quasi-coherent.

Since X is affine. Apply the functor $\Gamma(X, \cdot)$, there is an exact sequence of A-modules

$$o \to \Gamma(X, \mathcal{I}) \to A \to \Gamma(X, i_*\mathcal{O}_Y) \to 0. \tag{4.2.2}$$

it follows that $I = \Gamma(X, \mathcal{I})$ is a submodule of A thus an ideal. And $\Gamma(X, i_*\mathcal{O}_Y) \xrightarrow{\sim} A/I$. Moreover, $\mathcal{I} \xrightarrow{\sim} \tilde{I}$. Clearly $Y \xrightarrow{\sim} \mathrm{Spec}A/I$ □

Remark 4.2.7. If X is any scheme, and $Y \subset X$ is a closed subscheme, then the ideal sheaf corresponding to Y is also a quasi-coherent sheaf of ideals.

Remark 4.2.8. If as topological space, $Y|_{\mathrm{top}}$ is a closed subspace of a scheme X, then Y may have many closed subscheme structures. If we define $\mathcal{J}$ to be the sheaf associated to the presheaf

$$U \mapsto \{s \in \mathcal{O}_X(U) |\ s = 0 \text{ on } Y\}, \tag{4.2.3}$$

where U is an open subset of X. And

$$Y = (Y|_{\mathrm{top}}, \mathcal{O}_X/\mathcal{J}), \qquad Y_n = (Y|_{\mathrm{top}}, \mathcal{O}_X/\mathcal{J}^n). \tag{4.2.4}$$

Then Y is a closed subscheme of the Y_n's. Actually, Y is the smallest subscheme of X with given $Y|_{\mathrm{top}}$.

Definition 4.2.9. *We say a scheme X is* reduced *if for any open subset U, the ring $\mathcal{O}_X(U)$ has no nilpotent elements.*

Corollary 4.2.10. *A scheme X is reduced if and only if the topological space $X|_{top}$ has no smaller closed subscheme structure.*

Remark 4.2.11. Let X be a reduced scheme. Then for any open affine subset $U = \mathrm{Spec}A$, suppose that the section $f \in \mathcal{O}_X(U)$ has zero value at every point $x \in U$, then we have for any prime ideal $p \subset A$, $f \in p$. It follows that $D(f) = \varnothing$, thus f can only be zero since X is reduced.

Definition 4.2.12. *A scheme X is called* irreducible *if $X|_{top}$ is irreducible; is called* connected *if $X|_{top}$ is connected.*

Definition 4.2.13. *We say a scheme X is* integral *if for any open affine cover $\{\mathrm{Spec}A_i\}$, the rings A_i are integral domains. Or equivalently, for any open set $U \subset X$, the ring $\mathcal{O}_X(U)$ is an integral domain.*

Proposition 4.2.14. *An integral scheme X is integral if and only if it is both reduced and irreducible.*

Proof. Clearly an integral scheme is reduced. Since this is a local question we assume that $X = \mathrm{Spec}A$ is affine. If X is not irreducible, then there is two open subset U_1 and U_2 such that $X = U_1 \bigcup U_2$. There exists nonzero elements $e_1, e_2 \in A$ such that $e_1|_{U_1} = 1$, $e_2|_{U_1} = 0$ in $\mathcal{O}(U_1)$ and $e_1|_{U_2} = 0$, $e_2|_{U_2} = 1$ in $\mathcal{O}(U_2)$ by the property of sheaves. Thus clearly $e_1 e_2 = 0$. So $A = \mathcal{O}(U_1) \times \mathcal{O}(U_2)$ which is not an integral domain.

Conversely assume that X is reduced and irreducible. If there exists $f, g \in A$ such that $fg = 0$, we want to show that $f = 0$ or $g = 0$. Consider

$$X_f^c = \{x \in X |\ f_x \in \mathfrak{m}_x\}, \qquad X_g^c = \{x \in X |\ g_x \in \mathfrak{m}_x\}$$

where for any $x \in X$, $\mathfrak{m}_x$ is the maximal ideal of $\mathcal{O}_x$. We have shown that X_f is open, then both X_f^c and X_g^c are closed and $X = X_g^c \bigcup X_f^c$. Without loss of generity we assume that $X = X_g^c$. Then $D(g) = X_g = \varnothing$, it follows that g is nilpotent. But X is reduced, thus $g = 0$. □

4.3 Fibre product and Base change

Now we start with the category of schemes. This category has a final object $\mathrm{Spec}\mathbb{Z}$. Indeed, since for any scheme X, there is a morphism

$$\mathbb{Z} \longrightarrow \Gamma(X, \mathcal{O}_X). \tag{4.3.1}$$

This induces a morphism of topological spaces:

$$X \longrightarrow \mathrm{Spec}\mathbb{Z}. \tag{4.3.2}$$

Let S be a scheme. A scheme X over S means there is a morphism of schemes $X \to S$. Let Sch/S denote the category of schemes over S. Then the object of this category is $X \to S$. Let Y be any other scheme. The morhism in Sch/S is naturally defined by a commutative diagram:

$$\begin{array}{ccc} X & \longrightarrow & S \\ \downarrow & & \| \\ Y & \longrightarrow & S \end{array} . \tag{4.3.3}$$

Then we show that the category of schemes has fibre product.

Definition 4.3.1. *Let $\mathcal{C}$ be a category. Let X, Y be two objects in $\mathcal{C}$. A product of X and Y is defined as an object Z with two morphisms*

$$\begin{array}{ccc} X & & Y \\ & \nwarrow \quad \nearrow & \\ & Z & \end{array}, \tag{4.3.4}$$

which saitisfy the universal property: if there is an object Z' with morphisms $Z' \to X$ and $Z' \to Y$, then the diagram below commutes

$$\begin{array}{ccc} & Z' & \\ \swarrow & \downarrow & \searrow \\ X & & Y \\ \nwarrow & & \nearrow \\ & Z & \end{array} \tag{4.3.5}$$

Definition 4.3.2. *Let $\mathcal{C}$ be a category. Let X, Y, Z be objects in $\mathcal{C}$. Suppose we have morphisms $X \to Z$ and $Y \to Z$. Then the* fibered product *of X and Y over Z, denoted $X \times_Z Y$, is an object such that the diagram*

$$\begin{array}{ccc} X \times_Z Y & \longrightarrow & Y \\ \downarrow & & \downarrow \\ X & \longrightarrow & Z \end{array}. \tag{4.3.6}$$

commutes, and has the universal property: if there is an object W saitisfies the same condition, then there is a unique morphism $W \to X \times_Z Y$ such that the following diagram commutes:

$$\begin{array}{cccc} W & & & \\ & \overset{\exists!}{\searrow} & & \searrow \\ & X \times_Z Y & \longrightarrow & Y \\ \searrow & \downarrow & & \downarrow \\ & X & \longrightarrow & Z \end{array} \tag{4.3.7}$$

Theorem 4.3.3. *There exists fibered product in the category* Sch/S*, and the fibered product is unique under universal property.*

Proof. By glueing property we need only prove the theorem for affine schemes. Let $S = \mathrm{Spec}R$, $X = \mathrm{Spec}A$, $Y = \mathrm{Spec}B$, where R, A, B are commutative rings. Since we have morphism $X \to S$ and $Y \to S$, by considering the underlying homomorphism of rings $R \to A$ and $R \to B$ we have the following commutative diagram:

$$\begin{array}{ccccccccc} & a \otimes 1 & & & A \otimes_R B & & & 1 \otimes b & \\ \nearrow & & & \nearrow & & \nwarrow & & & \nwarrow \\ a & & A & & & & B & & b \\ & & & \nwarrow & & \nearrow & & & \\ & & & & R & & & & \end{array}.$$

By definition of tensor product it has universal property. This diagram induces a commutative diagram of schemes:

$$\begin{array}{ccccc} & & \mathrm{Spec} A \otimes_R B & & \\ & \swarrow & & \searrow & \\ \mathrm{Spec} A & & & & \mathrm{Spec} B \\ & \searrow & & \swarrow & \\ & & \mathrm{Spec} R & & \end{array}.$$

To show that $\mathrm{Spec} A \otimes_R B$ is a fiber product of $\mathrm{Spec} B$ and $\mathrm{Spec} A$ over S is equivalent to showing that $A \otimes_R B$ is the co-product of A, B over R in the category of algebras over R. But this is exactly the universal property of the tensor product $A \otimes_R B$. □

Example 4.3.4. Let $X = Y = \mathrm{Spec}\mathbb{Q}[\sqrt{2}]$, $S = \mathrm{Spec}\mathbb{Q}$. Then

$$X \times_S Y = \mathrm{Spec}\mathbb{Q}[\sqrt{2}] \otimes_{\mathbb{Q}} \mathrm{Spec}\mathbb{Q}[\sqrt{2}] \tag{4.3.8}$$
$$= \mathrm{Spec}\mathbb{Q}[\sqrt{2}] \otimes_{\mathbb{Q}} \frac{\mathbb{Q}[x]}{x^2 - 2} \tag{4.3.9}$$
$$= \mathrm{Spec}\frac{\mathbb{Q}[\sqrt{2}][x]}{x^2 - 2} = \mathrm{Spec}\mathbb{Q}[\sqrt{2}] \oplus \mathbb{Q}[\sqrt{2}] \tag{4.3.10}$$
$$= \mathrm{Spec}\mathbb{Q}[\sqrt{2}] \coprod \mathbb{Q}[\sqrt{2}] \tag{4.3.11}$$

Remark 4.3.5. Let X, Y be schemes over a scheme S. If $S = \mathrm{Spec}\mathbb{C}$, let $X(\mathbb{C}) = \mathrm{Hom}(\mathrm{Spec}\mathbb{C}, X)$. Then

$$(X \times_S Y)(\mathbb{C}) \xrightarrow{\sim} X(\mathbb{C}) \times Y(\mathbb{C}).$$

Definition 4.3.6. *Let S, S' be schemes and suppose there is a morphism $S' \to S$. Let X be a scheme over S. A* base change *of X is defined to be the fibre product $X' = X \times_S S'$. In other words, we have the commutative diagram with niversal property:*

$$\begin{array}{ccc} X \times_S S' & \longrightarrow & S' \\ \downarrow & & \downarrow \\ X & \longrightarrow & S \end{array}$$

Notice that the base change defines a fuctor from the category Sch/S *to the category* Sch/S' *which is the right adjoint of the forgetful functor* Sch/$S' \to$ Sch/S

Proposition 4.3.7. *Let $\mathcal{C}$ be any category. If we have morphisms $X_1 \to Y$, $X_2 \to Y$ and $Y \to Z$. Then $X_1 \times_Y X_2$ is the fibered product of Y and $X_1 \times_Z X_2$ over $Y \times_Z Y$:*

$$\begin{array}{ccc} X_1 \times_Y X_2 & \longrightarrow & X_1 \times_Z X_2 \\ \downarrow & & \downarrow \\ Y & \longrightarrow & Y \times_Z Y \end{array}$$

Proof. The map $X_1 \times_Y X_2 \to X_1 \times_Z X_2$ corresponds to a pair of maps $X_1 \times_Y X_2 \to X_1$ and $X_1 \times_Y X_2 \to X_2$ whose composition to Z are the same. Take the projections from the fibred product and notice that there composition to Y are the same, then the same is true for Z.

The map $X_1 \times_Z X_2 \to Y \times_Z Y$ is induced by two maps $X_1 \to Z$ and $X_2 \to Z$. The map $Y \to Y \times_Z Y$ is the diagonal map which is defined to be Id_Y in both factors. Finally, the morphism $X_1 \times_Y X_2 \to Y$ is the natural map factor through X_1 and X_2. Thus every morphism in the diagram is canonical and the diagram commutes. □

Finiteness conditions

Definition 4.3.8. *A morphism $f : X \to Y$ of schemes is called* finite type *if for every open affine subset $V = \mathrm{Spec}B$ of Y, $f^{-1}(V)$ can be covered by a finite number of open affines $U_i = \mathrm{Spec}A_i$ and A_i is finitely generated B-algebra.*

Remark 4.3.9. A scheme X is called finite type over a field k if the morphism $f : X \to \mathrm{Spec}k$ is of finite type.

Definition 4.3.10. *A morphism $f : X \to Y$ of schemes is called* finite morphism *if for every open affine subset $V = \mathrm{Spec}B$ of Y, $f^{-1}(V) = \mathrm{Spec}A$ is affine and A is finite B-module.*

Example 4.3.11. The morphism $f : \mathrm{Spec}\mathbb{Z}_p \to \mathrm{Spec}\mathbb{Z}$ is not of finite type.

Definition 4.3.12. *Let X, Y be schemes. A morphism $f : X \to Y$ is called* quasi-compact *if for any open subset $V \subset Y$, $f^{-1}(V)$ is quasi-compact. i.e. every cover has a finite subcover.*

Proposition 4.3.13. *A morphism of finite type is stable under base change.*

Proof. Let $f : Y \to X$ be morphism of schemes. Suppose there is a morphism of schemes $\varphi : X' \to X$. Let $Y' = Y \times_X X'$. We only need to show that the morphism $f' : Y' \to X'$ is of finite type.

First we have the following commutative diagram:

$$\begin{array}{ccc} Y' & \xrightarrow{f'} & X' \\ \phi \downarrow & & \downarrow \varphi \\ Y & \xrightarrow{f} & X \end{array}, \tag{4.3.12}$$

where Y' has the universal property. For any affine open subset $V = \mathrm{Spec}A \subset X$, $f^{-1}(V)$ can be covered by finite $U_i = \mathrm{Spec}B_i \subset Y$, where B_i is finitely generated A-module. Assume that $\mathrm{Spec}A' = V' \subset \varphi(V)$ is an affine open subset of X'. Since ϕ, f' are projections and

$$f'^{-1}(V') = f^{-1}(V) \times_X V', \qquad \phi(U_i) = U_i \times_X V' = \mathrm{Spec}(B_i \otimes_A A').$$

Since $\{U_i\}$ cover $f^{-1}(V)$ thus $\{U_i \times_X V'\}$ cover $f^{-1} \times_X V'$.

The rest part is to show that $B_i \times_A A'$ is finitely generated A-module. Assume that the generators of B_i over A is $b_1, \cdots, b_n$, then $b_1 \otimes 1, \cdots, b_n \otimes 1$ generate $B_i \otimes_A A'$ over A'. □

4.4 Separaness and Properness

We will only consider Noetherian schemes in this section.

Separaness

Definition 4.4.1. *Let $f: X \to S$ be a morphism of schemes. We say f is* separated *if*

$$\Delta = (id, id): X \longrightarrow X \times_S X$$

is a closed morphism.

Proposition 4.4.2. *Let $f: X \to S$ be a morphism of affine schemes. The f if separated.*

Proof. Let $X = \operatorname{Spec} A$, $S = \operatorname{Spec} B$ then $X \times_S X = \operatorname{Spec} A \otimes_B A$. The diagonal moprhism is induced by $A \otimes_B A \to A$.

Separatedness is equivalent to surjectivity of $A \otimes_B A \to A$. This is obvious. □

A criterion of separatedness. Let $f: X \to S$ be a morphism of schemes. Then f is separated if and only if for any discrete valuation ring R with a fraction field K and any commutative diagram morphism

$$\begin{array}{ccc} \operatorname{Spec}(K) & \longrightarrow & X \\ \downarrow & & \downarrow \\ \operatorname{Spec}(R) & \longrightarrow & S \end{array}$$

there is at most one extension $\operatorname{Spec}(R) \to X$.

Proposition 4.4.3. *Separatness is closed under the base change. In other words, for any base change diagram ($X' = X \times_S S'$)*

$$\begin{array}{ccc} X' & \longrightarrow & X \\ \downarrow{\scriptstyle sep} & & \downarrow{\scriptstyle sep} \\ S' & \longrightarrow & S \end{array}$$

if X/S is separated, then X'/S' is separated.

Proof. We apply the separatedness criterion. Consider the following diagram with R a DVR with fraction field K,

$$\begin{array}{ccccc} \operatorname{Spec}(K) & \longrightarrow & X \times_S S' & \longrightarrow & X \\ \downarrow & & \downarrow & & \downarrow \\ \operatorname{Spec}(R) & \longrightarrow & S' & \longrightarrow & S \end{array}$$

Since X/S is separated, there is at most one extension $\operatorname{Spec} R \to X$. Thus there is at most one extension $\operatorname{Spec} R \to X'$. □

Proof. This is a local question. So we suppose that $f : X = \mathrm{Spec}A \to Y = \mathrm{Spec}B$ is finite morphism. Let R be a valuation ring and K be its quotient field. consider the base change

$$\begin{array}{ccc} X \times_Y \mathrm{Spec}R & \longrightarrow & \mathrm{Spec}R \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}.$$

Then reduce to the case $Y = \mathrm{Spec}R$, thus A is free of finite rank over R. □

Proposition 4.4.4. *1) Closed immersions and open immersions are separated.*
2) Composition of separated morphism is separated.

Exercise 4.4.5. Prove the above proposition.

Exercise 4.4.6. Prove that if a scheme X is separable, then for any affine open subset $U, V \subset X$, $U \bigcap V$ is also affine.

Properness

Definition 4.4.7. *a)* $f{:}\, X \to S$ *is called* closed *if for any closed subscheme* $Z \hookrightarrow X$, $f(Z) \subset S$ *is closed.*
b) f *is called* universally closed *if for any* $g{:}\, S' \to S$ *the base change* $f'{:}\, X' \to S'$ *of* f *is closed.*

Definition 4.4.8. *Let* $f{:}\, X \to S$ *be a morphism. We say* f *is* proper *if*
1) f *is separated;*
2) f *is universally closed.*

Theorem 4.4.9 (Criterion for properness)**.** *Properness of* f *is equivalent to the the fact that any diagramm of the following type*

$$\begin{array}{ccc} \mathrm{Spec}(K) & \longrightarrow & X \\ \downarrow & & \downarrow f \\ \mathrm{Spec}(R) & \longrightarrow & S \end{array}$$

has a unique extension: $\mathrm{Spec}R \to X$.

Proposition 4.4.10. *1) Properness is closed under the base change.*
2) Closed immersion is proper.

Recall $\mathbb{P}^n = \mathrm{Proj}\,\mathbb{Z}[x_1, \ldots, x_n]$. For S-scheme, the projective space of $\dim n$ over S is $\mathbb{P}^n = S \times_{\mathrm{Spec}\mathbb{Z}} \mathbb{P}^n$.

Theorem 4.4.11. *Let* S *be a scheme. Then the morphism* $\mathbb{P}^n \to S$ *is proper.*

Proof. It suffies to show $\mathbb{P}^n \to \mathrm{Spec}\mathbb{Z}$ is proper.

$$\begin{array}{ccc} \mathbb{P}^n & \longrightarrow & \mathbb{P}^n \\ \downarrow & & \downarrow \\ S & \longrightarrow & \mathrm{Spec}\mathbb{Z} \end{array}$$

It reduce to the following situation. Let R be a discrete valuation ring, K the fraction field of R. Let

$$\phi : R \to \mathbb{P}^n_R = \mathrm{Proj}\, R[x_0, \dots, x_n]$$

be a morphism over R. Then ϕ extends uniquely to $\mathrm{Spec}R$. Now ϕ is given by an R-homomorphism.

$$\begin{array}{ccc} \mathrm{Spec}(K) & \longrightarrow & \mathbb{P}^n \\ \downarrow & & \downarrow \\ \mathrm{Spec}(R) & \longrightarrow & \mathrm{Spec}\mathbb{Z} \end{array}$$

Replace n by some $m < n$ if necessary, we may assume that $\phi(\mathrm{Spec}K)$ is not included any hyperplane $x_i = 0$. In otherwords, $\phi(\mathrm{Spec}K)$ is represented by homogeneous coordinates

$$[a_0, \cdots, a_n], \qquad a_i \in K - \{0\}.$$

Let $\min(v_R(a_i)) = m$, π be a uniformizer and $b_i = a_i \pi^{-m}$. Then $[b_0, \cdots, b_n]$ will define a morphism

$$\mathrm{Spec}R \to \mathbb{P}^n_R.$$

□

Exercise 4.4.12. Show that a finite morphism is proper.

osed proper *Exercise* 4.4.13. Let $f : X \to Y$ be a morphism of separated schemes of finite type over a noetherian scheme S. Show that the image of the proper closed subscheme $Z \subset X$ is also closed in Y and proper over S.

4.5 Affine morphism

Definition 4.5.1 (Affine morphisms.)**.** *A morphism $f : X \to Y$ of schemes is called* affine morphism *if there is an open affine cover $\{V_i\}$ of Y such that $f^{-1}(V)$ is affine for each i.*

Remark 4.5.2. There is an equivalent definition of affine morphisms. A morphism $f : X \to Y$ is an affine morphism if and only if for any open affine subset $V \subset Y$, $f^{-1}(V)$ is affine.

Proposition 4.5.3. *A morphism between affine schemes is affine.*

Proof. Let $f : \mathrm{Spec}A \to \mathrm{Spec}B$ be a morphism of affine schemes. Suppose that the corresponding morphism of rings is $\varphi : B \to A$. This question is local. Then let $b \in B$, since

$$f^{-1}(D(b)) = D(\varphi(b)) \cong \mathrm{Spec}A_{\varphi(b)},$$

thus f is affine. □

Lemma 4.5.4. *Let Y be a closed subscheme of a scheme X, then the closed embedding $i : Y \to X$ is an affine morphism.*

Proof. This result comes directly from 4.2.6. □

Corollary 4.5.5. *A finite morphism is affine.*

Example 4.5.6. Let Y be a closed subscheme of an affine scheme $X = \mathrm{Spec}A$. Let $i : Y \hookrightarrow X$ be the inclusion morphism. If there is an ideal $I \subset A$ such that $\tilde{I}$ corresponding to the ideal sheaf of Y, then $i_*\mathcal{O}_Y \cong \widetilde{(A/I)}$.

Proposition 4.5.7. *An affine morphism is quasi-compact and separated.*

Proof. Let $f : X \to Y$ be an affine morphism. For any affine open subset $V \subset Y$, since $f^{-1}(V)$ is affine, thus quasi-compact, then f is quasi-compact.

For separaness, choose an open affine cover $\{V_i = \mathrm{Spec}B_i\}$ of Y such that $U_i = f^{-1}(V_i) = \mathrm{Spec}A_i$ are all affine. Then $U_i \times_{V_i} U_i = \mathrm{Spec}A_i \otimes_{B_i} A_i$. Since the morphism

$$\begin{aligned} A_i \otimes_{B_i} A_i &\longrightarrow A_i \\ a \otimes a' &\longmapsto aa' \end{aligned}$$

is surjective, then $U_i \to V_i$ is separated. Since we have gluing condition, it follows that f is separated. □

Remark 4.5.8. Proposition 4.5.7 shows that if $f : X \to Y$ is a morphism of schemes, then for any quasi-coherent sheaf $\mathcal{F}$ on X, $f_*\mathcal{F}$ is also quasi-coherent

Lemma 4.5.9. *Let X be a scheme and let*

$$0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{G} \to 0$$

be an exact sequence of quasi-coherent sheaves on X. If $f : X \to Y$ is an affine morphism, then the sequence of quasi-coherent sheaves on Y

$$0 \to f_*\mathcal{E} \to f_*\mathcal{F} \to f_*\mathcal{G} \to 0$$

is also exact.

Proof. □

5 Projective Schemes

5.1 Graded ring and modules

Definition 5.1.1. *A* graded ring *is a ring together with a decomposition $A = \oplus_{d \geq 0} A_d$ of A into a direct sum of abelian groups A_d, such that for any $d, e \geq 0$, $A_d \times A_e \subset A_{d+e}$. An element in A_d is called a homogenuous element of degree d. An ideal $I \subset A$ is a homogeneous ideal if $I = \oplus_{d \geq 0} I \cap A_d$.*

Then A_0 is a subring. For any integer d, $A_{\geq d} = \oplus_{n\geq d} A_n$ is an ideal. Then A is an algebra of A_0.

Definition 5.1.2. *Let A be a graded ring. A graded A-module is an A-module M, together with a decomposition $M = \oplus_{n\in\mathbb{Z}} M_n$ such that $A_d \times M_n \subset M_{n+d}$. For any graded A-module M, and for any integer l, we define the twisted module $M(l)$ by shifting l places to the left, i.e, $M(l)_n = M_{l+n}$.*

For a given graded ring A, we have a category $\mathrm{GradMod}_A$ of graded A-modules with morphisms given by usual homomorphisms which respect the graded. In this category, we can define the usual notion of kernel, image, and cokernel.

For two graded modules M and N, we can define a graded module $\mathrm{Hom}^*(M,N)$ with $\mathrm{Hom}^*(M,N)_n = \mathrm{Hom}(M, N(n))$. we can also define the tensor product $M \otimes_A N$ with degree n given by image of $\sum_{i+j=n} M_i \otimes_{A_0} N_j$.

Exercise 5.1.3. Prove that for a graded ring A and graded modules M, N, L, the isomorphism

$$\mathrm{Hom}(M \otimes N, L) = \mathrm{Hom}(M, \mathrm{Hom}^*(N, L)).$$

Example 5.1.4. Let R be a ring, and A be the polynomial ring $R[x_1, ..., x_n]$. Then $A = \oplus_{d\geq 0} R[x_1, ..., x_n]_{\deg d}$.

Proposition 5.1.5. *Let $\varphi : A \to B$ be a morphism of graded rings. Then $I = \ker\varphi$ is a graded homogeneous ideal.*

Now we want to define a scheme $\mathrm{Proj}\, A$

- Set: We define the set $\mathrm{Proj}\, A$ to be the set of all homogeneous prime ideals of A not containing all of $A_+ = \oplus_{d>0} A_d$.
- Topology: For any homogeneous ideal $I = \oplus I_d$, the closed subset of $\mathrm{Proj}\, A$ is

$$V(I) := \{p \in \mathrm{Proj}\, A \mid p \text{ contains } I\} = \bigcup_{f\in\bigcup_{d\geq 0} I_d} V(f). \tag{5.1.1}$$

For any homogeneous element $f \in A_+ = \oplus_{d>0} A_d$, define

$$V(f) := \{p \in \mathrm{Proj}\, A \mid p \text{ contains } f\}. \qquad D_+(f) = \mathrm{Proj}\, A - V(f). \tag{5.1.2}$$

For each $p \in \mathrm{Proj}\, A$, we define the ring $A_{(p)}$ to be the ring of elements of degree zero in the localized ring $S^{-1}A$, where S is the multiplicative system consisting of all homogeneous elements of A which are not in p.

Proposition 5.1.6. *For $f \in \bigcup_{d\geq 0} A_d$, $D_+(f)$ is affine as topological set and form fundamental system of open sets.*

Consider two homogeneous elements $f_1, f_2 \in \operatorname{Proj} A$ with $deg f_1 = deg f_2$.

$$\begin{array}{ccccc} D_+(f_1) & \xleftarrow{j_1} & D_+(f_1 f_2) & \xrightarrow{j_2} & D_+(f_2) \\ \| & & \| & & \| \\ \operatorname{Spec} A_{(f_1)} & \xleftarrow{i_1} & \operatorname{Spec} A_{(f_1 f_2)} & \xrightarrow{i_2} & \operatorname{Spec} A_{(f_2)} \end{array} \tag{5.1.3}$$

Exercise 5.1.7. Show that $\operatorname{Im} i_1 = \operatorname{Spec} A_{f_1}[\frac{f_2}{f_1}]$, $\operatorname{Im} i_2 = \operatorname{Spec} A_{f_2}[\frac{f_1}{f_2}]$.

By the exercise we can define structure sheaf on $\mathcal{O}_X$ for $X = \operatorname{Proj} A$ such that $\mathcal{O}_X|_{D_+(f)} \cong \varphi^{-1}\mathcal{O}_{\operatorname{Spec} A_{(f)}}$, where $\varphi : D_+(f) \xrightarrow{\sim} \operatorname{Spec} A_{(f)}$.
For any open subset $U \subset \operatorname{Proj} A$, we define $\mathcal{O}(U)$ be the set of maps

$$s : U \to \coprod A_{(p)}$$

such that for any $p \in U$, there is a neighborhood $V \subset U$ of p and homogeneous elements $a, f \in A$ of same degree such that for all $q \in V, f \notin q$,

$$s(q) = a/f \in A_{(q)}.$$

Proof of the proposition. Suppose f is homogeneous of degree d. $D_+(f) = \{p \in \operatorname{Proj} A |\ f \notin p\}$. Since elements of $\operatorname{Proj} A$ are those homogeneous prime ideals p of A which do not cantain all of A_+, it follows that the open sets $D_+(f)$ cover $\operatorname{Proj} A$.

Refine A by $A[\frac{1}{f}]$. Then $A[\frac{1}{f}]_n = \left\{ \frac{a}{f^k} \middle| \deg a - kd = n \right\}$. Since localization is flat, for homogeneous ideal $I \in \operatorname{Proj} A$, there is a natural ring homomorphism

$$\begin{aligned} \varphi :\ A &\longrightarrow A[\tfrac{1}{f}], \\ I &\longmapsto I[\tfrac{1}{f}]_{\deg 0} = I_{(f)} \end{aligned} \tag{5.1.4}$$

In particular, for $p \in D_+(f)$, $\varphi(p) \in \operatorname{Spec} A_{(f)}$. The properties of localization show that φ is bijective as a map from $D_+(f)$ to $\operatorname{Spec} A_{(f)}$.

If I is a homogeneous ideal of A, then $I \subset p$ if and only if $\varphi(I) \subset \varphi(p)$. Hence φ is a homeomorphism. Note that if $p \in D_+(f)$, then the local ring $A_{(p)} \cong (A_{(f)})_{\varphi(p)}$. Then φ induces a morphism between sheaves

$$\varphi^\sharp : \mathcal{O}_{\operatorname{Spec} A_{(f)}} \to \varphi_*(\mathcal{O}_{\operatorname{Proj} A}|_{D_+(f)}) \tag{5.1.5}$$

which is an isomorphism. □

Question

Let $A = \oplus_{d \geq 0} A_d$ be a graded ring. We know that there is a morphism

$$\begin{aligned} f :\ \operatorname{Proj} A &\longrightarrow \operatorname{Spec} A_0 \\ p &\longmapsto p \cap A_0 \end{aligned}$$

obtained from $\varphi : A_0 \to A$. But if there is a graded ring B and a graded ring homomorphism $\varphi : A \to B$, can we obtain a morphism $f : \operatorname{Proj} B \to \operatorname{Proj} A$? The answer is no. Since in general not every point of $\operatorname{Proj} B$ have a well-defined image. But what we know is $\varphi(A_+) \hookrightarrow B_+$. Let A^+ denote the union of homogeneous elements of A of positive degree $\bigcup_{d>0} A_d$.

Proposition 5.1.8. *Let $\varphi : A \to B$ be a homomorphism of graded rings. Let $U = \bigcup_{g\in A^+} D_+(\varphi(g))$, then there is a morphism*

$$f : U \to \operatorname{Proj} A \tag{5.1.6}$$

which is well-defined.

Proof. Take covering $\operatorname{Proj} A = \bigcup_{f\in A^+} D_+(f)$, $\operatorname{Proj} B = \bigcup_{g\in B^+} D_+(g)$. φ induces morphism of affine schemes $A_{(f)} \to B_{\varphi(f)}$ for each homogeneous element $f \in A_+$. Then we have a morphism of affine schemes:

$$D_+(\varphi(f)) \cong \operatorname{Spec} B_{\varphi(f)} \to D_+(f) = \operatorname{Spec} A_{(f)}.$$

Since $D_+(f_1 f_2) = D_+(f_1) \cap D_+ f_2$, $D_+(\varphi(f_1 f_2)) = D_+(\varphi(f_1)) \cap D_+\varphi(f_2)$, these morphisms can be glued together to get a morphism

$$\bigcup_{f\in A^+} D_+(\varphi(f)) \to \operatorname{Proj} A$$

□

Exercise 5.1.9. Suppose that $\varphi_d : A_d \to B_d$ is an isomorphism for all $d \gg 0$, then show that $U = \operatorname{Proj} B$ and the morphism $f : \operatorname{Proj} B \to \operatorname{Proj} A$ is an isomorphism. This exercise shows that f can be an isomorphism even when φ is not.

Proposition 5.1.10. *Let R be a ring and let A be a graded ring with $A_0 = R$, $A_n = 0$ for some $n \gg 0$. Then* $\operatorname{Proj} A = 0$. *More general,* $\operatorname{Proj} A = \varnothing$ *if and only if every element in A_+ is nilpotent.*

Proof. Assume $\operatorname{Proj} A = \varnothing$, then for any homogeneous element $f \in A_+$,

$$D_+(f) = \{p \in \operatorname{Proj} A |\ f \notin p\} = \varnothing.$$

Then $\operatorname{Spec} A_{(f)} = \varnothing$, it follows that $A_{(f)} = 0$. For any homogeneous prime ideal $p \in A$, $\sum_{d\geq 0} p \bigcap A_d \subset p$ is prime. so all homogeneous prime ideal contains f, thus f is nilpotent.

Conversely, if $\forall f \in A_+$ is nilpotent then $f^n = 0$ fro some integer n. Then the element f^n is of degree 0. For every $p \in \operatorname{Proj} A$, $f \in \sqrt{p} = p$, thus $\operatorname{Proj} A = \varnothing$. □

Example 5.1.11. Let k be a field. Then $\operatorname{Proj} k[T] = \{0\}$. If R is a ring, then $\operatorname{Proj} R[T] = \operatorname{Spec} R$

Remark 5.1.12. Let R be a ring, $\operatorname{Proj} R[x_0, ..., x_n] = \mathbb{P}^n_R$ is Projective space of dimension n on R. Let A be a graded ring, then $\operatorname{Proj} A = \bigcup_{f\in A^+} D_+(f) = \bigcup_{f\in\Sigma} D_+(f)$ if A_+ is generated by a set Σ of homogeneous element f. If k is an algebraically closed field, then $\operatorname{Proj} k[x_0, ..., x_n] = \bigcup_{i=0}^n D_+(x_i)$, where $D_+(x_i) \cong \operatorname{Spec} k[x_0, ..., x_n]_{x_i} = \operatorname{Spec} k[\frac{x_0}{x_i}, ..., \frac{x_0}{x_n}] = \mathbb{A}^n$. Each x_i is called homogeneous coordinates and each $\frac{x_j}{x_i}$ is called affine coordinates.

Example 5.1.13. Let $\mathbb{CP}^n = (\mathbb{C}^{n+1} - \{0\})/\mathbb{C}^\times$ be the complex projective n-space, then there is a natural embedding

$$\mathbb{CP}^n \hookrightarrow \operatorname{Proj} \mathbb{C}[x_0, ..., x_n] = \mathbb{P}^n_{\mathbb{C}} \tag{5.1.7}$$

such that the image of this map is the set of closed points in $\mathbb{P}^n_{\mathbb{C}}$. The point $(a_0, ..., a_n) \in \mathbb{CP}^n$ corresponds to the maximal ideal $\langle x_i a_j - x_j a_i \rangle$ in $\operatorname{Proj} \mathbb{C}[x_0, ..., x_n]$.

Example 5.1.14. If R is a ring, we define projective n-space over R to be the scheme $\mathbb{P}^n_R = \operatorname{Proj} R[x_0, ..., x_n]$. In particular, if R is an algebraically closed field k, then $\mathbb{P}^n_R$ is a scheme whose subspace of closed points is naturally homeomorphic to the variety called projective n-space.

Remark 5.1.15. $\operatorname{Proj} A$ is not necessarily compact.

5.2 Quasi-coherent Sheaf

qc sheaf **Definition 5.2.1.** *Let X be a scheme, and let $\mathcal{F}$ be an $\mathcal{O}_X$ module. Then $\mathcal{F}$ is* quasi-coherent *if on every affine open subset $U = \operatorname{Spec} A \subset X$, $\mathcal{F}|_U = \widetilde{M}$ for some A-module M.*

Remark 5.2.2. There is a equivalent definition for quasi-coherent sheaf on a general scheme: Let X be a scheme, and let $\mathcal{F}$ be an $\mathcal{O}_X$ module. Then $\mathcal{F}$ is quasi-coherent if there is an open affine covering $\{U_i = \operatorname{Spec} A_i\}$ such that for each i there is an A_i-module M_i with $\mathcal{F}|_{U_i} \cong \widetilde{M}_i$.

Exercise 5.2.3. Prove the equivalence of these two definitions of quasi-coherent sheaf on a general scheme. (Hint: use 3.7.5 [M=gamma tilde] and reduce to the case when X is affine)

Let A be a graded ring. For any graded module M on A, there is a quasi-coherent sheaf $\widetilde{M}$ on $\operatorname{Proj} A$ using some constructions on $\mathcal{O}_X$. Let f be a homogeneous element in A.

$$\begin{array}{ccc} A & \longrightarrow & A[\frac{1}{f}] \\ \downarrow & & \uparrow \\ A_{(f)} & = & A[\frac{1}{f}]_{\deg 0} \end{array}$$

If M is an A-module, then the diagram above induces

$$\begin{array}{ccc} M & \longrightarrow & M \otimes A[\frac{1}{f}] \\ \downarrow & & \uparrow \\ M_{(f)} & = & M[\frac{1}{f}]_{\deg 0} \end{array}$$

In this way we get sheaves $\widetilde{M_{(f)}}$ on $D_+(f)$. Since $D_+(f_i f_j) = D_+(f_i) \cap D_+(f_j)$, then we can glue these sheaves to get a sheaf $\widetilde{M}$ on $\operatorname{Proj} A$.

From definition, it is easy to see that the functor $M \mapsto \widetilde{M}$ from $\operatorname{GradMod}(A)$ to the $\operatorname{QCoh}(X)$ is exact and respect to tensor product.

Definition 5.2.4. *Let A be a graded ring, and let $X = \operatorname{Proj} A$. For any integer n, we define the sheaf $\mathcal{O}_X(n)$ to be $\widetilde{A(n)}$. We call $\mathcal{O}_X(1)$ the* twisting sheaf of Serre. *For any sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$, denote $\mathcal{F}(n)$ to be the sheaf $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(n)$.*

Proposition 5.2.5. *Let S be a graded ring and let $X = \operatorname{Proj} S$. Assume that S is generated by S_1 as an S_0-algebra.*

1. *The sheaf $\mathcal{O}_X(m)$ is locally free of rank 1.*
2. *For any graded S-module M, $\widetilde{M(d)} = \tilde{M} \otimes \mathcal{O}_X(d)$. In partikular, $\mathcal{O}_X(d_1) \otimes \mathcal{O}_X(d_2) = \mathcal{O}_X(d_1 + d_2)$*
3. *For homogeneous degree 1 element $f \in S$, $\mathcal{O}_X(d)|_{D_+(f)} = \widetilde{S(d)_{(f)}}$, where*

$$\widetilde{S(d)_{(f)}} = \left\{ \frac{a}{f^n} \middle| \; dega = n + d \right\} = S_{(f)} \cdot f^d$$

Proof. 1. Cover X by X_f, $f \in S_1$. $X_f = \operatorname{Spec} S_{(f)}$. We have

$$\mathcal{O}_X(n)|_{X_f} = \widetilde{S(n)_{(f)}},$$

where

$$\begin{aligned} S(n)_{(f)} &= \left\{ \frac{a}{f} \middle| \; \deg_S\left(\frac{a}{f^n}\right) - n = 0 \right\} && (5.2.1) \\ &= \left\{ \frac{a}{f^m} \cdot f^n \middle| \; \deg a = m \right\} && (5.2.2) \\ &= S(f) \cdot f^n. && (5.2.3) \end{aligned}$$

$S(n)_{(f)} \cong S(f)$ as an $S(f)$-module, therefore $\mathcal{O}_X(n)|_{X_f} \cong \widetilde{S}(f) = \mathcal{O}_{X_f}$.

2. This follows from the fact that $(M \otimes_S N)^\sim \cong \tilde{M} \otimes_{\mathcal{O}_X} \tilde{N}$ for any two graded S-modules M and N, when S is generated by S_1. Indeed, for any $f \in S_1$, we have $(M \otimes_S N)_{(f)} = M_{(f)} \otimes_{S_{(f)}} N_{(f)}$.

$$\mathcal{O}_X(n) \otimes_{\mathcal{O}_X} \mathcal{O}_X(m) = \widetilde{S}(n) \otimes_{\widetilde{S}} \widetilde{S}(m) = \widetilde{S}(m+n).$$

Note that $\mathcal{O}_X(0) = \mathcal{O}_X$.

□

Example 5.2.6. Let $X = \operatorname{Proj} A[t_0, \ldots, t_r]$ and let $S = A[t_0, \ldots, t_r]$. Compute $\Gamma(X, \mathcal{O}_X(n))$.

$$\mathcal{O}_X(n)|_{X_{t_i}} = \widetilde{S}(n)_{(t_i)}, \qquad \Gamma(x_{t_i}, \mathcal{O}_X(n)) = S(n)_{(t_i)},$$

which is equal to the $\deg n$ part of S_{t_i}. Let $s \in \Gamma(X, \mathcal{O}(n))$,

$$s|_{X_{t_i}} = \sum_{i_0 + \ldots + i_r = n} a_{i_0 \ldots i_r} t_0^{i_0} \cdot \ldots \cdot t_r^{i_r}, \qquad i_k \geq 0$$

(Laurent polynomial). Therefore $\Gamma(X, \mathcal{O}(n))$ = polynomials of $t_0, \ldots, t_r$ of degree n. It follows that $\Gamma(X, \mathcal{O}(n)) = 0$ if $n < 0$. $\Gamma(X, \mathcal{O}_X) = A$,

$$\operatorname{rank}_A \Gamma(X, \mathcal{O}_X(n)) = \binom{n+r}{n}, \qquad n > 0.$$

In particular $\mathcal{O}_X(n)$ are not isomorphic to each other.

5.3 Noetherian Condition

Proposition 5.3.1.

1. *If a graded ring A is noetherian, then* $\operatorname{Proj} A$ *is noetherian.*
2. *If a graded ring A is noetherian if and only if A_0 is noetherian and A_+ is finitely generated A_0-algebra.*

Proof of 2. (⇐) Hilbert base theorem

(⇒) if A is noetherian, then A_+ is finitely generated over A_0 and $A_0 = A/A_+$ is noetherian □

Exercise 5.3.2. Show that if a graded ring A is noetherian, then $A = A_0[x_1, ..., x_n]$ for some integer n.

Remark 5.3.3.

1. Let S be a Noetherian graded ring which is generated by homogeneous elements $x_0, \ldots, x_d$. Assume $S_0 = k$ is a field. Then one can show

$$\sum_n \dim S_n \cdot T^n = \frac{Q(T)}{\prod_{i=0}^n (1 - T^{\deg x_i})}.$$

2. Assume then $\deg(x_i) = 1$. That is $x_i \in S_1$. Then $\dim S_n = P(n)$ for $n \gg 0$, where P is a polynomial.

Exercise 5.3.4. Let S be a noetherian graded ring and let $X = \operatorname{Proj} S$. Show that $\dim X = \dim S - 1$.

We want to show $\deg P = \dim X$
We also want to define a third invariant $\delta(S)$.

Definition 5.3.5. *$\delta(S)$ is the minimal number of elements $y_1, \ldots, y_l \in S_1$ such that $S/\sum y_i S$ is a finite dimensional k-vector space.*

Theorem 5.3.6. $\dim S = \deg P + 1 = \delta(S)$.

A generalization: Let M/S be a module of finite type with M graded, we define S_M to be $S/Ann(M)$ where $Ann(M) = \{x \in S : xM = 0\}$, and $\delta(M)$ to be the minimal number of $x_1, \ldots, x_m \in S$ such that $M/\sum x_i M$ is of finite length.

Theorem 5.3.7. $\dim S_M = \deg P_M + 1 = \delta(M)$.

Lemma 5.3.8. *There is a filtration $0 \subsetneq M_1 \subsetneq M_2 \subsetneq \ldots \subsetneq M$ such that $M_i/M_{i-1} \cong S/\mathfrak{p}$, where $\mathfrak{p}$ is a prime ideal.*

Proof. We'll do the proof in three steps:

Step 1: $\deg P_M + 1 \geq \dim S_M$.

Step 2: $\delta(M) \geq \deg P_M + 1$.

Step 3: $\dim S_M \geq \delta(M)$.

Step 1 $\deg P_M + 1 \geq \dim S_M$: First let us consider the case $M = S$. We use induction on $\deg P_M$. If $\deg P_M = -1$ (i.e., P_M=0), $\dim S_n = 0$ for large enough values of n. Therefore

$$S = \bigoplus_{n=0}^{\infty} S_n = \bigoplus_{n=0}^{k} S_n, \quad S_+ = \bigoplus_{n>0} S_n, \quad S_+^k = 0.$$

So S_+ is nilpotent and this is the only prime ideal so $\dim S = 0$. Now assume $\deg P_M \geq 0$. If $\dim S = 0$ we are done. Assume $\dim S > 0$. Let $\mathfrak{p}_1 \subsetneq \mathfrak{p}_2 \subsetneq \ldots \subsetneq \mathfrak{p}_r$ be a chain of prime ideals in S_1, $r - 1 = \dim S > 0$. Let $x \in \mathfrak{p}_2 - \mathfrak{p}_1$,

$$0 \longrightarrow S/\mathfrak{p}_1 \xrightarrow{x} S/\mathfrak{p}_1 \longrightarrow S/(xS + \mathfrak{p}_1) \longrightarrow 0.$$

Because $r - 1 = \dim S$,

$$\dim S/(\mathfrak{p}_1 + xS) = \dim S - 1 = \dim S/\mathfrak{p}_1 - 1.$$

From the above exact sequence,

$$\dim\big(S/\mathfrak{p}_1\big)_n = \dim\big(S/\mathfrak{p}_1\big)_{n-\deg x} + \dim\big(S/(xS + \mathfrak{p}_1)\big)_n.$$

Thus

$$P_{S/\mathfrak{p}_1}(T) = P_{S/\mathfrak{p}_1}(T - \deg x) + P_{S/(xS+\mathfrak{p}_1}(T).$$

So

$$\deg P_{S/(\mathfrak{p}_1+xS)} \leq \deg P_{S/\mathfrak{p}_1} - 1 \leq \deg P_S - 1.$$

Now by induction $\deg S/(\mathfrak{p}_1 + xS) \geq \dim S/(\mathfrak{p}_1 + xS)$. So now

$$\dim S = 1 + \dim S/(\mathfrak{p}_1 + xS) \leq 1 + \deg S/(\mathfrak{p}_1 + xS) \leq 1 + \deg S - 1 = \deg S.$$

So step 1 works for the case $M = S$. For general M we use the exact sequence $0 \to M_1 \to M \to M_2 \to 0$. If Step 1 works for M_1, M_2, then it works for M as well. Notice that $\dim S_M = \max(\dim S_{M_1}, \dim S_{M_2})$ and $\deg P_M = \max(\deg P_{M_1}, \deg P_{M_2})$. (Recall that $S_M = S/Ann(M)$). This will reduce to the case $M = S/\mathfrak{p}$.

Step 2: $\delta(M) \geq \deg P_M + 1$. We use induction on $\delta(M)$. If $\delta(M) = 0$, M is of finite length so $M_n = 0$ for n large enough, so $P_M = 0$ and $\deg P_M = -1$. We are done. Now assume $\delta(M) > 0$. There are elements $x_1, \ldots, x_{\delta(M)} \in S_+$ such that $M/\sum x_i M$ has finite length. Consider the exact sequence:

$$M \xrightarrow{x} M \longrightarrow \big(M/xM\big) \longrightarrow 0$$

We have $\delta(M/x_1M) = \delta(M) - 1$ and

$$\dim\big(M/x_1M\big)_n \geq \dim M_n - \dim M_{n-\deg x}.$$

It follows that $\deg P_{M/xM} \geq \deg P_M - 1$. From induction $\delta(M/x_1) \geq \deg P_{M/x_1} + 1$, we have

$$\delta(M) = \delta(M/x_1M) + 1 \geq \deg P_{M/x_1} + 2 \geq \deg P_M + 1.$$

Step 3: $\dim S_M \geq \delta(M)$.

Again we use induction on $\dim S_M$. If $\dim S_M = 0$ then $S_{M,+}$ is the only maximal ideal. So it is own nill radical. Thus some power of $S_{M,+}$ is zero, as $S_{M,+}$ is finitely generated. Thus $S_{M,+} = 0$ for n sufficiently large. So S_M, therefore M, has finite length. It follows that $\delta(M) = 0$.

So now assume $\dim S_M > 0$. Let $\mathfrak{p}_i$ $(i = 1, \cdot n)$ all minimal ideals of S_M.

Exercise 5.3.9.

1. Show that there is an $x \in S_{M,+} - \bigcup_{i=1}^n \mathfrak{p}_i$.
2. For such x, $\dim S_M \geq \dim S_{M/xM} + 1$ and $\delta(M/x_1M) \geq \delta(M) - 1$.

Now the inequality follows from the exercise and the induction $\dim S_{M/x_1M} \geq \delta(M/x_1M)$.

□

Definition 5.3.10. *The degree of X is a number such that the leading coefficient of $P(T)$ has the form $\dfrac{\deg(X)}{d!}T^d$.*

Exercise 5.3.11. Compare the degree and leading coefficient of the Hilbert polynomial for $k[x_0, \ldots, x_n]/(F) = S$, where F is a homogeneous polynomial of degree d, $\dim S_n = P(n)$.

5.4 Sheaves on Projective Schemes

Given a graded ring $S = \oplus_{d\geq 0} S_d$, which is generated by S_1 as an S_0-algebra. Then S gives a pair $(\operatorname{Proj} S, \mathcal{O}(1))$. In other words , there exists a special sheaf on projective schemes which has many saitisfying properties. We want to recover S from $(\operatorname{Proj} S, \mathcal{O}(1))$.

An important fact is that if $S^{(d)} = \oplus_{n\geqslant 0} S_n^{(d)}$ is the graded ring with $S_n^{(d)} = S_{nd}$, then there is an isomorphism

$$(\operatorname{Proj} S, \mathcal{O}(d)) \xrightarrow{\sim} (\operatorname{Proj} S^{(d)}, \mathcal{O}(1)) \tag{5.4.1}$$

We will state this fact as a theorem later.

Question

Let A be a ring and let S be a graded ring. In previous chapters we have shown the equivalence between the category of A-modules and the category of quasi-coherent sheaves on affine scheme Spec$A$. Later we will see in 5.5.3 that we can prove a similar result about quasi-coherent sheaves on projective schemes.i.e given a quasi-coherent sheaf$\mathcal{F}$ on $\operatorname{Proj} S$ we can find an S-module M such that $\tilde{M} \cong \mathcal{F}$. Last section we have proved that Then the question is that can we prove a similar result about coherent sheaves on $\operatorname{Proj} S$ and finitely generated S-modules? Unfortunately it is not a trivial question. The difficult part is, if M, N are two graded S-modules, $\tilde{M} \cong \tilde{N}$ can not imply $M \cong N$. But if M and N is finitely generated, then $\tilde{M} \cong \tilde{N}$ implies that $M_n \cong N_n$ for $n \gg 0$.

Remark 5.4.1. We can not easily conclude the equivalence of category of S-modules and category of quasi-coherent sheaves on $\operatorname{Proj} S$. Even if a quasi-coherent sheaf can be recovered from a graded S-module, we will then show that even different S-modules can induce a same quasi-coherent sheaf.

5.5 Quasi-coherent sheaves on projective schemes

Let A be a graded ring. In this section we can show that the category of quasi-coherent sheaves on projective schemes is equivalent to the category of graded A-modules.

Definition 5.5.1. *Let A be a graded ring, and let $X = \operatorname{Proj} A$. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-omdules. We define the grade A-module associated to $\mathcal{F}$ as a group, to be $\Gamma_*(\mathcal{F}) = \oplus_{n\in\mathbb{Z}}\Gamma(X,\mathcal{F}(n))$. Then it becomes a graded A-module naturally: any section in A_d determines a global section in $\Gamma(X,\mathcal{O}_X(d))$; and since $\mathcal{F}(n)\otimes\mathcal{O}_X(d) = \mathcal{F}(n+d)$, then we can difine the product in $\Gamma(X,\mathcal{F}(n+d))$ by taking the tensor product in $\mathcal{F}(n)\otimes\mathcal{O}_X(d)$*

Example 5.5.2. Let R be a ring and let A be the polynomial ring $R[x_0,...,x_n]$. By definition of projective space, $\operatorname{Proj} A = \mathbb{P}^n_R$. Then $\operatorname{Proj} A$ is covered by affine open subsets $D_+(x_i) \cong$ Spec$A_{(x_i)}$, $i = 0,1,...,n$. Now consider $\Gamma(\mathbb{P}^n_R,\mathcal{O}(d))$ for a given integer d, since $A_{(x_i)} = R[\frac{x_0}{x_i},...,\frac{x_n}{x_i}]$, from the definition of $\mathcal{O}(d)$ we get

$$\mathcal{O}(d)|_{D_+(x_i)} = (A_{(x_i)}\otimes x_i^d)^{\sim}. \tag{5.5.1}$$

Then

$$\begin{aligned}\Gamma(\mathbb{P}^n_R,\mathcal{O}(d)|_{D_+(x_i)}) &= A_{(x_i)}\bigotimes x_i^d &(5.5.2)\\ &= \left\{ f\in R[x_0,...,x_n] \;\middle|\; \begin{array}{l} f=\sum a_{k_0...k_n}x_0^{k_0}...x_n^{k_n}, \\ \sum k_j = d, k_j \geqslant 0 \text{ if } j\neq i\end{array}\right\}. &(5.5.3)\end{aligned}$$

By the property of sheaf we have the exact sequence

$$\Gamma(X,\mathcal{O}(d)) \to \prod_{i=0}^{n}\Gamma(D_+(x_i),\mathcal{O}(d)) \to \prod_{i,j}\Gamma(D_+(x_ix_j),\mathcal{O}(d)), \tag{5.5.4}$$

which implies $\Gamma(X, \mathcal{O}(d)) = R[x_0, ..., x_n]_{\deg d}$.

Finally,

$$\oplus_{m=0}^{\infty}\Gamma(X, \mathcal{O}(d+m)) = \oplus_{m=0}^{\infty}R[x_0, ..., x_n]_{\deg d+m}.$$

v for proj **Theorem 5.5.3.** *Let A be a graded ring which is generated by A_1 as A_0-algebra and X = Proj A. Let $\mathcal{F}$ be a quasi-coherent sheaf on X. Then there is a canonical isomorphism $\Gamma_*(\mathcal{F})^{\sim} \xrightarrow{\sim} \mathcal{F}$*

Proof. **Step 1**

Let $M = \Gamma_*(\mathcal{F})$, cover X by $D_+(x)$, where x runs through the elements of A_1. Then

$$\tilde{M}|_{D_+(x)} = \widetilde{M_{(x)}}|_{\operatorname{Spec} A_{(x)}}, \qquad M_{(x)} = \left\{ \frac{m}{x^n} \middle| \, m \in M_n \right\} \mapsto M[\frac{1}{x}]_{\deg 0}. \tag{5.5.5}$$

Since $\mathcal{F}$ is quasi-coherent, then by 3.7.5, $\mathcal{F}|_{D_+(x)} \cong \Gamma(D_+(x), \mathcal{F})$in any case (locally $\mathcal{F}|_{D_+(x)} \cong \tilde{N}$ for some $A_{(x)}$-module N). We want to construct a canonial isomorphism

M=gamma tilde

$$M_{(x)} \xrightarrow{\sim} \Gamma(D_+(x), \mathcal{F}), \qquad M_{(x)} = \sum_{n=0}^{\infty} M_n \otimes \frac{1}{x^n}, \quad M_n = \Gamma(X, \mathcal{F}(n)). \tag{5.5.6}$$

Remember that $\mathcal{F}(n)$ is the sheaf $\mathcal{O}(n) \otimes \mathcal{F}$ and $\mathcal{F}(n)|_{D_+(x)} \cong \mathcal{F}|_{D_+(x)} \otimes x^n$

Then we have

$$M_n = \Gamma(D_+(x), \mathcal{F}(n)) = \Gamma(D_+(x), \mathcal{F}) \otimes x^n.$$

So at least we have a natural map

$$\sum_{n=0}^{\infty} M_n \otimes \frac{1}{x^n} \to \Gamma(D_+(x), \mathcal{F}) \tag{5.5.7}$$

Step 2

We need to show that the natural map 5.5.7 is bijective. This means that :

(a) For any n, let $s \in \Gamma(X, \mathcal{F}(n))$ be a global section. If $\frac{s}{x^n} = 0$ in $\Gamma(D_+(x), \mathcal{F})$, then there exists an integer n' such that $x^{n'}s = 0$. This shows the injectivity of 5.5.7.

(b) Given a section $s \in \Gamma(D_+(x), \mathcal{F})$, there exists an integer n such that $x^n s$ extends to a section of $\Gamma(X, \mathcal{F}(n))$. This shows the surjectivity of 5.5.7.

We will later state these two facts as a lemma. But first let us give some definitions and properties of the sheaf $\mathcal{F}(n)$. □

Definition 5.5.4 (Globally generated)**.** *Let X be a scheme, and let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. We say that $\mathcal{F}$ is generated by global sections if there is a family of global sections $\{s_i\}_{i \in I}$, $s_i \in \Gamma(X, \mathcal{F})$, such that for each $x \in X$, the images of s_i in the stalk $\mathcal{F}_x$ generate that stalk as an $\mathcal{O}_X$-module.*

Note that $\mathcal{F}$ is generated by global sections if and only if $\mathcal{F}$ can be written as a quotient of a free sheaf. Indeed, the generating secions $\{s_i\}_{i \in I}$ define a surjective morphism of sheaves $\oplus_{i \in I}\mathcal{O}_X \to \mathcal{F}$, and conversely.

ugly lemma **Lemma 5.5.5** (Ugly lemma)**.** *Let X be a noetherian separable scheme. Let $\mathcal{L}$ be invertible sheaf on X and $\mathcal{F}$ a quasi-coherent sheaf on X. Let $f \in \Gamma(X, \mathcal{L})$ and X_f be the nonzero locus of X, i.e. the set of points $x \in X$ such that the stalk $f_x \notin m_x \mathcal{L}_x$ where m_x is the maximal ideal of $\mathcal{L}_x$. Then:*

(a) If the restriction of $s \in \Gamma(X, \mathcal{F})$ in $\Gamma(X_f, \mathcal{F})$ is zero, then there exists a positive integer n such that $sf^n = 0$ in $\Gamma(X, \mathcal{F} \otimes \mathcal{L}^n)$.

(b) Given a section $s \in \Gamma(X_f, \mathcal{F})$, there exists a positive integer n and a section $t \in \Gamma(X, \mathcal{F} \otimes \mathcal{L}^n)$ such that $t|_{X_f} = sf^n$.

Proof. (a) Let $U_i = \mathrm{Spec} A_i$ be an affine open cover of X. Since X is noetherian, it follows that X is quasi-compact. Then suppose a finite number of U_i cover X. Since $\mathcal{L}$ is invertible then for any U_i we can pick an isomorphism:

$$\mathcal{L}|_{U_i} \to \mathcal{O}_X|_{U_i} \tag{5.5.8}$$

and suppose $a_i \in A_i$ is the image of $f|_{U_i} = f_i$ by this isomorphism. Then $a_i \frac{f}{f_i}$.

Since we have showed in X_f 3.4.11 that $X_f \bigcap U_i = \mathrm{Spec} A_i[\frac{1}{a_i}]$, then $X_f = \bigcup \mathrm{Spec}\mathrm{Spec} A_i[\frac{1}{a_i}]$.

Since $\mathcal{F}$ is quasi-coherent, by qc sheaf 5.2.1 there is an A_i-module M_i such that $\mathcal{F}|_{U_i} \cong \tilde{M}_i$ for each i. let $s \in \Gamma(X, \mathcal{F})$ be a global section, then restrict s to U_i give an element $s \in M_i$. Similarly, since $\mathcal{F}|_{X_f \cap U_i} \cong \widetilde{(M_i)_{(a_i)}}$ and s restrict to X_f is zero. Then the image of s in $(M_i)_{(a_i)}$ is 0. Thus there exists an integer n_i such that $sa_i^{n_i} = 0$ for each i. Take $n = \max(n_i)$ we get $s \otimes f^n = s(\frac{f}{f_i})^n \otimes f_i^n = sa_i^n \otimes f_i^n = 0$

To prove (b), consider the commutative diagram

$$\begin{array}{ccc} \Gamma(X, \mathcal{F}) & \xrightarrow{f^n} & \Gamma(X, \mathcal{F} \otimes \mathcal{L}^n) \\ \downarrow & & \downarrow \\ \Gamma(X_f, \mathcal{F}) & \xrightarrow{\sim} & \Gamma(X_f, \mathcal{F}) \otimes f^n \end{array} \tag{5.5.9}$$

Given a section $s \in \Gamma(X_f, \mathcal{F})$ a section $a_i^n \otimes s$ of $\Gamma(X_f, \mathcal{F}) \otimes f^n$ can be lift to a section $t_i \in \Gamma(U_i, \mathcal{F} \otimes \mathcal{L}^n)$ for n sufficiently large. Then we only need to check the gluing condition: given $s_i \in \Gamma(U_i, \mathcal{F})$, suppose that $s_i|_{U_i \cap U_j} = s_j|_{U_j \cap U_j}$

By (a) since $U_i \cap U_j$ is quasi-compact. $s_i|_{U_i \cap U_j} - t_j|_{U_i \cap U_j} = 0$ in $\Gamma(U_j \cap U_j, \mathcal{F})$, then there exists $m \gg 0$ such that $f^m(s_i|_{U_i \cap U_j} - t_j|_{U_i \cap U_j}) = 0$ in $\Gamma(X, \mathcal{F} \otimes \mathcal{L}^m)$, glue together to give a global section t of $\mathcal{F} \otimes \mathcal{L}^{n+m}$, whose restriction to X_f is sf^{n+m} □

Remark 5.5.6. Let A be a graded ring which is generated by A_1 as A_0-algebra. We can only show that given a quasi-coherent sheaf $\mathcal{F}$ on $\mathrm{Proj}\, A$, we can find a graded A-module M such that $\tilde{M} \cong \mathcal{F}$. But the choice of such M is not unique. To explain it more precisely, we give some propositions and examples.

Proposition 5.5.7. *Let A be a graded ring which is generated by A_1 as A_0-algebra, let M be a graded A-module, and let $X = \mathrm{Proj}\, A$. Show that there is a natural homomorphism $\alpha_M : M \to \Gamma_*(\tilde{M})$.*

Proof. Since A_1 generate A, then cover X by affine open subsets $\{D_+(a_i)\}_{a_i\in A_1}$. For all $m \in M$, it must be contained in some M_d, hence m has degree 0 in $M(d)_{(a_i)} = \tilde{M}(d)(D_+(a_i))$ for all $a_i \in A_1$. Since $D_+(a_i)\bigcap D_+(a_j) = D_+(a_ia_j)$, then the sections in $\tilde{M}D_+(a_i)$ and $\tilde{M}D_+(a_j)$ agree on $\tilde{M}D_+(a_ia_j)$. Hence they can be glued together to be a section of $\oplus_{n\in\mathbb{Z}}\Gamma(X,\tilde{M})$. Then we have a morphism

$$\alpha_M : M \to \Gamma_*(\tilde{M}).$$

For any $m \in M$ of degree d_1, $a \in A$ of degree d_2, we have $a \times \alpha(m)$ is the image of $m \times a$ in $\Gamma(X, \tilde{M}(d_1) \otimes \mathcal{O}_X(d_2) = \Gamma(X, \tilde{M}(d_1 + d_2))$. Then α is a homomorphism. □

By the above theorem and proposition, we see that Γ_* is the right adjoint of the functor $M \longrightarrow \widetilde{M}$. Thus for any A-module M and quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$, we have a canonical isomorphism

$$\mathrm{Hom}_A(M, \Gamma_*(\mathcal{F})) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{O}_X}(\widetilde{M}, \mathcal{F}), \qquad \varphi \mapsto \varphi.$$

The inverse is given by $\psi \mapsto \Gamma_*(\psi) \circ \alpha$.

From the proposition above, α induces an exact sequence:

$$0 \to \ker\alpha \to M \xrightarrow{\alpha} \Gamma_*(\tilde{M}) \to \mathrm{coker}\alpha \to 0. \tag{5.5.10}$$

Apply the functor ~ on this exact sequence:

$$0 \to \widetilde{\ker\alpha} \to \tilde{M} \xrightarrow{\sim} \widetilde{\Gamma_*(\tilde{M})} \to \widetilde{\mathrm{coker}\alpha} \to 0. \tag{5.5.11}$$

we get $\widetilde{\ker\alpha} = 0$ and $\widetilde{\mathrm{coker}\alpha} = 0$, but this cannot imply $\ker\alpha = 0$ or $\mathrm{coker}\alpha$.

Example 5.5.8. Let $A = k[x_0, ..., x_n]$ where k is a field. Let $M^i = A/(x_0, ..., x_n)^i$ be the graded A-module. Then $\widetilde{\oplus M^i} = 0$ but $\oplus M^i \neq 0$.

Exercise 5.5.9. If $\tilde{M} = 0$, describe M. (answer: $A_+ \subset \sqrt{AnnM}$)

Embeddings between projective spaces

Note that we have defined projective n-space over a ring to be $\mathrm{Proj}\, A[x_0, ..., x_n]$, denoted $\mathbb{P}^n_A$. If $A \to B$ is a homomorphism of rings, and $\mathrm{Spec}B \to \mathrm{Spec}A$ is the corresponding morphism of affine schemes. Then we can easily see that $\mathbb{P}^n_B \cong \mathbb{P}^n_A \times_{\mathrm{Spec}A} \mathrm{Spec}B$. In particular, for any ring A, we have $\mathbb{P}^n_A \cong \mathbb{P}^n_{\mathbb{Z}} \times_{\mathrm{Spec}\mathbb{Z}} \mathrm{Spec}A$. This motivates the following definition for any scheme Y.

Definition 5.5.10. *If Y is any scheme, we define projective n-space over Y, denoted $\mathbb{P}^n_Y$ to be $\mathbb{P}^n_{\mathbb{Z}} \times_{\mathrm{Spec}\mathbb{Z}} Y$.*

Definition 5.5.11. *For any scheme Y, let $g : \mathbb{P}^r_Y \to \mathbb{P}^r_{\mathbb{Z}}$ be the natural map. We define the twisting sheaf $\mathcal{O}(1)$ on $\mathbb{P}^r_Y$ to be $g^*(\mathcal{O}(1))$. Note that if $Y = \mathrm{Spec}A$, this is the same as the $\mathcal{O}(1)$ already defined on $\mathbb{P}^r_A = \mathrm{Proj}\, A[x_0, ..., x_r]$. We call a morphism $f : X \to Y$ is projective morphism if it factors into a closed immersion $i : X \to \mathbb{P}^n_Y$ for some n, followed by the projection $\mathbb{P}^n_Y \to Y$.*

Definition 5.5.12 (The Veronese Embedding)**.** *The Veronese embedding of degree $d > 0$ is the map*

$$\begin{aligned} \rho_d : \quad \mathbb{P}^n &\longrightarrow \mathbb{P}^N \\ (x_0, \cdots, x_n) &\longmapsto (\cdots, \prod_{\substack{0 \le r_i \le d \\ \sum_{i=0}^n r_i = d}} x_i^{r_i}, \cdots) \end{aligned} \tag{5.5.12}$$

where $N = \binom{n+d}{n} - 1$.

Theorem 5.5.13. *Let S be a graded ring, generated by S_1 as an S_0-algebra. For any integer $d \geq 0$, let $S^{(d)} = \oplus_{n \geqslant 0} S_n^{(d)}$ be the graded ring where $S_n^{(d)} = S_{nd}$. Then $\operatorname{Proj} S^{(d)} \cong \operatorname{Proj} S$ and the sheaf $\mathcal{O}(1)$ on $\operatorname{Proj} S^{(d)}$ corresponds via this isomorphism to $\mathcal{O}(d)$ on $\operatorname{Proj} S$*

Proof. If S_1 is generate by $x_0, ..., x_r$, corresponding to an embedding $X \to \mathbb{P}^r_{S_0}$. Then consider the Veronese embedding:

$$\begin{aligned} \varphi : \quad \mathbb{P}^r_{S_0} &\longrightarrow \mathbb{P}^N_{S_0} \\ (x_0, ..., x_r) &\longmapsto (M_0, ..., M_N) \end{aligned} \tag{5.5.13}$$

where $N = \begin{pmatrix} n+d \\ n \end{pmatrix}$-1 and M_i are monic polynomials of degree d. Then $M_0, ..., M_N$ is a set of generators of $S_1^{(d)} = S_d$ corresponding to the embbeding $\operatorname{Proj} S^{(d)} \hookrightarrow \mathbb{P}^N_{S_0}$.

Since $\mathbb{P}^r_{S_0} \cong \operatorname{Im}\varphi$, then $\operatorname{Proj} S \cong \operatorname{Im}\varphi \bigcap \operatorname{Proj} S^{(d)} = \operatorname{Proj} S^{(d)}$. Now we have an isomorphism given by the Veronese embedding $\phi : X = \operatorname{Proj} S \to \operatorname{Proj} S^{(d)} = Y$. Choose $f \in S_1$ then f^d is a homogeneous degree 1 element in $\operatorname{Proj} S^{(d)}$, and $D_+(f^d)$ form a basis for Y, $D_+(f)$ form a basis for X.

Since

$$\mathcal{O}_X(D_+(f^d)) = \mathcal{O}_X(d)(\phi^{-1}(D_+(f^d))) = \phi_*(\mathcal{O}_X(d))(D_+(f^d)),$$
$$\mathcal{O}_Y(1)(D_+(f^d)) = S^{(d)}(1)(f^d) = S^{(d)}_{(f)}$$

it follows that $\mathcal{O}_Y(1)(D_+(f^d)) \cong \phi_* \mathcal{O}_X(d)(D_+(f^d))$, thus $\phi_* \mathcal{O}_X(d) \cong \mathcal{O}_Y(1)$. □

5.6 Coherent sheaves

Question

Let A be a graded ring, generated by A_1 as an A_0-algebra and let $X = \operatorname{Proj} A$. If $\mathcal{F}$ is coherent sheaf on X, can we prove that $M = \Gamma_*(\mathcal{F})$ is finitely generated and $\tilde{M} = \mathcal{F}$? First consider a weak condition: if $\mathcal{F} = \tilde{M}$ then $\mathcal{F}$ is generated by all $M(n)$. Now state the main theorem of this section.

Definition 5.6.1. *Let A be a graded ring, We call an A-module M the quasi-finitely generated module if there exists some finitely generated graded A-module N such that $M_n \cong N_n$ for $n \gg 0$. We define a equivalence relation denoted by $' \approx '$ as follows: we say graded A-modules $M \approx N$ if $M_n \cong N_n$ for $n \gg 0$.*

Theorem 5.6.2. *Let A be a graded ring, generated by A_1 as an A_0-algebra. Then the category of coherent sheaves on $X = \operatorname{Proj} A$ is equivalent to the category of quasi-finitely generated A-modules module the relation $' \approx '$ by the functor $\Gamma_*(\cdot)$, and the converse functor is $\sim$.*

We will return to this after introducing cohomology.

coh fg

Theorem 5.6.3. *Let R be a ring and let $A = R[x_0, .., x_r]$ be a noetherian graded ring, let $X = \operatorname{Proj} A$. If $\mathcal{F}$ is a coherent sheaf on X, then there exists $n_0 \geq 0$ such that for any $n > n_0$, the sheaf $\mathcal{F}(n)$ can be generated by finite number of sections in $\Gamma(X, \mathcal{F}(n))$.*

Proof. Since $\mathcal{F}$ is coherent, then for any x_i, $\mathcal{F}|_{D_+(x_i)}$ is generated by finitely many sections s_{ij}. Applying ugly lemma ugly lemma 5.5.5 by replacing f in the lemma with x_i, it follows that there is an $n \gg 0$ such that $x_i s_{ij}$ can be lifted to global section in $\Gamma(X, \mathcal{F}(n))$. □

Remark 5.6.4. We say that $\mathcal{F}$ is generated by global sections is equivalent to say there is a surjective morphism of sheaves $\bigoplus_{i \in I} \mathcal{O}_X \to \mathcal{F}$. Indeed, the generating sections define this morphism

Corollary 5.6.5. *Let $X = \operatorname{Proj} A$ be a projective scheme, where A is a noetherian graded ring. Assume that $\mathcal{F}$ is a coherent sheaf on X and n_i are integers, then there is a surjective morphism:*

$$\bigoplus_{n_i} \mathcal{O}(n_i) \longrightarrow \mathcal{F}. \tag{5.6.1}$$

Definition 5.6.6 (Very ample)**.** *If X is any scheme over Y, i.e. there is a morphism $X \to Y$. An invertible sheaf $\mathcal{L}$ on X is called very ample relative to Y if there is an immersion $i : X \to \mathbb{P}^r_Y$ for some r, such that $i^*(\mathcal{O}(1)) \cong \mathcal{L}$. We say that a morphism $i : X \to Z$ is an immersion if it gives an isomorphism of X with an open subscheme of a closed subscheme of Z.*

Remark 5.6.7. The theorem coh fg 5.6.3 is actually true for any projective scheme X over a noetherian A. Since there is closed immersion $i : X \to \mathbb{P}^r_A$ such that $i^*(\mathcal{O}(1)) \cong \mathcal{O}_X(1)$. Then the push forward of the coherent sheaf $\mathcal{F}$ through i is still coherent on $\mathbb{P}^r_A$, and $i_*(\mathcal{F}(n)) = (i_*\mathcal{F})(n)$, then $\mathcal{F}$ is generated by global sections if and only if $i_*(\mathcal{F}(n))$ is. So we can reduce to the case when X is projective n-space.

Definition 5.6.8 (Ample line bundle)**.** *Let X be a scheme over a ring A. Let $\mathcal{L}$ be a line bundle (invertible $\mathcal{O}_X$-module). We say $\mathcal{L}$ is ample if for any coherent sheaf $\mathcal{F}$ on X there is an integer $n > 0$ such that $\mathcal{F} \otimes \mathcal{L}^n$ is generated by global sections.*

This means $\widetilde{\Gamma}(X, \mathcal{F} \otimes \mathcal{L}^n) \to \mathcal{F} \otimes \mathcal{L}^n$ is surjective or equivalently for any $x \in X$, the following morphism is surjective:

$$\Gamma(X, \mathcal{F} \otimes \mathcal{L}^n) \twoheadrightarrow \mathcal{F}(x), \quad \left(\mathcal{F}(x) = \mathcal{F}_x \big/ m_x \mathcal{F}_x \right).$$

coh fg

Remark 5.6.9. The theorem 5.6.3 shows that a very ample sheaf $\mathcal{L}$ on a projective scheme X over a noetherian ring A is ample. But the converse is not true.

Example 5.6.10. 1) $\mathcal{O}_X(n)$ is not generated by global sections if $n < 0$. Therefore, $\mathcal{O}_X(n)$ can't be ample.

2) $\mathcal{O}_X(0) = \mathcal{O}_X$ is generated by global section 1. But $\mathcal{O}_X^n \otimes \mathcal{O}_X(-1) = \mathcal{O}_X^n(-1)$ can not be globally generated. Thus, $\mathcal{O}_X$ can't be ample.

Proposition 5.6.11. *Let X be a projective scheme. Then $\mathcal{O}_X(n)$ is ample for any $n > 0$.*

Proof. First reduction. We need only show $n = 1$. Let $X = \operatorname{Proj}(S)$, define $S^n = \oplus_{d\geq 0} S_{nd}$, where $\deg S^n = \deg S/n$. Then $X = \operatorname{Proj}(S^n)$, $\mathcal{O}_{X,S}(n) = \mathcal{O}_{X,S^n}(1)$. □

Theorem 5.6.12. *Let X be a scheme of finite type over a noetherian ring A, and let $\mathcal{L}$ be an invertible sheaf on X. Then $\mathcal{L}$ is ample if and only if $\mathcal{L}^m$ is very ample over* Spec$A$ *for some integer m.*

6 Cohomology

6.1 Čech cohomology

Let X be a topological space, $\mathcal{F}$ be a sheaf on X. The functor Γ maps sheaves over X into Abelian groups, i.e.$\mathcal{F} \mapsto \Gamma(X, \mathcal{F})$. this is a left exact functor. If $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ is exact then

$$0 \longrightarrow \Gamma(X, \mathcal{F}_1) \longrightarrow \Gamma(X, \mathcal{F}_2) \longrightarrow \Gamma(X, \mathcal{F}_3)$$

is exact as well. If X is affine and $\mathcal{F}_i$ are quasi-coherent then Γ is exact.

$$0 \to \Gamma(\mathcal{F}_1) \to \Gamma(\mathcal{F}_2) \to \Gamma(\mathcal{F}_3) \to 0.$$

Cover X by a set of open subsets $\mathcal{U} = \{U_i,\ i \in I\}$, I is ordered.

Let $\underline{i}_p = \{i_0 < i_1 < \ldots < i_p\}$ be any ordered subset of I whose cardinality is $p+1$, let $U_{\underline{i}_p} = U_{i_0 \ldots i_p} = \bigcap_{k=0}^{p} U_{i_k}$. Thus $U_{ij} = U_i \cap U_j$, $U_{ijk} = U_i \cap U_j \cap U_k$ and so on. Then we get a sequence:

$$0 \to \Gamma(X, \mathcal{F}) \longrightarrow \prod_i \Gamma(U_i, \mathcal{F}) \xrightarrow{d^0} \prod_{i<j} \Gamma(U_{ij}, \mathcal{F}) \xrightarrow{d^1} \prod_{i<j<k} \Gamma(U_{ijk}, \mathcal{F}) \to \ldots . \tag{6.1.1}$$

Let $s_{\underline{i}_p} \in \Gamma(U_{\underline{i}_p}, \mathcal{F})$ be a section, let $s = (s_{\underline{i}_p})_{\underline{i}_p \subset I}$ denote a section in$\prod_{\underline{i}_p} \Gamma(U_{\underline{i}_p}, \mathcal{F})$. The maps d^α, $\alpha = 0, 1, \ldots.$ are defined as follows:

$$\begin{aligned} d^0 :\ \textstyle\prod_i \Gamma(U_i, \mathcal{F}) &\longrightarrow \textstyle\prod_{i<j} \Gamma(U_{ij}, \mathcal{F}) \ ; \\ (s_i)_{i \in I} &\longmapsto ((s_i - s_j)|_{U_{ij}})_{i<j} \end{aligned} \tag{6.1.2}$$

$$\begin{aligned} d^p :\ \textstyle\prod_{\underline{i}_p} \Gamma(U_{\underline{i}_p}, \mathcal{F}) &\longrightarrow \textstyle\prod_{\underline{i}_{p+1}} \Gamma(U_{\underline{i}_{p+1}}, \mathcal{F}) \quad . \\ (s_{\underline{i}_p})_{\underline{i}_p \subset I} &\longmapsto (\textstyle\sum_{k=0}^{p+1} (-1)^k s_{i_0 \ldots \hat{i_k} \ldots i_{p+1}}|_{U_{\underline{i}_p}})_{\underline{i}_{p+1} \subset I} \end{aligned} \tag{6.1.3}$$

Here the notation $\hat{i_k}$ means omit i_k. Then since $s_{i_0\ldots\hat{i_k}\ldots i_{p+1}}$ is an element of $\mathcal{F}(U_{i_0\ldots\hat{i_k}\ldots i_{p+1}})$, we restrict to $U_{i_0\ldots i_{p+1}}$ to get an element of $\mathcal{F}(U_{i_0\ldots i_{p+1}})$.

Notation. *Let $C^p(\mathcal{U},\mathcal{F}) = \prod_{\underline{i}_p} \Gamma(U_{\underline{i}_p},\mathcal{F})$, then we have a complex of abelian groups*

$$C^0(\mathcal{U},\mathcal{F}) \xrightarrow{d^0} C^1(\mathcal{U},\mathcal{F}) \xrightarrow{d^1} C^2(\mathcal{U},\mathcal{F}) \to \ldots. \tag{6.1.4}$$

The complex defined above, denoted by $(C^\bullet(\mathcal{U},\mathcal{F}), d^\bullet)$ is called Čech complex of abelian groups associated to $\mathcal{U}$ and $\mathcal{F}$. The fact is that $\ker d^0 = \Gamma(X,\mathcal{F})$.

Exercise 6.1.1. Prove that $d^{p+1}d^p = 0$

It is clear that $d^1 \circ d^0 = 0$, but $d^1(s_{ij}) = 0$ does not necessary imply $s_{ij} \in \mathrm{Im}(d^0)$.

Definition 6.1.2. *Define the pth* Čech cohomology *of $(X,\mathcal{F})$ with respect to the covering $\mathcal{U}$, to be*

$$H^i_{\mathcal{U}}(X,\mathcal{F}) = \ker(d^i)\big/\mathrm{Im}(d_{i-1}).$$

If $\mathcal{V}$ is a refinement of $\mathcal{U}$, then we have a morphism of topological subset $V_i \to U_i$, it induces homomorphisms:

$$C^i(\mathcal{U},\mathcal{F}) \to C^i(\mathcal{V},\mathcal{F}) \rightsquigarrow H^i_{\mathcal{U}}(X,\mathcal{F}) \to H^i_{\mathcal{V}}(X,\mathcal{F}) \tag{6.1.5}$$

Lemma 6.1.3. *For any X, $\mathcal{U}$, $\mathcal{F}$ as above, we have $H^0_{\mathcal{U}}(X,\mathcal{F}) \cong \Gamma(X,\mathcal{F})$.*

Proof. $H^0_{\mathcal{U}}(X,\mathcal{F}) = \ker d^0$. Let $s = (s_i)_{i\in i}$ be a section in C^0, then for each $i < j$, $(d^0(s))_i j = s_j - s_i$. So $d^0(s) = 0$ implies the sections s_i and s_j agree on $U_i \bigcap U_j$. From the sheaf axioms it follows that $\ker d^0 = \Gamma(X,\mathcal{F})$. □

Example 6.1.4. Let X be the circle $\mathbb{S}^1$ covered by two open sets U_1 and U_2 and let $\mathcal{F} = \mathbb{Z}$. We have $H^0_{\mathcal{U}}(X,\mathbb{Z}) = \Gamma(X,\mathcal{F}) = \ker(d^0)$,

$$\Gamma(U_1,\mathbb{Z}) \oplus \Gamma(U_2,\mathbb{Z}) \xrightarrow{d^0} \Gamma(U_1 \cap U_2,\mathbb{Z}) \xrightarrow{d_1} 0.$$

In this example $\Gamma(U_1,\mathbb{Z}) = \Gamma(U_2,\mathbb{Z}) = \mathbb{Z}$ and $\Gamma(U_1 \cap U_2,\mathbb{Z}) = \mathbb{Z}^2$. Since

$$\begin{array}{ccccc} \mathbb{Z}^2 & \longrightarrow & \mathbb{Z}^2 & \longrightarrow & 0 \\ (a,b) & \longmapsto & (b-a, b-a) & & \end{array} \tag{6.1.6}$$

Therefore $H^0(X,\mathcal{F}) = \mathbb{Z}$ and $H^1(X,\mathcal{F}) = \mathbb{Z}^2/\mathbb{Z}(1,1) = \mathbb{Z}$.

Example 6.1.5. Let $X = \mathbb{P}^1$, covered by $U_0 = \mathbb{P}^1 \setminus \{\infty\}$ and $U_1 = \mathbb{P}^1 \setminus \{0\}$ and let $\mathcal{F} = \Omega^1_X$, where for open subset $U \subset \mathbb{P}^1$ we denote by $\Omega^1_X(U)$ the space of holomorphic 1-forms on U. We have $U_0 \cong \mathbb{C}^1 = \mathbb{A}^1 = \mathrm{Spec}\mathbb{C}[x]$ and $U_1 \cong \mathbb{C}^1 = \mathbb{A}^1 = \mathrm{Spec}\mathbb{C}\left[\frac{1}{x}\right]$, so $\Omega^1(U_0) = \mathbb{C}[x]\,dx$, while $\Omega^1(U_1) = \mathbb{C}\left[\frac{1}{x}\right] d\left(\frac{1}{x}\right)$. Note also that $\Omega^1(U_0 \cap U_1) = \mathbb{C}\left[x, \frac{1}{x}\right] dx$. Since $d\left(\frac{1}{x}\right) = -\frac{1}{x^2}\,dx$ we can choose dx as a generator. The sequence becomes

$$\mathbb{C}[x]\,dx \oplus \mathbb{C}\left[\frac{1}{x}\right] d\left(\frac{1}{x}\right) \xrightarrow{d^0} \mathbb{C}\left[x, \frac{1}{x}\right] dx,$$

where $d^0 : (\alpha, \beta) \mapsto (\beta - \alpha)$. Thus $H^0(X, F) = \ker(d^0) = \{(\alpha, \beta),\ \beta = \alpha\} = 0$. Indeed, if $\alpha = P(x)\,dx$ and $\beta = Q\left(\frac{1}{x}\right) d\left(\frac{1}{x}\right)$ then the condition

$$P(x)\,dx = Q\left(\frac{1}{x}\right)\left(-\frac{1}{x^2}\,dx\right)$$

implies $P(x) = -\frac{1}{x^2} Q\left(\frac{1}{x}\right)$ which may be true only for $P(x) = Q(x) = 0$. Next,

$$H^1(X, \mathcal{F}) = \frac{\mathbb{C}\left[x, \frac{1}{x}\right] dx}{\mathbb{C}[x]\,dx + \mathbb{C}\left[\frac{1}{x}\right]\left(-\frac{1}{x^2}\right) dx} = \mathbb{C}$$

(generated by $\frac{dx}{x}$). Finally, $H^0(X, \Omega^1_X) = 0$, $H^1(X, \Omega^1_X) \cong \mathbb{C}$. Remark: $\Omega^1_X \cong \mathcal{O}_X(-2)$.

`coh thm` **Theorem 6.1.6.** *Let X be a separated scheme, $\mathcal{F}$ be a quasi-coherent sheaf on X. Let $\mathcal{U}$ be an affine cover of X. Then:*

1. *If X is affine, then $H^i_{\mathcal{U}}(X, \mathcal{F}) = 0$ for $i > 0$.*
2. *If $0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{G} \to 0$ is an exact sequence of quasi-coherent sheaf, then it induces a long exact sequence*

$$\begin{aligned} 0 &\longrightarrow \Gamma(X, \mathcal{E}) \longrightarrow \Gamma(X, \mathcal{F}) \longrightarrow \Gamma(X, \mathcal{G}) \longrightarrow \\ &\longrightarrow H^1(X, \mathcal{E}) \longrightarrow H^1(X, \mathcal{F}) \longrightarrow H^1(X, \mathcal{G}) \longrightarrow \\ &\longrightarrow H^2(X, \mathcal{E}) \longrightarrow H^2(X, \mathcal{F}) \longrightarrow H^2(X, \mathcal{G}) \longrightarrow \\ &\longrightarrow H^3(X, \mathcal{E}) \longrightarrow \dots. \end{aligned} \tag{6.1.7}$$

3. *$H^i_{\mathcal{U}}(X, \mathcal{F})$ doesn't depend on $\mathcal{U}$.*
4. *If X is noetherian, then $H^i_{\mathcal{U}}(X, \mathcal{F}) = 0$ for i sufficiently large.*

6.2 Čech complex of sheaves

Let X be a topological space. Let $\mathcal{F}$ be a sheaf on X. Let $\mathcal{U} = \{U_i\}_{i \in I}$ be a cover of X, where I is an ordered set. We want to study is the Čech complex resolution.

Step 1: Construct complex of sheaves

Similarly, Let $\underline{i}_p = \{i_0 < i_1 < \dots < i_p\}$ be any ordered subset of I whose cardinality is $p+1$, let $U_{\underline{i}_p} = U_{i_0 \dots i_p} = \bigcap_{k=0}^p U_{i_k}$, $U_{ij} = U_i \cap U_j$. Thus $U_{ijk} = U_i \cap U_j \cap U_k$ and so on. Then we always have the inclusions:

$$j_{\underline{i}_p} : U_{\underline{i}_p} \hookrightarrow X. \tag{6.2.1}$$

Restrict $\mathcal{F}$ on the open subset $U_{\underline{i}_p}$. Then $(j_{\underline{i}_p})_* \mathcal{F}|_{U_{\underline{i}_p}}$ is a sheaf on X.

Definition 6.2.1. *Define the Čech complex of sheaf $\mathcal{F}$ on a scheme X associated to $\mathcal{U}$, denoted $\mathcal{C}^\bullet(\mathcal{U},\mathcal{F})$, as follows:*

$$\mathcal{C}^p(\mathcal{U},\mathcal{F}) := \prod_{\underline{i}_p} (j_{\underline{i}_p})_* \left(\mathcal{F}|_{U_{\underline{i}_p}}\right). \tag{6.2.2}$$

Thus a section of $\mathcal{C}^p(\mathcal{U},\mathcal{F})$ can be written as $s = (s_{\underline{i}_p})_{\underline{i}_p \subset I}$ where $s_{\underline{i}_p} \in (j_{\underline{i}_p})_* \left(\mathcal{F}|_{U_{\underline{i}_p}}\right)$.

Define the morphisms as before:

$$\mathcal{C}^0(\mathcal{U},\mathcal{F}) \xrightarrow{d^0} \mathcal{C}^1(\mathcal{U},\mathcal{F}) \xrightarrow{d^1} \mathcal{C}^2(\mathcal{U},\mathcal{F}) \cdots \to \mathcal{C}^p(\mathcal{U},\mathcal{F}) \xrightarrow{d^p} \mathcal{C}^{p+1}(\mathcal{U},\mathcal{F}) \cdots \tag{6.2.3}$$

$$\begin{aligned} d^p : \mathcal{C}^p(\mathcal{U},\mathcal{F}) &\longrightarrow \mathcal{C}^{p+1}(\mathcal{U},\mathcal{F}) \\ (s_{\underline{i}_p})_{\underline{i}_p \subset I} &\longmapsto \left(\textstyle\sum_{k=0}^{p+1} (-1)^k s_{i_0 \ldots \hat{i_k} \ldots i_{p+1}}|_{U_{\underline{i}_p}}\right)_{\underline{i}_{p+1} \subset I} \end{aligned} \tag{6.2.4}$$

Exercise 6.2.2. Prove that $d^{p+1} d^p = 0$ and $d^2 = 0$.

Remark 6.2.3. Recall that in last section we have defined $C^p(\mathcal{U},\mathcal{F}) = \prod_i \Gamma(U_{\underline{i}_p},\mathcal{F})$, this is an abelian group. In this section $\mathcal{C}^p(\mathcal{U},\mathcal{F})$ is a sheaf.

Remark 6.2.4. $C^p(\mathcal{U},\mathcal{F}) = \Gamma(X, \mathcal{C}^p(\mathcal{U},\mathcal{F}))$.

Remark 6.2.5. The sequence

$$\Gamma(X,\mathcal{F}) \longrightarrow C^0(\mathcal{U},\mathcal{F}) \longrightarrow C^1(\mathcal{U},\mathcal{F}) \longrightarrow C^2(\mathcal{U},\mathcal{F}) \longrightarrow \ldots$$

may not be exact and the cohomology groups may be obstructions with obstructions.

Ftoc CC exact

Proposition 6.2.6. *Let X be a topological space and $\mathcal{F}$ be a sheaf on X, $\mathcal{C}^\bullet(\mathcal{U},\mathcal{F})$ is the Čech complex of $\mathcal{F}$, together with the chain of complex*

$$\mathcal{C}^0(\mathcal{U},\mathcal{F}) \xrightarrow{d^0} \mathcal{C}^1(\mathcal{U},\mathcal{F}) \xrightarrow{d^1} \mathcal{C}^2(\mathcal{U},\mathcal{F}) \cdots \to \mathcal{C}^p(\mathcal{U},\mathcal{F}) \xrightarrow{d^p} \mathcal{C}^{p+1}(\mathcal{U},\mathcal{F}) \cdots \tag{6.2.5}$$

then $\ker d^0 = \mathcal{F}$ *and the following sequence*

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{C}^0(\mathcal{U},\mathcal{F}) \longrightarrow \mathcal{C}^1(\mathcal{U},\mathcal{F}) \longrightarrow \mathcal{C}^2(\mathcal{U},\mathcal{F}) \longrightarrow \ldots$$

is exact. In other words, $H^0(\mathcal{C}^\bullet(\mathcal{U},\mathcal{F})) = \mathcal{F}$, and $H^p(\mathcal{C}^\bullet(\mathcal{U},\mathcal{F})) = 0$ for $p > 0$.

Proof. The exactness at $\mathcal{F}$ and $\mathcal{C}^0(\mathcal{U},\mathcal{F})$ is clear. We need only check the exact sequence at $\mathcal{C}^p$ for $p > 0$. So we have to study morphisms of two complexes. □

Step 2: Homotopy of morphisms of complexes

Definition 6.2.7. *Given two Čech complexes of abelian groups $C^\bullet$ and $D^\bullet$ and two morphisms of complexes $f^\bullet : C^\bullet \to D^\bullet$, $g^\bullet : C^\bullet \to D^\bullet$. Then we have a diagram:*

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & C^0 & \xrightarrow{d^0} & C^1 & \xrightarrow{d^1} & C^2 & \xrightarrow{d^2} & \cdots & \longrightarrow & C^p \\ & & \downarrow & \overset{k^1}{\swarrow} & \downarrow & \overset{k^2}{\swarrow} & \downarrow & & & \overset{k^p}{\swarrow} & \downarrow \\ 0 & \longrightarrow & D^0 & \xrightarrow{d^0} & D^1 & \xrightarrow{d^1} & D^2 & \xrightarrow{d^2} & \cdots & \longrightarrow & D^p \end{array} \tag{6.2.6}$$

A homotopy *of $f^\bullet - g^\bullet$ is a collection of maps $C^i \to D^{i-1}$ such that $f^i - g^i = k^{i+1} \circ d_C^i + d_D^{i-1} \circ k^i$, i.e. $f - g = dk + kd$.*

Exercise 6.2.8. If $f^\bullet$ is homotopic to $g^\bullet$, then $f^\bullet$ and $g^\bullet$ induces a same homomorphism of cohomological group:

$$H^i(f) \colon H^i(C^\bullet) \longrightarrow H^i(D^\bullet), \qquad H^i(g) \colon H^i(C^\bullet) \longrightarrow H^i(D^\bullet). \tag{6.2.7}$$

We would like to ask ourselves when does $H^i(f^\bullet) = H^i(g^\bullet)$ for all i? The answer is given by the following:

Lemma 6.2.9. *$H^i(f) = H^i(g)$ for all i if $f^\bullet$ and $g^\bullet$ are homotopic. This means that there are homomorphisms $k^i \colon C^i \to D^{i-1}$ such that $f^i - g^i = k^{i+1} \circ d_C^i + d_D^{i-1} \circ k^i$.*

Proof of the Lemma. $H^i(f^\bullet - g^\bullet)$ defines $H^i(C^\bullet) \to H^i(D^\bullet)$, $x \in H^i(C^\bullet)$ represented by $\widetilde{x} \in \ker d_i$, while $(H^i(f^i) - H^i(g^i))(x)$ is represented by

$$(f^i - g^i)(\widetilde{x}) = (k^{i+1} d^i + d^{i-1} k^i)(\widetilde{x}) = -d^{i-1} k^i \widetilde{x} = d^{i-1}(-k^i \widetilde{x}) \in \mathrm{Im}(d^{i-1}),$$

therefore $H^i(f^\bullet - g^\bullet)(x) = 0$. □

Now we can go back to prove the proposition CFtoCC exact 6.2.6

Step 3: Proof of the proposition CFtoCC exact 6.2.6

To prove the exactness of $\mathcal{F} \to \mathcal{C}^\bullet(\mathcal{U}, \mathcal{F})$, it is sufficient to show that its stalk is exact at any point $x \in X$. Fix a point $x \in X$, consider $\mathcal{F}_x \to \mathcal{C}_x^\bullet$. We need only show shat two morphisms Id: $\mathcal{C}_x^\bullet \to \mathcal{C}_x^\bullet$ and $0 : \mathcal{C}_x^\bullet \to \mathcal{C}_x^\bullet$ are homotopic. In other words we need to construct $k^i : \mathcal{C}^i \to \mathcal{C}^{i-1}$ such that $kd + dk = Id$.

Note that $\mathcal{C}_x^p = \prod_{\underline{i}_p} (\mathcal{F}_x)_{U_{\underline{i}_p}}$, where

$$(\mathcal{F}_x)_{U_{\underline{i}_p}} = \begin{cases} \mathcal{F}_x & \text{if } x \in U_{\underline{i}_p} \\ 0 & \text{if } x \notin U_{\underline{i}_p} \end{cases} \tag{6.2.8}$$

Suppose $x \in U_r$ for some open subset $U_r \subset X$. For $s_x \in \mathcal{C}^p(\mathcal{U}, \mathcal{F}_x)$, there exists a small neighborhood $V \subset U_r$ of x such that s_x is represented by a section $s \in \Gamma(V, \mathcal{C}^p(\mathcal{U}, CF))$. For any p, define

$$k^p : \mathcal{C}^p(\mathcal{U}, \mathcal{F}_x) \longrightarrow \mathcal{C}^{p-1}(\mathcal{U}, \mathcal{F}_x), \tag{6.2.9}$$

by

$$k^p(s)_{i_0\dots i_{p-1}} = s_{ri_0\dots i_{p-1}}. \tag{6.2.10}$$

This is well defined since $V\bigcap U_{\underline{i}_{p-1}} = V\bigcap Uri_0\dots i_{p-1}$. Then talk the stalk of $k^p(s)$ at x to get the required map k.

Exercise 6.2.10. Check that for any $p>0$, $s\in\mathcal{C}^p_x$, $(kd+dk)(s) = s$.

Thus k is a homotopy operator for the complex $\mathcal{C}^\bullet_x$, and the identity map is homotopic to the zero map. Showing that the sequence is exact.

Now we have shown a very important consequence: Suppose $\mathcal{C}^\bullet(\mathcal{U},\mathcal{F})$ is the Čech complex of sheaf $\mathcal{F}$ on a scheme X with respect to affine covering $\mathcal{U}$, then the sequence

$$0\longrightarrow\mathcal{F}\longrightarrow\mathcal{C}^0(\mathcal{U},\mathcal{F})\longrightarrow\mathcal{C}^1(\mathcal{U},\mathcal{F})\longrightarrow\mathcal{C}^2(\mathcal{U},\mathcal{F})\longrightarrow\dots \tag{6.2.11}$$

is exact.

Corollary 6.2.11. *Let X be a noetherian separated scheme and let $\mathcal{F}$ be a quasi-coherent sheaf on X. Let $\mathcal{E} = \mathcal{C}^0(\mathcal{U},\mathcal{F})$. Suppose $\mathcal{U} = \{U_i\}_{i\in I}$ is any affine cover of X, then:*

(a) If X is affine, then $H^i_{\mathcal{U}}(X,\mathcal{F}) = 0$ for $i>0$.

(b) Let $\mathcal{W}$ be any another affine cover, then $H^i_{\mathcal{W}}(X,\mathcal{E}) = 0$ for $i>0$.

Proof. (a) Since X is separated, then for any $\underline{i}_p$, $U_{\underline{i}_p}$ is affine. Since X is noetherian, then $\mathcal{C}^p = 0$ for p sufficiently large. If X is affine, then the exactness of $\mathcal{F}\longrightarrow\mathcal{C}^\bullet$ implies the exactness of global sections $\Gamma(X,\mathcal{F})\longrightarrow\Gamma(X,\mathcal{C}^\bullet) = C^\bullet$. This means no cohomology when $i>0$.

(b) Since $\mathcal{E} = \prod_k (j_k)_*\mathcal{F}|_{U_k}$, to show $\mathcal{E}$ has trivial cohomology for $i>0$, it is sufficient to show that every piece has no cohomology when $i>0$. Suppose $\mathcal{W}$ is any another affine cover of X, then $\mathcal{W}\bigcap U_k$ is an affine cover of U_k. By (a), for any $i>0$, we have

$$H^i_{\mathcal{W}}(X,(j_k)_*\mathcal{F}|_{U_k}) = H^i_{\mathcal{W}\bigcap U_k}(U_k,\mathcal{F}|_{U_k}) = 0. \tag{6.2.12}$$

Then we have

$$H^i_{\mathcal{W}}(X,\mathcal{E}) = \prod_k H^i_{\mathcal{W}}(X,(j_k)_*\mathcal{F}|_{U_k}) = 0. \tag{6.2.13}$$

□

6.3 Long exact sequence

Let X be a noetherian separated scheme and $\mathcal{F}$ be a quasi-coherent sheaf. Let $\mathcal{U}$ be an affine covering of X.

In this section we will show the second property in coh thm 6.1.6: $H^i_{\mathcal{U}}(X,\mathcal{F})$ form a long exact sequence for every exact sequence of quasi-coherent sheaves on X:

$$0\to\mathcal{E}\to\mathcal{F}\to\mathcal{G}\to 0. \tag{6.3.1}$$

Let $0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{G} \to 0$ be any exact sequence of quasi-coherent sheaves on X. Then we have the following diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{E} & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{G} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{C}^0(\mathcal{E}) & \longrightarrow & \mathcal{C}^0(\mathcal{F}) & \longrightarrow & \mathcal{C}^0(\mathcal{G}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{C}^1(\mathcal{E}) & \longrightarrow & \mathcal{C}^1(\mathcal{F}) & \longrightarrow & \mathcal{C}^1(\mathcal{G}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \cdots & & \cdots & & \cdots & &
\end{array} \tag{6.3.2}$$

Apply the functor $\Gamma(X, \cdot)$ we have the diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Gamma(X,\mathcal{E}) & \longrightarrow & \Gamma(X,\mathcal{F}) & \longrightarrow & \Gamma(X,\mathcal{G}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & C^0(\mathcal{E}) & \longrightarrow & C^0(\mathcal{F}) & \longrightarrow & C^0(\mathcal{G}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & C^1(\mathcal{E}) & \longrightarrow & C^1(\mathcal{F}) & \longrightarrow & C^1(\mathcal{G}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \cdots & & \cdots & & \cdots & &
\end{array} \tag{6.3.3}$$

Remark 6.3.1.

1. The vertical sequences of 6.3.3 defines cohomology of $\mathcal{E}$, $\mathcal{F}$ and $\mathcal{G}$.
2. $0 \longrightarrow \Gamma(X,\mathcal{E}) \longrightarrow \Gamma(X,\mathcal{F}) \longrightarrow \Gamma(X,\mathcal{G}) \longrightarrow 0$ is in general not exact.
3. $\mathcal{U}$ is an affine cover and X is noetherian and separated, then for all $\underline{i}_p$, $U_{\underline{i}_p}$ is affine subscheme of X, the product is also affine.

Claim. $0 \longrightarrow \mathcal{C}^i(\mathcal{U},\mathcal{E}) \longrightarrow \mathcal{C}^i(\mathcal{U},\mathcal{F}) \longrightarrow \mathcal{C}^i(\mathcal{U},\mathcal{G}) \longrightarrow 0$ is exact for all $i \geq 0$.

Proof. The restriction of the exact sequence 6.3.1 on each $U_{\underline{i}_p}$ is also exact. From the remark we know that for all $\underline{i}_p$,

$$j_{\underline{i}_p} : U_{\underline{i}_p} \hookrightarrow X \tag{6.3.4}$$

are affine morphisms. Thus $(j_{\underline{i}_p})_*$ is an exact functor. □

Claim. $0 \longrightarrow C^i(\mathcal{U},\mathcal{E}) \longrightarrow C^i(\mathcal{U},\mathcal{F}) \longrightarrow C^i(\mathcal{U},\mathcal{G}) \longrightarrow 0$ is exact for all $i \geq 0$.

Proof. For all i, $\mathcal{C}^i(\mathcal{E})$, $\mathcal{C}^i(\mathcal{F})$ and $\mathcal{C}^i(\mathcal{G})$ are quasi-cohernet sheaves on affine schemes. We have proved that

$$0 \longrightarrow \mathcal{C}^i(\mathcal{E}) \longrightarrow \mathcal{C}^i(\mathcal{F}) \longrightarrow \mathcal{C}^i(\mathcal{G}) \longrightarrow 0$$

is exact for all $i \geq 0$, then by the equivalence between category of quasi-coherent sheaves on affine schemes and category of $\mathcal{O}_{U_{\underline{i}_p}}$-modules,

$$0 \longrightarrow C^i(\mathcal{E}) \longrightarrow C^i(\mathcal{F}) \longrightarrow C^i(\mathcal{G}) \longrightarrow 0$$

is also exact. $\square$

Now we introduce the general way to define long exact sequences.

Lemma 6.3.2 (Snake lemma)**.** *Let A, B, C and A_1, B_1, C_1 be abelian groups. Assume there are homomorphisms $\alpha : A \to A_1$, $\beta : B \to B_1$ and $\gamma : C \to C_1$. Suppose we have the following commutative diagram:*

$$\begin{array}{ccccccc}
 & & \ker\alpha & & \ker\beta & & \ker\gamma & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \\
0 & \longrightarrow & A_1 & \longrightarrow & B_1 & \longrightarrow & C_1 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \mathrm{coker}\alpha & & \mathrm{coker}\beta & & \mathrm{coker}\gamma & &
\end{array} \quad . \tag{6.3.5}$$

Then there is an exact sequence relating the kernel and cokernels of α, β and γ.

$$\ker\alpha \to \ker\beta \to \ker\gamma \xrightarrow{d} \mathrm{coker}\alpha \to \mathrm{coker}\beta \to \mathrm{coker}\gamma, \tag{6.3.6}$$

where d is a homomorphism, called connecting homomorphism

Exercise 6.3.3. Prove the snake lemma.

Proposition 6.3.4. *Let $A^\bullet$, $B^\bullet$ and $C^\bullet$ be complexes in the same abelian category which fits in an exact sequence*

$$0 \to A^\bullet \to B^\bullet \to C^\bullet \to 0 \tag{6.3.7}$$

Then we have long exact sequence

$$\begin{aligned}
0 &\longrightarrow H^0(A^\bullet) \longrightarrow H^0(B^\bullet) \longrightarrow H^0(C^\bullet) \longrightarrow \\
&\longrightarrow H^1(A^\bullet) \longrightarrow H^1(B^\bullet) \longrightarrow H^1(C^\bullet) \longrightarrow \\
&\longrightarrow H^2(A^\bullet) \longrightarrow H^2(B^\bullet) \longrightarrow H^2(C^\bullet) \longrightarrow \\
&\longrightarrow H^3(A^\bullet) \longrightarrow \dots .
\end{aligned} \tag{6.3.8}$$

Proof. Since for any i we have $d_A^i : A^i \longrightarrow A^{i+1}$, consider the morphism

$$\alpha : A^i \big/ \mathrm{Im} d_A^{i-1} \longrightarrow \ker(d_A^{i+1}) \tag{6.3.9}$$

Notice that $\ker \alpha = H^i(A^\bullet)$, $\mathrm{coker} \alpha = H^{i+1}(A^\bullet)$. The same is for $B^\bullet$, $C^\bullet$. Then apply the snake lemma to the following commutative diagram:

$$\begin{array}{ccccccccc} & & H^i(A^\bullet) & & H^i(B^\bullet) & & H^i(C^\bullet) & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & A^i \big/ \mathrm{Im} d_A^{i-1} & \longrightarrow & B^i \big/ \mathrm{Im} d_B^{i-1} & \longrightarrow & C^i \big/ \mathrm{Im} d_C^{i-1} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} & & \\ 0 & \longrightarrow & \ker(d_A^{i+1}) & \longrightarrow & \ker(d_B^{i+1}) & \longrightarrow & \ker(d_C^{i+1}) & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & H^{i+1}(A^\bullet) & & H^{i+1}(B^\bullet) & & H^{i+1}(C^\bullet) & & \end{array} . \tag{6.3.10}$$

Thus we get

$$\cdots \to H^i(A^\bullet) \to H^i(B^\bullet) \to H^i(C^\bullet) \xrightarrow{d} H^{i+1}(A^\bullet) \to H^{i+1}(B^\bullet) \to H^{i+1}(C^\bullet) \to \cdots. \tag{6.3.11}$$

□

Let X be a noetherian separated scheme and $\mathcal{F}$ be a quasi-coherent sheaf. Let $\mathcal{U}$ be an affine covering of X. Assume that $0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{G} \to 0$ is any exact sequence of quasi-coherent sheaves on X. Then we have

$$0 \longrightarrow C^i(\mathcal{U}, \mathcal{E}) \longrightarrow C^i(\mathcal{U}, \mathcal{F}) \longrightarrow C^i(\mathcal{U}, \mathcal{G}) \longrightarrow 0$$

is exact for all $i \geq 0$. Let $A^\bullet$, $B^\bullet$ and $C^\bullet$ in the above proposition be $C^\bullet(\mathcal{U}, \mathcal{E})$, $C^\bullet(\mathcal{U}, \mathcal{F})$ and $C^\bullet(\mathcal{U}, \mathcal{G})$ respectively, it follows that $0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{G} \to 0$ induces a long exact sequence of cohomology. In other words, we have proved the second property in coh thm 6.1.6.

6.4 Independence of $\mathcal{U}$

In this section we will prove the third property in coh thm 6.1.6:

Proposition 6.4.1. *The Čech cohomology $H^i_{\mathcal{U}}(X, \mathcal{F})$ doesn't depend on $\mathcal{U}$ whenever $\mathcal{U}$ is finite cover of affine open subsets.*

If $\mathcal{U}' = \{U_i'\}_{i\in I'}$ is another affine covering, then we have a common refinement

$$\mathcal{W} := \{U_i \cap U'_{i'}\}_{(i,i')\in I\times I'}$$

of coverings which actually an affine cover since X is separated. Thus we have morphisms of cohomology groups:

$$H^i_{\mathcal{U}}(X,\mathcal{F}) \to H^i_{\mathcal{W}}(X,\mathcal{F}), \qquad H^i_{\mathcal{U}'}(X,\mathcal{F}) \to H^i_{\mathcal{W}}(X,\mathcal{F}). \tag{6.4.1}$$

We will prove that for any i,

$$H^i_{\mathcal{U}}(X,\mathcal{F}) \xrightarrow{\sim} H^i_{\mathcal{W}}(X,\mathcal{F}).$$

Step 1: construct morphism

Let $\mathcal{W} = \{W_j\}_{j\in J}$, $\mathcal{U} = \{U_i\}_{i\in J}$ and $\sigma : J \to I$ such that $W_j \hookrightarrow U_{\sigma(j)}$. Let s be a section in $C^p(\mathcal{U},\mathcal{F})$. Define

$$\phi : C^p(\mathcal{U},\mathcal{F}) \longrightarrow C^p(\mathcal{W},\mathcal{F}) \tag{6.4.2}$$

by restriction

$$\phi(s)|(W_{j_0\dots j_p}) = s(U_{\sigma(j_0)\dots\sigma(j_p)})|_{W_{j_0\dots j_p}}. \tag{6.4.3}$$

This defines a morphism:

$$\phi^\bullet : C^\bullet(\mathcal{U},\mathcal{F}) \longrightarrow C^\bullet(\mathcal{W},\mathcal{F}) \tag{6.4.4}$$

which induces

$$H^\bullet(\phi^\bullet) : H^\bullet(\mathcal{U},\mathcal{F}) \longrightarrow H^\bullet(\mathcal{W},\mathcal{F}). \tag{6.4.5}$$

We need to show that $H^\bullet(\phi^\bullet)$ is an isomorphism.

Step 2: induction

Let $\mathcal{V}$ be any affine cover and let $\mathcal{E} = \mathcal{C}^0(\mathcal{V},\mathcal{F})$. We have proved that $\mathcal{E}$ is quasi-coherent and $H^i_{\mathcal{U}}(X,\mathcal{E}) = 0$ for any affine cover $\mathcal{U}$ and $i > 0$.

Let $\mathcal{G}$ be the quotient sheaf $\mathcal{E}/\mathcal{F}$, then $\mathcal{G}$ is also quasi-coherent and there is an exact sequence of quasi-coherent sheaves on X:

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{E} \longrightarrow \mathcal{G} \longrightarrow 0. \tag{6.4.6}$$

This induces a long exact sequence:

$$\begin{array}{ccccccc} 0 = H^{i-1}_{\mathcal{U}}(X,\mathcal{E}) & \longrightarrow & H^{i-1}_{\mathcal{U}}(X,\mathcal{G}) & \longrightarrow & H^i_{\mathcal{U}}(X,\mathcal{F}) & \longrightarrow & H^i_{\mathcal{U}}(X,\mathcal{E}) = 0 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ 0 = H^{i-1}_{\mathcal{W}}(X,\mathcal{E}) & \longrightarrow & H^{i-1}_{\mathcal{W}}(X,\mathcal{G}) & \longrightarrow & H^i_{\mathcal{W}}(X,\mathcal{F}) & \longrightarrow & H^i_{\mathcal{W}}(X,\mathcal{E}) = 0 \end{array} \tag{6.4.7}$$

for any $i > 1$. It follows that $H^{i-1}_{\mathcal{W}}(X,\mathcal{G}) \xrightarrow{\sim} H^i_{\mathcal{W}}(X,\mathcal{F})$ and $H^{i-1}_{\mathcal{U}}(X,\mathcal{G}) \xrightarrow{\sim} H^i_{\mathcal{U}}(X,\mathcal{F})$.

We can induct on i:

(1) Note that $H^0_{\mathcal{U}}(X,\mathcal{F}) = \Gamma(X,\mathcal{F})$ for any affine cover $\mathcal{U}$.

(2) If $i = 1$, In this case we have the exact sequence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Gamma(X,\mathcal{F}) & \longrightarrow & \Gamma(X,\mathcal{E}) & \longrightarrow & \Gamma(X,\mathcal{G}) & \longrightarrow & H^1_{\mathcal{U}}(X,\mathcal{F}) & \longrightarrow & 0 \\ & & \| & & \| & & \| & & \downarrow & & \\ 0 & \longrightarrow & \Gamma(X,\mathcal{F}) & \longrightarrow & \Gamma(X,\mathcal{E}) & \longrightarrow & \Gamma(X,\mathcal{G}) & \longrightarrow & H^1_{\mathcal{W}}(X,\mathcal{F}) & \longrightarrow & 0 \end{array} \tag{6.4.8}$$

Then it is easy to see that $H^1_{\mathcal{U}}(X,\mathcal{F}) \xrightarrow{\sim} H^1_{\mathcal{W}}(X,\mathcal{F})$.

(3) If $i > 1$, suppose that $H^{i-1}_{\mathcal{U}}(X,\mathcal{F}) \xrightarrow{\sim} H^{i-1}_{\mathcal{W}}(X,\mathcal{F})$. By the diagram 6.4.7 we have $H^i_{\mathcal{U}}(X,\mathcal{F}) \xrightarrow{\sim} H^i_{\mathcal{W}}(X,\mathcal{F})$.

6.5 Cohomology on projective schemes

Let R be a ring and $X = \operatorname{Proj} R[x_0,...,x_n] = \mathbb{P}^n_R$. Recall that $\mathcal{O}(d) = (R[x_0,...,x_n](d))^\sim$ is vector bundle on X, where

$$(R[x_0,...,x_n](d))_{\deg i} = R[x_0,...,x_n]_{\deg i+d}.$$

Let $D_+(x_i)$ be principle open subset of X, then $\mathcal{O}(d)|_{D_+(x_i)} \xrightarrow{\sim} \widetilde{A[\frac{1}{x_i}]_{\deg d}}$

Theorem 6.5.1. *Let $X = \mathbb{P}^n_R$, then*

$$H^p(X,\mathcal{O}(d)) = \begin{cases} \sum_{r_i \geq 0} R \cdot x_0^{r_0} \dots x_n^{r_n}, & \text{if} \quad p = 0 \\ \sum_{r_i < 0} R \cdot x_0^{r_0} \dots x_n^{r_n}, & \text{if} \quad p = n \\ 0 & \text{if} \quad p \neq 0, n \end{cases}$$

In all sums above we assume that $\sum r_i = d$, and there is a duality pairing

$$H^0(X,\mathcal{O}(d)) \times H^n(X,\mathcal{O}(-1-n-d)) \longrightarrow H^n(X,\mathcal{O}(-1-n)) \xrightarrow{\sim} R \cdot x_0^{-1} \cdots x_n^{-1}. \tag{6.5.1}$$

Proof. (1) If $p = 0$, $H^0(X,\mathcal{O}(d)) = \Gamma(X,\mathcal{O}(d)) = R[x_0,...,x_n]_{\deg d}$.

(2) If $p = n$, take standard affine cover $\mathcal{U} = \{D_+(x_i)\}_{i=0}^n$, and let $A = R[x_0,...,x_n]$

$$\mathcal{O}(d)|_{D_+(x_i)} = (A_{(x_i)} \bigotimes x_i^d)^\sim. \tag{6.5.2}$$

The Čech complex of abelian groups is given by

$$C^0(\mathcal{U},\mathcal{O}(d)) \longrightarrow C^1(\mathcal{U},\mathcal{O}(d)) \longrightarrow \cdots \longrightarrow C^{n-1}(\mathcal{U},\mathcal{O}(d)) \xrightarrow{d^{n-1}} C^n(\mathcal{U},\mathcal{O}(d)) \longrightarrow 0. \tag{6.5.3}$$

Then

$$H^n(X,\mathcal{O}(d)) = C^n(\mathcal{U},\mathcal{O}(d)) \big/ \operatorname{Im} d^{n-1}.$$

Note that

$$\begin{array}{ccc} C^{n-1}(\mathcal{U},\mathcal{O}(d)) & = & \bigoplus_{i=0}^{n}\Gamma(D_+(x_0...\hat{x}_i...x_n),\mathcal{O}(d)) \\ \downarrow & & \downarrow d^{n-1} \\ C^{n}(\mathcal{U},\mathcal{O}(d)) & = & \Gamma(D_+(x_0...x_n),\mathcal{O}(d)) \end{array} \quad (6.5.4)$$

Suppose that $(s_i)_{i=0}^n$ is a section in $C^{n-1}(\mathcal{U},\mathcal{O}(d))$. Then the map d^{n-1} is given by

$$\begin{array}{ccc} C^{n-1}(\mathcal{U},\mathcal{O}(d)) & \xrightarrow{d^{n-1}} & C^{n}(\mathcal{U},\mathcal{O}(d)) \\ \| & & \| \\ \prod_{j=0}^{n} R[x_0,...,x_n,x_0^{-1},...,\hat{x_j^{-1}},...,x_n^{-1}]_{\deg d} & & R[x_0,...,x_n,x_0^{-1},...,x_n^{-1}]_{\deg d} \\ (s_i)_{i=0}^n & \longmapsto & \sum_{i=0}^{n}(-1)^i s_i \end{array} \quad . \quad (6.5.5)$$

Since for every i, x_i has non-negative power in s_i. Thus $\mathrm{Im} d^{n-1}$ is the set of linear combinations of degree-d monomials with no totally negative powers(i.e. $\prod_{j=0}^n x_j^{r_j}$, $r_j < 0$). Then

$$H^n(X,\mathcal{O}(d)) = \mathrm{coker} d^{n-1} = \sum_{r_i<0} R\cdot\prod_{j=0}^{n} x_j^{r_j}. \quad (6.5.6)$$

(3) If $p \neq 0, n$, we will using induction on n.

Let $\mathcal{F} = \bigoplus_{d\in\mathbb{Z}}\mathcal{O}(d) = \bigoplus_{d\in\mathbb{Z}}\widetilde{A(d)}$, we know that $\mathcal{F}$ is quasi-coherent. Consider the hyperplane $V(x_n) = \{x_n = 0\}$ in $\mathbb{P}^n$. Since there is an isomorphism

$$\begin{array}{ccc} \mathbb{P}^{n-1} & \xrightarrow{\sim} & V(x_n) \\ (x_0,...,x_{n-1}) & \longmapsto & (x_0,...,x_{n-1},0) \end{array} \quad (6.5.7)$$

Then let $i:\mathbb{P}^{n-1}\hookrightarrow\mathbb{P}^n$ be the embedding defined above. Note that $\mathcal{F}(-1) = \bigoplus_{d\in\mathbb{Z}}\mathcal{O}(d-1)\cong\mathcal{F}$, let $x_n\in\mathcal{O}_{\mathbb{P}^n}(1)$ we have an exact sequence:

$$0\longrightarrow\mathcal{F}(-1)\xrightarrow{\cdot x_n}\mathcal{F}\longrightarrow\mathcal{F}/x_n\mathcal{F}\longrightarrow 0. \quad (6.5.8)$$

Since $\Gamma_*(\mathcal{O}_X)\xrightarrow{\sim}A$, apply the functor $\Gamma(X,\cdot)$ we get the sequence

$$0\longrightarrow A\xrightarrow{\cdot x_n}A\longrightarrow\Gamma(X,\mathcal{F}/x_n\mathcal{F}). \quad (6.5.9)$$

Thus $\mathcal{F}/x_n\mathcal{F}$ is a quasi-coherent sheaf of A/x_nA-module.

Exercise 6.5.2. Let $\mathcal{F}$ be a quasi-coherent sheaf on a scheme X, then there is a natural isomorphism:

$$\begin{aligned} \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{F}) &\longrightarrow \Gamma(X, \mathcal{F}) \\ \varphi &\longmapsto \varphi(1) \end{aligned} \tag{6.5.10}$$

Claim:

$$i_*(i^*\mathcal{F}) = \mathcal{F}\Big/_{x_n\mathcal{F}} \tag{6.5.11}$$

Now let $\mathcal{E} = \mathcal{O}_{\mathbb{P}^{n-1}} = \widetilde{A/x_nA}$ and $\mathcal{F} = \widetilde{M}$. Then projection formula 2.5.4,

$$i_*i^*(\mathcal{F}) = \mathcal{F} \otimes_{\mathcal{O}_X} i_*\mathcal{O}_{\mathbb{P}^{n-1}} = \widetilde{M} \otimes_{\widetilde{A}} \widetilde{A/x_nA} = \widetilde{M/x_nM} = \mathcal{F}/x_n\mathcal{F}.$$

Denote $i^*\mathcal{F}$ by $\mathcal{F}_{\mathbb{P}^{n-1}}$. Now the exact sequence 6.5.8 becomes

$$0 \longrightarrow \mathcal{F}(-1) \xrightarrow{\cdot x_n} \mathcal{F} \longrightarrow i_*\mathcal{F}_{\mathbb{P}^{n-1}} \longrightarrow 0. \tag{6.5.12}$$

Exercise 6.5.3. Let $f : X \to Y$ be an affine morphism and $\mathcal{F}$ be a quasi-coherent sheaf on X. Show that there is an natural isomorphism for each $i \geq 0$:

$$H^i(X, \mathcal{F}) \xrightarrow{\sim} H^i(Y, f_*\mathcal{F}). \tag{6.5.13}$$

Since $i : \mathbb{P}^{n-1} \hookrightarrow \mathbb{P}^n$ is affine morphism, then the functor i_* is an exact functor. The cohomology of $\mathcal{F}$ on X is stable under i_*.

$$H^p(X, i_*\mathcal{F}|_{\mathbb{P}^{n-1}}) = H^p(\mathbb{P}^{n-1}, \mathcal{F}|_{\mathbb{P}^{n-1}}). \tag{6.5.14}$$

So we have an long exact sequence induced by 6.5.12:

$$H^{p-1}(\mathbb{P}^{n-1}, \mathcal{F}_{\mathbb{P}^{n-1}}) \to H^p(X, \mathcal{F}(-1)) \xrightarrow{x_n} H^p(X, \mathcal{F}) \to H^p(\mathbb{P}^{n-1}, \mathcal{F}_{\mathbb{P}^{n-1}}). \tag{6.5.15}$$

We want to show $H^p(X, \mathcal{F}) = 0$ for $0 < p < n$. We can use induction on n.
Claim: $H^p(X, \mathcal{F}(-1)) \xrightarrow{x_n} H^p(X, \mathcal{F})$ is an isomorphism for $0 < p < n$.
(1) $1 < p < n-1$, by induction $H^{p-1}(\mathbb{P}^{n-1}, \mathcal{F}_{\mathbb{P}^{n-1}}) = H^p(\mathbb{P}^{n-1}, \mathcal{F}_{\mathbb{P}^{n-1}}) = 0$. Thus

$$H^p(X, \mathcal{F}(-1)) \xrightarrow{\sim} H^p(X, \mathcal{F}). \tag{6.5.16}$$

(2) $p = 1$, consider the exact sequence

$$\begin{aligned} 0 \longrightarrow H^0(\mathbb{P}^n, \mathcal{F}(-1)) \xrightarrow{x_n} H^0(\mathbb{P}^n, \mathcal{F}) \longrightarrow H^0(\mathbb{P}^{n-1}, \mathcal{F}_{\mathbb{P}^{n-1}}) \longrightarrow {}, \\ \longrightarrow H^1(\mathbb{P}^n, \mathcal{F}(-1)) \xrightarrow{x_n} H^1(\mathbb{P}^n, \mathcal{F}) \longrightarrow H^1(\mathbb{P}^{n-1}, \mathcal{F}_{\mathbb{P}^{n-1}}) \longrightarrow \end{aligned} \tag{6.5.17}$$

bu induction $H^1(\mathbb{P}^{n-1}, \mathcal{F}_{\mathbb{P}^{n-1}}) = 0$. Therefore $H^1(\mathbb{P}^n, \mathcal{F}(-1)) \to H^1(\mathbb{P}^n, \mathcal{F})$ is surjective. Note that the corresponding morphism of rings of the first row is

$$0 \to A(-1) \xrightarrow{x_n} A \to A/x_nA \to 0, \tag{6.5.18}$$

which is already exact. Thus $H^1(\mathbb{P}^n, \mathcal{F}(-1)) \to H^1(\mathbb{P}^n, \mathcal{F})$ is injective.

(3) $p = n-1$, there is an exact sequence

$$0 \longrightarrow H^{n-1}(\mathbb{P}^n, \mathcal{F}(-1)) \xrightarrow{x_n} H^{n-1}(\mathbb{P}^n, \mathcal{F}) \longrightarrow H^{n-1}(\mathbb{P}^{n-1}, \mathcal{F}_{\mathbb{P}^{n-1}}) \longrightarrow \quad , \tag{6.5.19}$$

$$\longrightarrow H^n(\mathbb{P}^n, \mathcal{F}(-1)) \xrightarrow{x_n} H^n(\mathbb{P}^n, \mathcal{F}) \longrightarrow H^n(\mathbb{P}^{n-1}, \mathcal{F}_{\mathbb{P}^{n-1}}) \longrightarrow$$

Note that $\mathcal{F} = \bigoplus \mathcal{O}(d)$, then

$$H^0(X, \mathcal{F}) = \Gamma(X, \mathcal{F}) = A, \qquad H^n(X, \mathcal{F}) = R[x_0^{-1}, \cdots, x_n^{-1}]x_0^{-1} \cdots x_n^{-1}. \tag{6.5.20}$$

Moreover, $\mathcal{F}^\vee = \bigoplus_{d \in \mathbb{Z}} \mathcal{O}(-d) \simeq \mathcal{F}$ and $\mathcal{F} \otimes \mathcal{O}(-1-n) \simeq \mathcal{F}$. Thus the duality paring for $\mathcal{F}$ is

$$H^0(X, \mathcal{F}) \times H^n(X, \mathcal{F}) \longrightarrow H^n(X, \mathcal{O}(-1-n)) \xrightarrow{\sim} R \cdot x_0^{-1} \cdots x_n^{-1}. \tag{6.5.21}$$

And this paring is also hold for $\mathbb{P}^{n-1}$

$$H^0(\mathbb{P}^{n-1}, \mathcal{F}) \times H^{n-1}(\mathbb{P}^{n-1}, \mathcal{F}) \longrightarrow H^{n-1}(\mathbb{P}^{n-1}, \mathcal{O}(-1-n)) \xrightarrow{\sim} R \cdot x_0^{-1} \cdots x_{n-1}^{-1}. \tag{6.5.22}$$

Then the exact sequence 6.5.19 becomes

$$0 \longrightarrow H^{n-1}(\mathbb{P}^n, \mathcal{F}(-1)) \xrightarrow{x_n} H^{n-1}(\mathbb{P}^n, \mathcal{F}) \longrightarrow H^0(\mathbb{P}^{n-1}, \mathcal{F}_{\mathbb{P}^{n-1}})^\vee \longrightarrow \quad . \tag{6.5.23}$$

$$\longrightarrow H^0(X, \mathcal{F})^\vee \xrightarrow{x_n} H^0(X, \mathcal{F})^\vee \longrightarrow 0$$

This implies the last three terms is exact, thus x_n must be surjective.

The claim implies x_n is invertible in $H^p(X, \mathcal{F})$ when $0 < p < n$ thus for all x_i. Since in every affine chart $D_+(x_i)$ the cohomology is trivial for $p > 0$, then we have an exact sequence of Čech complex:

$$C^{p-1}(\mathcal{U}, \mathcal{F})_{D_+(x_i)} \to C^p(\mathcal{U}, \mathcal{F})_{D_+(x_i)} \to C^{p+1}(\mathcal{U}, \mathcal{F})_{D_+(x_i)}. \tag{6.5.24}$$

But since x_i is invertible, localizing at x_i does't change anything, then

$$H^p(X, \mathcal{F}) = H^p(X, \mathcal{F})_{D_+(x_i)} = 0.$$

□

Remark 6.5.4. The above theorem implies that if $\mathcal{F}$ equals $\mathcal{O}(d)$ or the direct sums of $\mathcal{O}(d)'s$, and if $\mathcal{L}$ is an ample line bundle over $X = \mathcal{P}_R^n$, then

1. $H^i(X, \mathcal{F})$ is finitely generated over A.

2. $H^i(X, \mathcal{F} \otimes \mathcal{L}^N) = 0$ for $i > 0$ and N sufficiently large.

But we want to get the general results for the case when $\mathcal{F}$ is coherent sheaf on a noetherian projective scheme X:

Theorem 6.5.5 (Serre)**.** *Let A be a graded Noetherian ring generated by A_1 as an A_0-algebra. Assume A_1 is finitely generated A_0-module and let $X = \operatorname{Proj} A$. Let $\mathcal{F}$ be a coherent sheaf on X, and suppose that $\mathcal{L}$ is an ample line bundle over X. Then:*

1. *$H^i(X, \mathcal{F})$ is finitely generated over A.*
2. *$H^i(X, \mathcal{F} \bigotimes \mathcal{L}^N) = 0$ for $i > 0$ and N sufficiently large.*
3. *$\Gamma_*(X, \mathcal{F}) = \oplus_{N \geq 0} H^0(X, \mathcal{F} \otimes \mathcal{L}^N)$ is a finitely generated module over A.*

Proof. **Step 1:**

To prove the theorem, note that it sufficients to reduce to the case $X = \mathbb{P}^n$, $\mathcal{L} = \mathcal{O}(1)$.

(1) Let $R = A_0$, assume that $a_i \in A_1$ generate R, $i = 0, 1, \cdots, n$. Then $A = R[a_0, \cdots, a_n]/I$ for some homogeneous ideal I, there is a surjective morphism:

$$\begin{array}{ccc} R[x_0, \cdots, x_n] & \longrightarrow & A \\ x_i & \longmapsto & a_i \end{array} . \tag{6.5.25}$$

It induces an affine morphism:

$$i : \operatorname{Proj} A \longrightarrow \mathbb{P}^n_R. \tag{6.5.26}$$

Thus we know that for any p and any coherent sheaf $\mathcal{F}$ on X,

$$H^p(X, \mathcal{F}(d)) = H^p(\mathbb{P}^n_R, i_* \mathcal{F}(d)) \tag{6.5.27}$$

So $H^p(X, \mathcal{F})$ is finitely generated A-module if and only if $H^p(\mathbb{P}^n_R, i_* \mathcal{F}(d))$ is finitely generated A-module. Thus reduce to the case $X = \mathbb{P}^n_R$.

(2) By definition $\mathcal{L}$ is ample if and only if there is an embedding $i : X \to \mathbb{P}^n$ such that $i^* \mathcal{O}(1) \cong \mathcal{L}^m$ for some $m > 0$. Let $N = mt + r$, where $0 < r < m$. Then

$$\begin{aligned} H^p(X, \mathcal{F} \bigotimes \mathcal{L}^N) &= H^p(\mathbb{P}^n, i_*(\mathcal{F} \bigotimes \mathcal{L}^N)) & (6.5.28) \\ &= H^p(\mathbb{P}^n, i_*(\mathcal{F} \bigotimes \mathcal{L}^r \bigotimes (i^* \mathcal{O}(1))^t)) & (6.5.29) \\ &= H^p(\mathbb{P}^n, i_*(i^* \mathcal{O}(t) \bigotimes \mathcal{F} \bigotimes \mathcal{L}^r)) & (6.5.30) \\ &= H^p(\mathbb{P}^n, \mathcal{O}(t) \bigotimes i_*(\mathcal{F} \bigotimes \mathcal{L}^r)). & (6.5.31) \end{aligned}$$

Since $N \gg 0$ is equivalent to $r \gg 0$, let $\mathcal{G} = i_*(\mathcal{F} \otimes \mathcal{L}^r)$, then the theorem for $(\mathcal{P}^n, \mathcal{O}(1), \mathcal{G})$ is equivalent to the theorem for $(X, \mathcal{L}, \mathcal{F})$.

Step 2:

Since $\mathcal{F}$ is coherent, then there is an integer d such that $\mathcal{F}(d)$ is generated by finitely many global sections $s_i \in \Gamma(X, \mathcal{F}(d))$, $i = 1, 2, ..., m$. So we have an exact sequence:

$$0 \longrightarrow \ker\phi \longrightarrow \mathcal{O}_X^m \xrightarrow{\phi} \mathcal{F}(d) \longrightarrow 0. \tag{6.5.32}$$

Tensor the exact sequence with $\mathcal{O}_X(-d)$ we have the exact sequence

$$0 \longrightarrow \mathcal{G} \longrightarrow \mathcal{O}_X^m(-d) \longrightarrow \mathcal{F} \longrightarrow 0, \tag{6.5.33}$$

where $\mathcal{G} = \ker\phi(-d)$ is also coherent over X.

This exact sequence induces a long exact sequence

$$... \to H^i(X, \mathcal{O}_X^m(-d)) \overset{\alpha}{\to} H^i(X, \mathcal{F}) \to H^{i+1}(X, \mathcal{G}) \overset{\beta}{\to} H^{i+1}(X, \mathcal{O}_X^m(-d)) \to ... \tag{6.5.34}$$

Step 3

Now we try to use this exact sequence to prove all assertion in the theorem.

For part 1, since A_0 is Noetherian, $H^i(X, \mathcal{O}_X^m(-d))$ and $H^{i+1}(X, \mathcal{O}_X^m(-d))$ are both finitely generated over A_0, both $\mathrm{Im}(\alpha)$ and $\mathrm{Im}(\beta)$ are finitely generated A_0-modules. Thus $H^i(X, \mathcal{F})$ is finitely generated over A_0 if and only if $H^{i+1}(X, \mathcal{G})$ is finitely generated. Thus we can use backward induction on i to reduce to the case $i > n$ where $H^i(X, \mathcal{F}) = 0$.

For part 2, we twisted the above sequence by $\mathcal{O}(N)$ to obtain

$$... \to H^i(X, \mathcal{O}_X^m(N-d)) \overset{\alpha}{\to} H^i(X, \mathcal{F}(N)) \to H^{i+1}(X, \mathcal{G}(N)) \overset{\beta}{\to} H^{i+1}(X, \mathcal{O}_X^m(N-d)) \to ...$$

If $N > d$, $i > 0$, both $H^i(X, \mathcal{O}_X^m(N-d))$ and $H^i(X, \mathcal{O}_X^m(N-d))$ vanishes, Thus $H^i(X, \mathcal{F}(N)) = 0$ if and only if $H^{i+1}(X, \mathcal{G}) = 0$. Again we use backward induction to reduce to the case $i > n$ where all cohomology vanishes.

For part 3, we twisted the above sequence by all $\mathcal{O}(N)$ for $N \geq 0$ and then take the sum. Then we get an exact sequence:

$$... \to \Gamma_*(X, \mathcal{O}_X(-d))^{\oplus m} \overset{u}{\to} \Gamma_*(X, \mathcal{F}) \overset{v}{\to} \oplus_N H^1(X, \mathcal{G}(N)) \to ...$$

From computation before, we know that $\Gamma_*(X, \mathcal{O}_X(-d)) = A(-d)$. Thus $\Gamma_*(X, \mathcal{O}_X(-d))^{\oplus m}$ is finitely generated over A. Also by part 1 and 2, $\oplus_N H^1(X, \mathcal{G}(N))$ is finitely generated over A_0 and thus over A. Since A is Noetherian, both $\mathrm{Im}u$ and $\mathrm{Im}v$ are finitely generated A-modules. Thus $\Gamma_*(X, \mathcal{F})$ is finitely generated over A, as it fits in an short exact sequence

$$0 \longrightarrow \mathrm{Im}u \longrightarrow \Gamma_*(X, \mathcal{F}) \longrightarrow \mathrm{Im}v \longrightarrow 0.$$

□

From part 3 of the theorem, we have a functor $\Gamma_* : \mathrm{Coh}(X) \longrightarrow \mathrm{FMod}_A$ if $X = \mathrm{Proj}\, A$. It is clear that Γ_* is the right adjoint of the functor $M \longrightarrow \widetilde{M}$, and the the functor $M \longrightarrow \widetilde{M}$ is the left inverse of Γ_*. Moreover we have a morphism $\alpha_M : M \longrightarrow \Gamma_*(\widetilde{M})$ with $\ker \alpha_M$ and cokerα_M with only finitely many non-vanishing components. Let define a category CMod$A$ of coherent A-modules with same objects as the category of finitely generated A-modules but homomorphism is given by

$$\mathrm{Hom}_{\mathrm{CMod}_A}(M, N) := \varinjlim_{d} \mathrm{Hom}_{\mathrm{FMod}_A}(M_{\geq d}, N_{\geq d})$$

where for a graded A-module M, $M_{\geq d}$ denote the sub module $\oplus_{n \geq d} M_n$. Then Γ_* and $M \longrightarrow \widetilde{M}$ define an equivalence of categories between $\mathrm{Coh}(X)$ and CMod_A.

By theorem, we can define charateristic in the Grothendieck group $K_0(X)$:

Definition 6.5.6. *Let X be a projective scheme over a field k. Let $\mathcal{F}$ be a coherent sheaf on X. We define the* Euler characteristic *of $\mathcal{F}$ by*

$$\chi(\mathcal{F}) = \sum (-1)^i \dim H^i(X, \mathcal{F}).$$

Proposition 6.5.7. *If there is an exact sequence of coherent sheaves on projective scheme X/k*

$$0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{G} \to 0, \tag{6.5.35}$$

then we have $\chi(\mathcal{F}) = \chi(\mathcal{E}) + \chi(\mathcal{G})$.

Proof. Since we have long exact sequence

$$H^0(X, \mathcal{E}) \to H^0(X, \mathcal{F}) \to H^0(X, \mathcal{G}) \to H^1(X, \mathcal{E}) \to H^1(X, \mathcal{F}) \to \cdots,$$

and since $\mathcal{F}$, $\mathcal{G}$, $\mathcal{E}$ are all coherent, X is finite dimensional, then the sequence is finite. So the sum

$$\sum_i (-1)^i (\dim H^i(X, \mathcal{F}) - \dim H^i(X, \mathcal{E}) - \dim H^i(X, \mathcal{G}))$$

is a finite sum and equals 0. □

6.6 Relative cohomology

Let $f : X \longrightarrow Y$ be a morphism of schemes and $\mathcal{F}$ a sheaf of quasi-coherent $\mathcal{O}_X$-module. For each open subset $U = \mathrm{Spec} A$ of Y, let $X_U = f^U$ denote the open subset of X which is an A-scheme. Let $\mathcal{F}_U$ denote the restriction of $\mathcal{F}$ on X_U. Then we can define Čech cohomology groups $H^i(X_U, \mathcal{F}_U)$ which are A-modules.

Lemma 6.6.1. *Let $f \in A$. The inclusion $U_f \subset U$ induces an isomorphism*

$$H^i(U_f, \mathcal{F}|_{U_f}) = H^i(U, \mathcal{F})_f.$$

Sketch of proof. By definition, $H^i(X_U, \mathcal{F}_U)$ is calculated using a Čech complex $C^\bullet(\mathcal{U}, \mathcal{F})$ for an affine covering $\mathcal{U} = \{U_i = \mathrm{Spec} A_i\}$ of X_U. Since $\mathcal{U}_f$ is an affine covering of U_f, we may use it to calculate $H^i(X_{U_f}, \mathcal{F}_{U_f})$. In other words, $H^i(X_{U_f}, \mathcal{F}_{U_f})$ is calculated using $C^\bullet(\mathcal{U}, \mathcal{F}) \otimes_A A_f$. Since A_f is flat over A, the cohomology is calculated by localization. □

quasi-coherent

Now we are applying Theorem 4.1.4 to get quasi-coherent sheaves $R^i f_* \mathcal{F}$ such that for affine subset U:

$$R^i f_* \mathcal{F}(U) = H^i(X_U, \mathcal{F}_U).$$

Notice that $R^0 f_* \mathcal{F} = f_* \mathcal{F}$, and that $R^i f_*(\mathcal{F}) = 0$ for $i > 0$ iff f is affine.

The functors $R^i f_* : \mathrm{QCoh}(X) \longrightarrow \mathrm{QCoh}(Y)$ forms a cohomolgy theory: for any short exact sequence of quasi-coherent sheaves on X:

$$0 \longrightarrow \mathcal{E} \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow 0$$

we have a long exact sequence:

$$0 \longrightarrow f_* \mathcal{E} \longrightarrow f_* \mathcal{F} \longrightarrow f_* \mathcal{G} \longrightarrow R^1 f_* \mathcal{E} \longrightarrow R^1 f_* \mathcal{F} \longrightarrow R^1 f_* \mathcal{G} \longrightarrow \cdots.$$

Let $\mathrm{Bun}(X)$ denote the category of locally free sheaves on X. Then we have pull-back:

$$f^* : \mathrm{Bun}(Y) \longrightarrow \mathrm{Bun}(X).$$

This functor is exact.

The two functors are related as follows: for a quasi-coherent sheaf $\mathcal{F}$ on X and locally free sheave on Y, we have an canonical isomorphism

$$R^i f_*(f^* \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{F}) = \mathcal{E} \otimes_{\mathcal{O}_Y} R^i f_* \mathcal{F}.$$

projection formula

This follows from projection formula 2.5.4 and definition of Čech cohomology.

Now we assume that both X and Y are Noetherian, and that f is projective with a relative ample line bundle $\mathcal{L}$, and that $\mathcal{F}$ is coherent. Then we have Serre's theorem:

1. $R^i f_*(\mathcal{F})$ is coherent;
2. $R^i f_*(\mathcal{F}) = 0$ for $i >> 0$;
3. $R^i f_*(\mathcal{F} \bigotimes \mathcal{L}^N) = 0$ for $i > 0$ and N sufficiently large.

Thus we have a homomorphism $f_* : K_0(X) \longrightarrow K_0(Y)$ defined by

$$f_*[\mathcal{F}] = \sum_i (-1)^i [R^i f_* \mathcal{F}].$$

Notice that we can define pull-back $f^* : K^0(Y) \longrightarrow K^0(X)$ on locally free sheaves. We have the following projection formula:

$$f_*[f^* \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{F}] = [\mathcal{E}] \cdot f_*[\mathcal{F}].$$

7 Curves

In general a curve means a projective subvariety of dimension 1. So first we study schemes of dimension 1 and their propositions.

7.1 Grothendieck groups over Dedekind domains

Now we fix a field k. Assume that X is a noetherian separated integral regular scheme of dimension 1 which is finite type over k. Then there is a structure morphism $X \to \mathrm{Spec} k$ and the corresponding morphism of rings is $k \to \mathcal{O}_X(X)$. Note that $\mathcal{O}_X(X)$ is a finitely generated integral k-algebra. We assume that k is the maximal subfield in $\mathcal{O}_X(X)$.

Question

1. How to complete X?
2. How to describe the structure of $K_0(X)$ and $K^0(X)$?
3. How to define the Euler characteristic map $\chi : K_0(X) \to K^0(k)$?

Consider a simple case: Let A be a Dedekind domain and Let $X = \mathrm{Spec} A$. We will study the Grothendieck K-group of X. We need to study the structure of Dedekind domain and the structure of finitely generated modules on Dedekind domain.

Grothendieck groups of a Dedekind domain

Let A be a Dedekind domain. We are going to describe the structure of Grothendieck groups in terms its ranks and determinants. First, we show that two Grothendicek groups are isomorphic to each other as a consequence of structure theorem of modules.

Lemma 7.1.1. *Let A be a Dedekind domain. Then $K_0(A) \xrightarrow{\sim} K^0(A)$.*

Proof. For any prime ideal $\mathfrak{p} \hookrightarrow A$, there is an exact sequence

$$0 \longrightarrow \mathfrak{p}^{n_i} \longrightarrow A \longrightarrow A/\mathfrak{p}^{n_i} \longrightarrow 0.$$

Then we have $[A/\mathfrak{p}^{n_i}] = [A] - [\mathfrak{p}^{n_i}]$. Using the structure theorem, any finitely generated A-module M has the decomposition

$$A^r \oplus I \oplus (\oplus_{i=1}^n A/\mathfrak{p}^{n_i}),,$$

for some integer r, n and ideal I. Thus

$$[M] = [A^r] + [I] + \sum_{i=1}^n [A] - [\mathfrak{p}^{n_i}] = [A^{r+n}] + [I] - \sum [\mathfrak{p}^{n_i}].$$

Note that $[A/\mathfrak{p}^{n_i}] \in K_0(A)$ is equal to $[A] - [\mathfrak{p}^{n_i}] \in K^0$. $\square$

Determinants of modules

Definition 7.1.2. *Let A be a ring, and let M be an A-module. We define the* tensor algebra *of M, denoted $T(M)$, to be the algebra of tensors on M with multiplication being the tensor product. In other words, $T(M) = \bigoplus_{k=0}^{\infty} T^k(M)$, where $\bigoplus_{k=0}^{\infty} T^k(M)$ is the tensor product $M \otimes M \otimes \cdots \otimes M$ of M with itself k times.*

We define the exterior algebra $\bigwedge(M) = \bigoplus_{k \geqslant 0} \bigwedge^k(M)$ *to be the quotient algebra of $T(M)$ by the two-sided ideal I generated by all elements of the form $x \otimes x$ for $x \in M$. The ideal I contains the ideal J generated by elements of the form*

$$x \otimes y + y \otimes x = (x+y) \otimes (x+y) - x \otimes x - y \otimes y.$$

Remark 7.1.3. Note that the k-th component $\bigwedge^k(M)$ of $\bigwedge(M)$ is called the k-th exterior power of M. When $k = 2$, $\bigwedge^2(M)$ is the wedge product defined earlier. If $u \in \bigwedge^k(M)$ and $v \in \bigwedge^s(M)$, denote the multiplication in this algebra by $\wedge$, then $u \wedge v = (-1)^{ks} v \wedge u$. Let $m_1, \cdots, m_k \in M$, then the image of $m_1 \otimes \cdots \otimes m_k$ in $\bigwedge^k(M)$ is denoted by $m_1 \wedge \cdots \wedge m_k$.

Definition 7.1.4. *Let K be a field and let V be a K-vector space of dimension k. We define the determinant of V, denoted* $\det(V)$ *to be the k-th component $\bigwedge^k(V)$ of $\bigwedge(V)$. Assume that V has a K-basis $e_1, \cdots, e_k$, and if*

$$v_i = \sum_{j=1}^{k} a_{ij} e_j \qquad i = 1, 2 \cdots, k \qquad a_{ij} \in K,$$

then

$$v_1 \wedge \cdots \wedge v_k = \det(a_{ij}) e_1 \wedge \cdots \wedge e_k.$$

Where $\det(a_{ij})$ *is the determinant of the matrix (a_{ij}).*

Let A be a Dedekind domain, then $K_0(A) \xrightarrow{\sim} K^0(A)$. We want to describe the structure of $K_0(A)$, first we have two invariants of $K^0(A)$:

- Let K be the fractional field of A. Let M be a projective A-module of rank r. Then $M \bigotimes K$ is a K-vector space of dimension r. Define the rank of M, denoted $\mathrm{rank} M$, to be the dimension of $M \bigotimes K$ over K. Then we have a map:

$$\begin{aligned} c_0 : K^0(A) &\longrightarrow \mathbb{Z} \\ [M] &\longmapsto \mathrm{rank} M \end{aligned} \tag{7.1.1}$$

  Let M be a projective A-module. Then $M \oplus L = E$ for some free A-module. Then $[M] = [E] - [L]$, $\mathrm{rank} M = \mathrm{rank} E - \mathrm{rank} L$. We can see that c_0 is surjective.

- Since $M \bigotimes K$ is a K-vector space of dimension r, suppose that $M \bigotimes K = \sum_{i=1}^{r} k_i e_i$, where $k_i \in K$ and e_i is a basis of $M \bigotimes K$ over K. Then we have

$$\det(M \bigotimes K) = K e_1 \bigwedge \cdots \bigwedge K e_r.$$

Consider the natural map $M \longrightarrow M \otimes K$. We can define the determinant of M, denoted $\det M$ as follows:

$$\det M = \bigwedge^r M = \{m_1 \wedge \cdots \wedge m_r | \ m_1, \cdots, m_r \in M\} \hookrightarrow \det(M \bigotimes K).$$

Lemma 7.1.5. *If there is an exact sequence*

$$0 \longrightarrow M_1 \longrightarrow M \longrightarrow M_2 \longrightarrow 0$$

then there is an isomorphism $\det M \xrightarrow{\sim} \det M_1 \otimes \det M_2$.

Proof. Let M_1, M, M_2 have ranks r_1, r, r_2 respectively. Since M_2 is projective, the above sequence is split. Thus $M = M_1 \oplus M_2$. It follows that

$$\bigwedge^r M = \bigwedge^r (M_1 \oplus M_2) = \bigoplus_{n=0}^{r} (\bigwedge^n M_1 \otimes \bigwedge^{r-n} M_2).$$

But if $n > r_1$ or $r - n > r_2$ then we have $\bigwedge^n M_1 = 0$. Then $\det M \xrightarrow{\sim} \det M_1 \otimes \det M_2$. □

Lemma 7.1.6. *The* $\det M$ *is a invertible projective module of rank 1.*

Proof. Reduce to the case $M = I$ for some fractional ideal $I \hookrightarrow A$. Since I is invertible, we have

$$R = I^{-1} I = \det(R \oplus I I^{-1}) = \det(I \oplus I^{-1}) = \det I \otimes \det I^{-1}.$$

□

Recall that $\mathrm{Pic}(A) = \mathrm{Cl}(A) = \mathbb{I}_A / \mathcal{P}(A)$ and two fractional ideals $I, J \in \mathbb{I}_A$ are isomorphic if and only if $I = (x)J$ for some $x \in K$. Then

$$\mathrm{Pic}(A) = \{\text{isomorphism classes of non-zero fractional ideals of } A\}.$$

Since for any finitely generated A-module M we have the structure theorem:

$$M \xrightarrow{\sim} A^r \bigoplus I \bigoplus (\bigoplus_{\mathfrak{p}} A/\mathfrak{p}^{n_{\mathfrak{p}}}),$$

and every surjective map $M \to I$ has a section. Then we have

$$\mathrm{Pic}(A) \xrightarrow{\sim} \{\text{isomorphism classes of invertible projective } A-\text{modules}\}.$$

Finally we get a map

$$\begin{aligned} c_1 : \ K^0(A) &\longrightarrow \mathrm{Pic}(A) \\ [M] &\longmapsto \det M \end{aligned} \tag{7.1.2}$$

Finally, we are going to state the structure theorem for Grothendieck group $K^0(A)$. First, we define a group structure on $\mathbb{Z} \bigoplus \mathrm{Pic}(A)$ by

$$(a, [I]) \cdot (b, [J]) = (ab, b[I] + a[J]), \qquad a, b \in \mathbb{Z}, \qquad I, J \in \mathbb{I}_A.$$

Theorem 7.1.7. *The map* $(c_0, c_1) : K^0(A) \longrightarrow \mathbb{Z} \bigoplus \mathrm{Pic}(A)$ *is an isomorphism.*

Proof. Since for any $I, J \in \mathbb{I}_A$ we have $I \bigoplus J = A \bigoplus IJ$, then denote by $[I]$ the isomorphic class of I, we have

$$[I] + [J] = [A] + [IJ].$$

Define

$$\begin{aligned} \phi : \mathbb{Z} &\longrightarrow K^0(A) \\ n &\longmapsto [A^n] \end{aligned} \tag{7.1.3}$$

$$\begin{aligned} \psi : \mathrm{Pic}(A) &\longrightarrow K^0(A) \\ [I] &\longmapsto [I] - [A] \end{aligned} \tag{7.1.4}$$

It is easy to check that $c_0 \circ \psi = 0$, $c_0 \circ \phi = \mathrm{Id}_{\mathbb{Z}}$, $c_1 \circ \psi = \mathrm{Id}_{\mathrm{Pic}(A)}$., $c_1 \circ \phi = 0$. Thus (ϕ, ψ) is an inverse of (c_0, c_1). So we have $(c_0, c_1) : K^0(A) \longrightarrow \mathrm{Pic}(A) \oplus \mathbb{Z}$ is an isomorphism. □

7.2 Regular projective curves

In this section, we want to construct projective regular curves from their function fields as a coverings of projective line. Here by a curve over field we mean a scheme X over a field of finite type dimension 1. We will assume that X is geometrically integral in the sense that $X \otimes_k \bar{k}$ is integral.

Let X is geometrically integral scheeme over k of dimension 1. Then it is clear that X is integral. Let $k(X) = K = \mathcal{O}_{X,\zeta}$ be the function field of X. Then it has transcendence degree 1 over k, and since X is geometrically integral, $k(X_{\bar{k}}) = K \otimes_k \bar{k}$ is still a field.

Conversely, if given a transcendence degree 1 field K over k such that $K \otimes_k \bar{k}$ is still a field, can we find a suitable curve C such that the function field of C is K? Last time we proved that for separated scheme $X = \mathrm{Spec} R$ of finite type over k, there is an injection

$$\left\{ \begin{matrix} \text{closed points} \\ \text{in } \mathrm{Spec} R \end{matrix} \right\} \hookrightarrow \left\{ \begin{matrix} \text{non-trivial discrete } k\text{-valuation} \\ \text{on } K \end{matrix} \right\} \tag{7.2.1}$$

Here a valuation on K is a k-valuation it is vanishes on $k^\times$. But we wish to find a regular scheme X such that the points of X are in 1-1 correspondence to the discrete valuations of K.

Example 7.2.1. Let K be a finite extension of $\mathbb{Q}$ and let $\mathcal{O}_K$ be the ring of integers. Then

1. $\mathcal{O}_K$ is a Dedekind domain.

2. the closed points in $\mathrm{Spec}\mathcal{O}_K$ are in 1-1 correspondence to discrete valuations of K.

n thm curve **Theorem 7.2.2** (Main Theorem). *Let K be a transcendence degree 1 field over k such that $K \otimes_k \bar{k}$ is still a field. Then there exists a regular projective curve C/k which is unique up to canonical isomorphism such that $k(C) = K$. Moreover, the following assertions hold:*

1. *The correspondence in* point-val 7.2.1 *is injective with image consisting of all discrete valuations with trivial restriction on k.*

2. *If X is a regular curve such that $k(X) = K$, then there exists a unique open embedding $X \to C$ which induces $k(C) = k(X) = K$.*

We wish to construct a one to one corresponding

$$\{\text{ points of } C\} \xrightarrow{\sim} \{\text{all } k\text{-valuations } v \text{ on } K\}. \tag{7.2.2}$$

Now consider the right hand side set $V = \{\text{all valuations } v \text{ on } K\}$, define the open subsets of V to be the finite intersections of the sets of form $\{v|\ v(f) \geqslant 0 \text{ for some } f \in K^\times\}$ It is easy to check that under this construction, V becomes a topological space. For any $v \in V$, define

$$\mathcal{O}_v := \{f \in K \mid v(f) \geqslant 0\},$$

for any open subset $U \subset V$ is an open subset, define $\mathcal{O}$ to be the sheaf associated to the presheaf

$$U \mapsto \mathcal{O}(U) := \bigcap_{v \in U} \mathcal{O}_v.$$

We will prove that $(V, \mathcal{O})$ forms a scheme.

Example 7.2.3. Let K be a finite separable extension of $k(T)$, then K is transcendence degree 1 over k. The main theorem is true for k with $\mathbb{P}^1 = \operatorname{Proj} k[x, y]$ and $T = y/x$.

The ideal of proof of main thm curve 7.2.2 is to make a finite separated morphism $C \xrightarrow{f} \mathbb{P}^1$, where $f \in K$.

separa tran

Theorem 7.2.4. *Let K be a transcendence degree 1 field over k such that $K \otimes_k \overline{k}$ is still a field. Then there exists an element $t \in K$ which is transcendental over k such that $K/k(t)$ is separable extension.*

Proof. We can choose $t \in K$ transcendental over k such that the inseprable degree $[K : k(t)_{\text{sep}}]$ is minimal. If $[K : k(t)_{\text{sep}}] > 1$, then k has positive characteristic p, and then there exists an element $\alpha \in K$ such that $g'(\alpha) = 0$, where $g(X) \in k(t)[X]$ is the minimal polynomial of α over $k(t)$. After clearing denominators we may assume that

$$g(X) = \Phi(t, X) = \sum a_{ij} t^i X^j$$

note that Φ is chosen so as to have minimal degree in X. If Φ contains an nonzero term $a_{ij} t^i X^j$ where $p \nmid j$, then $\frac{\partial \Phi(t,X)}{\partial X}$ is not identically 0. So α is separable over $k(t)$. Thus we can supose that $\Phi(t, X) = \Psi(t, X^p)$.

Claim that Ψ is not a polynomial in t^p: otherwise

$$\Psi(t, X^p) = h(t^p, X^p) = h(t, X)^p$$

for some polynomial $h(t, X) \in \overline{k}[t, X]$. Thus in $\overline{k} \otimes_k K$, we have a nilpotent element $h(t, \alpha)$, thus t is separable over $k(\alpha)$. We have $k(t)_{\text{sep}} \subset k(\alpha)_{\text{sep}}$ and

$$[K : k(\alpha)_{\text{sep}}] < [K : k(t)_{\text{sep}}],$$

contradiction. □

Exercise 7.2.5. Let k be a field. Let $f(X,Y) \in k[X,Y]$ be an irreducible polynomial. Then $f(X,Y)$ is also irreducible in $k(X)[Y]$.

Therefore we obtain a finite separable field extension $k(t) \hookrightarrow K$. Note that if $K = k(t)$, then we can choose $C = \mathbb{P}^1_k$ in separa tran 7.2.4 and it saitisfies all the assertions in the main theorem. And note that Spec$k[t]$ and Spec$k[1/t]$ are affine covers of $\mathbb{P}^1_k$.

Proof of main thm curve 7.2.2. Let A and B be the integral closure of $k[t]$ and $k[1/t]$ in K, then they are Dedekind domains. We obtain finite separated morphisms

$$\mathrm{Spec}A \to \mathrm{Spec}k[t]$$

and

$$\mathrm{Spec}B \to \mathrm{Spec}k[1/t].$$

Let $C = \mathrm{Spec}A \bigcup \mathrm{Spec}B$, then the morphism $f : C \to \mathbb{P}^1_k$ is separated. By Theorem dedekind-extension A.4.21, both A and B are Dedekind domain. Thus C is regular.

To prove that C is projective, we consider the diagram

$$\begin{array}{ccc} S & \longrightarrow & K[x] \\ \uparrow & & \uparrow \\ k[x,y] & \longrightarrow & k(T)[x] \end{array}$$

where S is the integral closure of $k[x,y]$ in $K[x]$. Since $K/k(T)$ is separable, one can show that S is noetherian.

Define a grading on $k[x]$ where $\deg a = 0$ if $a \in k$, and $\deg x = 1$. Then S is a graded ring. We claim C is isomorphic to $\mathrm{Proj}\, S$. In deed let

$$f : \mathrm{Proj}\, S \longrightarrow \mathbb{P}^1 = \mathrm{Proj}\, k[x,y]$$

be the natural projection and cover $\mathbb{P}^1$ by affine open subsets $U = \mathrm{Spec}k[T]$ and $V = \mathrm{Spec}k[1/T]$. From the construction we know that since $S[x^{-1}]_{\deg 0}$ is the integral closure of $k[T]$ in K, and $S[y^{-1}]_{\deg 0}$ is the integral closure of $k[1/T]$ in K,. Thus we have $f^{-1}(U) = \mathrm{Spec}S[x^{-1}]_{\deg 0}$ and $f^{-1}(V) = \mathrm{Spec}S[y^{-1}]_{\deg 0}$. It is clear that $S[x^{-1}]_{\deg 0} = A$, and $S[y^{-1}]_{\deg 0} = B$. Thus $\mathrm{Proj}\, S = C$.

To prove that part 1 about valuation, we use exercise valuation-extenstion A.4.22. The points of Spec$A$ (resp. Spec$B$) correspond to discrete valuations v of K such that $v(x) \geq 0$ (resp. $v(x) \leq 0$). Combined together, points of C correspond to whole set of k-valuations image.

To prove the last part, from the identity $k(X) = k(C)$, we have a rational morphism $f : X \dashrightarrow C$. We claim that this is defined at every point of X. Let x be a point of X corresponding to valuation v_x of K. Then f actually sends x to the point in C corresponds to valuation v_x.

□

7.3 Grothendieck groups over curves

Now we consider the structure of $K^0(C)$ and $K_0(C)$ for a projective nonsingular curve C over a field k. Let η be the generic point of C and $K = \mathcal{O}_{C,\eta}$ the function field of X. First we claim that $K^0(C) = K_0(C)$. In fact, if $\mathcal{F}$ is a coherent sheaf over C, we define

$$\mathcal{F}_{\text{tors}} := \ker(\mathcal{F} \to \mathcal{F} \otimes_\eta K)$$

to be the torsion subsheaf of $\mathcal{F}$. Equivalently, $\mathcal{F}_{\text{tors}}(U) := \{s \in \mathcal{F}(U) \mid s_\eta = 0\}$ for any open subset U of C. Then we have a short exact sequence

$$0 \to \mathcal{F}_{\text{tors}} \to \mathcal{F} \to \mathcal{F}/\mathcal{F}_{\text{tors}} \to 0$$

with $\mathcal{F}/\mathcal{F}_{\text{tors}}$ torsion-free sheaf.

Exercise 7.3.1. The $\mathcal{F}/\mathcal{F}_{\text{tors}}$ is locally free.

Therefore $[\mathcal{F}/\mathcal{F}_{\text{tors}}] \in K^0(C)$. On the other hand, the $\mathcal{F}_{\text{tors}}$ is of form

$$\sum_{\substack{x \in C \\ \text{closed point}}} (i_x)_* \widetilde{M_x}$$

for some M_x finitely generated torsion $\mathcal{O}_{C,x}$-module, which must be of the form $\bigoplus_{i=1}^n \mathcal{O}_{C,x}/\mathfrak{m}_x^{n_i}$ since $\mathcal{O}_{C,x}$ is a discrete valuation ring. This means that $[\mathcal{F}_{\text{tors}}] \in K^0(C)$. So $[\mathcal{F}] = [\mathcal{F}_{\text{tors}}] + [\mathcal{F}/\mathcal{F}_{\text{tors}}] \in K^0(C)$.

We define the group of divisors of C

$$\mathrm{Div}(C) := \bigoplus_{\substack{x \in C \\ \text{closed point}}} \mathbb{Z} \cdot x$$

which consists of formal $\mathbb{Z}$-linear combinations of closed points of C. For a closed point x of C, there is a short exact sequence

$$0 \to \mathcal{I}_x \to \mathcal{O}_C \to (i_x)_* \mathcal{O}_x \to 0,$$

here $\mathcal{O}_x$ is the structure sheaf of $\mathrm{Spec}(k_x)$, $\mathcal{I}_x$ is a sheaf of ideals, locally generated by π_x a uniformizer of $\mathcal{O}_{C,x}$. The $\mathcal{I}_x$ is also denoted by $\mathcal{O}_C(-x)$, which is an invertible sheaf. In general, if $D = \sum_x n_x \cdot x$ is a divisor of C, then we may define the sheaf $\mathcal{O}_C(D)$ to be the subsheaf of the sheaf $\mathcal{K}$ of rational functions on C by

$$\mathcal{O}_C(D)(U) := \{f \in \mathcal{K}(U) \mid v_x(f) \geq -n_x \text{ for all } x \in U\}.$$

Similar to $\mathcal{I}_x$, For an integer $n \geq 1$, the $\mathcal{O}_C(-nx)$ fits into the following short exact sequence

$$0 \to \mathcal{O}_C(-nx) \to \mathcal{O}_C \to (i_x)_*(\mathcal{O}_{C,x}/\mathfrak{m}_x^n) \to 0,$$

which means that $[(i_x)_*(\mathcal{O}_{C,x}/\mathfrak{m}_x^n)] = [\mathcal{O}_C] - [\mathcal{O}_C(-nx)]$ in $K_0(C)$.

Similar to the affine case, we define

$$\begin{aligned} c_0 : K^0(C) &\to \mathbb{Z}, \\ [\mathcal{F}] &\mapsto \dim(\mathcal{F} \otimes_\eta K), \end{aligned}$$

and

$$\begin{aligned} c_1 : K^0(C) &\to \operatorname{Pic}(C), \\ [\mathcal{F}] &\mapsto \det(\mathcal{F}) := \wedge^{\dim(\mathcal{F}\otimes_\eta K)}(\mathcal{F}), \end{aligned}$$

here $\operatorname{Pic}(C)$ is the group of isomorphism classes of invertible sheaves on C.

Theorem 7.3.2. *The map* $(c_0, c_1) : K^0(C) \to \mathbb{Z} \oplus \operatorname{Pic}(C)$ *is an isomorphism of rings.*

Exercise 7.3.3. Prove it.

7.4 Riemann-Roch theorem

Still let C be the projective nonsingular curve in the last talk. A divisor of C is called a *principal divisor* if it is of form $\operatorname{div}(f) := \sum_{x\in C} v_x(f)\cdot x$ for some $f \in K^\times$. The divisor class group $\operatorname{Cl}(C)$ of C is the $\operatorname{Div}(C)$ modulo the principal divisors. We have the isomorphism $\operatorname{Cl}(C) \cong \operatorname{Pic}(C)$ induced by the natural map $\operatorname{Div}(C) \to \{\text{invertible sheaves on } C\}$, $D \mapsto \mathcal{O}_C(D)$.

If $\mathcal{L}$ is an invertible sheaf on C, then we have $c_0(\mathcal{L}) = 1$ and $c_1(\mathcal{L}) = [\mathcal{L}]$. Therefore the preimage of $(0, [\mathcal{L}]) \in \mathbb{Z} \oplus \operatorname{Pic}(C)$ under the map (c_0, c_1) is $[\mathcal{L}] - [\mathcal{O}_C]$.

We define the degree map

$$\begin{aligned} \deg : \operatorname{Div}(C) &\to \mathbb{Z}, \\ \sum_x n_x \cdot x &\mapsto \sum_x n_x \cdot \deg x, \end{aligned}$$

where $\deg x := [k_x : k]$. It's known that a principal divisor have degree zero, hence the degree map factors through $\operatorname{Cl}(C)$. If $[\mathcal{F}] \in K^0(C)$, define $\deg([\mathcal{F}]) := \deg(c_1([\mathcal{F}]))$. If $[\mathcal{F}]$ is of form $[\mathcal{F}] = [\mathcal{O}_C]\cdot \operatorname{rank}\mathcal{F} + \sum_x n_x[\mathcal{O}_C/\pi_x\mathcal{O}_{C,x}]$, then we have $\deg([\mathcal{F}]) = \sum_x n_x \cdot \deg x$.

Theorem 7.4.1 (Riemann-Roch)**.** *Let*

$$\begin{aligned} \chi : K^0(C) &\to K^0(k) = \mathbb{Z}, \\ [\mathcal{F}] &\mapsto \dim_k H^0(C,\mathcal{F}) - \dim_k H^1(C,\mathcal{F}). \end{aligned}$$

Then $\chi([\mathcal{F}]) = \chi([\mathcal{O}_C]) \cdot \operatorname{rank}\mathcal{F} + \deg([\mathcal{F}])$.

Proof. This is by the above discussion and $\chi([\mathcal{O}_C/\pi_x\mathcal{O}_{C,x}]) = \deg x$. □

Definition 7.4.2. *The (arithmetic)* genus *of* C *is* $g(C) := \dim_k H^1(C, \mathcal{O}_C)$.

Example 7.4.3. If $C = \mathbb{P}^1_k$ then $g(C) = 0$.

7.5 Serre duality

Definition 7.5.1 (Differential form)**.** *Let $A \hookrightarrow B$ be rings. The set of differentials $\Omega_{B/A}$ is a free B-module generated by "dB" modulo the following Leibniz relations: for all $a, b \in B$,*

$$d(ab) = a\,db + b\,da, \quad d(a+b) = da + db, \quad dx = 0 \text{ if } x \in A$$

Now let C be a projective regular curve as above. We define the *canonical bundle* $\Omega^1_{C/k}$ on C, or called sheaf of *differential forms* , to be $\Omega^1_{C/k}|_{\mathrm{Spec}(A_i)} := \widetilde{\Omega^1_{A_i/k}}$, if $C = \bigcup_i \mathrm{Spec}(A_i)$ is an affine open cover of C.

Definition 7.5.2. *A curve C over a field k is* smooth *if $\Omega^1_{C/k}$ is locally free of rank one.*

Theorem 7.5.3. *Let C be the curve as in Theorem* main thm curve *7.2.2. Then C is smooth.*

Proof. Consider $C \xrightarrow{f} \mathbb{P}^1_k$ in the theorem main thm curve 7.2.2 It induces the short exact sequence

$$0 \to \Omega^1_{k(t)/k} \otimes K \to \Omega^1_{K/k} \to \Omega^1_{K/k(t)} \to 0.$$

Note that $\Omega^1_{K/k(t)} = 0$, so we have $\Omega^1_{C/k} \otimes K = \Omega^1_{K/k} = K \cdot \mathrm{d}f$. On the other hand, $\Omega^1_{C/k}$ is torsion-free, hence it is locally free of rank one. $\square$

This means that $\Omega^1_{C/k}$ is an invertible sheaf, and $\chi(\Omega^1_{C/k}) = 1 - g + \deg \Omega^1_{C/k}$.

Example 7.5.4. Let $C = \mathbb{P}^1_k$. Then we have $\mathbb{Z} \cong \mathrm{Pic}(\mathbb{P}^1_k)$, $n \mapsto [\mathcal{O}(n)]$. Now it's easy to see that $\Omega^1_{\mathbb{P}^1_k/k} \cong \mathcal{O}(-2)$ and $H^1(\mathbb{P}^1_k, \Omega^1_{\mathbb{P}^1_k/k}) \cong k$.

In the following we fix an isomorphism $H^1(\mathbb{P}^1_k, \Omega^1_{\mathbb{P}^1_k/k}) \cong k$.

Theorem 7.5.5 (Serre duality)**.** *Let k be a field. Let C be a projective smooth curve over k, we have $\dim H^1(C, \Omega_C) = 1$. Moreover any coherent sheaf $\mathcal{F}$ on C, we have a canonical perfect pairing*

$$\mathrm{Hom}(\mathcal{F}, \Omega^1_{C/k}) \times H^1(C, \mathcal{F}) \to H^1(C, \Omega^1_{C/k}) \cong k.$$

Sketch of proof. **Step 0.** Reduce to the case that $\mathcal{F}$ is torsion-free. In this case we have

$$\mathrm{Hom}(\mathcal{F}, \Omega^1_{C/k}) = H^0(C, \mathcal{F}^\vee \otimes \Omega^1_{C/k}).$$

Step 1. We prove that it holds for $C = \mathbb{P}^1_k$. By induction we only need to prove it for $\mathcal{F}$ invertible sheaf case. The invertible sheaf on $\mathbb{P}^1_k$ is of form $\mathcal{O}(n)$, $n \in \mathbb{Z}$, it's easy to prove that the theorem holds for it.

Step 2. Consider $C \xrightarrow{f} \mathbb{P}^1_k$ in the last talk. Then we have

$$H^0(C, \mathcal{F}^\vee \otimes \Omega^1_{C/k}) \cong H^0(\mathbb{P}^1_k, f_*(\mathcal{F}^\vee \otimes \Omega^1_{C/k})), \qquad H^1(C, \mathcal{F}) \cong H^1(\mathbb{P}^1_k, f_*\mathcal{F}).$$

Apply Serre duality for $\mathbb{P}^1$, we are reduced to construct a perfect pairing of sheaves on $\mathbb{P}^1$:

$$f_*(\mathcal{F}^\vee \otimes \Omega^1_{C/k}) \otimes f_*\mathcal{F} \longrightarrow \Omega^1_{\mathbb{P}^1_k/k}.$$

First we define this map at the generic point of $\mathbb{P}^1$ using identity $\Omega^1_{k(C)/k} = k(C)\Omega^1_{k(\mathbb{P}^1)/k}$ and trace map $\mathrm{tr}_{k(C)/k(\mathbb{P}^1)} : k(C) \to k(\mathbb{P}^1)$. To show that this induces a perfect pairing as above, we need only work on local case: let $x \in C$ with image $y \in \mathbb{P}^1$ with local rings B and A respectively. Then we may assume that $\mathcal{F}_x = B^n$. Thus we are reduce to the case where $\mathcal{F}_x = B$. It remains to prove that the trace map induces a perfect pairing of A-modules:

$$\Omega_{B/k} \otimes_A B \longrightarrow \Omega_{A/k}.$$

Let s, t be a local parameter of C and $\mathbb{P}^1$ at x, y respectively, such that $B = A[s]$. Then $\Omega_{A/k} = Adt$ and $\Omega_{B/k} = Bds = Ddt$ for a fractional ideal $D = B/(\frac{dt}{ds})$ of B in $k(C)$. Thus the above pairing is

$$D \otimes_A B \longrightarrow A, \qquad (x, y) \mapsto \mathrm{tr}(xy).$$

To show this pairing is perfect, it suffices to show that D is equal to the dual $B^\vee$ of B under the trace pairing. Now we use two exact sequences.

The first one is

$$0 \to \mathfrak{d}_{B/A} \to B \overset{\cdot ds}{\to} \Omega_{B/A} \to 0,$$

where $\mathfrak{d}_{B/A} \subset B$ is the relative different of B/A, namely, the inverse of $B^\vee$. To see the exactness, let f be the minimal polynomial of α over A, then we have $\mathfrak{d}_{B/A} = f'(\alpha) \cdot B$.

On the other hand, we have an exact sequence

$$0 \longrightarrow \Omega_{A/k} \otimes_A B \longrightarrow \Omega_{B/k} \longrightarrow \Omega_{B/A} \longrightarrow 0.$$

From these two exact sequence, we get an identity of two ideals of B:

$$\mathfrak{d}_{B/A} = \Omega_{A/k} \otimes_A \Omega^{-1}_{B/k}.$$

This identity is equivalent to the perfectness of the pairing

$$\Omega_{B/k} \otimes_A B \longrightarrow \Omega_{A/k}.$$

□

Exercise 7.5.6. Fill in the details and complete the proof.

The Serre duality has the following consequences: Let C be a smooth curve over a field k with genus g.

- Take $\mathcal{F} = \Omega^1_{C/k}$ we obtain
$$H^1(C, \Omega^1_{C/k}) \cong H^0(C, \mathcal{O}_C)^\vee$$
which is of dimension 1.

- Take $\mathcal{F} = \mathcal{O}_C$ we obtain
$$H^1(C, \mathcal{O}_C) \cong H^0(C, \Omega^1_{C/k})^\vee$$
which is of dimension g.

- If $\mathcal{F}$ is torsion free, we have

$$\dim H^0(C,\mathcal{F}) = \dim H^0(C,\mathcal{F}^\vee\otimes\omega_C) = (1-g)\mathrm{rank}\mathcal{F} + \deg\mathcal{F},$$

$$\dim H^0(C,\mathcal{F}\otimes\omega_C) - \dim H^0(C,\mathcal{F}) = (1-g)\mathrm{rank}(\mathcal{F}^\vee\otimes\omega_C) + \deg(\mathcal{F}^\vee\otimes\omega_C).$$

Therefore take $\mathcal{F} = \Omega^1_{C/k}$ in Riemann-Roch theorem we obtain

$$\deg\Omega^1_{C/k} = 2g-2.$$

Applications of Serre's Duality

The first application of serre's duality is Hurwitz genus formula.

Let $f: X\to Y$ be a finite morphism between smooth projective curves over k. Denote the function field of X by $K(X)$. Then f induced an injection of function fields fixing k:

$$f^*:\ K(Y)\longrightarrow K(X)\ .\qquad t\longmapsto t\circ f$$

Theorem 7.5.7. *Let $f: X\to Y$ be a non-constant map between smooth projective curves over k, then $K(X)$ is a finite field extension of $K(Y)$.*

Proof. Since X is smooth, then X is regular, $f(X)$ must be closed in Y and proper over Spec$k$ by [closed proper] 4.4.13. And since $f(X)$ is irreducible, Thus if $f(X)=Y$ since f is non-constant.

Now f induced an inclusion $K(Y)\subset K(X)$. Since $K(X)$, $K(Y)$ are of transcendence degree 1 over k, $K(X)$ must be a finite algebraic extension of $K(Y)$. We only need to show that f is a finite morphism. Let Spec$B\subset Y$ be an open affine subset. Let A be the integral closure of B in $K(X)$. Then A is a finite B-module by [dedekind-extension] A.4.21, consider the following diagram

$$\begin{array}{ccc} B & \hookrightarrow & A \\ \downarrow & & \downarrow \\ K(Y) & \hookrightarrow & K(X) \end{array}\ .$$

We have Spec$A = f^{-1}(\mathrm{Spec}B)$, thus f is a finite morphism. ☐

Definition 7.5.8. *Let $f: X\to Y$ be a finite morphism of curves over k. Define the degree of f, denoted* $\deg f$*, to be the degree of field extension $[K(X):K(Y)]$. If f is a constant map, then definte* $\deg f = 0$*.*

If the field extension $K(X)/f^*K(Y)$ is separable or has other field extension properties, then we also say that f is separable or has other field extension properties.

Definition 7.5.9. *Let $f: X\to Y$ be a finite morphism of smooth curves. Let $Q\in Y$ be a closed point and suppose that $t\in K(Y)$ has order 1 at Q. (i.e $\mathrm{ord}_Q(t)=1$ where ord_Q is the valuation corresponding to the discrete valuation ring $\mathcal{O}_Q$.), and we call t the local parameter at Q. We define the* ramification index *of f at P, denoted $e_f(P)$, to be*

$$e_f(P) = \mathrm{ord}_P(t).$$

Note that $e_f(P)\geqslant 1$. We say f is unramified at P if $e_f(P)=1$, say f is unramified if it is unramified at all points of X; we say f is tamely ramified at P if $\mathrm{char}(k)\nmid e_f(P)$.

Remark 7.5.10. Note that $f^*(Q)$ is independent of the choice of t. Indeed, if $t' \in K(Y)$ also has order 1 in Q, then suppose that $t' = ut$, where u is an unit. For any point $P \in f^{-1}(Q)$, since u is also a unit in $K(X)$, so $\mathrm{ord}_p(t) = \mathrm{ord}_P(t')$.

Corollary 7.5.11. *Suppose that $f(P) = Q$. Let $t_P \in K(X)$, $t_Q \in K(Y)$ such that $\mathrm{ord}_P(t_P) = 1$ and $\mathrm{ord}_Q(t_Q) = 1$. A direct calculation shows that*

$$\mathrm{ord}_P\left(\frac{\mathrm{d}t_Q}{\mathrm{d}t_P}\right) \geq e_f(P) - 1,$$

and if x is tamely ramified, then "=" holds.

Exercise 7.5.12. Let $f : X \to Y$ be a finite morphism of smooth curves. Show that for every $Q \in Y$, $\deg f = \sum_{P \in f^{-1}(Q)} e_f(P)$.

Definition 7.5.13. *Let $f : X \to Y$ be a finite morphism of smooth curves. Then f also induces a map on the divisor groups as follows:*

$$\begin{aligned} f^* :\ \mathrm{Div}(Y) &\longrightarrow \mathrm{Div}(X) \\ (Q) &\longmapsto \textstyle\sum_{P \in f^{-1}(Q)} e_f(P) \cdot P \end{aligned}$$

Note that f is a finite morphism then f^* is well-defined.

Exercise 7.5.14. Show that f^* preserves linear equivalence of divisors, so it induces a homomorphism $f^* : \mathcal{L}(Y) \to \mathcal{L}(X)$.

Exercise 7.5.15. Show that for any divisor $D \in \mathrm{Div}(Y)$, we have $\deg f^* D = \deg f \cdot \deg D$.

In the following we denote the local parameter at a point P by t_P.

Proposition 7.5.16. *Let $f : X \to Y$ be a finite separable morphism between smooth projective curves over k. Then we have the induced map*

$$f^* : \mathrm{Div}(Y) \to \mathrm{Div}(X)$$

and the exact sequence:

$$0 \to f^*\Omega^1_{Y/k} \to \Omega^1_{X/k} \to \Omega^1_{X/Y} \to 0. \tag{7.5.1}$$

Remark 7.5.17. Note that for any $P \in X$ and $Q \in Y$, we have dt_Q is a generator of the free $\mathcal{O}_Q$-module $(\Omega^1_{Y/k})_Q$ and dt_P is a generator of the free $\mathcal{O}_P$-module $(\Omega^1_{X/k})_P$. then there is a unique element $u \in \mathcal{O}_P$ such that $f^* dt_Q = u \cdot dt_P$.

Proposition 7.5.18. *Let $f : X \to Y$ be a finite separable morphism between smooth projective curves over k. Then*

1. *The sheaf $\Omega^1_{X/Y}$ is a torsion sheaf on X with support equal to the set of ramification points of f.*

2. The sheaf $\Omega^1_{X/Y}$ is a finite $\mathcal{O}_X$-module and

$$c_1(\Omega^1_{X/Y}) = \sum_{P\in X} \mathrm{ord}_x\left(\frac{\mathrm{d}t_{f(P)}}{\mathrm{d}t_P}\right)\cdot P.$$

note that for each $P \in X$, the stalk $(\Omega^1_{X/Y})_P$ is a principle $\mathcal{P}_P$-module of length $\frac{\mathrm{d}t_{f(P)}}{\mathrm{d}t_P}$.

Proof. 1. Since $f^*\Omega^1_{Y/k}, Omega^1_{X/k}$ are locally free of rank 1, then from the exact sequence diff exact seq 7.5.1 we have $\Omega^1_{X/Y}$ is torsion sheaf. Now we need to prove that for all $P \in X$ such that f is unramified at P, $(\Omega^1_{X/Y})_P = 0$. Suppose that $e_f(P) = 1$, we have $t_P = t_{f(P)}$, then $f^*dt_(f(P))$ is a generater of $(\Omega_{X/k})_P$, thus $(\Omega^1_{X/Y})_P = 0$. 2. Let $Q = f(P)$, then from the exact sequence diff exact seq 7.5.1 we have an exact sequence on stalks, which implies

$$(\Omega^1_{X/Y})_P \cong (\Omega^1_{X/k})_P / f^*(\Omega^1_{Y/k})_Q \xrightarrow{\sim} \mathcal{O}_P \Big/ {dt_Q/dt_P}.$$

□

From the exact sequence above we have $\deg \Omega^1_{X/k} = \deg \Omega^1_{Y/k} \cdot \deg f + \deg \Omega^1_{X/Y}$, that is,

Theorem 7.5.19 (Hurwitz Formula)**.**

$$2g(X) - 2 = (2g(Y) - 2)\cdot \deg f + \sum_{P\in X} \mathrm{ord}_x\left(\frac{\mathrm{d}t_Q}{\mathrm{d}t_P}\right)\cdot \deg P.$$

The Riemann-Roch theorem allows us to do the classification of curves.

For example, if C is a curve of genus 0 with $C(k) = \{\text{points on C whose values are in } k\} \neq \varnothing$, fix a point $p \in C(k)$ and take line bundle $\mathcal{L} = \mathcal{O}_C(p)$, then $\dim_k H^0(C, \mathcal{L}) = 2$. The global sections of $\mathcal{L}$ allows us to construct an isomorphism $C \cong \mathbb{P}^1_k$. In general, if C is a curve of genus 0, then we can take $\mathcal{L} = (\Omega^1_{C/k})^{-1}$, it allows us to find a closed embedding of C into $\mathbb{P}^2_k$.

Remark 7.5.20. If C is a smooth projective curve of genus g, $\mathcal{L}$ is an invertible sheaf on C. If $\deg \mathcal{L} \geq 2g - 1$, then $H^1(C, \mathcal{L}) = 0$ and $\dim_k H^0(C, \mathcal{L}) = 1 - g + \deg \mathcal{L}$. If $\deg \mathcal{L} \geq 2g$ then it is generated by global sections. If $\deg \mathcal{L} \geq 2g + 1$ then it is "very ample", i.e. gives an embedding of C into a projective space.

8 Grothendieck–Riemann–Roch Theorem

As a generalization, we define the divisors on schemes whose dimension is higher than 1.

Definition 8.0.1. *We say a scheme X is regular in codimension 1 if every 1-dimensional local ring of X is regular local ring.*

Definition 8.0.2 (Weil divisor)**.** *Let X be a noetherian integral separated scheme which is regular in codimension 1. We call a subscheme $Y \subset X$ the prime divisor if Y is a closed integral subscheme of codimension 1. We define the* Weil divisor *on X to be the finite free sum of prime divisors*

$$D = \sum_Y n_Y Y,$$

where n_Y are integers. If n_Y are all positive then we call D effective. We define the divisor group of X, denoted by $\mathrm{Div}(X)$ to be the free abelian group generated by the prime divisors.

Let X be any scheme. For each open subset $U \subset X$, let $S(U)$ denote the elements that are not zero divisors in $\Gamma(U, \mathcal{O}_X)$. Let $K(U)$ denote the localization of $\Gamma(U, \mathcal{O}_X)$ by the multiplication system $S(U)$. Then let $\mathcal{K}$ be the sheaf associated to the presheaf $U \mapsto K(U)$. So we have an exact sequence

$$0 \longrightarrow \mathcal{O}_X^\times \longrightarrow \mathcal{K}^\times \longrightarrow \mathcal{K}^\times/\mathcal{O}_X^\times \longrightarrow 0.$$

Definition 8.0.3. *A* Cartier divisor *on a scheme X is a global section of the sheaf $\mathcal{K}^\times/\mathcal{O}_X^\times$. We say a Cartier divisor is principle if it is in the image of the natural map*

$$\Gamma(X, \mathcal{K}^\times) \longrightarrow \Gamma(X, \mathcal{K}^\times/\mathcal{O}_X^\times).$$

8.1 Preliminaries

In this section we are going to introduce Grothendieck-Riemann-Roch theorem. Reference: BorelSerre [2]. In the following a scheme is always Noetherian, separated, finite type over a field k or $\mathbb{Z}$.

Theorem 8.1.1. *Let X be a Noetherian separated regular scheme. Then $K^0(X) = K_0(X)$.*

Recall that a scheme is *regular* if for any $x \in X$ we have $\dim_{k_x} \mathfrak{m}_x/\mathfrak{m}_x^2 = \dim \mathcal{O}_{X,x}$.

Proof. Let $X = \bigcup_i U_i$ be a cover of X by finitely many affine open subsets. We may assume that for each i, the $X \setminus U_i$ is a union of codimension one integral subschemes ("effect Cartier divisor" D_i). Let $\mathcal{F}$ be a coherent sheaf on X. We only need to find a finite length resolution of $\mathcal{F}$ by locally free coherent sheaves.

Let $U_i = \mathrm{Spec}(A_i)$ and let $\mathcal{F}|_{U_i} = \widetilde{M_i}$ for some finitely generated A_i-module M_i. Let $\{m_{ij}\}_{i=1}^{n_i}$ be a set of generators of M_i. We may write $U_i = X_{s_i}$ the non-zero locus of s_i for some $s_i \in \Gamma(X, \mathcal{O}_X(D_i))$. Then by Lemma ugly lemma 5.5.5, we may find a (common) $\ell_i \gg 0$ such that $s_i^{\ell_i} m_{ij}$ extends to a section of $\mathcal{F} \otimes \mathcal{O}_X(\ell_i D_i)$ for any $1 \le j \le n_i$. Therefore we defined a morphism $\mathcal{O}_X(\ell_i D_i)^{\oplus n_i} \to \mathcal{F}$ which is surjective on U_i. Hence we can define $\mathcal{E}_0 := \bigoplus_i \mathcal{O}_X(\ell_i D_i)^{\oplus n_i}$ with surjection $\mathcal{E}_0 \twoheadrightarrow \mathcal{F}$. Repeat this process on its kernel, we obtain a resolution of $\mathcal{F}$ by locally free coherent sheaves

$$\cdots \to \mathcal{E}_2 \to \mathcal{E}_1 \to \mathcal{E}_0 \to \mathcal{F} \to 0.$$

A theorem in commutative algebra tells us that $\ker(\mathcal{E}_{n-1} \to \mathcal{E}_{n-2})$ is a locally free sheaf, here $n = \dim X$. Hence we can find a finite length resolution of $\mathcal{F}$ by locally free coherent sheaves. □

Recall that if A is a ring, let M be an A-module. The tensor algebra of M, denoted $T(M) = \oplus_{k=0}^{\infty} T^k(M)$, is defined to be the algebra of tensors on M with multiplication being the tensor product.

Definition 8.1.2. *Let A be a ring and M an A-module. We define the* symmetric algebra $\mathrm{Sym}^*(M) = \bigoplus_{d\geq 0} \mathrm{Sym}^d(M)$ *to be the quotient of $T(M)$ by the two-sided ideal generated by all expressions $x \otimes y - y \otimes x$, for all $x, y \in M$.*

Remark 8.1.3. If M is a free A-module of rank r, then $\mathrm{Sym}^*(M) \cong A[x_1, \cdots, x_r]$.

Definition 8.1.4. *Let X be a scheme and $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$ modules. We define the tensor algebra, symmetric algebra, and exterior algebra of $\mathcal{F}$ by taking the sheaves associated to the presheaf, which to each open set U assigns the corresponding tensor operation applied to $\mathcal{F}(U)$ as an $\mathcal{O}_X(U)$-module. The results are $\mathcal{O}_X$-algebras, and their components in each degree are $\mathcal{O}_X$-modules.*

Exercise 8.1.5. Let X be a noetherian scheme and $\mathcal{E}$ be a locally free sheaf on X. Then the symmetric algebra $\mathrm{Sym}^*(\mathcal{E})$ is a quasi-coherent sheaf of $\mathcal{O}_X$-modules which has a graded $\mathcal{O}_X$-algebra structure. And $\mathrm{Sym}^*(\mathcal{E})$ is locally generated by $\mathrm{Sym}^1(\mathcal{E})$ as an $\mathcal{O}_X$-algebra.

Definition 8.1.6. *Let $\mathcal{E}$ be a locally free coherent sheaf of rank $n+1$ on a noetherian scheme X. Let $\mathrm{Sym}^*\mathcal{E} := \bigoplus_{d\geq 0} \mathrm{Sym}^d \mathcal{E}$ be the sheaf of symmetric algebra of $\mathcal{E}$, which is a sheaf of graded $\mathcal{O}_X$-algebras on X. Then define the* projective space bundle $\mathbb{P}(\mathcal{E})$ *to be the projective space* $\mathrm{Proj}\,(\mathrm{Sym}^*\mathcal{E})$ *with a morphism*

$$\pi : \mathbb{P}(\mathcal{E}) \to X$$

and an invertible sheaf $\mathcal{O}(1)$.

In other words, if $X = \bigcup_i U_i$ is an affine open cover, $U_i = \mathrm{Spec}(A_i)$, define

$$\mathrm{Proj}\,(\mathrm{Sym}^*(\mathcal{E})(U_i)) \to U_i$$

(note that $\mathrm{Sym}^*(\mathcal{E})(U_i)$ is a graded ring with subring of degree 0 equals A_i), and glue them together. For each $i \in \mathbb{Z}$ there is a sheaf $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(i)$ on it, defined by the glueing of $\mathcal{O}(i)$ on each $\mathrm{Proj}\,(\mathrm{Sym}^*(\mathcal{E})(U_i))$.

Remark 8.1.7. If $\mathcal{E}$ is locally free of rank $n+1$ over an open subset U, then $\pi^{-1}(U) \cong \mathbb{P}^n_U$.

Remark 8.1.8. $\mathbb{P}(\mathcal{E})$ is the moduli space of quotient line bundles of $\mathcal{E}$.

proj bundle **Theorem 8.1.9.** *We have*

(i) $\pi_*[\mathcal{O}_{\mathbb{P}(\mathcal{E})}(i)] = [\mathrm{Sym}^i \mathcal{E}]$ *for* $i \geq 0$,

(ii) $\pi_*[\mathcal{O}_{\mathbb{P}(\mathcal{E})}(i)] = 0$ *for* $-1-n < i < 0$.

Exercise 8.1.10. Prove it.

Exercise 8.1.11. Can you calculate $\pi_*[\mathcal{O}_{\mathbb{P}(\mathcal{E})}(i)]$ for $i \leq -1-n$? You may need to study relative Serre duality.

of K group

Theorem 8.1.12. *Let X be a Noetherian scheme.*

(1) *Let $i : Y \hookrightarrow X$ be a closed subscheme, $U := X \setminus Y$, $j : U \hookrightarrow X$ be an open subscheme, then*

$$K_0(Y) \xrightarrow{i_*} K_0(X) \xrightarrow{j^*} K_0(U) \to 0$$

is exact.

(2) *Let $\pi : Y \to X$ be an* affine bundle *(i.e. locally of form* $\mathrm{Spec}(A[X_1, \cdots, X_n]) \to \mathrm{Spec}(A)$*), then*

$$\pi^* : K_0(X) \to K_0(Y)$$

is an isomorphism.

(3) *Let $\mathcal{P} = \mathbb{P}(\mathcal{E}) \xrightarrow{\pi} X$ be a projective bundle with* $\mathrm{rank}\mathcal{E} = n + 1$*, then*

$$\bigoplus_{i=0}^{n} K_0(X) \to K_0(\mathcal{P}),$$

$$([\mathcal{F}_i])_{i=0}^n \mapsto \sum_{i=0}^{n} [\mathcal{O}_{\mathcal{P}}(i)] \cdot \pi^*[\mathcal{F}_i]$$

is an isomorphism.

Sketch of proof. (1) j^* is surjective: note that coherent sheaf on U can be extended to coherent sheaf on X by using Lemma 5.5.5 [ugly lemma]. For the details, see first few pages of [2] [BorelSerre].

$\mathrm{Im}(i_*) \subset \ker(j^*)$: trivial.

$\ker(j^*) \subset \mathrm{Im}(i_*)$: if $\mathcal{F}$ is a coherent sheaf on X such that $j^*[\mathcal{F}] = 0$, i.e. $[\mathcal{F}|_U] = 0$, then $\mathcal{F}_U$ is a finite sum of short exact sequences of sheaves on U, each of them can be extended to coherent sheaf on X by using Lemma 5.5.5 [ugly lemma]. More precisely, we can find coherent sheaves $\mathcal{E}_i, \mathcal{F}_i, \mathcal{G}_i$ on X with injective morphisms $\mathcal{E}_i \hookrightarrow \mathcal{F}_i$ and surjective morphisms $\mathcal{F}_i \twoheadrightarrow \mathcal{G}_i$, such that $[\mathcal{F}] = \sum_i([\mathcal{F}_i] - [\mathcal{E}_i] - [\mathcal{G}_i])$ and such that $0 \to \mathcal{E}_i|_U \to \mathcal{F}_i|_U \to \mathcal{G}_i|_U \to 0$ is exact. Then we have $[\mathcal{F}_i] - [\mathcal{E}_i] - [\mathcal{G}_i] = [\ker(\mathcal{F}_i \to \mathcal{G}_i)/\mathrm{Im}(\mathcal{E}_i \to \mathcal{F}_i)] \in \mathrm{Im}(i_*)$.

(2) We prove π^* is surjective. By (1) and diagram chasing, we may assume X is affine and $Y = \mathbb{A}_X^n$. By induction, we may assume $n = 1$. By intersect all open subsets, we may assume $X = \mathrm{Spec}(A)$ for A Artinian. Then $A = \prod_\eta \mathcal{O}_{X,\eta}$ for η runs over generic points of X. Hence $K_0(Y) = \prod_\eta K_0(\mathbb{A}^1_{k_\eta}) = \mathbb{Z}[\pi_* \mathcal{O}_{\mathrm{Spec}(k_\eta)} \mid \eta \in X]$.

The injectivity of π^* is easy when $X = \mathrm{Spec}(k)$ for a field k and $Y = \mathbb{A}_k^n$. In the general case, it is a consequence of (3).

(3) Surjectivity: similar to (2) we may assume $X = \mathrm{Spec}(k)$ for a field k and $\mathcal{P} = \mathbb{P}_k^n$. Induction on n. Note that $j : \mathbb{A}_k^n \hookrightarrow \mathbb{P}_k^n$ is an open subscheme, with $\mathbb{P}_k^n \setminus \mathbb{A}_k^n = \mathbb{P}_k^{n-1} \xrightarrow{i} \mathbb{P}_k^n$, hence by (1) we have

$$K_0(\mathbb{P}_k^{n-1}) \xrightarrow{i_*} K_0(\mathbb{P}_k^n) \xrightarrow{j^*} K_0(\mathbb{A}_k^n) \to 0.$$

We have $K_0(\mathbb{A}_k^n) = K_0(k) \cong \mathbb{Z}$, and by the surjectivity of (2),

$$K_0(\mathbb{P}_k^n) = i_* K_0(\mathbb{P}_k^{n-1}) + \pi^* K_0(k).$$

By induction hypothesis,

$$K_0(\mathbb{P}_k^{n-1}) = \sum_{d=0}^{n-1} \mathbb{Z} \cdot \mathcal{O}_{\mathbb{P}_k^{n-1}}(d),$$

note that $\mathcal{O}_{\mathbb{P}_k^{n-1}}(d) = i_*\mathcal{O}_{\mathbb{P}_k^n}(d)$, and $0 \to \mathcal{O}_{\mathbb{P}_k^n}(-1) \to \mathcal{O}_{\mathbb{P}_k^n} \to i_*\mathcal{O}_{\mathbb{P}_k^{n-1}} \to 0$ with $i_*\mathcal{O}_{\mathbb{P}_k^{n-1}} = i_*i^*\mathcal{O}_{\mathbb{P}_k^n}$, we have

$$i_*K_0(\mathbb{P}_k^{n-1}) = \sum_{d=0}^{n-1} \mathbb{Z} \cdot i_*i^*\mathcal{O}_{\mathbb{P}_k^n}(d) = \sum_{d=0}^{n-1} \mathbb{Z} \cdot \mathcal{O}_{\mathbb{P}_k^n}(d+1).$$

Injectivity: if $([\mathcal{F}_i])_{i=0}^n$ are not all zero such that $\sum_{i=0}^n [\mathcal{O}_P(i)] \cdot \pi^*[\mathcal{F}_i] = 0$, let j be the maximal integer such that $[\mathcal{F}_j] \neq 0$, taking $\pi_*(- \otimes \mathcal{O}_P(-j))$ of this element, utilizing Theorem 8.1.9 `cohomology of proj bundle` we may obtain a contradiction. □

Corollary 8.1.13. *Let n be a positive integer. Then $K_0(\mathbb{P}^n)$ is generated by the classes of line bundles $[\mathcal{O}(d)]$.*

8.2 Chow group

Let X be a Noetherian scheme of pure dimension n. For an integer $p \geq 0$ let $Z_p(X)$ be the free abelian group generated by all closed integral subscheme Y of X of dimension p. Let

$$Z_*(X) := \bigoplus_{p \geq 0} Z_p(X).$$

Note that such Y is determined by its generic point, hence $Z_*(X)$ is also the free abelian group generated by all the points of X. An element of $Z_*(X)$ is called a *cycle* in X; and element of $Z_p(X)$ is called a *p-cycle* in X.

If Y is a closed subscheme of X, with $Y = \bigcup_i Y_i$ the irreducible components, let $Y_{i,\mathrm{red}} \subset Y_i$ be the maximal reduced part of Y_i, then $Y_{i,\mathrm{red}}$ is a closed integral subscheme of X. Let η_i be the generic point of Y_i, then $\mathcal{O}_{Y_i,\eta_i}$ is an Artinian local ring with residue field $k(\eta_i)$, and that the length $\mathrm{legnth}_{k(\eta_i)} \mathcal{O}_{Y_i,\eta_i}$ is finite. We define the class of Y in $Z_*(X)$ to be

$$[Y] := \sum_i \mathrm{legnth}_{k_{\eta_i}} \mathcal{O}_{Y_i,\eta_i} \cdot [Y_{i,\mathrm{red}}].$$

If Y is a closed integral subscheme of X, let η be the generic point of Y, then the function field of Y is $k(Y) = \mathcal{O}_{Y,\eta} = k(\eta)$. For a non-zero rational function $f \in k(Y)^\times$, define

$$\mathrm{div}(f) := \sum_{\substack{Z \subset Y \text{ closed integral} \\ \text{subscheme of codimension } 1}} \mathrm{ord}_Z(f) \cdot [Z],$$

where for $f \in \mathcal{O}_{Y,\eta_Z} \setminus \{0\}$ (note that $\mathcal{O}_{Y,\eta_Z}$ is of dimension 1), define $\mathrm{ord}_Z(f) := \mathrm{legnth}_{k(\eta_Z)} \mathcal{O}_{Y,\eta_Z}/(f)$.

Exercise 8.2.1. The definition of ord_Z can be extended to non-zero elements of the function field $k(Y) = \mathrm{Frac}(\mathcal{O}_{Y,\eta_Z})$, and which is well-defined.

Definition 8.2.2. *Let* $Z'_*(X) \subset Z_*(X)$ *be the subgroup generated by all such* $\operatorname{div}(f)$*. The elements of it are called "rational equivalent to zero". Define the* Chow group

$$\operatorname{Ch}_*(X) := Z_*(X)/Z'_*(X).$$

Remark 8.2.3. Let X be a scheme of dimension n. For integer $p \le n$, the Chow group of index p is defined by :

$$\operatorname{Ch}_p(X) := Z_p(X)/Z'_p(X)$$

and $Z'_p(X) \subset Z_p(X)$ is the subgroup generated by $\operatorname{div}(f)$, where $0 \neq f$ is in the function field of arbitrary closed subscheme $Y' \subset X$ of dimension $p-1$.

Example 8.2.4. If X is a variety of dimension n, the Chow group $\operatorname{Ch}_{n-1}(X) = \operatorname{Div}(X)$. When X is smooth over a field k, this is isomorphic to $\operatorname{Pic}(X)$.

Remark 8.2.5. If $f : X \to Y$ is a proper morphism, for $[Z] \in \operatorname{Ch}_*(X)$, then we can define $f_* : \operatorname{Ch}_*(X) \to \operatorname{Ch}_*(Y)$ by

$$f_*[Z] := \begin{cases} 0, & \text{if } \dim f(Z) < \dim Z, \\ [k(Z) : k(f(Z))] \cdot [f(Z)], & \text{if } \dim f(Z) = \dim Z. \end{cases}$$

If f is a flat morphism, i.e. for any $x \in X$, the local ring $\mathcal{O}_{X,x}$ is flat over the local ring $\mathcal{O}_{Y,f(x)}$. Then we can define $f^* : \operatorname{Ch}_*(Y) \to \operatorname{Ch}_*(X)$ by

$$f^*[Z] := [f^{-1}(Z)].$$

If X is regular then we have intersection theory on $\operatorname{Ch}_*(X)$.

8.3 Chern class for line bundle

Let $\mathcal{L}$ be a line bundle on X. Let $Z \overset{i}{\hookrightarrow} X$ be a closed integral subscheme of X, and denote the generic point of Z by η_Z. Assume that ℓ is any non-zero rational section of $i^*\mathcal{L}_{\eta_Z}$. If the dimension of Z is p, then by trivilization $\operatorname{div}(\ell) \hookrightarrow Z$ has dimension $p-1$. So we get a map

$$Z_p(X) \longrightarrow Z_{p-1}(X)$$

which preserves $Z'_p(X) \longrightarrow Z'_{p-1}(X)$. Then for each p, we can define the first Chern chass to be a map

$$\begin{aligned} c_1(\mathcal{L}) :\ \operatorname{Ch}_p(X) &\longrightarrow \operatorname{Ch}_{p-1}(X)\ , \\ [Z] &\longmapsto [\operatorname{div}(\ell)] \end{aligned} \tag{8.3.1}$$

The Chern class has the following properties:

- Projection formula: if $f : X \to Y$ is proper, $\mathcal{L}$ is a line bundle on Y, then for $\alpha \in \operatorname{Ch}_*(X)$,

  $$f_*(c_1(f^*\mathcal{L})(\alpha)) = c_1(\mathcal{L})(f_*(\alpha)).$$

  Note that here f_* is the map $\operatorname{Ch}(X) \to \operatorname{Ch}(Y)$.

- If $\mathcal{L}$ and $\mathcal{L}'$ are line bundles on X, then

$$c_1(\mathcal{L}) \circ c_1(\mathcal{L}') = c_1(\mathcal{L}') \circ c_1(\mathcal{L}).$$

The Chow group has the properties similar to K_0-group (Theorem prop of K group 8.1.12):

Theorem 8.3.1. *Let X be a Noetherian scheme.*

(1) *Let $i : Y \hookrightarrow X$ be a closed subscheme, $U := X \setminus Y$, $j : U \hookrightarrow X$ be an open subscheme, then*

$$\mathrm{Ch}_*(Y) \xrightarrow{i_*} \mathrm{Ch}_*(X) \xrightarrow{j^*} \mathrm{Ch}_*(U) \to 0$$

is exact.

(2) *Let $\pi : Y \to X$ be an affine bundle of relative dimension n, then for each p, $\pi^* : \mathrm{Ch}_p(X) \to \mathrm{Ch}_{p+n}(Y)$ is an isomorphism.*

(3) *Let $\mathcal{P} = \mathbb{P}(\mathcal{E}) \xrightarrow{\pi} X$ be a projective bundle with $\mathrm{rank}\mathcal{E} = n+1$, then*

$$\bigoplus_{i=0}^{n} \mathrm{Ch}_*(X) \to \mathrm{Ch}_*(\mathcal{P}),$$

$$(\alpha_i)_{i=0}^n \mapsto \sum_{i=0}^{n} c_1(\mathcal{O}_{\mathcal{P}}(1))^i(\pi^*(\alpha_i))$$

is an isomorphism.

Proof. (1) Obvious.

(2) We first prove that π^* is surjective. Let $U \hookrightarrow X$ be an affine open subset and $Z = X - U$ the closed subscheme of X. By (1) we need only to prove the assertion for $Y \bigcap U \to U$ and $Y \bigcap Z \to Z$. By induction on $\dim X$ we may assume that X is affine and $Y = X \times \mathbb{A}^n$. By induction on n we may assume that $n = 1$ and Y is a trivial line bundle over X. Let $Y' \subsetneqq Y$ be a closed integral subscheme. We want to prove that Y' is rationally equivalent to $\pi_*(\alpha)$ for some cycle α in X, assume that the closure of $\pi(Y')$ is X. Let η be a generic point of X and Y_η the generic fibre of Y over η. Then $Y'_\eta = Y' \times Y_\eta$ is a divisor of Y_η, it must be equivalent to 0. We can extend Y'_η to a cycle β in Y which is equivalent to 0. Now the closure of the image of support of $\pi(Y' - \beta)$ is a proper subscheme of X.

The injectivity will follow from (3).

(3) First we prove this map is surjective. By (1) and induction on $\dim X$ we may assume that Y is affine and $\mathcal{E}$ is trivial. So we have a projective subbundle $i : \mathcal{P}' \to \mathcal{P}$ if rank $n-1$ such that $\mathcal{P} - \mathcal{P}'$ is an affine bundle over X. The assertion follows from (2) and the fact

$$c_1(\mathcal{O}(1))\mathrm{Ch}_*(\mathcal{P}) \subset i_*(\mathrm{Ch}_*(\mathcal{P}')).$$

The injectivity follows from the fact that for $\alpha \in \mathrm{Ch}_*(X)$, $\pi(c_1(\mathcal{O}(1))^n\pi^*(\alpha)) = \alpha$ and $\pi_*(c_1(\mathcal{O}(1))^i\pi^*(\alpha)) = 0$ for $0 \leqslant i < n$. We may assume that $\alpha = [X]$ and X is a point. The assertion follows from the fact that $c_1(\mathcal{O}(1))^n[\mathcal{P}]$ is represented by any section of $\mathcal{P}$ over X. □

8.4 Chern classes for a vector bundle

Let $\mathcal{E}$ be a vector bundle on a noetherian scheme X of rank $n+1$, and

$$\mathbb{P}(\mathcal{E}) \xrightarrow{\pi} X$$

the corresponding projective bundle of relative dimension n, endowed with

$$\mathcal{O}(1) := \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1).$$

Recall that there is an isomorphism

$$\bigoplus_{i=0}^{n} \mathrm{Ch}_*(X) \to \mathrm{Ch}_*(\mathcal{P}),$$

$$(\alpha_i)_{i=0}^{n} \mapsto \sum_{i=0}^{n} c_1(\mathcal{O}_{\mathcal{P}}(1))^i(\pi^*(\alpha_i))$$

Then we have

$$\mathrm{Ch}_*(\mathbb{P}(\mathcal{E})) \xrightarrow{\sim} \bigoplus_{i=0}^{n} c_1(\mathcal{O}(1))^i(\pi^*\mathrm{Ch}_*(X)).$$

Hence for an element $\alpha \in \mathrm{Ch}_*(X)$, we have

$$c_1(\mathcal{O}(1))^{n+1}(\pi^*(\alpha)) = \sum_{i=0}^{n}(-1)^i c_1(\mathcal{O}(1))^{n-i}(\pi^*(\alpha_{i+1}))$$

for unique $\alpha_1, \cdots, \alpha_{n+1} \in \mathrm{Ch}_*(X)$.

Definition 8.4.1. *For each p and $1 \le i \le n+1$, we define the map*

$$c_i(\mathcal{E}) : \mathrm{Ch}_p(X) \to \mathrm{Ch}_{p-i}(X)$$

by $\alpha \mapsto \alpha_i$, and define

$$c(\mathcal{E})[t] := t^{n+1} + \sum_{i=0}^{n}(-1)^{i+1} c_i(\mathcal{E}) t^{n-i} \in \mathrm{End}(\mathrm{Ch}_*(X))[t].$$

We have $c(\mathcal{E})[c_1(\mathcal{O}(1))] = 0$, namely, $c(\mathcal{E})[t]$ is the characteristic polynomial of $c_1(\mathcal{O}(1))$ acting on $\mathrm{Ch}_*(\mathbb{P}(\mathcal{E}))$.

Definition 8.4.2. *Let $\mathcal{E}$ be a vector bundle on a noetherian scheme X of rank $n+1$. We can define the* flag variety $\mathrm{Flag}(\mathcal{E})$ *which classifies the filtrations of $\mathcal{E}$*

$$0 = \mathcal{E}_n \subset \mathcal{E}_{n-1} \subset \cdots \subset \mathcal{E}_1 \subset \mathcal{E}_0 = \mathcal{E}$$

such that $\mathcal{E}_i/\mathcal{E}_{i+1}$ are line bundles.

Remark 8.4.3. The flag variety is constructed by the following way:

1. First construct the projective bundle $\mathbb{P}(\mathcal{E}) \xrightarrow{\pi} X$, then there is a surjective morphism of sheaves
$$\pi^*\mathcal{E} \twoheadrightarrow \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$$
on $\mathbb{P}(\mathcal{E})$, whose kernel is a vector bundle $\mathcal{E}_1$ of rank n, then we get a projective space $\mathbb{P}(\mathcal{E}_1)$;

2. repeat this process on its kernel, then the rank of the bundle decreases, finally we obtain
$$\mathrm{Flag}(\mathcal{E}) \to \cdots \to \mathbb{P}(\mathcal{E}_2) \to \mathbb{P}(\mathcal{E}_1) \to X.$$
Note that $\mathrm{Flag}(\mathcal{E}) \to X$ is a flat morphism.

bundle ch2 **Theorem 8.4.4.** *Let $\mathcal{E}$ be a vector bundle on a noetherian scheme X. If*
$$0 = \mathcal{E}_n \subset \mathcal{E}_{n-1} \subset \cdots \subset \mathcal{E}_0 = \mathcal{E}$$
is a filtration of $\mathcal{E}$ such that $\mathcal{E}_i/\mathcal{E}_{i+1}$ is line bundle for every i, then
$$c(\mathcal{E})[t] = \prod_{i=0}^{n-1}(c(\mathcal{E}_i/\mathcal{E}_{i+1})[t]) = \prod_{i=0}^{n-1}(t - c_1(\mathcal{E}_i/\mathcal{E}_{i+1})).$$

Proof. The above equation is equivalent to
$$\prod_{i=0}^{n-1}(c(\mathcal{E}_i/\mathcal{E}_{i+1})^{\otimes(-1)} \otimes \mathcal{O}(1)) = 0.$$
We prove this equiation by induction on n.

Consider the map $\phi : \mathcal{E}^{n-1} \to \mathcal{O}(1)$, for any cycle α in $\mathcal{P} = \mathbb{P}(\mathcal{E}/\mathcal{E}_{n-1})$,
$$c_1(\mathcal{O}(1) \otimes (\mathcal{E}_{n-1})^{\otimes(-1)})(\alpha)$$
represents an element β in $\mathrm{Ch}_*(\mathcal{P})$. By induction we have
$$\prod_{i=0}^{n-1}(c(\mathcal{E}_i/\mathcal{E}_{i+1})^{\otimes(-1)} \otimes \mathcal{O}(1))(\alpha) = \prod_{i=0}^{n-2}(c(\mathcal{E}_i/\mathcal{E}_{i+1})^{\otimes(-1)} \otimes \mathcal{O}(1))|_{\mathcal{P}}(\beta) = 0.$$
□

bundle ch1 **Corollary 8.4.5.** *If $0 \to \mathcal{E}_1 \to \mathcal{E}_2 \to \mathcal{E}_3 \to 0$ is a short exact sequence of vector bundles on X, then*
$$c(\mathcal{E}_2)[t] = c(\mathcal{E}_1)[t] \cdot c(\mathcal{E}_3)[t].$$

Proof. Let $\mathcal{E}$ be a vector bundle of rank $n+1$ on a noetherian scheme X, the construction of flag variety implies that $\pi^*\mathcal{E}$ to $\mathrm{Flag}(\mathcal{E})$ has universal filtration. If we have a short exact sequence
$$0 \to \mathcal{E}_1 \to \mathcal{E}_2 \to \mathcal{E}_3 \to 0,$$

we consider the flat morphism

$$f : \mathrm{Flag}(\mathcal{E}_1) \times \mathrm{Flag}(\mathcal{E}_3) \to X,$$

which induces

$$f^* : \mathrm{Ch}_*(X) \hookrightarrow \mathrm{Ch}_*(\mathrm{Flag}(\mathcal{E}_1) \times \mathrm{Flag}(\mathcal{E}_3)),$$

then the pullback of $\mathcal{E}_1$ and $\mathcal{E}_3$ have filtrations, hence the pullback of $\mathcal{E}_2$ also have filtration. This technique is called "splitting principle". □

Remark 8.4.6. Let X be a noetherian scheme over a field k. From the above theorem we know that c is a group homomorphism from $K^0(X)$ to $\mathrm{End}(\mathrm{Ch}(X))[t]$ by $\mathcal{E} \mapsto c(\mathcal{E})[t]$.

Remark 8.4.7. In particular, using the splitting principle to reduce to the case that the bundles have complete filtrations. So in $K(X)$, $\mathcal{E} = \bigoplus_{i=1}^n \mathcal{L}_i$ is a direct sum of line bundles, then $c(\mathcal{E})[t] = \prod_{i=1}^n (t - c_1(\mathcal{L}_i))$, hence by simple calculation

$$c_d(\mathcal{E}) = \sum_{1 \le i_1 < \cdots < i_d \le n} \prod_{j=1}^d c_1(\mathcal{L}_{i_j}).$$

8.5 Chern character

Let X be a regular scheme. We define the *chern class* as a ring homomorphism

$$\mathrm{ch} : K^0(X) \to \mathrm{End}(\mathrm{Ch}_*(X)) \otimes \mathbb{Q},$$

which is contravariant in X, such that if $\mathcal{E}$ has a filtration

$$0 = \mathcal{E}_n \subset \mathcal{E}_{n-1} \subset \cdots \subset \mathcal{E}_0 = \mathcal{E}$$

of line bundles,

$$\mathrm{ch}(\mathcal{E}) := \sum_{i=0}^{n-1} \exp(c_1(\mathcal{E}_i/\mathcal{E}_{i+1})) = \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{i=0}^{n-1} c_i(\mathcal{L}_i)^k$$

with coefficients in $\mathbb{Q}$.

Note that the right hand side is symmetric in i, hence the chern class $\mathrm{ch}(\mathcal{E})$ can be expressed by $c_d(\mathcal{E})$ for $d \ge 1$.

Theorem 8.5.1. *If X is regular and is of finite type over a field or $\mathbb{Z}$, then there is an isomorphism*

$$K^0(X) \otimes \mathbb{Q} \xrightarrow[\cong]{\mathrm{ch}} \mathrm{Ch}_*(X) \otimes \mathbb{Q},$$
$$[\mathcal{F}] \mapsto \mathrm{ch}([\mathcal{F}])([X]).$$

8.6 Todd class

Let $\mathcal{E}$ be a vector bundle on a noetherian regular scheme X such that $[\mathcal{E}] = \sum_{i=1}^{n}[\mathcal{L}_i]$ in $K^0(X)$. We define the *Todd class* to be the map

$$\mathrm{Td} : K^0(X) \to \mathrm{End}(\mathrm{Ch}_*(X)) \otimes \mathbb{Q},$$
$$[\mathcal{E}] \mapsto \prod_{i=1}^{n} \frac{c_1(\mathcal{L}_i)}{1 - \exp(-c_1(\mathcal{L}_i))},$$

Remark 8.6.1. Note that if let $x = c_1(\mathcal{L}_i)$ we have

$$\frac{x}{1 - e^{-x}} = 1 + \frac{1}{2}x + \sum_{k=1}^{\infty}(-1)^{k-1}\frac{\mathcal{B}_k}{(2k)!}x^{2k},$$

where $\mathcal{B}_k$ is the Bernoulli number.

Remark 8.6.2. If we have an exact sequence $0 \to \mathcal{E}_1 \to \mathcal{E}_2 \to \mathcal{E}_3 \to 0$, then

$$\mathrm{Td}(\mathcal{E}_2) = \mathrm{Td}(\mathcal{E}_1)\mathrm{Td}(\mathcal{E}_3).$$

8.7 Grothendieck-Riemann-Roch theorem

In this section we will state the Grothendieck -Riemann-Roch theorem(namely GRR), and then reduce GRR to the case for projective space. Finally we finish the proof for general cases.

Definition 8.7.1. *Let $f : X \to Y$ be a flat morphism of schemes of finite type over k. Then we have a $\mathcal{O}_X$-module $\Omega_{X/Y}$ of relative differentials. We say f is a smooth morphism if $\Omega_{X/Y}$ is locally free of rank* $\dim X - \dim Y$.

Definition 8.7.2. *Let $f : X \to Y$ be a morphism of schemes. We define the* relative tangent bundle *to be the dual space of the relative differential form, namely*

$$T_{X/Y} = (\Omega^1_{X/Y})^\vee.$$

Proposition 8.7.3. *If both of X and Y are regular varieties over a field k, let $f : X \to Y$ be a smooth morphism, then there is an exact sequence*

$$0 \to T_{X/Y} \to T_X \to f^*T_Y \to 0. \tag{8.7.1}$$

Note that

$$\mathcal{H}om(T_{X/Y}, \mathcal{O}_X) \cong \Omega_{X/Y}.$$

Now we state the Grothendieck-Riemann-Roch theorem:

GRR **Theorem 8.7.4** (Grothendieck-Riemann-Roch)**.** *Let $f : X \to Y$ be a smooth projective morphism of regular schemes. We have induced map $f_* : \mathrm{Ch}_*(X) \otimes \mathbb{Q} \to \mathrm{Ch}_*(Y) \otimes \mathbb{Q}$. Then for any $\alpha \in K^0(X)$,*

$$f_*(\mathrm{ch}(\alpha)\mathrm{Td}(T_{X/Y})) = \mathrm{ch}(f_*(\alpha)).$$

Example 8.7.5. Let X be a smooth projective curve over a field k. Let $\mathcal{E}$ be a vector bundle on X. Suppose that $f : X \to Y = \mathrm{Spec}(k)$ is a smooth morphism. Then we have

$$\mathrm{ch}(\mathcal{E}) = \mathrm{rank}\mathcal{E} + c_1(\mathcal{E})$$

since

$$c_i(\mathcal{L}) = 0 \qquad \text{for } i > 1, \mathcal{L} \text{ is line bundle}$$

and

$$\mathrm{Td}(T_{X/Y}) = 1 + c_1(T_{X/Y})/2 = 1 - c_1(\Omega^1_{X/k})/2.$$

Hence by applying Hurwitz formula

$$\begin{aligned} f_*(\mathrm{ch}(\mathcal{E})\mathrm{Td}(T_{X/Y})) &= f_*([\mathrm{rank}\mathcal{E} + c_1(\mathcal{E})] \cdot [1 - c_1(\Omega^1_{X/k})/2]) && (8.7.2)\\ &= -\frac{\mathrm{rank}\mathcal{E}}{2} \deg \Omega^1_{X/k} + \deg \mathcal{E} && (8.7.3)\\ &= (1 - g(X))\mathrm{rank}\mathcal{E} + \deg \mathcal{E}, && (8.7.4) \end{aligned}$$

we can calculate the chow group of a point:

$$\mathrm{Ch}_*(\mathrm{Spec}k) = \mathrm{Ch}_0(\mathrm{Spec}k) = \mathbb{Z}[\mathrm{Spec}k] \xrightarrow{\sim} \mathbb{Z}.$$

Then on the left hand side we have

$$\mathrm{ch}(f_*\mathcal{E}) = \dim_k H^0(X, \mathcal{E}) - \dim_k H^1(X, \mathcal{E}),$$

which recovers Riemann-Roch theorem on curves.

Remark 8.7.6. From the exact sequence tangent seq 8.7.1 we have

$$\mathrm{Td}(T_X) = f^*\mathrm{Td}(T_Y) \cdot \mathrm{Td}(T_{X/Y}),$$

therefore the Grothendieck-Riemann-Roch theorem can be rewritten as

$$f_*(\mathrm{ch}(\alpha)\mathrm{Td}(T_X)) = \mathrm{ch}(f_*\alpha) \cdot \mathrm{Td}(T_Y),$$

which is more symmetric and is Grothendieck's original formulation. This also means the following diagram commutes:

$$\begin{array}{ccccc} K^0(X) \otimes \mathbb{Q} & \xrightarrow[\cong]{\mathrm{ch}} & \mathrm{Ch}_*(X) \otimes \mathbb{Q} & \xrightarrow{\mathrm{Td}(T_X)} & \mathrm{Ch}_*(X) \otimes \mathbb{Q} \\ \downarrow f_* & & & & \downarrow f_* \\ K^0(Y) \otimes \mathbb{Q} & \xrightarrow[\cong]{\mathrm{ch}} & \mathrm{Ch}_*(Y) \otimes \mathbb{Q} & \xrightarrow{\mathrm{Td}(T_Y)} & \mathrm{Ch}_*(Y) \otimes \mathbb{Q}. \end{array}$$

Remark 8.7.7. When X is a smooth projective variety over k and $Y = \mathrm{Spec}(k)$, this is Hirzebruch-Riemann-Roch theorem.

Lemma 8.7.8. *Given $f : X \to Y$ and $g : Y \to Z$ smooth morphism between regular schemes, if Grothendieck-Riemann-Roch theorem holds for f and g, then it also holds for $g \circ f$.*

Proof. Since

$$f_*(\mathrm{ch}(\alpha)\mathrm{Td}(T_X)) = \mathrm{ch}(f_*\alpha) \cdot \mathrm{Td}(T_Y),$$

then apply GRR for g on this equation again we get

$$g_* f_*(\mathrm{ch}(\alpha)\mathrm{Td}(T_X)) = \mathrm{ch}(g_* f_*(\alpha)) \cdot \mathrm{Td}(T_Z).$$

Since $(g \circ f)_* = g_* f_*$, hence we have GRR for $g \circ f$. □

Now we introduce important techniques to prove the theorem 8.7.4.

GRR for projective line bundles

Theorem 8.7.9. *Let X be a regular noetherian scheme over a field k. Let $\mathcal{E}$ be a vector bundle on X of rank 2. Then GRR is valid for $f : X = \mathbb{P}(\mathcal{E}) \to Y$.*

Proof. by theorem 8.1.12 we have $K_0(X) \otimes \mathbb{Q}$ is generated by $\mathcal{O}_X$ and $\mathcal{O}(-1)$ as a module over $K_0(Y) \otimes \mathbb{Q}$. We have $f_*\mathcal{P}_X = \mathcal{O}_Y$, and all other higher direct image vanish. Note that in $K^0(X)$, $\mathcal{E} = \mathcal{L}_1 \oplus \mathcal{L}_2$ is a direct sum of line bundles.

Let $h = c_1(\mathcal{O}(1)) - \frac{1}{2}c_1(\mathcal{E})$. Since we have the exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{E}^\vee \otimes \mathcal{O}(1) \to T_{X/Y} \to 0$$

and $c_1(\mathcal{E}) = c_1(\mathcal{L}_1) + c_1(\mathcal{L}_2)$, by simple calculation we get $TX/Y \cong \det(\mathcal{E})^{-1} \otimes \mathcal{O}(2)$ and $c_1(T_{X/Y}) = 2h$. Since

$$h = \frac{h}{1 - e^{-2h}} - \frac{-h}{1 - e^{2h}},$$

so

$$\mathrm{Td}(T_{X/Y}) = \frac{2h}{1 - e^{-2h}} = 1 + h + \text{even powers of } h.$$

Since $h^2 = c_2(\mathcal{E}) + \frac{1}{4}c_1(\mathcal{E})^2$, so $f_* h^{2n} = 0$ for any non-negative integer n and $f_* h = 1$. Then finally

$$f_*(\mathrm{Td}(T_{X/Y})) = 1.$$

Since $\frac{h}{1-e^{-2h}}$ contains only even powers of h, we have

$$f_*[(\mathcal{O}(-1))] = 0 = f_*(\mathrm{Ch}(\mathcal{O}(-1)) \cdot \mathrm{Td}(T_{X/Y}).$$

□

Idea of proof of Theorem 8.7.4. Utilizing normal cone construction, we can reduce it to the case that $X = \mathbb{P}(\mathcal{E})$ with $\mathcal{E}$ a vector bundle on Y of rank 2, hence $f : X \to Y$ is of relative dimension 1. In this case we have the isomorphisms induced by the projective bundle $X \xrightarrow{\pi} Y$:

$$K^0(X) \xrightarrow{\sim} K^0(Y) + K^0(Y)[\mathcal{O}(-1)],$$
$$\mathrm{Ch}_*(X) \xrightarrow{\sim} \mathrm{Ch}_*(Y) + c_1(\mathcal{O}(-1))\mathrm{Ch}_*(Y).$$

The X is the moduli space of quotient bundles of $\mathcal{E}$ of rank 1, and the tangent space $T_{X/Y}$ classifies the deformation of $0 \to \mathcal{E}_1 \to \mathcal{E} \to \mathcal{E}/\mathcal{E}_1 \to 0$. We have

$$0 \to \mathcal{O}_X \to f^*\mathcal{E}^\vee(1) \to T_{X/Y} \to 0,$$

hence

$$T_{X/Y} = (f^* \det \mathcal{E})^\vee(2).$$

Now the Grothendieck-Riemann-Roch theorem can be checked in this case by explicit computation. □

A Appendix

A.1 Category theory

First we review some basic in category theory.

A *category* $\mathcal{C}$ contains a collection of objects $\mathrm{Ob}(\mathcal{C})$ and a set of morphisms $\mathrm{Hom}(X, Y)$ for every pair of objects (X, Y) of $\mathcal{C}$, and it has two basic properties: the ability to compose the morphisms associatively, and the existence of an identity arrow for each object. A morphism $f : X \to Y$ in $\mathcal{C}$ is called an isomorphism if there exsits a morphism $g : Y \to X$ such that $gf = \mathrm{Id}_X$ and $fg = Id_Y$. The morphism g is unique and is called the inverse of f, denoted by f^{-1}.

Example A.1.1. Here are two basic examples:

1. $\mathcal{S}et$: the category of sets. The objects are sets, and morphisms are maps between sets.

2. $\mathcal{A}b$: the category of abelian groups. The objects are abelian groups, and morphisms are homomorphisms between abelian groups.

Given two category $\mathcal{C}$ and $\mathcal{D}$, we can define the product category $\mathcal{C} \times \mathcal{D}$ in the obvious way, and a category $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ of follows.

A *functor* $\mathcal{F} : \mathcal{C} \longrightarrow \mathcal{D}$ is a pair of assignments on objects

$$\mathrm{Ob}(\mathcal{C}) \longrightarrow \mathrm{Ob}(\mathcal{D}), \qquad X \mapsto \mathcal{F}(X) \in \mathrm{Ob}\mathcal{D}$$

and on sets of morphisms $\mathcal{F}_{X,Y}$ for two objects X, Y of $\mathcal{C}$:

$$\mathcal{F}_{X,Y} : \mathrm{Hom}(X, Y) \longrightarrow \mathrm{Hom}(\mathcal{F}(X), \mathcal{F}(Y)), \qquad f \mapsto \mathcal{F}(f)$$

such that for three objects X, Y, Z of $\mathcal{C}$

1. $\mathcal{F}(\mathrm{id}_X) = \mathrm{id}_{\mathcal{F}(X)}$ for every object X in $\mathcal{C}$,

2. $\mathcal{F}(g \circ f) = \mathcal{F}(g) \circ \mathcal{F}(f)$ for all morphisms $f : X \to Y$ and $g : Y \to Z$ in $\mathcal{C}$

The functor $\mathcal{F}$ is called *faithful* (resp.*full*, resp. *fully faithful*) , if $\mathcal{F}_{X,Y}$ are all injective (resp. surjective, resp. bijective).

For two functors $\mathcal{F}, \mathcal{G}$ from $\mathcal{C}$ to $\mathcal{D}$, a *natural transformation* $T(X) \in \mathrm{Hom}(\mathcal{F}(X), \mathcal{G}(X))$ is collection for objects X in $\mathcal{C}$ so that the following diagram is commutative for any $\varphi \in \mathrm{Hom}(X, Y)$ of two objects in $\mathcal{C}$:

$$\begin{array}{ccc} \mathcal{F}(X) & \xrightarrow{T(X)} & \mathcal{G}(X) \\ \downarrow{\scriptstyle \mathcal{F}(\varphi)} & & \downarrow{\scriptstyle \mathcal{G}(\varphi)} \\ \mathcal{F}(Y) & \xrightarrow{T(Y)} & \mathcal{G}(Y) \end{array} .$$

All natural transforms from $\mathcal{F}$ and $\mathcal{G}$ form a set $\mathrm{Hom}(\mathcal{F}, \mathcal{G})$.

All functors from $\mathcal{C}$ to $\mathcal{D}$ form a category $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ with homomorohisms given by natural transformations.

If $\mathcal{E}$ is a third category, then we can define the composition of functors

$$\circ : \mathrm{Fun}(\mathcal{D}, \mathcal{E}) \times \mathrm{Fun}(\mathcal{C}, \mathcal{D}) \longrightarrow \mathrm{Fun}(\mathcal{D}, \mathcal{E}) : \qquad \mathcal{G} \times \mathcal{F} \mapsto \mathcal{G} \circ \mathcal{F}$$

in an obvious way. A functor $\mathcal{F} \in \mathrm{Fun}(\mathcal{C}, \mathcal{D})$ is called an *equivalence functor* if there is a functor $\mathcal{G} \in \mathrm{Hom}(\mathcal{D}, \mathcal{C})$ such that the compositions

$$\mathcal{G} \circ \mathcal{F} \in \mathrm{Fun}(\mathcal{C}, \mathcal{C}), \qquad \mathcal{F} \circ \mathcal{G} \in \mathrm{Fun}(\mathcal{D}, \mathcal{D})$$

are isomorphic to the identity functors on $\mathcal{C}$ and $\mathcal{D}$ respectively. The $\mathcal{F}$ and $\mathcal{G}$ are inverse to each other.

The *opposite category* of a category $\mathcal{C}$, denoted by $\mathcal{C}^{op}$ is defined by $\mathrm{Ob}(\mathcal{C}^{op}) = \mathrm{Ob}(\mathcal{C})$ and $\mathrm{Hom}_{\mathcal{C}^{op}}(X, Y) = \mathrm{Hom}_{\mathcal{C}}(Y, X)$. A *cotravariant functor* from category $\mathcal{C}$ to $\mathcal{D}$ is a functor $\mathcal{C}^{op} \to \mathcal{D}$.

A.2 Homological algebra

Abelian category

By an additive category , we mean a category $\mathcal{C}$ with the following properties:

1. For $X, Y \in \mathrm{Ob}(X)$, $\mathrm{Hom}(X, Y)$ has an abelian group structure such that the composition law is bilinear.

2. There is a zero object O in $\mathcal{C}$ such that both $\mathrm{Hom}(X, 0)$ and $\mathrm{Hom}(0, X)$ are both zero group.

3. For any two object X, Y, we have a direct sum $X \oplus Y$ with projections and embeddings

$$p_X \in \mathrm{Hom}(X \oplus Y, X), \quad p_Y \in \mathrm{Hom}(X \oplus Y, Y)$$

which induces an isomorphism of abelian groups for any $Z \in \mathrm{Ob}\mathcal{C}$:

$$\mathrm{Hom}(Z, X \oplus Y) = \mathrm{Hom}(Z, X) \oplus \mathrm{Hom}(Z, Y) : \qquad \varphi \mapsto (p_X \circ \varphi, p_Y \circ \varphi).$$

A functor $\mathcal{F}:\mathcal{C}\longrightarrow\mathcal{D}$ between two additive categories is called an *additive functor* , if it preserve the zero element, fiinite direct sums, and such that for any $X,Y\in \mathrm{Ob}(\mathcal{C})$, the map

$$\mathcal{F}_{X,Y}:\mathrm{Hom}(X,Y)\longrightarrow\mathrm{Hom}(\mathcal{F}(X),\mathcal{F}(Y))$$

is a group homomorphism.

An additive category $\mathcal{C}$ is called an *abelian category* if for any morphism $f:X\longrightarrow Y$ between objects X and Y, there are kernel, kernel, and images as follows.

By the *kernel* of f, we mean a morphism $\alpha:\mathrm{Ker}(f)\longrightarrow X$ such that for any object Z of $\mathcal{C}$ the induced morphism $\mathrm{Hom}(Z,\mathrm{Ker}(f))\longrightarrow\mathrm{Hom}(Z,X)$ is the kernel of the homomorphism of $\mathrm{Hom}(Z,X)\longrightarrow\mathrm{Hom}(Z,Y)$ of abelian groups.

By the *cokernel* of f, we mean a morphism $\beta:Y\longrightarrow\mathrm{coker}(f)$ such that for any object Z of $\mathcal{C}$, $\mathrm{Hom}(\mathrm{coker}(f),Z)\longrightarrow\mathrm{Hom}(Y,Z)$ is the kernel of $\mathrm{Hom}(Y,Z)\longrightarrow\mathrm{Hom}(X,Z)$.

If both kernel and cokernel exits, then there is unique morphism

$$\gamma:\mathrm{coker}(\alpha)\longrightarrow\ker(\beta)$$

such that f is the composition

$$X\longrightarrow\mathrm{coker}(\alpha)\stackrel{\gamma}{\longrightarrow}\ker(\beta)\longrightarrow Y.$$

An additive category $\mathcal{C}$ is called *abelian categpry*, if the following two conditions hold:

1. for any morphism $f:X\longrightarrow Y$, the kernel $\alpha:\ker(f)\longrightarrow X$ and cokernel $\beta:Y\longrightarrow\mathrm{coker}(f)$ exit.
2. the induced morphism γ is an isomorphism.

The $\ker(\beta)$ is called the *image* of f.

Example A.2.1. Let $\mathcal{V}ect_{\mathbb{R}}$ be the category of $\mathbb{R}$-vector spaces, and let V_1, V_2 be two objects inside it. Then $\mathrm{Hom}_{\mathbb{R}}(V_1,V_2)$ is the group of $\mathbb{R}$-linear maps from V_1 to V_2 as we have studied in linear algebra. Moreover, one can see that

$$\mathrm{Hom}_{\mathbb{R}}(\mathbb{R}^n,\mathbb{R}^m)\cong M_{n\times m}(\mathbb{R}) \tag{A.2.1}$$

A.3 Commutative algebra

Integral dependence

gral closed

Theorem A.3.1. *[Finiteness of Integral Closure] Let k be a field. Let A be an integral domain which is finitely generated k-algebra. Let L be a finite algebraic extension of the fractional field of A. Then the integral closure $\tilde{A}$ of A in L is a finitely generated A-module, and is also a finitely generated k-algebra.*

Proof. Zariski-Samuel [1, vol. 1, Ch. V., Thm. 9, p. 267.] □

Preliminary field theory

Definition A.3.2. *Let k be a field. A polynomial $f(x) \in k[x]$ is called separable if it has no multiple root. In other words, it is coprime to its formal derivative $f'(x)$.*

Definition A.3.3. *Let k be a field. An algebraic extension $L \supseteq k$ is called separable if for every $x \in L$, the minimal polynomial $f(X) \in k[X]$ of x is separable. Otherwise, it is said to be inseparable. It is called purely inseparable if $f(X)$ has only one distinct root on k*

Remark A.3.4. By definition, it is easy to see that the following are equivalent:

(1) L/k is purely inseparable.

(2) If $x \in L$ is separable over k, then $x \in k$.

(3) Suppose that k has characteristic p, where p is a prime. Then the minimal polynomial $f(X)$ of any element $\alpha \in L$ has the form $X^{p^n} - a$ for some positive integer n and $a \in k$.

Example A.3.5. The extension $\mathbb{F}_p(x) \hookrightarrow \mathbb{F}_p(x^{1/p})$ is purely inseparable.

Exercise A.3.6. If L/k is finite separable field extension of degree n, then $L \otimes_k \overline{k} = \overline{k}^n$.

Proposition A.3.7. *Separability is stable under base change.*

Definition A.3.8. *Let k be a field and $\overline{k}$ an algebraic closure of k. The separable closure k_s of k is defined by*

$$k_s = \left\{x \in \overline{k} \;\middle|\; x \text{ is separable over } k\right\}$$

Lemma A.3.9. *Let L/k be an algebraic field extension. Then there is a unique intermediate field*

$$k_{sep} = \{x \in L \mid x \text{ is separable over } k\}$$

such that k_{sep} is separable over k and L purely inseparable over k_{sep}.

From the lemma we know that an arbitrary algebraic extension L/k can be decoposed into a separable extension followed by a purely inseparable one. Then we can define the degree $[L : k_{sep}]$ to be the inseparable degree and $[k_{sep} : k]$ to be the separable degree.

Definition A.3.10. *A field k is called perfect if any algebraic extension of k is separable.*

Remark A.3.11. A field which has characteristic zero is perfect. But pefect does not imply characteristic zero. For example, the finite field $\mathbb{F}_p$ is pefect, but has characteristic p, where p is a prime.

Lemma A.3.12. *Let k be any field, then there exists a purely inseparable extension k_p which is unique up to isomorphism such that k_p is perfect. We call k_p the perfect closure of k.*

Proof. Suppose that $\mathrm{chara}(k) = 0$. Then k is perfect, $k_p = k$ is unique.

If $\mathrm{chara}(k) = p > 0$. For every positive integer n, consider $k^{\frac{1}{p^n}} = k(x^{\frac{1}{p^n}}, x \in k)$ where n is a positive integer. Claim that the extension $k^{\frac{1}{p^n}} \supseteq k$ is purely inseparable. Indeed, the p^n-th power of every element $x \in k^{\frac{1}{p^n}}$ lies in k and for any $a \in k$, its p-th root $a^{1/p^n} \in k^{\frac{1}{p^n}}$. Thus the minial polynomial of $x \in k^{\frac{1}{p^n}}$ has the form $X^{p^m} - \alpha$ for some $m \leqslant n$ and $\alpha \in k$. Since we have

$$\begin{array}{ccc} k^{\frac{1}{p^n}} & \longrightarrow & k^{\frac{1}{p^{n+1}}} \\ x & \longmapsto & x^p \end{array}$$

So just take $k_p = \bigcup_{n>0} k^{\frac{1}{p^n}}$ □

Remark A.3.13. By definition, if k is a perfect field then $k_s = \overline{k}$. In particular, if k has charateristic zero then $k_s = \overline{k}$.

Remark A.3.14. Let k be a field. From the discussion above we can conclude two chains of field extensions:

1. $k \hookrightarrow k_s \hookrightarrow \overline{k}$, where $k \hookrightarrow k_s$ is separable and $k_s \hookrightarrow \overline{k}$ is purely inseparable.
2. $k \hookrightarrow k_p \hookrightarrow \overline{k}$, where $k \hookrightarrow k_p$ is purely inseparable and $k_p \hookrightarrow \overline{k}$ is separable.
3. $\overline{k} = k_s \otimes_k k_p$.

Recall that a field extension L/k is separable if and only if for any $\alpha \in L$, let $f(x)$ be the minimal polynomial of α over k, then $f'(\alpha) \neq 0$.

Definition A.3.15. *Let K be Galois extension of k. Denote the Galois group of K/k by $\mathrm{Gal}(K/k)$. For any $\alpha \in L$, define the trace of α to be the sum of Galois conjugates of α:*

$$Tr_{L/k}(\alpha) = \sum_{\sigma \in \mathrm{Gal}(K/k)} \sigma(\alpha)$$

Define a pairing:

$$\begin{array}{rccc} (\quad,\quad): & L \times L & \longrightarrow & k \\ & (x,y) & \longmapsto & Tr_{L/k}(xy) \end{array}.$$

It is called the trace form of L/k.

Proposition A.3.16. *For any $\alpha \in L$, $Tr_{L/k}(\alpha) \in k$ and the trace is an additive map.*

Proof. Let K be a Galois extension of k containing L, Let $G = \mathrm{Gal}(K/k)$. Then there is a subgroup $H \subset G$ corresponding to L. Let $\alpha \in L$. The sum $\sum_\sigma \sigma(\alpha)$ runs over representatives σ for the left cosets of H in G. And $\sigma : L \hookrightarrow K$ is the embeddings of L in K by the fundamental theorem of Galois theory. Note that $\sigma(\alpha)$ depends only on the left H-cosets of σ: Since H fix L it follows that if $\sigma H = \sigma' H$, then $\sigma' \sigma(\alpha) = h(\alpha) = \alpha$ for some $h \in H$. Thus $\sigma(\alpha) = \sigma'(\alpha)$

From above we know that the action of G by left multiplication permutes the left H-cosets. For all $\sigma \in G$, $\sigma(\mathrm{Tr}(\alpha)) = \mathrm{Tr}(\alpha)$. Since the elements that fixed by any $\sigma \in \mathrm{Aut}(K/k)$ all belong to k, we have $\mathrm{Tr}(\alpha) \in k$.

By definition of trace,

$$\mathrm{Tr}(\alpha+\beta) = \sum_{\sigma} \sigma(\alpha+\beta) = \sum_{\sigma} \sigma(\alpha) + \sigma(\beta) \tag{A.3.1}$$

$$= \sum_{\sigma} \sigma(\alpha) + \sum_{\sigma} \sigma(\beta) = \mathrm{Tr}(\alpha) + \mathrm{Tr}(\beta) \tag{A.3.2}$$

□

Lemma A.3.17. *Let L/k be a finite extension of fields, and $[L:k] < \infty$. The extension L/k is separable if the trace form is non-degenerate, i.e. the pair $(x,y) = 0$ for all y implies $x = 0$.*

Proof. Assume the trace form is non-degenerate. If $\alpha \in L$ is separable, Let f be the minimal polynomial of α over k. then $f' \neq 0$. Consider $f = (x-a_1)\cdots(x-a_n)$, where a_i are the roots of f in $\overline{k}$.

$$\frac{f'}{f} = \frac{1}{x-a_1} + \cdots + \frac{1}{x-a_n} \tag{A.3.3}$$

$$= \frac{1}{x} \sum_{r=0}^{\infty} \left(\frac{a_1^r}{x^r} + \cdots + \frac{a_n^r}{x^r}\right) \tag{A.3.4}$$

$$= \frac{1}{x} \sum_{r=0}^{\infty} \frac{\mathrm{Tr}_{k(\alpha)/k}(\alpha^r)}{x^r}. \tag{A.3.5}$$

So $f' \neq 0$ if and only if $\mathrm{Tr}_{k(\alpha)/k}(\alpha^r) \neq 0$ for every integer r. Now suppose that the trace form is non-degenerate, if L/k is not separable, choose an inseparable element $\alpha \in L$, then we have $\mathrm{Tr}_{L/k} = \mathrm{Tr}_{L/k(\alpha)} \circ \mathrm{Tr}_{k(\alpha)/k} = 0$, contradiction. □

Corollary A.3.18. *If L/k is finite separable field extension then $Tr_{L/k}(L) = k$.*

Exercise A.3.19. Let L/k be a finite extension of fields, and $[L:k] < \infty$. Then

1. $\mathrm{chara}(L) = p > 0$ where p is a prime.

2. $p | [L:k]$.

Theorem A.3.20. *Let K/k be a finitely generated extension of a perfect field k. Then there exists a transcendence basis T such that the finite extension $K/k(T)$ is separable. In this situation, we call the transcendence basis T the separating transcendence basis.*

Proof. Dummit and Foote[Ch. 14.9]. □

Remark A.3.21. From the discussion above we can conclude that any field extension K/k can be decomposed into a purely transcendental extension $k(T)$ of k followed by a separable extension K_1 of $k(T)$ followed by a purely inseparable extension K/K_1. But K_1 is depended on the choice of the transcendence base T.

A.4 Dedekind domain

Local rings

We begin with the local rings of dimension 1.

Definition A.4.1. *Let R be a noetherian local ring and m be its maximal ideal with residue field $k = R/m$. We say R is a* regular local ring *if*

$$\dim_k {}^{m}\!/_{m^2} = \dim R.$$

Our first main result about a regular local ring is the following:

egular-prin **Proposition A.4.2.** *An Noetherian local ring A is regular if and only if its maximal ideal m is principle. Moreover if π is a generator of m.*

To prove this theorem, we first need the following

Lemma A.4.3 (Nakayama's lemma)**.** *Let A be a local ring with maximal ideal m. Let M be a finitely generated A-module such that $mM = M$. Then $M = 0$.*

Proof. Let x_i $(i = 1, \cdots, n)$ be some generates of M over A. Since $mM = M$, each x_i can be written as a linear combination of x_j's with coefficients in m: thus there are $a_{i,j} \in m$ such that $x_i = \sum_j a_{ij} x_j$. Let A denote the matrix of (a_{ij}). Then

$$(I - A)\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = 0.$$

Multiplying both sides by the adjoint matrix of $(I - A)$, we obtain

$$\det(I - A) x_i = 0, \qquad i = 1, \cdots, n.$$

Now $\det(I - A)$ is of form $1 - a$ with $a \in m$. As m is maximal, $1 - m$ is invertible, then $x_i = 0$ for every i. It means that $M = 0$. ☐

regular-prin

Proof of Theorem A.4.2. Assume A is regular of dimension 1, then $\dim {}^{m}\!/_{m^2} = \dim A = 1$. Let $x \in m$ whose image $\mod m^2$ generates m/m^2. Then $m = Ax + m^2$. So $m/Ax = m \cdot m/Ax$ where we consider m/Ax as an A-module. It follows from Nakayama's lemma that $m/Ax = 0$ which implies $m = Ax$. Thus m is principle.

Conversely, assume that m is principle, then m/m^2 has dimension at most 1 over $k = A/m$. If it is zero, then $m = m^2$. By Nakayama's lemma, $m = 0$. Thus A is a filed and $\dim A = 0$. A contradiction! Thus $\dim_k m/m^2 = 1$ and A is regular. ☐

egular-unif **Proposition A.4.4.** *Let A be a reglar local ring of dimension 1. Let m be its maximal ideal generated by an element π. Then every nonzero element $x \in A$ can be written uniquely in the form $x = u \cdot \pi^n$, where u is an invertible element in A and n is a positive integer.*

We first need the following:

Lemma A.4.5. *Let A be a noetherian local ring and m be its maximal ideal. Then $\bigcap_{n=1}^{\infty} m^n = 0$.*

Proof. Let $N = \bigcap_{n=1}^{\infty} m^n \subset A$. Then N must be finitely generated since A is noetherian. By Nakayama's lemma, $mN = N$ implies $N = 0$. ☐

Proof of Proposition A.4.4. Let a be a nonzero element in A. Then there is an n such that $a \in m^n$, and $a \notin m^{n+1}$. Since A is regular, it follows that m is principle. Suppose $m = (x)$ for $0 \neq x \in A$, since $\bigcap_{n=1}^{\infty} m^n = 0$, then $a = b \cdot x^n$ for some $b \notin m$ (if $b \in m$ then $a \in m^{n+1}$, contraiction), thus b is invertible and we are done.

Now suppose that there is a $c \notin m$ such that $a = c \cdot m^{n'}$ for some integer n'. Then $m^n = m^{n'}$. If $n < n'$, we have a filtration

$$m^n \subset m^{n+1} \subset \cdots \subset m^{n'}. \tag{A.4.1}$$

Thus $m^n = m^{n+1} = m \cdot m^n$, by Nakayama's lemma, $m^n = 0$, contradiction. ☐

By Proposition A.4.4, A is integral and we have a map:

$$\begin{array}{rccc} v: & A - \{0\} & \longrightarrow & \mathbb{Z}_{\geq 0} \\ & x = u\pi^n & \longmapsto & n \end{array} . \tag{A.4.2}$$

Let $K = \mathrm{Frac}(A)$ then we can extend v to K: $v : K \twoheadrightarrow \mathbb{Z}$. This map has the following property

1. $v(xy) = v(x) + v(y)$.
2. $v(x + y) \geq \min\{v(x), v(y)\}$.
3. v is surjective.

Definition A.4.6. *Let F be a field. A map from $F^* = F - \{0\}$ to $\mathbb{Z}$ is called a* discrete valuation *if it saitisfies the above three properties.*

Lemma A.4.7. *If $v : F \to \mathbb{Z}$ is a discrete valuation then*

1. *$R := \{x \in F : v(x) \geq 0\}$ is a local ring;*
2. *$m := \{x \in F : v(x) > 0\}$ is a maximal ideal of R. Moreover m is principle.*

We call R the discrete valuation ring for the valuation v. With all the facts above we can write:

regular-dvr **Corollary A.4.8.** *A local ring of dimension 1 is regular if and only if it is a discrete valuation ring.*

In the following we will give another equivalent condition which is very useful when we compare different regular rings.

Definition A.4.9. *Let A be an integral ring. Let $R \subset A$ be a subring. Let $x \in A$. We say x is integral over R if x saitisfies an equation*

$$x^n + a_1 x^{n-1} + \cdots + a_n = 0, \qquad a_i \in R. \tag{A.4.3}$$

Definition A.4.10. *Let A be an integral ring and $R \subset A$ a subring. Then the set of elements that are integrally over on R forms a subring of A. We call this subring the* integral closure *of R in A.*

Exercise A.4.11. Let A be an integral ring and $R \subset A$ a subring. Show that the element x which is integral over R if and only if the polynomial ring $R[x]$ is an R-module of finite type.

Definition A.4.12. *Let R be an integral ring. We say R is* integrally closed *or normal if R coincides with its integral closure in its field of fractions.*

Here is the main theorem about equivalent different descriptions of regular local ring of dimension one.

Theorem A.4.13. *Let R be an integral local ring of dimension 1. Then the following conditions are equivalent:*

1. *R is regular.*
2. *R is a discrete valuation ring.*
3. *R is integrally closed.*

Proof. (1) We already shown 1 is equivalent to 2 in Corollary regular-dvr A.4.8.

(2) We now show that 2⇒3. Let K be the fraction field of R, suppose $x \in K$ saitisfies an equation

$$x^n + a_1 x^{n-1} + \cdots + a_n = 0, \qquad a_i \in R. \tag{A.4.4}$$

We want to show that $x \in R$, or equivalently, if we write $x = u\pi^n$, where $n \in \mathbb{Z}$, $u \in R^\times$, we need to show that $n \geq 0$.

Assume that $n < 0$, $x = \frac{u}{\pi^{|n|}}$. So:

$$\left(\frac{u}{\pi^{|n|}}\right)^d + a_1\left(\frac{u}{\pi^{|n|}}\right)^{d-1} + \cdots + a_d = 0. \tag{A.4.5}$$

Multiply both sides by $\pi^{|n|d}$ we have $u^d + \pi^{|n|} \cdot a = 0$, where $a \in R$. Now u is a unit, therefore $\pi^{|n|} \cdot a$ is a unit which is impossible.

(3) Now show that 3⇒1. We need only show that the maximal ideal m of R is principle. Let $0 \neq a \in m$, then we know that $R/(a)$ is dimension 0, because $m/(a)$ is the only prime

ideal. Thus it is clear that the nilpotent radical of $R/(a)$ is $m/(a)$, it follows that $m/(a)$ as an ideal of $R/(a)$ is nilpotent.

So we know $(m/(a))^n = 0$ for some integer n or $m^n \subset (a)$. Taking n minimal with this property, i.e. $m^{n-1} \subsetneq (a)$. Let $b \in m^{n-1} - (a)$. Let $x = \frac{a}{b}$. We want to show that $m = (x)$, or $x^{-1}m = R$ as an identity of subsets in K. Notice that

$$x^{-1}m = \frac{b}{a}m \subset m^{n-1} \cdot m/a = m^n/a \subset R \tag{A.4.6}$$

Thus we have two choices:

(a) $x^{-1}m = R$ which implies $m = xR$ and we are done.

(b) $x^{-1}m \subset m$.

If $x^{-1}m \subset m$, we claim that x^{-1} is integral over R. Thus $x^{-1} \subset R$ which is a contradiction since $b/a \in R$ implies $b \in Ra$ which contradicts our choice of a, b. To prove the claim we need the following lemma:

Lemma A.4.14. *Let Q be an endomorphism of a finitely generated module over a ring R. Then Q saitisfies an equation of the form*

$$Q^n + a_1 Q^{n-1} + \cdots + a_n = 0, \qquad a_i \in R. \tag{A.4.7}$$

The lemma above completes our proof. □

Exercise A.4.15. Prove the lemma A.4.14.

Dedekind domain

Definition A.4.16. *An integral noetherian ring R of dimension 1 is called a* Dedekind domain *if R saitisfies one of the following equivalent conditions:*

1. *Let p be the maximal ideal of R, R_p is discrete valuation ring.*

2. *R is integrally closed.*

Exercise A.4.17. Let R be a Noetheriam integral ring of dimension 1. Show that the following conditions are equivalent:

1. R is regular.

2. R is integrally closed.

3. R_p is integrally closed for every maximal ideal p.

Exercise A.4.18. Let R be a regular ring of dimension 1 with fractional field K Define a map from the set of primes p in R to valuations v_p of K using equivalence in Corollary A.4.8. Show this is map is injective with image consisting all discrete valuations v such that $v(R) \subset \mathbb{Z}_{\geq 0}$.

Example A.4.19. Let k be a field, then

$$k[x]_{(x)} = \left\{ \frac{f(x)}{g(x)} \middle| \, g(0) \neq 0 \right\} \tag{A.4.8}$$

is a discrete valuation ring.

Example A.4.20. Let p be a prime ideal in $\mathbb{Z}$, then the localized ring $\mathbb{Z}_p$ is a discrete valuation ring.

One important property using integral closed condition is about construction of Dedekind domains in a separable extension of fields L/K. Recall that a field extension L/K is separated if one of the following three conditions holds:

1. $\mathrm{tr} : L \longrightarrow K$ is surjective;

2. every element $x \in L$ have a minimal polynomial P with $P'(x) \neq 0$.

l-extension **Theorem A.4.21.** *Let L/K be a finite separable field extension. Let A be a Dedekind domain with fractional field K. Let B be the integral closure of A in L. Then B is a Dedekind domain.*

Proof. Let $n = [L : K]$ and $e_1, \cdots, e_n \in L$ be a base of L as K vector spaces. After an multiplication by an element in A, We may assume that all $e_i \in B$. Let $C = \sum A e_i$ be the free module of B of full rank. Now we have a pairing

$$C \otimes_A B \longrightarrow A, \qquad (c, a) \mapsto \mathrm{tr}(ca).$$

This pairing defines an injective morphism of A-modules:

$$B \longrightarrow \mathrm{Hom}(C, A) \simeq A^n.$$

Since A is Noetherian, B is finitely generated A-module. Thus B is a Noetherian ring. □

-extenstion *Exercise* A.4.22. With notation in the above theorem. Define a map from the set of primes q in B to valuations v_q of L using equivalence in Corollary regular-dvr A.4.8. Show this is map is injective with image consisting all discrete valuations v such that $v(A) \subset \mathbb{Z}_{\geq 0}$.

Ideals

Theorem A.4.23 (Unique Factorization Theorem)**.** *Every ideal I in a Dedekind domain A can be decomposed in a unique way into a product of prime ideals.*

Proof. Let p be any prime ideal in A, denote the maximal ideal of the localized ring A_p by m_p. Since A_p is discrete valuation ring, we can suppose that n_p is the maximal integer such that $I_p \subset m_p^{n_p}$. If $n_p > 0$, then $I \subset p$. Consider A/I, the prime ideals in A/I corresponding to the prime ideals in A which contains I. If I is prime thus maximal, then A/I is a field; if I is not prime, then A/I is not integral. Thus $\dim A/I = 0$, A/I is Artinian. It follows that A/I has only finitely many prime ideals containing I. In summary we have proven:

(1) For every prime ideal $p \subset A$, $I_p \subset m_p^{n_p}$, $n_p \geqslant 0$.

(2) $n_p \neq 0$ for only finitely many p.

Let $J = \prod p^{n_p}$. Claim: $I = J$. By local property A.4.32, it is equivalent to $I_p = J_p$ for every p. To prove this, we need the following lemma:

Lemma A.4.24.

(1) If I_1, I_2 are two ideals of A then $(I_1 I_2)_p = (I_1)_p (I_2)_p$.

(2) If p, q are distinct prime ideals, then the discrete valuation of q at p is 0.

Thus $I = J = \prod p^{n_p}$. This proves the theorem.

□

Ideal classes

Definition A.4.25. *Let A be a Noetherian domain and let K be the fractional field of A. A* fractional ideal *of A is a finitely generated A-submodule of K. Thus a fractional ideal can be written as*

$$I = \sum_i a_i A, \qquad a_i = \frac{\alpha_i}{\beta_i}, \qquad \alpha_i, \beta_i \in A. \tag{A.4.9}$$

By taking a denominator $I = \frac{1}{\beta} \sum \alpha_i A$ so equivalently a fractional ideal of A is an A-submodule I of K with the form $I = \gamma J$ where $\gamma \in K$ and $J \subset A$ is an ideal.

Remark A.4.26. Let A be a domain and let K be the fractional field of A. Fractional ideals of A can be defined more generally. In this case they are A-submodules I of K for which there exists an element $\gamma \in A$ such that $\gamma I \subset A$. When A is noetherian this coincides with our definition.

Example A.4.27. Let $I = \{\frac{n}{3} : n \in \mathbb{Z}\}$, then I is a fractional ideal of $\mathbb{Z}$. It is generated by $\frac{1}{3} \in \mathbb{Q}$ as an $\mathbb{Z}$-submodule, and we have $3I \subset \mathbb{Z}$.

Define two operations of two fractional ideals I_1, I_2 of a noetherian domain A as follows:

- $I_1 + I_2 := \{a + b | a \in I_1, b \in I_2\}$.
- $I_1 \cdot I_2 := \{\sum a_i b_i | a_i \in I_1, b_i \in I_2\}$.

Definition A.4.28. *Let A be a noetherian domain. A fractional ideal I is called invertible if there is an unique fractional ideal J such that $I \cdot J = A$.*

Definition A.4.29. *Let A be a noetherian domain. And let $K = \mathrm{Frac}(A)$. Denote the set of invertible fractional ideals of A by $\mathbb{I}_A$. It is called the ideal group of A. A fractional ideal I of A is principle if it is generated by only one element $x \in K$. Denote the set of principal fractional ideals of A by $\mathcal{P}_A$.*

Remark A.4.30. Let A be a noetherian domain. And let $K = \mathrm{Frac}(A)$. Then it is easy to see that $\mathbb{I}_A$ form an abelian group under multiplication. Since every nonzero principle fractional ideal (x) has an inverse $(1/x)$, where $x \in K$, and a product of principle ideals is principle, so $\mathcal{P}_A$ form a subgroup.

Definition A.4.31. *The quotient* $\mathrm{Cl}(A) = \mathbb{I}_A/\mathcal{P}_A$ *is called that ideal class group of* A*, and also called the Picard group of* A *and denoted* $\mathrm{Pic}(A)$*.*

One can show that if A is a Dedekind domain, then acturally all the fractional ideals are invertible. Thus $\mathbb{I}_A$ is the free abelian group generated by its nonzero prime ideals. But first we need some localization theory:

l property **Proposition A.4.32** (Local property)**.** *Let* A *be a commutative ring and* M *a finitely generated* A*-module.*

(1) $M = 0$ *if and only if* $M_p = 0$ *for every prime ideal* p*.*

(2) Let $\varphi : M_1 \to M_2$ *be a homomorphism between* A*-modules of finite type, then* φ *is bijective if and only if* $\varphi_p : (M_1)_p \to (M_2)_p$ *is bijective for every prime ideal* p*.*

(3) Let N_1, N_2 *be finitely generated submodules of* M*, then* $N_1 = N_2$ *if and only if* $(N_1)_p = (N_2)_p$ *for every prime ideal* p*.*

Proof. (2) We have proved that the equivalence between the category of A-modules and the category of quasi-coherent $\mathcal{O}_{\mathrm{Spec}A}$-modules. Then the bijectivity of the morphism of sheaves $\phi : \tilde{M}_1 \to \tilde{M}_2$ is equivalent to the bijectivity of φ. Since $\tilde{M}_p \xrightarrow{\sim} M_p$, and $\tilde{M}_1 \to \tilde{M}_2$ is isomorphism if and only if $\phi_p : \tilde{M}_{1p} \to \tilde{M}_{2p}$ is isomorphism. Thus φ is bijective if and only if φ_p is bijective. The (1) and (3) strictly induced from (2). □

Lemma A.4.33. *Let* A *be a Dedekind domain and* K *the fractional field of* A*. Let* I *be an ideal of* A*. Then there is a fractional ideal* J *of* A *such that* $I \cdot J$ *is principle ideal in* A*.*

Proof. Let J be a fractional ideal defined by $J = \{x \in K : xI \subset A\}$. So $I \cdot J \subset A$. We want to show $I \cdot J = A$, we need only to show that $I_p J_p = A_p$ for every prime ideal p. Note that:

$$J_p = \left\{ \frac{a}{b}, a \in J, b \notin p \right\} = \{x \in K, xI_p \subset A_p\}.$$

Now A_p is discrete valuation ring. If π is a uniformizer of A_p then $I_p = (\pi^n)$ for some integer n. It follows that $J_p = \{x \in K : xI_p \subset A_p\}$. Let u be a unit in A_p. Suppose that $x = u\pi^m \in K$ for some integer m. Then we have $u\pi m + 1 \in A_p$, $m + n \geqslant 0$. Take $m = -n$ so $J_p = (\pi^{-n})$. Thus $I_p J_p = A_p$ and we are done. □

Corollary A.4.34. *If* A *is a discrete valuation ring with uniformizer* π *then its nonzero fractional ideals are the principal fractional ideals* (π^n)*, where* $n \in \mathbb{Z}$*. Then* $\mathbb{I}_A \xrightarrow{\sim} \mathbb{Z}$ *under addition. We have* $\mathcal{P}_A = \mathbb{I}_A$*, thus* $\mathrm{Pic}(A)$ *is a trivial group.*

Theorem A.4.35. *Let A be a Dedekind domain. Then the multiplication makes $\mathbb{I}_A$ a free group over prime ideals with unit A.*

Proof. Since for every prime ideal $p \hookrightarrow A$, A_p is a discrete valuation ring. Suppose each p defines a valuation v_p. Note that if $I, J \in \mathbb{I}_A$ and $v_p(I) = v_p(J)$ for every p, then $I_p = J_p$, using the local property we know $I = J$. Write $I = \prod_p p^{v_p(I)}$. Then it is easy to see that the map

$$\begin{aligned} \mathbb{I}_A &\longrightarrow \textstyle\bigoplus_p \mathbb{Z} \\ I &\longmapsto (\cdots, v_p(I), \cdots) \end{aligned} \tag{A.4.10}$$

has a inverse

$$\begin{aligned} \textstyle\bigoplus_p \mathbb{Z} &\longrightarrow \mathbb{I}_A \\ \textstyle\prod_p p^{n_p} &\longmapsto (\cdots, n_p, \cdots) \end{aligned} \tag{A.4.11}$$

such that their composition is identity. □

But when are two fractional ideals of a Dedekind domain A isomorphic? Let $\varphi : I_1 \xrightarrow{\sim} I_2$ be an isomorphism between two fractional ideals. Localize with respect to 0-ideal of A, then we have a commutative diagram:

$$\begin{array}{ccc} (I_1)_{(0)} & \xrightarrow{\varphi_{(0)}} & (I_2)_{(0)} \\ \downarrow & & \downarrow \\ K & \longrightarrow & K \end{array} ,$$

where K is the fractional field of A. And $\varphi_{(0)}$ is K-linear. So $\varphi_{(0)}$ is given by multiplication of $a = \varphi(1)$. Thus φ is also given by multiplication. Notice that $aI_1 = (a)I_1$ for some $a \in K$. So there is an exact sequence:

$$1 \longrightarrow \mathcal{P}_A \longrightarrow \mathbb{I} \longrightarrow \mathrm{Cl}(A) \longrightarrow 1. \tag{A.4.12}$$

Then $I_1 \xrightarrow{\sim} I_2$ if and only if there is a nonzero fractional principle ideal (a) such that $I_1 = (a)I_2$.

On sums of two ideals

Lemma A.4.36 (Moving lemma)**.** *Let A be a Dedekind domain. Let $p_1, \cdots, p_s$ be a finite set of prime ideals. Then for any ideal $I \hookrightarrow A$, there is an ideal $J \in \mathbb{I}_A$ which can not be divided by any p_i for $1 \leqslant i \leqslant s$, such that $I \xrightarrow{\sim} J$.*

Proof. By the Unique Factorization Theorem, write $I = \prod_p p^{n_p}$. By the Chinese remainder theorem, there is an $a \in K^\times$ such that $a \in I^{-1}$, $a \notin I^{-1}p_i$ for every i. Now it is easy to see that aI saitisfies the requirement of the lemma. □

Lemma A.4.37 (Projective property)**.** *Let A be a Dedekind domain. Let M be an A-module, I be an ideal. Let $\varphi : M \to I$ be a surjective map, then there is a homomorphis $\pi : I \to M$ such that $\varphi \circ \pi$ is the identity map and in this condition, we say φ has a section.*

Proof. It is easy to see that the lemma is equivalent to the surjectivity of the following homomorphism of A-modules:

$$\mathrm{Hom}_A(I, M) \longrightarrow \mathrm{Hom}_A(I, I).$$

Since locally $\mathrm{Hom}_{A_p}(I_p, M_p) \longrightarrow \mathrm{Hom}_{A_p}(I_p, I_p)$ is surjective for any prime ideal p, and localization is flat. We know from the local property local property A.4.32 that $\mathrm{Hom}_A(I, M) \longrightarrow \mathrm{Hom}_A(I, I)$ is also surjective. □

I+J=A+IJ **Lemma A.4.38.** *Let I and J be two fractional ideals of a Dedekind domain A, then $I \oplus J \simeq A \oplus IJ$.*

Using the moving lemma, we may reduce to the case that $I = \prod p_i^{n_i}$ is an ideal, where $n_i > 0$. And assume J is an ideal, with $p_i \nmid J$ for every i. Now $(I + J)_p = A_p$ for every prime ideal p, then $I + J = A$. Define a map:

$$\begin{aligned} \varphi : \ I \oplus J &\longrightarrow A \\ (x, y) &\longmapsto x + y \end{aligned} \quad . \tag{A.4.13}$$

Clearly this map is surjective, using the projective property it has a inverse homomorphism. Then $I \oplus J = A \oplus L$ for some ideal L.

So we reduce the lemma to proving that $L = IJ$. Here we need some small trick:

The wedge product

Definition A.4.39. *Let A be a ring. Let V be an A-module such that $V_p \xrightarrow{\sim} A_p^2$ for every prime ideal p. (We call V has rank n if it is locally a free A_p-module of rank n). Define a free A-module generated by $V \times V$, denoted by $\det(V) := V \wedge V$, with the following relations for all $x, y \in V, a \in A$:*

$$(ax, y) = (x, ay) = a(x, y); \tag{A.4.14}$$
$$(x, y) = -(y, x), \tag{A.4.15}$$

We call $V \times V$ the wedge product *of V.*

Remark A.4.40. The second relation just shows that $(x, ax) = 0$ for any $x \in V$, $a \in A$ in $\det(V)$.

Proposition A.4.41. *Let A be a ring. Let V be an A-module of rank 2. Then*

1. *There is a homomorphism $V \times V \xrightarrow{\phi} V \wedge V$ such that for all $x, y \in V, a \in A$,*

$$\phi(ax, y) = \phi(x, ay) = a\phi(x, y); \tag{A.4.16}$$
$$\phi(x, y) = -\phi(y, x), \tag{A.4.17}$$

2. *$V \wedge V$ has the universal property: if $\psi : V \times V \longrightarrow W$ saitisfies Property 1 then there is a unique $f : V \wedge V \longrightarrow W$ such that $\psi = f \circ \phi$, where W is an A-module.*

We use this proposition to show I+J=A+IJ A.4.38.

Proof of I+J=A+IJ *A.4.38.* We have already shown $I \oplus J = R \oplus L$ for some ideal L. If we show that $\det(I \oplus J) = IJ$, then the special case is when $I = A$, thus $\det(A \oplus L) = L$, we are done. Now we prove that there is an isomorphism from $\det(I \oplus J)$ to IJ. Define a morphism

$$\begin{aligned} \psi : \ (I \oplus J) \wedge (I \oplus J) &\longrightarrow I \cdot J \\ \sum((a,b),(c,d)) &\longmapsto \sum \det(\begin{pmatrix} a & b \\ c & d \end{pmatrix}) \end{aligned} \quad . \tag{A.4.18}$$

Where $\det(\begin{pmatrix} a & b \\ c & d \end{pmatrix})$ means the determinant of the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

Since if $\psi((a,b),(c,d)) = 0$, then the vector $(a,b) = r(c,d)$ for some $r \in A$, $((a,b),(c,d)) = (r(c,d),(a,b)) = 0$. Thus ψ is injective. The surjectivity is clear. □

Corollary A.4.42. *Let A be a Dedekind domain. Then every ideal I of A is generated by two elements.*

Proof. $I \oplus I^{-1} \xrightarrow{\sim} R \oplus II^{-1} = R \oplus R$. Thus there is a surjective morphism $R^2 \twoheadrightarrow I$. □

Modules of finite type over a Dedekind domain

Theorem A.4.43 (Structure Theorem of Modules)**.** *Let M be a finitely generated module over a Dedekind domain A. Then $M \cong N \oplus A^r \oplus I$ where N is a torsion module $N \xrightarrow{\sim} \bigoplus_p A/p^{n_p}$ (summation over prime ideals) and I is an ideal of A.*

Let M be a finitely generated module over a Dedekind domain A and let $K = \mathrm{Frac}(A)$. The torsion submodule of M is defined to be:

$$M_{\mathrm{tor}} = \{x \in M | \ \exists a \in A, a \neq 0, ax = 0\}.$$

Equivalently, the M_{tor} is the kernel of the natural homomorphism $M \to M_{(0)}$, we have an exact sequence:

$$0 \longrightarrow M_{\mathrm{tor}} \longrightarrow M \xrightarrow{\pi} M_{(0)}.$$

Then $M_{(0)}$ is a K-vector space of finite dimension, say n.

Let M' be the image of π, then we have the exact sequence:

$$0 \longrightarrow M_{\mathrm{tor}} \longrightarrow M \xrightarrow{\pi} M' \longrightarrow 0, \qquad M' \hookrightarrow K^n.$$

And M' has no torsion. If M is not torsion, then $n \neq 0$, then we have the projection $K^n \to K$ onto the first factor. Let N be the image of M' of this projection. Then N is a nonzero fractional ideal of R. We have

Lemma A.4.44. *If $M_{(0)} \neq 0$ (i.e. $M \neq M_{\mathrm{tor}}$) then there is a surjective map $M \twoheadrightarrow N$ with N a fractional ideal of R.*

Since N is projective, $M \twoheadrightarrow N$ has a section, whence $M \cong M_1 \oplus N$.

Continuing this argument for M_1, Eventually, after n steps ($n = \dim_K M_{(0)}$) we have

$$M \simeq M_{\text{tor}} \oplus I_1 \oplus I_2 \oplus \ldots \oplus I_n.$$

Now apply the fact $I_1 \oplus I_2 = R \oplus I_1 I_2$ then we may assume $M \simeq M_{\text{tor}} \oplus R^{n-1} \oplus I$, where I is an ideal of R (if M is not torsion).

What remains is to study the structure of M_{tor}. Since M_{tor} is finitely generated, there exists $x \in R$, $x \neq 0$ such that $xM_{tor} = 0$. Therefore $(M_{\text{tor}})_{\mathfrak{p}} = 0$ if $x \notin \mathfrak{p}$ (equivalently, $\mathfrak{p} \nmid (x)$). It follows that $(M_{\text{tor}})_{\mathfrak{p}} = 0$ for all but finitely many $\mathfrak{p}$ ($(x) = \prod \mathfrak{p}^{n_{\mathfrak{p}}}$). Now by localization principal, the natural homomorphism

$$M_{\text{tor}} \longrightarrow \bigoplus_{\mathfrak{p}} (M_{\text{tor}})_{\mathfrak{p}}$$

is actually an isomorphism. This is because $\mathfrak{p}_1 \neq \mathfrak{p}_2$ implies $\mathfrak{p}_1 + \mathfrak{p}_2 = R$ whence $((M_{tor})_{\mathfrak{p}_1})_{\mathfrak{p}_2} = M_{(0)} = 0$.

It remains to study the structure of $(M_{\text{tor}})_{\mathfrak{p}}$. It is an $R_{\mathfrak{p}}$-module, $R_{\mathfrak{p}}$=DVR.

Lemma A.4.45. *Let R be a discrete valuation ring. Let N be a finitely generated torsion R-module, then*

$$N \simeq \bigoplus_{i=0}^{k} R/\pi^{n_i}, \qquad n_i \geq 0.$$

Proof. Let $k = R/\pi$, (π) = maximal ideal of R, then $N/\pi N$ is a finite dimensional k-vector space. Let $x_1, \ldots, x_t$ be elements of N such that their images in $N/\pi N$ generate $N/\pi N$ over k. By Nakayama's lemma $x_1, \ldots, x_t$ generate N. Thus we have a surjective map

$$R^t \overset{\phi}{\twoheadrightarrow} N \to 0,$$

then $\ker \phi$ is an R-module without torsion.

By what we have proved for non-torsion modules over Dedekind domain: $\ker \phi \simeq R^{m+1} \oplus I$, I an R-ideal but R is PID, hence $I \simeq R$. Thus we have an exact sequence

$$0 \longrightarrow R^m \xrightarrow{\alpha} R^t \longrightarrow N \longrightarrow 0.$$

Claim: $m = t$

Proof. Localize at (0)-ideal. Since N is torsion, $N_{(0)} = 0$, $R^m_{(0)} = R^t_{(0)}$ and $R_{(0)} = k$ (vector spaces). Thus $m = t$. □

Putting $R^t = \sum_{i=1}^{t} Re_i$, α is given by a $t \times t$ matrix A with entries a_{ij} in R: $\alpha(e_i) = \sum a_{ij} e_j$. Now, for two automorphism u and v of R^t, one has commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & R^t & \xrightarrow{\alpha} & R^t & \longrightarrow & N & \longrightarrow & 0\,. \\ & & {\scriptstyle u}\downarrow{\scriptstyle \sim} & & {\scriptstyle v}\downarrow{\scriptstyle \sim} & & \downarrow{\scriptstyle \sim} & & \\ 0 & \longrightarrow & R^t & \xrightarrow{v\alpha u^{-1}} & R^t & \longrightarrow & N' & \longrightarrow & 0 \end{array} \qquad \text{(A.4.19)}$$

If u, v are given by invertible $t \times t$ matrices B and C in R, then $u\alpha v$ is given by BAC.

In summary,

1. The structure of N is determined by A (it is the cokernel).

2. The abstract structure of N doesn't change if A is replaced by BAC where B, C are invertible $t \times t$ matrices over R.

Three operations for rows (and columns) are thus allowed to determined the structure of N: (i) Switch row. (ii) Multiply one row by an element in R^*. (iii) Add R-multiple of one row to another row.

Lemma A.4.46. *There exist matrices B, C such that*

$$BAC = \begin{pmatrix} \pi^{n_1} & 0 & \dots & 0 \\ 0 & \pi^{n_2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \pi^{n_t} \end{pmatrix}, \qquad n_1 \le n_2 \le \ldots \le n_t.$$

Moreover, $\{n_1, \ldots, n_t\}$ are uniquely determined.

Proof.

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}$$

Note that A may not be invertible in R but it is invertible in K, where A has a non-zero entry.

Let $n_1 = \min v(a_{ij}) = v(a_{i_0 j_0})$. After switching rows and columns, we may assume that $i_0 = 1$, $j_0 = 1$. Thus every a_{ij} is a multiple of a_{11} (in R!), $a_{11} = u \cdot \pi^{n_1}$, multiply row 1 by n^{-1}, then may assume $a_{11} = \pi^{n_1}$. Performing operation (iii) on rows and columns, we may assume that $a_{1k} = 0 = a_{k1}$ for every k. So A is transformed to

$$\begin{pmatrix} \pi^{n_1} & 0 & \dots & 0 \\ 0 & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & a_{n2} & \dots & a_{nn} \end{pmatrix}$$

$a_{ij} \in R$, $\pi^{n_1} \mid a_{ij}$. Continue this argument.

Exercise A.4.47. Prove the uniqueness of $n_1 \le n_2 \le \ldots \le n_t$.

□

Apply the above lemma to find the structure of N

$$0 \longrightarrow R^t \xrightarrow{A} R^t \longrightarrow N \longrightarrow 0$$

with $A = \mathrm{diag}(\pi^{n_1}, \dots, \pi^{n_t})$, i.e. $Ae_i = \pi^{n_i} e_i$.

We obtain

$$N = \frac{R}{\pi^{n_1} R} \oplus \dots \oplus \frac{R}{\pi^{n_t} R}$$

and we are done. □

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