ON p-ADIC WALDSPURGER FORMULA

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1. Introduction

In this article, we study *p-adic torus periods* for certain $\mathbb{C}_p$-valued functions on Shimura curves coming from classical origin. We prove a *p-adic Waldspurger formula* for these periods, generalizing the recent work of Bertolini–Darmon–Prasanna [BDP13]. We may view it as the counterpart of the classical Waldspurger formula for (complex automorphic) torus periods. In pursuing such a formula, we construct a new anti-cyclotomic p-adic L-function of Rankin–Selberg type. At a character of positive weight, the p-adic L-function interpolates the central critical value of the complex Rankin–Selberg L-function. Its value at a Dirichlet character, which is outside the range of interpolation, essentially computes the corresponding p-adic torus period.

The nonvanishing of such period provides a new criterion for the nontriviality of Heegner points on modular abelian varieties of GL(2)-type over totally real number fields. This sort of result, in a slightly different form, was first obtained by Rubin in [Rub92], for CM elliptic curves over the rationals. The generalization to other elliptic curves or abelian varieties parameterized by modular curves is due to Bertolini–Darmon–Prasanna assuming the Heegner condition and other control of ramification (see [BDP13, Assumption 5.12] for a list of conditions). Also, recently we learn that Brooks, in his PhD thesis [Bro13], obtains a result similar to [BDP13] for classical new forms under certain control of ramification without assuming the Heegner condition, which is a special case of our formula when $F = \mathbb{Q}$. Their method uses p-adic differential operators, traced back to the work of Katz [Kat78], which is different from Rubin's. Our result generalizes all known results and is placed in the framework of Waldspurger formula ([Wal85], for the central value of the complex L-function), or Yuan–Zhang–Zhang's general version of Gross–Zagier formula ([YZZ13], for the central derivative of the complex L-function). The method we use is a global Mellin transform for the Lubin–Tate action on the Igusa tower of the Shimura curve at infinite level, which is closely related to p-adic differential operators in the classical situation.

We remark that the p-adic L-function we construct is a distribution, or equivalently, a rigid analytic function on certain "space of (twisted) anti-cyclotomic characters". Note that the rigid analyticity is crucial for developing the corresponding Iwasawa theory. Also, our construction assumes neither the Heegner condition (which is somehow apparent since we work over totally real fields), nor any control of ramification on either representations, characters, or test vectors. We only need one assumption – the prime $\mathfrak{p}$ in consideration of the totally real field is split in the CM extension we use to define torus periods/Heegner points.

1.1. **p-adic Maass functions and p-adic torus periods.** Throughout the article, we fix a prime p, a totally imaginary number field $E \subset \mathbb{C}_p$ with F the maximal totally real subfield whose degree is g. Thus we obtain a distinguished place $\mathfrak{p}$ (resp. $\mathfrak{P}$) of F (resp. E) above p. Denote by $\mathbb{A}$ (resp. $\mathbb{A}^\infty$) the ring of adèles (resp. finite adèles) of F. Let $\eta = \prod \eta_v \colon F^\times\backslash\mathbb{A}^\times \to \{\pm 1\}$ be the quadratic character associated to E/F. In particular, we have the L-function $L(s,\eta) = \prod_{v<\infty} L(s,\eta_v)$. Let $\delta_E \in \mathbb{Z}_{>0}$ be the absolute value of the discriminant of E.

Recall from [YZZ13, §1.2.1] that a quaternion algebra $\mathbb{B}$ over $\mathbb{A}$ is *incoherent* if $\Sigma_{\mathbb{B}}$, the set of places v of F where $\mathbb{B}$ is ramified, is a finite set of odd cardinality; $\mathbb{B}$ is *totally* *definite* if $\Sigma_{\mathbb{B}}$ contains all archimedean places of F. Let $\mathbb{B}$ be a totally definite incoherent quaternion algebra over $\mathbb{A}$, which gives rise to a projective system of Shimura curves $\{X(\mathbb{B})_U\}_U$ defined over F, indexed by compact open subgroups U of $\mathbb{B}^{\infty\times} := (\mathbb{B}\otimes_{\mathbb{A}}\mathbb{A}^\infty)^\times$. Put $X(\mathbb{B}) = \varprojlim_U X(\mathbb{B})_U$. For every $g \in \mathbb{B}^{\infty\times}$, we denote by $\mathrm{T}_g\colon X(\mathbb{B}) \to X(\mathbb{B})$ the induced Hecke morphism by right translation.

Definition 1.1 (p-adic Maass function)*.* We introduce the following objects and notation.

1. We say a function $X(\mathbb{B})(\mathbb{C}_p) \to \mathbb{C}_p$ is a *p-adic Maass function* if it is the pullback of some locally analytic $\mathbb{C}_p$-valued function on $X(\mathbb{B})_U \otimes_F \mathbb{C}_p$ for some U. Denote by $\mathscr{A}_{\mathbb{C}_p}(\mathbb{B}^\times)$ the $\mathbb{C}_p$-vector space spanned by all p-adic Maass functions on $X(\mathbb{B})(\mathbb{C}_p)$, which is a representation of $\mathbb{B}^{\infty\times}$ such that $g \in \mathbb{B}^{\infty\times}$ acts by T_g^*. For simplicity, we will write g^* instead of T_g^* in what follows.
2. Let A be an abelian variety over F, $f\colon X \to A$ be a morphism, and $\omega \in \mathrm{H}^0(A, \Omega^1_A)$ be a differential form. Then we have the function $f^*\log_\omega$ on $X(\mathbb{C}_p)$, where $\log_\omega\colon A(\mathbb{C}_p) \to \mathbb{C}_p$ is the p-adic logarithm on A along ω [Bou89], which is an element in $\mathscr{A}_{\mathbb{C}_p}(\mathbb{B}^\times)$. A p-adic Maass function is *(cuspidal) classical* if it is a finite linear combination of functions of the form $f^*\log_\omega$ for different (A, f, ω).
3. An irreducible $\mathbb{B}^{\infty\times}$-subrepresentation π of $\mathscr{A}_{\mathbb{C}_p}(\mathbb{B}^\times)$ is *classical* if π contains a nonzero classical p-adic Maass function.
4. If π is classical, then there exists a simple F-abelian variety A, unique up to isogeny, and an embedding $i\colon M := \mathrm{End}^0(A) \hookrightarrow \mathbb{C}_p$ such that π is the space generated by $f^*\log^i_\omega$ for $\omega \in \mathrm{H}^0(A, \Omega^1_A)$ and $f\colon X(\mathbb{B}) \to A$ a modular parametrization (see Notation 3.3). Here, $\log^i_\omega\colon A(\mathbb{C}_p) \to M\otimes_{\mathbb{Q}}\mathbb{C}_p \xrightarrow{i} \mathbb{C}_p$ is the M-linear logarithm. In particular, every function in π is classical. Moreover, we have a decomposition $\pi = \bigotimes'_v \pi_v$ such that π_v is a representation of $\mathbb{B}_v^\times$, which is unramified for all but finitely many v.
5. Suppose π is classical, and we may replace A by $A^\vee$ in (4). Then we obtain another classical representation $\pi^\vee$, called the *dual* of π, which is isomorphic to $\pi\otimes\omega_\pi^{-1}$ as a representation. Here, ω_π denotes the central character of π.

Remark 1.2. It is an interesting question to give a function-theoretical criterion for a p-adic Maass function to be classical.

Definition 1.3. An *E-embedding* of $\mathbb{B}$ is an embedding

$$\mathsf{e} = {\prod_v}' \mathsf{e}_v\colon \mathbb{A}_E^\infty = {\prod_{v<\infty}}' E\otimes_F F_v \hookrightarrow \mathbb{B}^\infty \tag{1.1}$$

of $\mathbb{A}^\infty$-algebras. We say $\mathbb{B}$ is *E-embeddable* if there exists an E-embedding of $\mathbb{B}$.

Now we take a quaternion algebra $\mathbb{B}$ together with an E-embedding. Put $X = X(\mathbb{B})$ for simplicity.

Definition 1.4 (CM-subscheme)*.* We define the *CM-subscheme* Y to be $X^{E^\times}$, the subscheme of X fixed by the action of $\mathsf{e}(E^\times)$. In fact, we have $Y = Y^+ \coprod Y^-$ such that $E^\times$ acts on the tangent space of points in $Y^\pm$ via the character $t \mapsto (t/t^c)^{\pm1}$, where c denotes the nontrivial element in $\mathrm{Gal}(E/F)$. Both $Y^+(\mathbb{C}_p)$ and $Y^-(\mathbb{C}_p)$ are equipped with the natural profinite topology, and admit a transitive action of $\mathbb{A}_E^{\infty\times}$ via Hecke morphisms.

For a p-adic Maass function $\phi \in \mathscr{A}_{\mathbb{C}_p}(\mathbb{B}^\times)$ and locally constant functions $\varphi_\pm\colon Y^\pm(\mathbb{C}_p) \to \mathbb{C}_p$, we define the *$p$-adic torus periods* to be

$$\mathscr{P}_{Y^\pm}(\phi, \varphi_\pm) = \int_{Y^\pm(\mathbb{C}_p)} \phi(y)\varphi_\pm(y)\,\mathrm{d}y,$$

where the Haar measures $\mathrm{d}y$ have total volume 1, and the integrals can be expressed as a finite sums.

1.2. **A p-adic Rankin–Selberg L-function.** From now on, we will assume $\mathfrak{p}$ is split in E. We fix a classical irreducible representation π contained in $\mathscr{A}_{\mathbb{C}_p}(\mathbb{B}^\times)$. Put $\pi^+ = \pi$ and $\pi^- = \pi^\vee$, both as subspaces of $\mathscr{A}_{\mathbb{C}_p}(\mathbb{B}^\times)$.

Definition 1.5. Let $\chi\colon E^\times\backslash\mathbb{A}_E^{\infty\times} \to \mathbb{C}_p^\times$ be a character.

(1) We say χ is a *p-adic character of weight* $w \in \mathbb{Z}$ if there exists a compact open subgroup V of $\mathbb{A}_E^{\infty\times}$ such that $\chi(t) = (t_{\mathfrak{P}}/t_{\mathfrak{P}^c})^w$ for $t \in V$.

(2) Let $\iota\colon \mathbb{C}_p \xrightarrow{\sim} \mathbb{C}$ be an isomorphism. For a locally algebraic character χ of weight w, we attach following local characters
   - $\chi_v^{(\iota)} = 1$ if $v|\infty$ but not equal to $\iota|_F$;
   - $\chi_v^{(\iota)}(z) = (z/z^c)^w$ for $v = \iota|_F$, where $z \in E \otimes_{F,\iota} \mathbb{R} \xrightarrow{\iota|_E} \mathbb{C}$;
   - $\chi_v^{(\iota)} = \iota \circ \chi_v$ for $v < \infty$ but $v \neq \mathfrak{p}$;
   - $\chi_{\mathfrak{p}}^{(\iota)}(t) = \iota\left((t_{\mathfrak{P}}/t_{\mathfrak{P}^c})^{-w}\chi_{\mathfrak{p}}(t)\right)$ for $t \in E_{\mathfrak{p}}^\times$.

   In particular, $\chi^{(\iota)} := \otimes_v \chi_v^{(\iota)}\colon \mathbb{A}_E^\times \to \mathbb{C}^\times$ is an automorphic character, which is called the *ι-avatar* of χ.

(3) We say a p-adic character of weight w is *π-related* if $\omega_\pi \cdot \chi|_{\mathbb{A}^{\infty\times}} = 1$, and

$$\epsilon(1/2, \pi_v, \chi_v) = \chi_v(-1)\eta_v(-1)\epsilon(\mathbb{B}_v)$$

   holds for every finite place $v \neq \mathfrak{p}$ of F, where $\epsilon(1/2, \pi_v, \chi_v)$ is the local Rankin–Selberg ϵ-factor and $\epsilon(\mathbb{B}_v) \in \{\pm 1\}$ is the Hasse invariant. Denote by $\Xi(\pi)_w$ be the set of all π-related p-adic characters of weight w.

(4) Define $\Omega_{X,Y^\pm}$ to be the restriction $\Omega_X^1|_{Y^\pm}$, which is an $\mathbb{A}_E^{\infty\times}$-equivariant sheaf on $Y^\pm$. For $\chi \in \Xi(\pi)_k$, we define $\sigma_\chi^\pm$ to be the subspace of $\mathrm{H}^0(Y^\pm, \Omega_{X,Y^\pm}^{\otimes -k}) \otimes_F \mathbb{C}_p$ consisting of φ such that $t^*\varphi = \chi(t)^{\pm 1}\varphi$ for all $t \in \mathbb{A}_E^{\infty\times}$.

There is an (ind-)proper rigid analytic curve over $\mathbb{C}_p$, which parameterizes locally analytic characters of $\mathbb{A}_E^{\infty\times}$ and contains $\bigcup_{\mathbb{Z}} \Xi(\pi)_w$ as a Zariski dense subset. We denote by $\mathscr{D}(\pi)$ the coordinate ring of such curve, which we call the *π-related distribution algebra*. It is a complete $\mathbb{C}_p$-algebra; see Definition 3.7 for its rigorous definition.

To state our result for p-adic L-function, we need to fix some p-adic pairing and archimedean pairing. For p-adic pairing, we need to make three choices:

- a *p-adic Petersson inner product* for π, which amounts to a nonzero $\mathbb{B}^{\infty\times}$-invariant bilinear pairing

$$(\ ,\ )_\pi\colon \pi^+ \times \pi^- \to \mathbb{C}_p.$$

  Such pairing exists uniquely up to a scalar in $\mathbb{C}_p^\times$;
- an *abstract conjugation*, that is, an $\mathbb{A}_E^{\infty\times}$-equivariant isomorphism

$$\mathbf{c}\colon Y^+(\mathbb{C}_p) \xrightarrow{\sim} Y^-(\mathbb{C}_p);$$

- an additive character $\psi\colon F_{\mathfrak{p}} \to \mathbb{C}_p^\times$ of level 0, that is, the kernel of ψ contains $O_{\mathfrak{p}}$ but not $\mathfrak{p}^{-1}$, where $O_{\mathfrak{p}}$ is the ring of integers in $F_{\mathfrak{p}}$.

The abstract conjugation $\mathbf{c}$ and ψ induce a $\mathbb{A}_E^{\infty\times}$-invariant pairing

$$(\ ,\ )_\chi\colon \sigma_\chi^+ \times \sigma_\chi^- \to \mathbb{C}_p$$

by the formula $(\varphi_+, \varphi_-)_\chi = (\varphi_+ \otimes \omega_{\psi+}^k) \cdot \mathbf{c}^*(\varphi_- \otimes \omega_{\psi-}^k)$, where the right-hand side is a constant function on Y^+, hence an element in $\mathbb{C}_p$. Here, $\omega_{\psi\pm}$ is a section of $\Omega_{X,Y^\pm} \otimes_F \mathbb{C}_p$ determined by ψ (see (3.1) for details).

For archimedean pairing, we assume $k \geq 1$. For every $\iota\colon \mathbb{C}_p \xrightarrow{\sim} \mathbb{C}$, we have a $\mathbb{B}^{\infty\times} \times \mathbb{A}_E^{\infty\times}$-invariant ι-linear pairing

$$(\ ,\ )_{\pi,\chi}^{(\iota)}\colon (\pi^+ \otimes \sigma_\chi^+) \times (\pi^- \otimes \sigma_\chi^-) \to \mathbb{C},$$

such that for $\phi_\pm := f_\pm^* \log_{\omega_\pm} \in \pi^\pm$ and $\varphi_\pm \in \sigma_\chi^\pm$,

$$(\phi_+ \otimes \varphi_+, \phi_- \otimes \varphi_-)_{\pi,\chi}^{(\iota)} = (\iota\varphi_+ \otimes \mathfrak{c}_\iota^* \iota\varphi_- \otimes \mu^k) \times \int_{X_\iota(\mathbb{C})} \frac{\Theta_\iota^{k-1} f_+^*\omega_+ \otimes \mathfrak{c}_\iota^* \Theta_\iota^{k-1} f_-^*\omega_-}{\mu^k}\, \mathrm{d}x,$$

where

- $\mathfrak{c}_\iota$ is the complex conjugation on (the underlying real analytic space of) $X_\iota := X \otimes_{F,\iota} \mathbb{C}$;
- μ is an arbitrary Hecke invariant hyperbolic metric on $X_\iota(\mathbb{C})$;
- $\iota\varphi_+ \otimes \mathfrak{c}_\iota^* \iota\varphi_- \otimes \mu^k$ is regarded as a complex number since it is a constant function on $(Y^+ \otimes_{F,\iota} \mathbb{C})(\mathbb{C})$;
- Θ_ι is the Shimura–Maass operator on X_ι (see §2.4 for details); and
- $\mathrm{d}x$ is the Tamagawa measure on $X_\iota(\mathbb{C})$.

Then we define the *period ratio* $\Omega_\iota(\chi)$ to be the unique element in $\mathbb{C}^\times$ such that

$$(\ ,\ )_{\pi,\chi}^{(\iota)} = \Omega_\iota(\chi) \cdot \iota(\ ,\ )_\pi \otimes \iota(\ ,\ )_\chi.$$

Theorem 1.6 (p-adic L-function)**.** *There is a unique element $\mathscr{L}(\pi) \in \mathscr{D}(\pi)$ such that for every π-related p-adic character $\chi \in \Xi(\pi)_k$ of weight $k \geq 1$, and every isomorphism $\iota\colon \mathbb{C}_p \xrightarrow{\sim} \mathbb{C}$,*

$$\iota\mathscr{L}(\pi)(\chi) = L(1/2, \pi^{(\iota)}, \chi^{(\iota)}) \cdot \frac{2^{g-3}\delta_E^{1/2}\zeta_F(2)\Omega_\iota(\chi)}{L(1,\eta)^2 L(1,\pi^{(\iota)},\mathrm{Ad})} \frac{\epsilon(1/2,\psi,\pi_{\mathfrak{p}}^{(\iota)} \otimes \chi_{\mathfrak{P}^c}^{(\iota)})}{L(1/2,\pi_{\mathfrak{p}}^{(\iota)} \otimes \chi_{\mathfrak{P}^c}^{(\iota)})^2},$$

where $\pi^{(\iota)} := \pi \otimes_{\mathbb{C}_p,\iota} \mathbb{C}$ and $\pi_{\mathfrak{p}}^{(\iota)} := \pi_{\mathfrak{p}} \otimes_{\mathbb{C}_p,\iota} \mathbb{C}$; and global L-functions do not include archimedean factors.

Remark 1.7. The element $\mathscr{L}(\pi)$ is our *anti-cyclotomic p-adic L-function* for π. It depends only on the choices of a p-adic Petersson inner product $(\ ,\ )_\pi$, an abstract conjugation $\mathbf{c}$, and an additive character ψ of $F_{\mathfrak{p}}$ of level 0. More precisely,

(1) if we change $(\ ,\ )_\pi$ to $(\ ,\ )'_\pi = c(\ ,\ )_\pi$ for some $c \in \mathbb{C}_p^\times$, then $\mathscr{L}(\pi)$ is multiplied by c^{-1};
(2) if we change $\mathbf{c}$ to $\mathbf{c}' = \mathrm{T}_t \circ \mathbf{c}$ for some $t \in \mathbb{A}_E^{\infty\times}$, then $\mathscr{L}(\pi)$ is multiplied by $[t]$, the Dirac distribution at t;
(3) if we change ψ to ψ_a for some $a \in O_{\mathfrak{p}}^\times$, where $\psi_a(x) = \psi(ax)$ for $x \in F_{\mathfrak{p}}$, then $\mathscr{L}(\pi)$ is multiplied by $\omega_{\pi_{\mathfrak{p}}}(a) \cdot [a]^2$, where a is regarded at the place $\mathfrak{P}^c$ in $[a]$.

In §3.2, we will state a version of Theorem 1.6 in terms of Heegner cycles on abelian varieties, which implies Theorem 1.6 by Lemma 3.9.

1.3. **A p-adic Waldspurger formula.** Note that the set of interpolation of $\mathscr{L}(\pi)$ is $\bigcup_{k\geq 1}\Xi(\pi)_k$. Thus, a natural question would be seeking the value of $\mathscr{L}(\pi)(\chi)$ at a locally constant character χ, that is, $\chi\in\Xi(\pi)_0$. The main theorem stated in this section answers this question, through a so called *p-adic Waldspurger formula*.

Having chosen a p-adic Petersson inner product $(\ ,\ )_\pi$ for π and an abstract conjugation $\mathbf{c}$, we may define a nonzero element $\alpha^\natural$, called the *local period*, in the space

$$\mathrm{Hom}_{\mathbb{A}_E^{\infty\times}}(\pi^+\otimes\sigma_\chi^+,\mathbb{C}_p)\otimes\mathrm{Hom}_{\mathbb{A}_E^{\infty\times}}(\pi^-\otimes\sigma_\chi^-,\mathbb{C}_p)$$

such that for $\phi_\pm\in\pi^\pm$, $\varphi_\pm\in\sigma_\chi^\pm$ and $\iota\colon\mathbb{C}_p\xrightarrow{\sim}\mathbb{C}$, the complex number $\iota\alpha^\natural(\phi_+,\phi_-;\varphi_+,\varphi_-)$ is a regularization of the following (formal) matrix coefficient integral

$$\int_{\mathbb{A}^{\infty\times}\backslash\mathbb{A}_E^{\infty\times}}\iota(t^*\phi_+,\phi_-)_\pi\cdot\iota(t^*\varphi_+,\varphi_-)_\chi\,\mathrm{d}t.$$

See Definition 4.4 for details.

Theorem 1.8 (p-adic Waldspurger formula)**.** *Let $\chi\in\Xi(\pi)_0$ be a p-adic character of weight 0, that is, a locally constant character. We have for every $\phi_\pm\in\pi^\pm$ and $\varphi_\pm\in\sigma_\chi^\pm$,*

$$\mathscr{P}_{Y^+}(\phi_+,\varphi_+)\mathscr{P}_{Y^-}(\phi_-,\varphi_-)=\mathscr{L}(\pi)(\chi)\cdot\frac{L(1/2,\pi_\mathfrak{p}\otimes\chi_{\mathfrak{P}^c})^2}{\epsilon(1/2,\psi,\pi_\mathfrak{p}\otimes\chi_{\mathfrak{P}^c})}\cdot\alpha^\natural(\phi_+,\phi_-;\varphi_+,\varphi_-).$$

We note that the ratio $\mathscr{L}(\pi)(\chi)/\epsilon(1/2,\psi,\pi_\mathfrak{p}\otimes\chi_{\mathfrak{P}^c})$ does not depend on ψ. In §3.3, we will state a version of Theorem 1.8 in terms of Heegner cycles on abelian varieties, which implies Theorem 1.8 by Lemma 3.11.

1.4. **Notation and conventions.**

- Denote by $F_\mathfrak{p}^{\mathrm{nr}}$ (resp. $F_\mathfrak{p}^{\mathrm{ab}}$) the completion of the maximal unramified (resp. abelian) extension of $F_\mathfrak{p}$ in $\mathbb{C}_p$ and $O_\mathfrak{p}^{\mathrm{nr}}$ (resp. $O_\mathfrak{p}^{\mathrm{ab}}$) its ring of integers. We denote by κ the residue field of $O_\mathfrak{p}^{\mathrm{nr}}$, which is isomorphic to $\mathbb{F}_p^{\mathrm{ac}}$.
- Denote by $F_{\mathrm{cl}}^\times$ (resp. $E_{\mathrm{cl}}^\times$) the closure of $F^\times$ (resp. $E^\times$) in $\mathbb{A}^{\infty\times}$ (resp. $\mathbb{A}_E^{\infty\times}$).
- Put $O_\mathfrak{p}^{\mathrm{anti}}=O_{E_\mathfrak{p}}^\times/O_\mathfrak{p}^\times$. We will write elements $t\in E_\mathfrak{p}^\times$ in the form $(t_\bullet,t_\circ)$ where $t_\bullet\in F_\mathfrak{p}^\times$ (resp. $t_\circ\in F_\mathfrak{p}^\times$) is the component at $\mathfrak{P}^c$ (resp. $\mathfrak{P}$).
- We fix an identification between $\mathbb{B}_\mathfrak{p}$ and $\mathrm{Mat}_2(F_\mathfrak{p})$ such that in (1.1),

$$\mathsf{e}_\mathfrak{p}(E\otimes_F F_\mathfrak{p})=\begin{pmatrix}E_{\mathfrak{P}^c}&\\&E_\mathfrak{P}\end{pmatrix}.$$

- Denote by J the matrix $\begin{pmatrix}&1\\-1&\end{pmatrix}$ in $\mathbb{B}_\mathfrak{p}=\mathrm{Mat}_2(F_\mathfrak{p})$.
- For $m\in\mathbb{Z}$, define the $\mathfrak{p}$-*Iwahori subgroup of level* m to be

$$U_{\mathfrak{p},m}=\left\{g\in\mathrm{GL}_2(O_\mathfrak{p})\mid g\equiv\begin{pmatrix}1&*\\0&1\end{pmatrix}\mod\mathfrak{p}^m\right\}\quad\text{if }m\geq 0,$$

$$U_{\mathfrak{p},m}=\left\{g\in\mathrm{GL}_2(O_\mathfrak{p})\mid g\equiv\begin{pmatrix}1&0\\ *&1\end{pmatrix}\mod\mathfrak{p}^{-m}\right\}\quad\text{if }m<0.$$

- $\mathbb{N} = \{m \in \mathbb{Z} \mid m \geq 0\}$. We write elements in $A^{\oplus m}$ in columns for an object A (with a well-defined underlying set) in an abelian category.
- The local or global Artin reciprocity maps send uniformizers to geometric Frobenii.
- If G is a reductive group over F, we always take the Tamagawa measure when integrating on the adèlic group $G(\mathbb{A})$. In particular, the total volume of $\mathbb{A}^\times E^\times \backslash \mathbb{A}_E^\times$ is 2 under such measure.
- For a relative (formal) scheme X/S, we will simply write Ω_X^1 instead of $\Omega_{X/S}^1$ for the sheaf of relative differentials if the base is clear in the context.
- Denote by $\widehat{\mathbf{G}}_m$ (resp. $\widehat{\mathbf{G}}_a$) the multiplicative (resp. additive) formal group. We also regard $\widehat{\mathbf{G}}_m$ as a formal p-divisible group. They have the coordinate T.
- Denote by $\mathcal{LT}$ the Lubin–Tate group over $O_{\mathfrak{p}}^{\mathrm{nr}}$, which is unique up to isomorphism.
- Denote by $F_{\mathfrak{p}}^{\mathrm{lt}}$ the field extension of $F_{\mathfrak{p}}^{\mathrm{nr}}$ by adding the "period" of the Lubin–Tate group $\mathcal{LT}$ (see [ST01, page 460]). It is not discrete unless $F_{\mathfrak{p}} = \mathbb{Q}_p$ by [ST01, Lemma 3.9].
- In this article, we will only use basic knowledge about rigid analytic varieties over complete p-adic fields in the sense of Tate. The readers may use the book [BGR84] for a reference. If $\mathscr{X}$ is an L-rigid analytic variety for some complete non-archimedean field L, we denote by $\mathscr{O}(\mathscr{X}, K)$ the complete K-algebra of K-valued rigid analytic functions on $\mathscr{X}$ for any complete field extension K/L.

We fix the Haar measure $\mathrm{d}t_v$ on $F_v^\times \backslash E_v^\times$ for every place v of F determined by the following conditions:

- When v is archimedean, the total volume of $F_v^\times \backslash E_v^\times \simeq \mathbb{R}^\times \backslash \mathbb{C}^\times$ is 1;
- When v is split, the volume of the maximal compact subgroup of $F_v^\times \backslash E_v^\times \simeq F_v^\times$ is 1;
- When v is nonsplit and unramified, the total volume is 1;
- When v is ramified, the total volume is 2.

Then the product measure $\prod_v \mathrm{d}t_v$ is $2^{-g}\delta_E^{-1/2} L(1, \eta)$ times the Tamagawa measure (compare with [YZZ13, 1.6]).

In the main part of the article, we will fix an additive character $\psi\colon F_{\mathfrak{p}} \to \mathbb{C}_p^\times$ of level 0. We choose a generator $\nu\colon \mathcal{LT} \to \widehat{\mathbf{G}}_m$ in the free $O_{\mathfrak{p}}$-module $\mathrm{Hom}(\mathcal{LT}, \widehat{\mathbf{G}}_m)$ of rank 1. Then there are unique isomorphisms

$$\nu_\pm\colon F_{\mathfrak{p}}/O_{\mathfrak{p}} \xrightarrow{\sim} \mathcal{LT}[\mathfrak{p}^\infty] \tag{1.2}$$

such that the induced composite map

$$\nu[\mathfrak{p}^\infty] \circ \nu_\pm\colon F_{\mathfrak{p}}/O_{\mathfrak{p}} \to \widehat{\mathbf{G}}_m[p^\infty] \subset \mathbb{C}_p^\times$$

coincides with $\psi^\pm$, where $\psi^+ = \psi$ and $\psi^- = \psi^{-1}$.

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2. Arithmetic of quaternionic Shimura curves

In this chapter, we study some p-adic arithmetic properties of quaternionic Shimura curves over a totally real field. We start from the local theory of some p-adic Fourier analysis on Lubin–Tate groups, following the work of [ST01], in §2.1. In §2.2, we study the Gauss–Manin connection and the Kodaira–Spencer isomorphism for quaternionic Shimura curves, followed by the discussion of universal convergent modular forms in §2.3. In particular, we prove Theorem 2.20, which is one of the most crucial technical results of the article. In §2.4, we prove some results involving comparison with transcendental constructions under a given complex uniformization. The last one §2.5 contains the proof of six claims in the previous ones, which requires an auxiliary use of unitary Shimura curves. In particular, no representations will show up in this chapter.

2.1. **Fourier theory on Lubin–Tate group.** Let G be a topologically finitely generated abelian locally $F_\mathfrak{p}$-analytic group. For a complete field K containing $F_\mathfrak{p}$, denote by $C(G,K)$ the locally convex K-vector space of locally analytic K-valued functions on G, and $D(G,K)$ its strong dual which is a topological K-algebra by convolution. Recall that the strong dual topology coincides with topology of uniform convergence on bounded sets in $C(G,K)$. We have a natural continuous injective homomorphism

$$[\ ]\colon G \to D(G,K)^\times$$

by taking Dirac distributions. Moreover, we have $D(G,K)\widehat{\otimes}_K K' \simeq D(G,K')$ for a complete field extension K'/K.

Definition 2.1 (Stable function). Let $\mathscr{B}$ be the generic fiber of (the underlying formal scheme of) $\mathcal{LT}$, which is isomorphic to the open unit disc over $F_\mathfrak{p}^{\mathrm{nr}}$. We have a map

$$\alpha\colon \mathscr{B} \times_{\mathrm{Spf}\, F_\mathfrak{p}^{\mathrm{nr}}} \mathscr{B} \to \mathscr{B}$$

induced by the formal group law. A function $\phi \in \mathscr{O}(\mathscr{B},K)$ is *stable* if

$$\sum_{\mathrm{Ker}[\mathfrak{p}]} \phi(\alpha(\ ,z)) = 0.$$

We denoted by $\mathscr{O}(\mathscr{B},K)^\heartsuit$ the subspace of $\mathscr{O}(\mathscr{B},K)$ of stable functions. The restricted map $\mathbf{M}_{\mathrm{loc}} := \alpha^* \,|\, \mathscr{O}(\mathscr{B},K)^\heartsuit$ is called the *local Mellin transform*, whose image is contained in $\mathscr{O}(\mathscr{B},K)^\heartsuit\widehat{\otimes}_K\mathscr{O}(\mathscr{B},K)^\heartsuit$.

From now on, we will assume K contains $F_\mathfrak{p}^{\mathrm{lt}}$. By [ST01, Theorems 2.3 & 3.6] (together with the remark after [ST01, Corollary 3.7]), we have a natural *Fourier transform* isomorphism

$$\lambda\colon \mathscr{O}(\mathscr{B},K) \xrightarrow{\sim} D(O_\mathfrak{p},K)$$

of topological K-algebras, using the homomorphism $\nu\colon \mathcal{LT} \to \widehat{\mathbf{G}}_m$. For $z \in \mathscr{B}(K)$, the assignment $\delta \mapsto (\lambda^{-1}\delta)(z)$ defines an element κ_z in the strong dual of $D(O_\mathfrak{p},K)$, that is, $C(O_\mathfrak{p},K)$. In fact, κ_z is a locally analytic character of $O_\mathfrak{p}$ (see [ST01, §3]) satisfying $\kappa_z(a) = \nu(a.z)$ for every $a \in O_\mathfrak{p}$.

Remark 2.2. We have an action of $O_\mathfrak{p}$ on $\mathscr{B}$ coming from the Lubin–Tate group, and hence on $D(O_\mathfrak{p},K)$ via λ. More precisely, $t \in O_\mathfrak{p}$ acts on $D(O_\mathfrak{p},K)$ by multiplying $[t]$.

If we identify $D(O_{\mathfrak{p}}^{\times}, K)$ as the closed subspace of $D(O_{\mathfrak{p}}, K)$ consisting of distributions supported on $O_{\mathfrak{p}}^{\times}$, then by [ST01, Lemma 4.6.5], λ induces an isomorphism between $\mathscr{O}(\mathscr{B}, K)^{\heartsuit}$ and $D(O_{\mathfrak{p}}^{\times}, K)$. The following diagram commutes

$$\begin{tikzcd} \mathscr{O}(\mathscr{B}, K)^{\heartsuit} \arrow[r, "\mathbf{M}_{\mathrm{loc}}"] \arrow[d, "\lambda", "\simeq"'] & \mathscr{O}(\mathscr{B}, K)^{\heartsuit} \widehat{\otimes}_K \mathscr{O}(\mathscr{B}, K)^{\heartsuit} \arrow[d, "\lambda", "\simeq"'] \\ D(O_{\mathfrak{p}}^{\times}, K) \arrow[r] & D(O_{\mathfrak{p}}^{\times}, K) \widehat{\otimes}_K D(O_{\mathfrak{p}}^{\times}, K), \end{tikzcd}$$

where the bottom arrow is the pushforward of distributions along the diagonal embedding $O_{\mathfrak{p}}^{\times} \to O_{\mathfrak{p}}^{\times} \times O_{\mathfrak{p}}^{\times}$.

We define the *Lubin–Tate differential operator* Θ on $\mathscr{O}(\mathscr{B}, K)$ by the formula

$$\Theta\phi = \frac{\mathrm{d}\phi}{\nu^*\mathrm{d}T/T}. \tag{2.1}$$

Consider the compact abelian locally $F_{\mathfrak{p}}$-analytic group $O_{\mathfrak{p}}^{\mathrm{anti}}$, which will be identified with $O_{\mathfrak{p}}^{\times}$ via $t \mapsto t/t^{c}$. We regard the range of $\mathbf{M}_{\mathrm{loc}}$ as $\mathscr{O}(\mathscr{B}, K)^{\heartsuit} \widehat{\otimes}_{F_{\mathfrak{p}}} D(O_{\mathfrak{p}}^{\mathrm{anti}}, F_{\mathfrak{p}})$. For each $w \in \mathbb{Z}$, we have a locally analytic $F_{\mathfrak{p}}$-valued character $t \mapsto t^{w}$ of $O_{\mathfrak{p}}^{\times}$ and hence $O_{\mathfrak{p}}^{\mathrm{anti}}$, denoted by $\langle w \rangle \in C(O_{\mathfrak{p}}^{\mathrm{anti}}, F_{\mathfrak{p}})$.

Lemma 2.3. *Let $\phi \in \mathscr{O}(\mathscr{B}, K)^{\heartsuit}$ be a stable function. We have for $k \geq 0$,*

$$\mathbf{M}_{\mathrm{loc}}(\phi)(\langle k \rangle) = \Theta^{k}\phi,$$

and $\Theta\mathbf{M}_{\mathrm{loc}}(\phi)(\langle -1 \rangle) = \phi$.

Proof. This follows from [ST01, Lemma 4.6 5&8]. □

Definition 2.4 (Admissible function). For $n \in \mathbb{N}$, a stable function $\phi \in \mathscr{O}(\mathscr{B}, K)^{\heartsuit}$ is *n-admissible* if $\phi(\alpha(\ ,\nu_{+}(x))) = \psi(x)\phi$ for every $x \in \mathfrak{p}^{-n}/O_{\mathfrak{p}}$, where ν_{+} is introduced in (1.2).

Lemma 2.5. *Let $\phi \in \mathscr{O}(\mathscr{B}, K)^{\heartsuit}$ be an n-admissible stable function. Then $\lambda(\phi)$ is supported on $1 + \mathfrak{p}^{n}$ if $n \geq 1$; in particular, $\mathbf{M}_{\mathrm{loc}}(\phi)(\langle k \rangle) = \mathbf{M}_{\mathrm{loc}}(\phi)(\chi\langle k \rangle)$ for any $k \in \mathbb{Z}$ and any (locally constant) character $\chi\colon O_{\mathfrak{p}}^{\mathrm{anti}} \to K^{\times}$ that is trivial on $(1 + \mathfrak{p}^{n})^{\times}$.*

Proof. This again follows from [ST01, Lemma 4.6.5] and the relation among ν, ν_{+} and ψ. □

2.2. **Kodaira–Spencer isomorphism.** Let $\mathbb{B}$ be a totally definition incoherent quaternion algebra over $\mathbb{A}$. Denote by Γ the set of all compact open subgroups $U^{\mathfrak{p}}$ of $\mathbb{B}^{\infty\mathfrak{p}\times} = (\mathbb{B} \otimes_{\mathbb{A}} \mathbb{A}^{\infty\mathfrak{p}})^{\times}$, which is a filtered partially ordered set under inclusion. Recall that we have the system of Shimura curves $\{X_{U}\}_{U}$ over $\operatorname{Spec} F$.

Definition 2.6 (Shimura pro-curve of Iwahori level). For a tame level $U^{\mathfrak{p}} \in \Gamma$ and $m \in \mathbb{Z}$, put $X(m, U^{\mathfrak{p}}) = X_{U^{\mathfrak{p}}U_{\mathfrak{p},m}} \otimes_{F} F_{\mathfrak{p}}^{\mathrm{nr}}$ where we recall that $U_{\mathfrak{p},m}$ is the $\mathfrak{p}$-Iwahori subgroup of level m. If we take the inverse limit with respect to m, we obtain

$$X(\pm\infty, U^{\mathfrak{p}}) = \varprojlim_{m \to \infty} X(\pm m, U^{\mathfrak{p}}).$$

For $m \in \mathbb{N} \cup \{\infty\}$, if we take the inverse limit with respect to $U^{\mathfrak{p}}$, we obtain

$$X(\pm m) = \varprojlim_{U^{\mathfrak{p}} \in \Gamma} X(\pm m, U^{\mathfrak{p}}),$$

which we call the *Shimura pro-curve of* $\mathfrak{p}$*-Iwahori level* m.

We have successive surjective morphisms

$$X(\pm\infty) \to \cdots \to X(\pm 1) \to X(0),$$

which are equivariant under the Hecke actions of $\mathbb{B}^{\infty\mathfrak{p}\times}$. By Carayol [Car86, §6], $X(0)$ admits a canonical smooth model (see [Mil92, Definition 2.2] for its meaning) $\mathcal{X}$ over $\operatorname{Spec} O_{\mathfrak{p}}^{\mathrm{nr}}$. Strictly speaking, Carayol assumed that $F \neq \mathbb{Q}$. But when $F = \mathbb{Q}$, one may take $\mathcal{X}$ to be the model defined by modular interpretation which is well-known.

We recall the construction in [Car86, 1.4] of an $O_{\mathfrak{p}}$-divisible group $\mathcal{G}$ on $\mathcal{X}$. For $m \geq 1$, denote the principal congruence subgroup of level $\mathfrak{p}^m$ by

$$U_{\mathfrak{p},m}^{\mathrm{pr}} = \{g \in U_{\mathfrak{p},0} \mid g \equiv 1 \mod \mathfrak{p}^m\}$$

and $X(m)^{\mathrm{pr}}$ the corresponding covering of $X(0)$. Consider the right action of $U_{\mathfrak{p},m}^{\mathrm{pr}}/U_{\mathfrak{p},0} \simeq \mathrm{GL}_2(O_{\mathfrak{p}}/\mathfrak{p}^m)$ on $(\mathfrak{p}^{-m}/O_{\mathfrak{p}})^{\oplus 2}$ sending $v \in (\mathfrak{p}^{-m}/O_{\mathfrak{p}})^{\oplus 2}$ to $g^{-1}v$. Then the quotient

$$\left(X(m)^{\mathrm{pr}} \times (\mathfrak{p}^{-m}/O_{\mathfrak{p}})^{\oplus 2}\right)/(U_{\mathfrak{p},m}^{\mathrm{pr}}/U_{\mathfrak{p},0})$$

defines a finite flat group scheme G_m, with strict $O_{\mathfrak{p}}$-action, over $X(0)$. The inductive system $\{G_m\}_{m\geq 1}$ defines an $O_{\mathfrak{p}}$-divisible group G over $X(0)$ (which is however denoted by E_∞ in [Car86]). In particular, over $X(+\infty)$ (resp. $X(-\infty)$), we have an exact sequence

$$0 \longrightarrow F_{\mathfrak{p}}/O_{\mathfrak{p}} \longrightarrow G \longrightarrow F_{\mathfrak{p}}/O_{\mathfrak{p}} \longrightarrow 0 \tag{2.2}$$

such that the second arrow is the inclusion into the first (resp. second) factor and the third arrow is the projection onto the second (resp. first) factor.

By [Car86, 6.4], the $O_{\mathfrak{p}}$-divisible group G extends uniquely to an $O_{\mathfrak{p}}$-divisible group $\mathcal{G}$ of height 2 over $\mathcal{X}$, together with an action by $\mathbb{B}^{\infty\mathfrak{p}\times}$ that is compatible with the Hecke action on the base.

For $m \geq 1$, put $\mathcal{X}^{(m)} = \mathcal{X} \otimes_{O_{\mathfrak{p}}} O_{\mathfrak{p}}/\mathfrak{p}^m$ and $\mathcal{G}^{(m)} = \mathcal{G}|_{\mathcal{X}^{(m)}}$. We have the following exact sequence

$$0 \longrightarrow \underline{\omega}_p^{\bullet(m)} \longrightarrow \mathcal{L}_p^{(m)} \longrightarrow (\underline{\omega}_p^{\circ(m)})^\vee \longrightarrow 0, \tag{2.3}$$

where if we put $h = [F_{\mathfrak{p}} : \mathbb{Q}_p]$,

- $\mathcal{L}_p^{(m)}$ is the Dieudonné crystal of $\mathcal{G}^{(m)}$ evaluated at $\mathcal{X}^{(m)}$, which is a locally free sheaf of rank $2h$;
- $\underline{\omega}_p^{\bullet(m)}$ is the sheaf of invariant differentials of $\mathcal{G}^{(m)}/\mathcal{X}^{(m)}$, which is a locally free sheaf of rank 1;
- $\underline{\omega}_p^{\circ(m)}$ is the sheaf of invariant differentials of $(\mathcal{G}^{(m)})^\vee/\mathcal{X}^{(m)}$, which is a locally free sheaf of rank $2h-1$.

They are equipped with actions of $O_{\mathfrak{p}}$ under which (2.3) is equivariant. The projective system of (2.3) for all $m \geq 1$ induces the following $O_{\mathfrak{p}}$-equivariant exact sequence

$$0 \longrightarrow \underline{\omega}_p^{\bullet} \longrightarrow \mathcal{L}_p \longrightarrow (\underline{\omega}_p^{\circ})^\vee \longrightarrow 0,$$

of locally free sheaves over $\mathcal{X}^\wedge$, the formal completion of $\mathcal{X}$ along its special fiber. Let $\mathcal{L}$ (resp. $\underline{\omega}^{\circ\vee}$) be the maximal sub-sheaf of $\mathcal{L}_p$ (resp. $(\underline{\omega}_p^\circ)^\vee$) where $O_\mathfrak{p}$ acts via the structure map. Then we have the following $\mathbb{B}^{\infty\mathfrak{p}\times}$-equivariant exact sequence

$$0 \longrightarrow \underline{\omega}^\bullet \longrightarrow \mathcal{L} \longrightarrow \underline{\omega}^{\circ\vee} \longrightarrow 0, \tag{2.4}$$

where $\underline{\omega}^\bullet = \underline{\omega}_p^\bullet$. We call (2.4) the *formal Hodge exact sequence*.

We have the *Gauss–Manin connection*

$$\nabla\colon \mathcal{L} \to \mathcal{L} \otimes \Omega^1_{\mathcal{X}^\wedge}, \tag{2.5}$$

which is equivariant under the Hecke action of $\mathbb{B}^{\infty\mathfrak{p}\times}$. We have the following two lemmas whose proofs will be given in §2.5.

Lemma 2.7. *The formal Hodge exact sequence* (2.4) *is algebraizable, that is, it is the formal completion of an exact sequence*

$$0 \longrightarrow \underline{\omega}^\bullet \longrightarrow \mathcal{L} \longrightarrow \underline{\omega}^{\circ\vee} \longrightarrow 0, \tag{2.6}$$

on $\mathcal{X}$. Here, we abuse of notation by adopting the same symbols for these sheaves. Moreover, the Gauss–Manin connection (2.5) *is algebraizable.*

We simply call (2.6) the *Hodge exact sequence*.

Remark 2.8. For $m \geq 1$, one may consider the right action of $U^{\mathrm{pr}}_{\mathfrak{p},m}/U_{\mathfrak{p},0} \simeq \mathrm{GL}_2(O_\mathfrak{p}/\mathfrak{p}^m)$ on $(O_\mathfrak{p}/\mathfrak{p}^m)^{\oplus 2}$ sending $v \in (O_\mathfrak{p}/\mathfrak{p}^m)^{\oplus 2}$ to $g^t v$. Then the quotient

$$\left(X(m)^{\mathrm{pr}} \times (O_\mathfrak{p}/\mathfrak{p}^m)^{\oplus 2}\right)/(U^{\mathrm{pr}}_{\mathfrak{p},m}/U_{\mathfrak{p},0})$$

defines an $O_\mathfrak{p}/\mathfrak{p}^m$-local system L_m on $X(0)$ of rank 2. Denote by L the $O_\mathfrak{p}$-local system over $X(0)$ defined by $(\mathrm{L}_m)_{m\geq 1}$. Then L is canonically isomorphic to the restriction of $\mathcal{L}$ on the generic fiber.

Proposition 2.9 (Kodaira–Spencer isomorphism)**.** *The composition of*

$$\underline{\omega}^\bullet \to \mathcal{L} \xrightarrow{\nabla} \mathcal{L} \otimes \Omega^1_{\mathcal{X}} \to \underline{\omega}^{\circ\vee} \otimes \Omega^1_{\mathcal{X}} \tag{2.7}$$

is an isomorphism of quasi-coherent sheaves, where $\underline{\omega}^\circ$ is the dual sheaf of $\underline{\omega}^{\circ\vee}$ and the tensor product is over the structure sheaf $\mathscr{O}_\mathcal{X}$.

The isomorphism (2.7) induces the following ($\mathbb{B}^{\infty\mathfrak{p}\times}$-equivariant) *Kodaira–Spencer isomorphism*

$$\mathrm{KS}\colon \underline{\omega}^\bullet \otimes \underline{\omega}^\circ \xrightarrow{\sim} \Omega^1_{\mathcal{X}}. \tag{2.8}$$

For $w \in \mathbb{N}$, put $\mathcal{L}^{[w]} = \mathrm{Sym}^w \mathcal{L} \otimes \mathrm{Sym}^w \mathcal{L}^\vee$. The Gauss–Manin connection $\nabla^\vee$ on the dual sheaf $\mathcal{L}^\vee$ and the original one ∇ induce a connection $\nabla^{[w]}\colon \mathcal{L}^{[w]} \to \mathcal{L}^{[w]} \otimes \Omega^1_{\mathcal{X}}$. Define $\Theta^{[w]}$ to be the composite map

$$(\Omega^1_{\mathcal{X}})^{\otimes w} \xrightarrow{\mathrm{KS}^{-1}} (\underline{\omega}^\bullet)^{\otimes w} \otimes (\underline{\omega}^\circ)^{\otimes w} \to \mathcal{L}^{[w]} \xrightarrow{\nabla^{[w]}} \mathcal{L}^{[w]} \otimes \Omega^1_{\mathcal{X}}. \tag{2.9}$$

Denote by $\mathcal{X}(0)$ the (dense) open subscheme of $\mathcal{X}$ with all points on the special fiber where $\mathcal{G}$ is supersingular removed. For $m \in \mathbb{N}$, denote by $\mathcal{X}(m)$ the functor classifying $O_\mathfrak{p}/\mathfrak{p}^m$-equivariant *frames*, that is, exact sequences

$$0 \longrightarrow \mathcal{LT}[\mathfrak{p}^m] \longrightarrow \mathcal{G}[\mathfrak{p}^m] \longrightarrow \mathfrak{p}^{-m}/O_\mathfrak{p} \longrightarrow 0$$

over $\mathcal{X}(0)$ with terms fixed. Then $\mathcal{X}(m)$ is representable by a scheme étale over $\mathcal{X}(0)$, which we again denote by $\mathcal{X}(m)$. Put $\mathcal{X}(\infty) = \varprojlim_{m\to\infty} \mathcal{X}(m)$. We define $\mathcal{G}$, $\nabla^{[w]}$, KS, $\Theta^{[w]}$, and the sequence (2.6) for $\mathcal{X}(m)$ ($m \in \mathbb{N} \cup \{\infty\}$) via restriction and denote them by the same notation. Over $\mathcal{X}(\infty)$, we have the *universal frame*

$$0 \longrightarrow \mathcal{LT} \xrightarrow{\varrho_\bullet^{\mathrm{univ}}} \mathcal{G} \xrightarrow{\varrho_\circ^{\mathrm{univ}}} F_\mathfrak{p}/O_\mathfrak{p} \longrightarrow 0\ . \tag{2.10}$$

There is a $\mathbb{B}^{\infty\mathfrak{p}\times}$-equivariant action of $O_{E_\mathfrak{p}}^\times$ on the morphism $\mathcal{X}(\infty) \to \mathcal{X}(0)$. More precisely, for $(t_\bullet, t_\circ) \in O_{E_\mathfrak{p}}^\times$, the pullback of (2.10) is the frame

$$0 \longrightarrow \mathcal{LT} \xrightarrow{\varrho_\bullet^{\mathrm{univ}} \circ t_\bullet^{-1}} \mathcal{G} \xrightarrow{\varrho_\circ^{\mathrm{univ}} \circ t_\circ} F_\mathfrak{p}/O_\mathfrak{p} \longrightarrow 0\ . \tag{2.11}$$

The trivialization (1.2) induces *transition isomorphisms*

$$\Upsilon_\pm \colon X(\pm\infty) \otimes_{F_\mathfrak{p}^{\mathrm{nr}}} F_\mathfrak{p}^{\mathrm{ab}} \xrightarrow{\sim} \mathcal{X}(\infty) \otimes_{O_\mathfrak{p}^{\mathrm{nr}}} F_\mathfrak{p}^{\mathrm{ab}} \tag{2.12}$$

such that the pullback of (2.10) coincide with (2.2). It is $\mathbb{B}^{\infty\mathfrak{p}\times}$-equivariant and $O_{E_\mathfrak{p}}^\times$-equivariant (resp. $O_{E_\mathfrak{p}}^\times$-anti-equivariant) for $X(+\infty)$ (resp. $X(-\infty)$). We have the following commutative diagram

$$\begin{tikzcd}
X \otimes_F F_\mathfrak{p}^{\mathrm{ab}} \arrow[rr, "\mathrm{T_J}"] \arrow[d] & & X \otimes_F F_\mathfrak{p}^{\mathrm{ab}} \arrow[d] \\
X(+\infty) \otimes_{F_\mathfrak{p}^{\mathrm{nr}}} F_\mathfrak{p}^{\mathrm{ab}} \arrow[rr, "\mathrm{T_J}"] \arrow[dr, "\Upsilon_+"'] & & X(-\infty) \otimes_{F_\mathfrak{p}^{\mathrm{nr}}} F_\mathfrak{p}^{\mathrm{ab}} \arrow[dl, "\Upsilon_-"] \\
& \mathcal{X}(\infty) \otimes_{O_\mathfrak{p}^{\mathrm{nr}}} F_\mathfrak{p}^{\mathrm{ab}}, &
\end{tikzcd}$$

where J is introduced in §1.4.

2.3. **Universal convergent modular forms.** For $m \in \mathbb{N} \cup \{\infty\}$, denote by $\mathfrak{X}(m)$ the formal completion of $\mathcal{X}(m)$ along its special fiber, which is an affine formal scheme over $O_\mathfrak{p}^{\mathrm{nr}}$, equipped with an $O_\mathfrak{p}$-divisible group $\mathfrak{G}$ induced from $\mathcal{G}$. The action of $O_{E_\mathfrak{p}}^\times$ (2.11) makes it the Galois group of the $\mathbb{B}^{\infty\mathfrak{p}\times}$-equivariant pro-étale Galois cover $\mathfrak{X}(\infty) \to \mathfrak{X}(0)$. Denote by $O_{E_\mathfrak{p},m}^\times$ the subgroup of $O_{E_\mathfrak{p}}^\times$ that fixes the sub-cover $\mathfrak{X}(m) \to \mathfrak{X}(0)$ for $m \in \mathbb{N}$.

The following lemma will be proved in §2.5.

Lemma 2.10. *There is a unique quasi-coherent formal sub-sheaf $\mathcal{L}^\circ$ of $\mathcal{L}$ (viewed as the formal sheaf induced from $\mathcal{X}(0)$) such that*

(1) *we have the following* unit-root decomposition
$$\mathcal{L} = \underline{\omega}^\bullet \oplus \mathcal{L}^\circ;$$
(2) $\nabla \mathcal{L}^\circ \subset \mathcal{L}^\circ \otimes \Omega^1_{\mathfrak{X}(0)}$;
(3) *for every closed point $x \in \mathfrak{X}(0)(\kappa)$, the restriction of $\mathcal{L}^\circ$ to $\mathfrak{X}(0)_{/x}$, the formal completion at x, is free of rank* 1.

We restrict the above decomposition to $\mathfrak{X}(m)$ for $m \in \mathbb{N} \cup \{\infty\}$. The splitting in the above lemma induces a map

$$\theta^{[w]}_{\mathrm{ord}} \colon \mathcal{L}^{[w]} \to (\underline{\omega}^{\bullet})^{\otimes w} \otimes (\underline{\omega}^{\circ})^{\otimes w} \xrightarrow{\mathrm{KS}} (\Omega^1_{\mathfrak{X}(m)})^{\otimes w}$$

for all $w \in \mathbb{N}$.

Definition 2.11 (Atkin–Serre operator). For $m \in \mathbb{N} \cup \{\infty\}$ and $w \in \mathbb{N}$, define the *Atkin–Serre operator* to be

$$\Theta^{[w]}_{\mathrm{ord}} \colon (\Omega^1_{\mathfrak{X}(m)})^{\otimes w} \xrightarrow{\Theta^{[w]}|_{\mathfrak{X}(m)}} \mathcal{L}^{[w]} \otimes \Omega^1_{\mathfrak{X}(m)} \xrightarrow{\theta^{[w]}_{\mathrm{ord}}} (\Omega^1_{\mathfrak{X}(m)})^{\otimes w+1},$$

where $\Theta^{[w]}$ is defined in (2.9). For $k \in \mathbb{N}$, define the *Atkin–Serre operator of degree k* to be

$$\Theta^{[w,k]}_{\mathrm{ord}} = \Theta^{[w+k-1]}_{\mathrm{ord}} \circ \cdots \circ \Theta^{[w]}_{\mathrm{ord}} \colon (\Omega^1_{\mathfrak{X}(m)})^{\otimes w} \to (\Omega^1_{\mathfrak{X}(m)})^{\otimes w+k}.$$

In what follows, w will always be clear from the text, and hence we will suppress w from notation; in other words, we simply write Θ_{ord} (resp. Θ^k_{ord}) for $\Theta^{[w]}_{\mathrm{ord}}$ (resp. $\Theta^{[w,k]}_{\mathrm{ord}}$) for all $w \in \mathbb{N}$.

Using Serre–Tate coordinates (Proposition B.1), the formal deformation space of $\mathcal{LT} \oplus F_{\mathfrak{p}}/O_{\mathfrak{p}}$ is canonically isomorphic to $\mathcal{LT}$. Thus, we have the classifying morphism

$$c \colon \mathfrak{X}(\infty) \to \mathcal{LT}$$

of $O^{\mathrm{nr}}_{\mathfrak{p}}$-formal schemes. It induces a morphism

$$c_{/x} \colon \mathfrak{X}(\infty)_{/x} \to \mathcal{LT}$$

for every closed point $x \in \mathfrak{X}(\infty)(\kappa)$, where $\mathfrak{X}(\infty)_{/x}$ denotes the formal completion of $\mathfrak{X}(\infty)$ at x. The following lemma and proposition will be proved in §2.5.

Lemma 2.12. *The morphism $c_{/x}$ is an isomorphism for every x.*

Proposition 2.13. *There is a morphism $\beta \colon \mathcal{LT} \times_{\mathrm{Spf}\, O^{\mathrm{nr}}_{\mathfrak{p}}} \mathfrak{X}(\infty) \to \mathfrak{X}(\infty)$ such that*

(1) *for every $x \in \mathfrak{X}(\infty)(\kappa)$, it preserves $\mathfrak{X}(\infty)_{/x}$ and the induced morphism*

$$\beta_{/x} \colon \mathcal{LT} \times_{\mathrm{Spf}\, O^{\mathrm{nr}}_{\mathfrak{p}}} \mathfrak{X}(\infty)_{/x} \to \mathfrak{X}(\infty)_{/x}$$

is simply the formal group law after identifying $\mathfrak{X}(\infty)_{/x}$ with $\mathcal{LT}$ via $c_{/x}$;

(2) *if we equip $\mathcal{LT}$ with the action of $O^{\times}_{E_{\mathfrak{p}}} \times \mathbb{B}^{\infty\mathfrak{p}\times}$ via the inflation $O^{\times}_{E_{\mathfrak{p}}} \to O^{\times}_{\mathfrak{p}}$ by $t \mapsto t/t^c$ and trivially on the second factor, then β is $O^{\times}_{E_{\mathfrak{p}}} \times \mathbb{B}^{\infty\mathfrak{p}\times}$-equivariant;*

(3) *for every $x \in F_{\mathfrak{p}}/O_{\mathfrak{p}}$, the following diagram*

$$\begin{array}{ccc}
\mathfrak{X}(\infty)\widehat{\otimes}_{O^{\mathrm{nr}}_{\mathfrak{p}}} F^{\mathrm{ab}}_{\mathfrak{p}} & \xrightarrow{\beta_{\nu_{\pm}(x)}} & \mathfrak{X}(\infty)\widehat{\otimes}_{O^{\mathrm{nr}}_{\mathfrak{p}}} F^{\mathrm{ab}}_{\mathfrak{p}} \\
\Big\downarrow{\scriptstyle \Upsilon^{-1}_{\pm}} & & \Big\downarrow{\scriptstyle \Upsilon^{-1}_{\pm}} \\
X(\pm\infty) \otimes_{F^{\mathrm{nr}}_{\mathfrak{p}}} F^{\mathrm{ab}}_{\mathfrak{p}} & \xrightarrow{\mathrm{T}_{\mathrm{n}^{\pm}(x)}} & X(\pm\infty) \otimes_{F^{\mathrm{nr}}_{\mathfrak{p}}} F^{\mathrm{ab}}_{\mathfrak{p}}
\end{array}$$

commutes, where

$$\mathrm{n}^{+}(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, \quad \mathrm{n}^{-}(x) = \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix};$$

and β_z is the restriction of β to a point z of $\mathscr{B}$.

In particular, $\mathcal{LT}$ acts trivially on the special fiber of $\mathfrak{X}(\infty)$ and such β is unique.

Definition 2.14. We put $\omega_\nu = c^*\nu^*\mathrm{d}T/T$, which is a nowhere zero global differential form on $\mathfrak{X}(\infty)$, that is, an element in $\mathrm{H}^0(\mathfrak{X}(\infty), \Omega^1_{\mathfrak{X}(\infty)})$. We call it a *global Lubin–Tate differential.* It is clear that the pullback $\Upsilon^*_\pm\omega_\nu$ depends only ψ (or rather $\psi^\pm$).

For $m \in \mathbb{N}\cup\{\infty\}$, $w \in \mathbb{Z}$ and a complete extension $K/F^{\mathrm{nr}}_{\mathfrak{p}}$, define the space of *$K$-valued convergent modular forms of weight w and $\mathfrak{p}$-Iwahori level m* to be

$$\mathscr{M}^w(m, K) = \mathrm{H}^0(\mathfrak{X}(m), (\Omega^1_{\mathfrak{X}(m)})^{\otimes w})\widehat{\otimes}_{O^{\mathrm{nr}}_{\mathfrak{p}}}K,$$

which is naturally a complete K-vector space. It has a natural action by $O^\times_{E_\mathfrak{p}} \times \mathbb{B}^{\infty\mathfrak{p}\times}$. A convergent modular form of weight 0 is simply called a *convergent modular function.* In particular, $\mathscr{M}^0(m, K)$ is the complete tensor product of the coordinate ring of $\mathfrak{X}(m)$ and the field K, and thus $\mathscr{M}^w(m, K)$ is naturally a topological $\mathscr{M}^0(m, K)$-module.

Define the subspace of *K-valued convergent modular forms of weight w and $\mathfrak{p}$-Iwahori level m and finite tame level* to be

$$\mathscr{M}^w_\flat(m, K) = \bigcup_{U^\mathfrak{p}\in\Gamma} \mathscr{M}^w(m, K)^{U^\mathfrak{p}} \subset \mathscr{M}^w(m, K),$$

which is stable under the action of $O^\times_{E_\mathfrak{p}} \times \mathbb{B}^{\infty\mathfrak{p}\times}$. The global Lubin–Tate differential ω_ν provides a canonical $\mathbb{B}^{\infty\mathfrak{p}\times}$-equivariant isomorphism $\mathscr{M}^w_\flat(\infty, K) \simeq \mathscr{M}^0_\flat(\infty, K)$ for $w \in \mathbb{Z}$. We will view $\mathscr{M}^w_\flat(m, K)$ as a subspace of $\mathscr{M}^0_\flat(\infty, K)$ for any $m \in \mathbb{N}\cup\{\infty\}$ and $w \in \mathbb{Z}$ via restriction. In particular for $w \in \mathbb{N}$, we have the *Atkin–Serre operator*

$$\Theta_{\mathrm{ord}}\colon \mathscr{M}^w_\flat(m, K) \to \mathscr{M}^{w+1}_\flat(m, K),$$

which is $O^\times_{E_\mathfrak{p}} \times \mathbb{B}^{\infty\mathfrak{p}\times}$-equivariant. Recall that both $O^\times_{E_\mathfrak{p}}$ and $\mathbb{B}^{\infty\mathfrak{p}\times}$ act on $\mathfrak{X}(\infty)$ and thus on $\mathscr{M}^0(\infty, K)$ for any complete extension $K/F^{\mathrm{nr}}_{\mathfrak{p}}$.

The morphism β induces a *translation map*

$$\beta^*_z\colon \mathscr{M}^0(\infty, K) \to \mathscr{M}^0(\infty, K)$$

for every $z \in \mathscr{B}(K)$.

Definition 2.15 (Stable convergent modular forms). A convergent modular function $f \in \mathscr{M}^0(\infty, K)$ is *stable* if

$$\sum_{z\in\mathscr{B}[\mathfrak{p}]} \beta^*_z f = 0.$$

Denote by $\mathscr{M}^0(\infty, K)^\heartsuit$ the subspace of $\mathscr{M}^0(\infty, K)$ of stable convergent modular functions. For every closed point $x \in \mathfrak{X}(\infty)(\kappa)$, we have the restriction map

$$\mathrm{res}_x\colon \mathscr{M}^0(\infty, K) \to \mathscr{O}(\mathscr{B}, K)$$

by Lemma 2.12. By Proposition 2.13, f is stable if and only if $\mathrm{res}_x f$ is stable (Definition 2.1) for all x.

For $m \in \mathbb{N}\cup\{\infty\}$ and $w \in \mathbb{Z}$, recall that we have viewed $\mathscr{M}^w(m, K)$ as a subspace of $\mathscr{M}^0(\infty, K)$ and thus we may define the space of *K-valued stable convergent modular forms of weight w, $\mathfrak{p}$-Iwahori level m and finite tame level* to be

$$\mathscr{M}^w_\flat(m, K)^\heartsuit = \mathscr{M}^w_\flat(m, K) \cap \mathscr{M}^0(\infty, K)^\heartsuit.$$

Remark 2.16. For $m \in \mathbb{N}$, the space $\mathscr{M}_\flat^w(m,K)$ (resp. $\mathscr{M}_\flat^w(m,K)^\heartsuit$) generalizes the notation of (the space of) convergent modular forms (resp. convergent modular forms of infinite slope) of weight $2w$ from modular curves to Shimura curves. Moreover, the space $\mathscr{M}_\flat^w(m,K)^\heartsuit$ does not depend on ψ or ν.

Definition 2.17 (Admissible convergent modular forms). For $n \in \mathbb{N}$, a stable convergent modular function $f \in \mathscr{M}^0(\infty,K)^\heartsuit$ is *n-admissible* if $\beta^*_{\nu_+(x)} f = \psi(x) f$ for all $x \in \mathfrak{p}^{-n}/O_\mathfrak{p}$. A stable convergent modular form $f \in \mathscr{M}_\flat^w(m,K)^\heartsuit$ is n-admissible if it is so, when regarded as an element in $\mathscr{M}^0(\infty,K)^\heartsuit$. By Proposition 2.13 (1), f is n-admissible if and only if $\mathrm{res}_x f$ is n-admissible (in the sense of Definition 2.4) for all x.

The following lemma is a comparison between the Atkin–Serre operator and the Lubin–Tate differential operator.

Lemma 2.18. *For a convergent modular form of weight w, $\mathfrak{p}$-Iwahori level m and finite tame level $f \in \mathscr{M}_\flat^w(m,K)$ for some $w, m \in \mathbb{N}$, we have*

$$\mathrm{res}_x(\Theta_{\mathrm{ord}} f) = \Theta(\mathrm{res}_x f)$$

for every $x \in \mathfrak{X}(\infty)(\kappa)$.

Proof. It follows from Lemma 2.12, Theorem B.5, and the definition of Θ (2.1). □

Definition 2.19 (Universal convergent modular form). A *universal convergent modular form* of depth $m \in \mathbb{N}$ and tame level $U^\mathfrak{p} \in \Gamma$ is an element $\mathbf{M} \in \mathscr{M}^0(\infty,K)\widehat{\otimes}_{F_\mathfrak{p}} D(O_\mathfrak{p}^{\mathrm{anti}}, F_\mathfrak{p})$ such that $\mathbf{M}$ is $U^\mathfrak{p}$-invariant and

$$t^*\mathbf{M} = [t]^{-1} \cdot \mathbf{M} \tag{2.13}$$

for $t \in O^\times_{E_\mathfrak{p},m}$.

Theorem 2.20. *Suppose K contains $F_\mathfrak{p}^{\mathrm{lt}}$. Let $f \in \mathscr{M}_\flat^w(m,K)^\heartsuit$ be a stable convergent modular form of weight w and $\mathfrak{p}$-Iwahori level m, for some $w, m \in \mathbb{N}$. Then there is a unique element $\mathbf{M}(f) \in \mathscr{M}^0(\infty,K)\widehat{\otimes}_{F_\mathfrak{p}} D(O_\mathfrak{p}^{\mathrm{anti}}, F_\mathfrak{p})$ such that for every $k \in \mathbb{N}$,*

$$\mathbf{M}(f)(\langle w+k\rangle) = \Theta^k_{\mathrm{ord}} f, \tag{2.14}$$

where in the target, we identify $\mathscr{M}_\flat^{w+k}(m,K)$ as a subspace of $\mathscr{M}^0(\infty,K)$. Moreover, we have

1. *if f is fixed by $U^\mathfrak{p} \in \Gamma$, so is $\mathbf{M}(f)$;*
2. *$\mathbf{M}(f)$ is a universal convergent modular form of depth m (Definition 2.19);*
3. *if $w \geq 1$, we have*
$$\Theta_{\mathrm{ord}}\mathbf{M}(f)(\langle w-1\rangle) = f;$$
4. *Suppose f is n-admissible. Then $\mathbf{M}(f)(\langle k\rangle) = \mathbf{M}(f)(\chi\langle k\rangle)$ any $k \in \mathbb{Z}$ and any (locally constant) character $\chi\colon O_\mathfrak{p}^{\mathrm{anti}} \to K^\times$ that is trivial on $(1+\mathfrak{p}^n)^\times$.*

We call $\mathbf{M}(f)$ the *global Mellin transform* of f.

Proof. The uniqueness is clear since the set $\{\langle w+k\rangle \mid k \geq 0\}$ spans a dense subspace of $C(O_\mathfrak{p}^{\mathrm{anti}}, F_\mathfrak{p})$. Regard f as an element in $\mathscr{M}^0(\infty,K)^\heartsuit$. Note that the map

$$\beta^*\colon \mathscr{M}^0(\infty,K) \to \mathscr{M}^0(\infty,K)\widehat{\otimes}_K \mathscr{O}(\mathscr{B},K)$$

sends $\mathscr{M}^0(\infty,K)^\heartsuit$ into $\mathscr{M}^0(\infty,K)^\heartsuit\widehat{\otimes}_K\mathscr{O}(\mathscr{B},K)^\heartsuit$. The element β^*f belongs to the space $\mathscr{M}^0(\infty,K)^\heartsuit\widehat{\otimes}_K\mathscr{O}(\mathscr{B},K)^\heartsuit$, which is isomorphic to $\mathscr{M}^0(\infty,K)^\heartsuit\widehat{\otimes}_{F_\mathfrak{p}}D(O_\mathfrak{p}^{\mathrm{anti}},F_\mathfrak{p})$ via λ since K contains $F_\mathfrak{p}^{\mathrm{lt}}$. Define a (continuous $F_\mathfrak{p}$-linear) translation map

$$\tau_w\colon D(O_\mathfrak{p}^{\mathrm{anti}},F_\mathfrak{p})\to D(O_\mathfrak{p}^{\mathrm{anti}},F_\mathfrak{p})$$

such that $(\tau_w\phi)(g)=\phi(g\cdot\langle -w\rangle)$ for every $g\in C(O_\mathfrak{p}^{\mathrm{anti}},F_\mathfrak{p})$. We take

$$\mathbf{M}(f)=\tau_w(\beta^*f).$$

The formula (2.14) follows from Lemma 2.3 and Lemma 2.18.

Property (1) follows from Proposition 2.13 (2). Property (3) and (4) follow from Lemma 2.3 and Lemma 2.5, respectively. For property (2), we only need to show that (2.13) holds for $\mathbf{M}=\mathbf{M}(f)$ and $t\in O_{E_\mathfrak{p},m}^\times$. In fact, since f is fixed by $O_{E_\mathfrak{p},m}^\times$, $\mathbf{M}(f)=\mathbf{M}(t^*f)$ which equals $[t]\cdot t^*\mathbf{M}(f)$ by Proposition 2.13 (2) and Remark 2.2. □

2.4. **Comparison of differential operators at archimedean places.** We equip $\mathbb{B}$ with an E-embedding (Definition 1.3). In particular, we have the CM-subscheme $Y^\pm$ of X (Definition 1.4). The projection $X\to X(\pm\infty)$ restricts to an isomorphic from $Y^\pm$ to its image. Thus we may regard $Y^\pm$ as a closed subscheme of $X(\pm\infty)$. Via the transition isomorphism (2.12), we obtain a closed subscheme $Y^\pm(\infty)=\Upsilon_\pm Y^\pm$ of $\mathcal{X}(\infty)\otimes_{O_\mathfrak{p}^{\mathrm{nr}}}F_\mathfrak{p}^{\mathrm{ab}}$, which is in fact a closed subscheme of $\mathcal{X}(\infty)\otimes_{O_\mathfrak{p}^{\mathrm{nr}}}F_\mathfrak{p}^{\mathrm{nr}}$. Finally for $m\in\mathbb{N}$, define $Y^\pm(m)$ to be the image of $Y^\pm(\infty)$ in $\mathcal{X}(m)\otimes_{O_\mathfrak{p}^{\mathrm{nr}}}F_\mathfrak{p}^{\mathrm{nr}}$.

Let S be a complex scheme locally of finite type. We denote by $\breve{S}$ the underlying real analytic space with the complex conjugation automorphism $\mathfrak{c}_S\colon\breve{S}\to\breve{S}$. In what follows, we will sometimes deal with a complex scheme S that is of the form $\varprojlim_I S_i$ where I is a filtered partially ordered set and each S_i is a smooth complex scheme, with a sheaf $\mathscr{F}$ that is the restriction of a quasi-coherent sheaf $\mathscr{F}_0$ on some S_0. Then we will write $\breve{S}=\{\breve{S}_i\}_{i\in I}$ for the projective system of the underlying real analytic spaces together with the complex conjugation $\mathfrak{c}_S$, and $\breve{\mathscr{F}}=\{\breve{\mathscr{F}}_i\}_{i\geq 0}$ the projective system of real analytification of the restricted sheaf $\mathscr{F}_i$ for $i\geq 0$. Moreover, we define

$$\mathrm{H}^0(\breve{S},\breve{\mathscr{F}}):=\varinjlim_{i\geq 0}\mathrm{H}^0(\breve{S}_i,\breve{\mathscr{F}}_i).$$

For $\iota\colon\mathbb{C}_p\xrightarrow{\sim}\mathbb{C}$, put $X_\iota=X\otimes_{F,\iota}\mathbb{C}$ and $\mathfrak{c}_\iota\colon\breve{X}_\iota\to\breve{X}_\iota$ the complex conjugation. Denote by $(\mathcal{L}_\iota,\nabla_\iota)$ the restriction of the pair $(\mathcal{L},\nabla)$ in Lemma 2.7 along $\pi_\iota\colon X_\iota\to\mathcal{X}\otimes_{O_\mathfrak{p}^{\mathrm{nr}},\iota}\mathbb{C}$. Denote the restriction of the sequence (2.6) along π_ι by

$$0\longrightarrow\underline{\omega}_\iota^\bullet\longrightarrow\mathcal{L}_\iota\longrightarrow\underline{\omega}_\iota^{\circ\vee}\longrightarrow 0.\tag{2.15}$$

The following lemma will be proved in §2.5.

Lemma 2.21. *The sequence (2.15) coincides with the Hodge filtration on $\mathcal{L}_\iota$.*

Let $\underline{\omega}_\iota^\circ$ be the restriction of $\underline{\omega}^\circ$ along π_ι, which is the dual sheaf of $\underline{\omega}_\iota^{\circ\vee}$. Then we have the Kodaira–Spencer isomorphism $\mathrm{KS}\colon\underline{\omega}_\iota^\bullet\otimes\underline{\omega}_\iota^\circ\xrightarrow{\sim}\Omega^1_{X_\iota}$. The Hodge decomposition

$$\mathcal{L}_\iota=\breve{\underline{\omega}}_\iota^\bullet\oplus\mathfrak{c}_\iota^*\breve{\underline{\omega}}_\iota^\bullet\tag{2.16}$$

on X_ι induces a map

$$\theta_\iota^{[w]}\colon\breve{\mathcal{L}}_\iota^{[w]}\to(\breve{\underline{\omega}}_\iota^\bullet)^{\otimes w}\otimes(\breve{\underline{\omega}}_\iota^\circ)^{\otimes w}\xrightarrow{\mathrm{KS}}(\breve{\Omega}^1_{X_\iota})^{\otimes w}$$

for all $w \in \mathbb{N}$. Similar to Definition 2.11, define the *Shimura–Maass operator* to be

$$\Theta_\iota^{[w]} \colon (\breve{\Omega}^1_{X_\iota})^{\otimes w} \xrightarrow{(2.9)} \breve{\mathcal{L}}^{[w]} \otimes \breve{\Omega}^1_{X_\iota} \xrightarrow{\theta_\iota^{[w]}} (\breve{\Omega}^1_{X_\iota})^{\otimes w+1}.$$

For $k \in \mathbb{N}$, define the *Shimura–Maass operator of degree k* to be

$$\Theta_\iota^{[w,k]} = \Theta_\iota^{[w+k-1]} \circ \cdots \circ \Theta_\iota^{[w]} \colon (\breve{\Omega}^1_{X_\iota})^{\otimes w} \to (\breve{\Omega}^1_{X_\iota})^{\otimes w+k}.$$

As for Θ_{ord}, we will suppress w from notation and write Θ_ι (resp. Θ_ι^k) for $\Theta_\iota^{[w]}$ (resp. $\Theta_\iota^{[w,k]}$). In particular, we have

$$\Theta_\iota \colon \mathrm{H}^0(\breve{X}_\iota, (\breve{\Omega}^1_{X_\iota})^{\otimes w}) \to \mathrm{H}^0(\breve{X}_\iota, (\breve{\Omega}^1_{X_\iota})^{\otimes w+1}).$$

Put

$$X(m)_\iota = X(m) \otimes_{F_\mathfrak{p}^{\mathrm{nr}},\iota} \mathbb{C}, \quad m \in \mathbb{Z} \cup \{\pm\infty\},$$
$$Y^\pm(m)_\iota = Y^\pm(m) \otimes_{F_\mathfrak{p}^{\mathrm{nr}},\iota} \mathbb{C}, \quad m \in \mathbb{N} \cup \{\infty\}.$$

Then $Y^\pm(m)_\iota$ is a closed subscheme of $X(\pm m)_\iota$ via the transition isomorphism (2.12). Denote by $\mathcal{Y}^\pm(m)$ the Zariski closure of $Y^\pm(m)$ in $\mathcal{X}(m)$.

Lemma 2.22. *For $m \in \mathbb{N} \cup \{\infty\}$, every morphism $\operatorname{Spec} F_\mathfrak{p}^{\mathrm{nr}} \to \mathcal{Y}^\pm(m)$ over $\operatorname{Spec} O_\mathfrak{p}^{\mathrm{nr}}$ extends uniquely to a section $\operatorname{Spec} O_\mathfrak{p}^{\mathrm{nr}} \to \mathcal{Y}^\pm(m)$.*

Proof. It suffices to show for $m = \infty$. Let $x^\pm \colon \operatorname{Spec} F_\mathfrak{p}^{\mathrm{nr}} \to \mathcal{Y}^\pm(\infty)$ be a morphism. It induces a unique morphism $\operatorname{Spec} O_\mathfrak{p}^{\mathrm{nr}} \to \mathcal{X}$ and we will regard $x^\pm$ as the latter one. Since $x^\pm$ is fixed by $E^\times$, there are actions of $E^\times \cap O_{E_\mathfrak{p}}^\times$ and hence $O_{E_\mathfrak{p}}$ on the $O_\mathfrak{p}$-divisible group $\mathcal{G}_{x^\pm}$. Therefore, the reduction of $\mathcal{G}_{x^\pm}$ is ordinary. To conclude, we only need to show that $x^\pm$ lifts the canonical subgroup of $\mathcal{G}_{x^\pm}$. This follows from the fact that $E^\times$ acts on the tangent space of $x^\pm$ via the character $t \mapsto (t/t^c)^{\pm 1}$. ☐

For $m \in \mathbb{N} \cup \{\infty\}$, denote by $\mathfrak{Y}^\pm(m)$ the formal completion of $\mathcal{Y}^\pm(m)$ along its special fiber, which is a closed (affine) formal subscheme of $\mathfrak{X}(m)$ by the above lemma.

et $F_\mathfrak{p}^{\mathrm{ab}} \subset K \subset \mathbb{C}_p$ be a complete intermediate field. Let f be an element in $\mathrm{H}^0(X(m), (\Omega^1_{X(m)})^{\otimes w}) \otimes_F K$ with $m \in \mathbb{Z} \cup \{\pm\infty\}$ and $w \in \mathbb{N}$. Then by the transition isomorphism (2.12) and restriction to ordinary locus, we have an element

$$f_{\mathrm{ord}} = \Upsilon_{\pm *} f \in \mathscr{M}_\flat^w(m, K). \tag{2.17}$$

On the other hand, we have the projection map $X_\iota \to X(m)_\iota$ for which Θ_ι descends. Thus, f induces another element

$$f_\iota \in \mathrm{H}^0(X(m)_\iota, (\Omega^1_{X(m)_\iota})^{\otimes w}). \tag{2.18}$$

We will freely regard f_ι as an element in $\mathrm{H}^0(X_\iota, (\Omega^1_{X_\iota})^{\otimes w})$ according to the context. The following lemma shows that the Atkin–Serre operator and the Shimura–Maass operator coincide on CM points.

Lemma 2.23. *Let the situation be as above. We have for $k \in \mathbb{N}$,*

$$\iota(\Theta_{\mathrm{ord}}^k f_{\mathrm{ord}})|_{\mathfrak{Y}^\pm(m)} = (\Theta_\iota^k f_\iota)|_{Y^\pm(m)_\iota}$$

as functions on $Y^\pm(m)_\iota$.

Proof. Generally, once we restrict to stalks, we can not apply differential operators anymore. Therefore, we need variant definitions of $\Theta_{\mathrm{ord}}^{[w,k]}$ and $\Theta_{\iota}^{[w,k]}$ (Here, we retrieve the original notation in order to be clear). In fact, since $\nabla\mathcal{L}^\circ \subset \mathcal{L}^\circ \otimes \Omega^1_{\mathfrak{X}}$ by Lemma 2.10, $\Theta_{\mathrm{ord}}^{[w,k]}$ is equal to the composition of the following sequence of maps

$$(\Omega^1_{\mathfrak{X}(n)})^{\otimes w} \xrightarrow{\mathrm{KS}^{-1}} (\underline{\omega}^\bullet)^{\otimes w} \otimes (\underline{\omega}^\circ)^{\otimes w} \to \mathcal{L}^{[w]} \xrightarrow{\Theta^{[w]}} \mathcal{L}^{[w]} \otimes \Omega^1_{\mathfrak{X}(n)} \xrightarrow{\mathrm{KS}^{-1}} \mathcal{L}^{[w]} \otimes (\underline{\omega}^\bullet \otimes \underline{\omega}^\circ)$$
$$\to \mathcal{L}^{[w+1]} \xrightarrow{\Theta^{[w+1]}} \mathcal{L}^{[w+1]} \otimes \Omega^1_{\mathfrak{X}(n)} \to \cdots \to \mathcal{L}^{[w+k]} \xrightarrow{\theta_{\mathrm{ord}}^{[w+k]}} (\Omega^1_{\mathfrak{X}(n)})^{\otimes w+k}.$$

Similarly, since $\nabla_\iota(\mathfrak{c}_\iota^*\breve{\underline{\omega}}_\iota^\bullet) \subset (\mathfrak{c}_\iota^*\breve{\underline{\omega}}_\iota^\bullet)\otimes\breve{\Omega}^1_{X_\iota}$, we have a similar description of $\Theta_\iota^{[w,k]}$. Therefore, to prove the lemma, we only need to show that the splitting (2.16) coincides with the restriction of the splitting $\mathcal{L} = \underline{\omega}^\bullet \oplus \mathcal{L}^\circ$ on $Y_\iota^\pm$. Pick up any point $y \in Y^\pm(m)_\iota(\mathbb{C})$. We have an action of $E^\times$ on both the splitting $\breve{\underline{\omega}}_\iota^\bullet|_y \oplus \mathfrak{c}_\iota^*\breve{\underline{\omega}}_\iota^\bullet|_y$ and $\underline{\omega}^\bullet|_y \oplus \mathcal{L}^\circ|_y$. By Lemma 2.21, $\breve{\underline{\omega}}_\iota^\bullet|_y$ and $\underline{\omega}^\bullet|_y$ coincide, which is one complex eigen-line of $E^\times$. We have that $\mathfrak{c}_\iota^*\breve{\underline{\omega}}_\iota^\bullet|_y$ and $\mathcal{L}^\circ|_y$ must also coincide, which is the other complex eigen-line. □

Definition 2.24 (ι-nearby data). For $\iota\colon \mathbb{C}_p \xrightarrow{\sim} \mathbb{C}$, an *$\iota$-nearby data* for $\mathbb{B}$ consists of

- a quaternion algebra $B(\iota)$ over F such that $B(\iota)_v$ is definite for archimedean places v other than $\iota|_F$,
- an isomorphism $B(\iota)_v \simeq \mathbb{B}_v$ for every finite place v other than $\mathfrak{p}$,
- an isomorphism $B(\iota)_\iota = B(\iota) \otimes_{F,\iota} \mathbb{R} \simeq \mathrm{Mat}_2(\mathbb{R})$,
- a uniformization
$$X_\iota(\mathbb{C}) \simeq B(\iota)^\times\backslash\mathcal{H} \times \mathbb{B}^{\infty\times}/F^\times_{\mathrm{cl}},$$
where $\mathcal{H} = \mathbb{C} \setminus \mathbb{R}$ denotes the union of Poincaré upper and lower half-planes,
- an embedding $\mathsf{e}(\iota)\colon E \hookrightarrow B(\iota)$ of F-algebras such that $\mathsf{e}(\iota)_v$ coincides with e_v under the isomorphism $B(\iota)_v \simeq \mathbb{B}_v$ for every finite place v other than $\mathfrak{p}$, and $\mathcal{H}^{E^\times} = \{\pm i\}$.

We now choose an ι-nearby data for $\mathbb{B}$. For every $w \in \mathbb{Z}$, denote by $\mathscr{A}^{(2w)}(B(\iota)^\times)$ (resp. $\mathscr{A}^{(2w)}_{\mathrm{cusp}}(B(\iota)^\times)$) the space of real analytic (resp. and cuspidal) automorphic forms on $B(\iota)^\times(\mathbb{A})$ of weight $2w$ at $\iota|_F$ and trivial at other archimedean places. There is a natural $\mathbb{B}^{\infty\times}$-equivariant map

$$\phi_\iota\colon \mathrm{H}^0(\breve{X}_\iota, (\breve{\Omega}^1_{X_\iota})^{\otimes w}) \to \mathscr{A}^{(2w)}(B(\iota)^\times) \tag{2.19}$$

such that for $g_\iota \in B(\iota)^\times_\iota = \mathrm{GL}_2(\mathbb{R})$,

$$\phi_\iota(f)([g_\iota, 1])j(g_\iota, i)^w = f(g_\iota(i)) \otimes \mathrm{d}z^{\otimes -w},$$

where $j(g_\iota, i) = (\det g_\iota)^{-1} \cdot (ci + d)^2$ is the square of the usual j-factor. We denote by $\mathrm{H}^0_{\mathrm{cusp}}(\breve{X}_\iota, (\breve{\Omega}^1_{X_\iota})^{\otimes w}) \subset \mathrm{H}^0(\breve{X}_\iota, (\breve{\Omega}^1_{X_\iota})^{\otimes w})$ the inverse image of $\mathscr{A}^{(2w)}_{\mathrm{cusp}}(B(\iota)^\times)$ under ϕ_ι.

We may recover the pair $(\mathcal{L}_\iota, \nabla_\iota)$ in the following way. Denote by L_ι the $\mathbb{C}$-local system on X_ι defined by the quotient $B(\iota)^\times\backslash\mathbb{C}^{\oplus 2} \times \mathcal{H} \times \mathbb{B}^{\infty\times}/F^\times_{\mathrm{cl}}$ where the action of $B(\iota)^\times$ is given by

$$\gamma[(a_1, a_2)^t, z, g] = [((a_1, a_2)\iota(\gamma)^{-1})^t, \iota(\gamma)(z), \gamma^\infty g].$$

Then L_ι is canonically isomorphic to the restriction of $\mathrm{L}\otimes_{O_\mathfrak{p},\iota}\mathbb{C}$ along the natural morphism π_ι, where L is the $O_\mathfrak{p}$-local system on $\mathcal{X}$ defined in Remark 2.8. Thus, $\mathcal{L}_\iota = \mathscr{O}_{X_\iota} \otimes_\mathbb{C} \mathrm{L}_\iota$ and $\nabla_\iota\colon \mathcal{L}_\iota \to \mathcal{L}_\iota \otimes \Omega^1_{X_\iota}$ is the induced connection.

The following lemma shows our definition of Shimura–Maass operators coincide with the classical one.

Lemma 2.25. *For every* $f \in \mathrm{H}^0(\breve{X}_\iota, (\breve{\Omega}^1_{X_\iota})^{\otimes w})$ *with some* $w \in \mathbb{N}$*, we have*

$$\Theta_\iota f \otimes \mathrm{d}z^{\otimes -w-1} = \left(\frac{\partial}{\partial z} + \frac{2w}{z-\overline{z}}\right) f \otimes \mathrm{d}z^{\otimes -w}.$$

Proof. We may pass to the universal cover $\mathcal{H} \times \mathbb{B}^{\infty\times}/F^\times_{\mathrm{cl}}$ and suppress the part $\mathbb{B}^{\infty\times}/F^\times_{\mathrm{cl}}$ in what follows. Over $\mathcal{H}$, the sheaf $\mathcal{L}_\iota$ is trivialized as $\mathbb{C}^{\oplus 2}$ and the sub-sheaf $\underline{\omega}^\bullet_\iota$ is generated by the section $\omega^\bullet_\iota$ whose value at z is $(z,1)^t$. Dually, the sheaf $\mathcal{L}^\vee_\iota$ is trivialized as two-dimensional complex row vectors and the sub-sheaf $\underline{\omega}^\circ_\iota$ is generated by the section ω°_ι whose value at z is $(1,-z)$. Then $\mathrm{KS}(\omega^\bullet_\iota \otimes \omega^\circ_\iota) = \mathrm{d}z$.

It is easy to see that

$$\Theta_\iota\left((\omega^\bullet_\iota)^{\otimes w} \otimes (\omega^\circ_\iota)^{\otimes w}\right) = \frac{2w}{z-\overline{z}}\left((\omega^\bullet_\iota)^{\otimes w} \otimes (\omega^\circ_\iota)^{\otimes w}\right) \otimes \mathrm{d}z$$

since $\mathfrak{c}^*_\iota\underline{\omega}^\bullet_\iota$ (resp. $\mathfrak{c}^*_\iota\underline{\omega}^\circ_\iota$) is generated by the section $(\overline{z},1)^t$ (resp. $(1,-\overline{z})$). The lemma follows. □

Denote by $\Delta_{\pm,\iota}$ the element

$$\Delta_\pm := \frac{1}{4i}\begin{pmatrix} 1 & \pm i \\ \pm i & 1 \end{pmatrix}$$

in $\mathfrak{gl}_{2,\mathbb{C}} = \mathrm{Mat}_2(\mathbb{C}) = \mathrm{Lie}_\mathbb{C}(B(\iota)\otimes_{F,\iota}\mathbb{C})$, and $\Delta^k_{\pm,\iota} = \Delta_{\pm,\iota}\circ\cdots\circ\Delta_{\pm,\iota}$ the k-fold composition.

Lemma 2.26. *For every* $f \in \mathrm{H}^0_{\mathrm{cusp}}(\breve{X}_\iota, (\breve{\Omega}^1_{X_\iota})^{\otimes w})$ *and* $k \in \mathbb{N}$*, we have*

$$\phi_\iota(\Theta^k_\iota f) = \Delta^k_{+,\iota}\phi_\iota(f).$$

Proof. This follows from Lemma 2.25, [Bum97] p.130, p.143, and Proposition 2.2.5 on p.155. □

2.5. **Proof of claims.** In this section, we prove six claims (2.7, 2.9, 2.10, 2.12, 2.13, and 2.21) left in previous sections. All these claims are natural extensions from their versions on the modular curve. The reader may skip this section for the first reading.

Our strategy is to use the unitary Shimura curves considered by Carayol in [Car86]. Thus we will fix an isomorphism $\iota\colon \mathbb{C}_p \xrightarrow{\sim} \mathbb{C}$. In particular, $F^{\mathrm{nr}}_\mathfrak{p}$ is a subfield of $\mathbb{C}$. We also fix an ι-nearby data for $\mathbb{B}$ (Definition 2.24) and put $B = B(\iota)$ for short.

Note that when $F = \mathbb{Q}$ there is no need to change the Shimura curve. In order to unify the argument, we will choose to do so in this case as well. We will also assume that we are not in the case of classical modular curves where all these statements are well-known.

Fix an element $\lambda \in \mathbb{C}$ such that $\mathrm{Im}\,\lambda > 0$, $-\lambda^2 \in \mathbb{N}$, $p \neq \lambda^2$, p splits in $\mathbb{Q}(\lambda) \subset \mathbb{C}$, and $\mathbb{Q}(\lambda)$ is not contained in E. Put $'F = F(\lambda)$ and $'E = E(\lambda)$ both as subfields $\mathbb{C}$. We identify the completion of $'F$ and $'E$ inside $\mathbb{C} \simeq \mathbb{C}_p$ with $F_\mathfrak{p}$. In [Car86, §2] (see also [Kas04, §2]), a reductive group $'G$ over $\mathbb{Q}$ is defined such that

$$'G(\mathbb{Q}_p) = \mathbb{Q}^\times_p \times \mathrm{GL}_2(F_\mathfrak{p}) \times (\mathbb{B}^\times_{\mathfrak{p}_2} \times \cdots \times \mathbb{B}^\times_{\mathfrak{p}_m}),$$

where $\mathfrak{p}_2, \ldots, \mathfrak{p}_m$ are primes of F over p over than $\mathfrak{p}$. Let $'G^\mathfrak{p} = \prod'_{q\neq p} {'G}(\mathbb{Q}_q) \times (\mathbb{B}^\times_{\mathfrak{p}_2} \times \cdots \times \mathbb{B}^\times_{\mathfrak{p}_m})$ and $'\Gamma$ the set of all (sufficiently small) compact open subgroups $'U^\mathfrak{p}$ of $'G^\mathfrak{p}$.

Then for each ${}'U^{\mathfrak{p}} \in {}'\Gamma$, there is a unitary Shimura curve ${}'X_{'U^{\mathfrak{p}}}$ over $\operatorname{Spec} F_{\mathfrak{p}}$ of the level structure $\mathbb{Z}_p^\times \times \mathrm{GL}_2(O_{\mathfrak{p}}^\times) \times {}'U^{\mathfrak{p}}$, which is smooth and projective. It has a canonical smooth model ${}'\mathcal{X}_{'U^{\mathfrak{p}}}$ over $\operatorname{Spec} O_{\mathfrak{p}}^{\mathrm{nr}}$ defined via a moduli problem [Car86, §6]. In particular, there is a universal abelian variety $\pi\colon \mathcal{A}_{'U^{\mathfrak{p}}} \to {}'\mathcal{X}_{'U^{\mathfrak{p}}}$ with a specific p-divisible subgroup ${}'\mathcal{G}_{'U^{\mathfrak{p}}} \subset \mathcal{A}_{'U^{\mathfrak{p}}}[p^\infty]$ that is naturally an $O_{\mathfrak{p}}$-divisible group of dimension 1 and height 2. Denote ${}'\mathcal{X}(0)_{'U^{\mathfrak{p}}}$ the (dense) open subscheme of ${}'\mathcal{X}_{'U^{\mathfrak{p}}}$ with all points on the special fiber where ${}'\mathcal{G}$ is supersingular removed. For $n \in \mathbb{N}$, define ${}'\mathcal{X}(n)_{'U^{\mathfrak{p}}}$ to be the functor classifying $O_{\mathfrak{p}}$-equivariant extensions

$$0 \longrightarrow \mathcal{LT}[\mathfrak{p}^n] \longrightarrow {}'\mathcal{G}[\mathfrak{p}^n] \longrightarrow \mathfrak{p}^{-n}/O_{\mathfrak{p}} \longrightarrow 0$$

of ${}'\mathcal{G}$ over ${}'\mathcal{X}(0)_{'U^{\mathfrak{p}}}$, which is a scheme étale over ${}'\mathcal{X}(0)_{'U^{\mathfrak{p}}}$.

Then the construction of Carayol amounts to saying that for every sufficiently small $U^{\mathfrak{p}} \in \Gamma$ and a connected component $\mathcal{X}'_{U^{\mathfrak{p}}}$ of $\mathcal{X}_{U^{\mathfrak{p}}}$, there exists a ${}'U^{\mathfrak{p}} \in {}'\Gamma$ such that

- $\mathcal{X}(n)'_{U^{\mathfrak{p}}} := \mathcal{X}'_{U^{\mathfrak{p}}} \times_{\mathcal{X}_{U^{\mathfrak{p}}}} \mathcal{X}(n)_{U^{\mathfrak{p}}}$ is isomorphic to the identity connected component of ${}'\mathcal{X}(n)_{'U^{\mathfrak{p}}}$ for $n \in \mathbb{N}$;
- under the above isomorphism $\mathcal{G}_{U^{\mathfrak{p}}}|_{\mathcal{X}(n)'_{U^{\mathfrak{p}}}}$ is isomorphic to the restriction of ${}'\mathcal{G}_{'U^{\mathfrak{p}}}$ to $\mathcal{X}(n)'_{U^{\mathfrak{p}}}$.

In what follows we may and will fix a sufficiently small $U^{\mathfrak{p}} \in \Gamma$, a connected component $\mathcal{X}'_{U^{\mathfrak{p}}}$ of $\mathcal{X}_{U^{\mathfrak{p}}}$, and a corresponding ${}'U^{\mathfrak{p}} \in {}'\Gamma$. The tame levels $U^{\mathfrak{p}}$ and ${}'U^{\mathfrak{p}}$ will be suppressed from the notation.

Consider the Hodge sequence

$$0 \longrightarrow \pi_*\Omega^1_{\mathcal{A}/{}'\mathcal{X}} \longrightarrow \mathscr{H}^1_{\mathrm{dR}}(\mathcal{A}/{}'\mathcal{X}) \longrightarrow \mathrm{R}^1\pi_*\mathscr{O}_{\mathcal{A}} \longrightarrow 0.$$

It has a direct summand

$$0 \longrightarrow (\pi_*\Omega^1_{\mathcal{A}/{}'\mathcal{X}})_1^{2,1} \longrightarrow \mathscr{H}^1_{\mathrm{dR}}(\mathcal{A}/{}'\mathcal{X})_1^{2,1} \longrightarrow (\mathrm{R}^1\pi_*\mathscr{O}_{\mathcal{A}})_1^{2,1} \longrightarrow 0 \tag{2.20}$$

which is $O_{\mathfrak{p}}$-equivariant (see [Car86, 4.2] for the meaning of $(-)_1^{2,1}$), in which the three sheaves are locally constant of rank 1, $2h$, and $2h-1$, respectively, where $h = [F_{\mathfrak{p}}\colon \mathbb{Q}_p]$.

If M is a projective $O_{\mathfrak{p}}$-module or a locally free sheaf on an $O_{\mathfrak{p}}$-scheme, equipped with an $O_{\mathfrak{p}}$-action $O_{\mathfrak{p}} \to \operatorname{End} M$, then we denote by $M^{O_{\mathfrak{p}}}$ the maximal submodule or subsheaf on which $O_{\mathfrak{p}}$ acts via the structure homomorphism. Then by construction, if we apply the functor $(-)^{O_{\mathfrak{p}}}$ and take formal completion (after restriction to $\mathcal{X}$) to (2.20), we will recover the the exact sequence (2.6) in Lemma 2.7. For later use, we denote the sequence (2.20) after $(-)^{O_{\mathfrak{p}}}$ by

$$0 \longrightarrow {}'\underline{\omega}^\bullet \longrightarrow {}'\mathcal{L} \longrightarrow {}'\underline{\omega}^{\circ\vee} \longrightarrow 0.$$

Moreover, we have the Gauss–Manin connection

$${}'\nabla_p\colon \mathscr{H}^1_{\mathrm{dR}}(\mathcal{A}/{}'\mathcal{X}) \to \mathscr{H}^1_{\mathrm{dR}}(\mathcal{A}/{}'\mathcal{X}) \otimes \Omega^1_{'\mathcal{X}}.$$

By the functoriality of the Gauss–Manin connection, we have an induced connection

$${}'\nabla\colon {}'\mathcal{L} \to {}'\mathcal{L} \otimes \Omega^1_{'\mathcal{X}},$$

whose formal completion (after restriction to $\mathcal{X}'$) coincides with (2.5). Therefore, Lemma 2.7 is proved.

Denote by ${}'\mathrm{KS}\colon {}'\underline{\omega}^\bullet \otimes {}'\underline{\omega}^\circ \to \Omega^1_{{}'\mathcal{X}}$ the induced Kodaira–Spencer map, where ${}'\underline{\omega}^\circ$ is the dual sheaf of ${}'\underline{\omega}^{\circ\vee}$. Proposition 2.9 follows from the following analogous one for ${}'\mathcal{X}$.

Lemma 2.27. *The Kodaira–Spencer map* ${}'\mathrm{KS}\colon {}'\underline{\omega}^\bullet \otimes {}'\underline{\omega}^\circ \to \Omega^1_{{}'\mathcal{X}}$ *is an isomorphism.*

Proof. The proof is similar to [DT94, Lemma 7]. Denote by $\mathcal{A}^\vee$ the dual abelian variety of $\mathcal{A}$. Then ${}'\underline{\omega}^{\circ\vee}$ is canonically isomorphic to $(\underline{\mathrm{Lie}}(\mathcal{A}^\vee/{}'\mathcal{X})_1^{2,1})^{O_\mathfrak{p}}$. We only need to show that for every closed point $t\colon \operatorname{Spec} k(t) \to {}'\mathcal{X}$, the induced map

$$ {}'\underline{\omega}^\bullet \otimes k(t) \to (\underline{\mathrm{Lie}}(\mathcal{A}^\vee/{}'\mathcal{X})_1^{2,1})^{O_\mathfrak{p}} \otimes \Omega^1_{{}'\mathcal{X}} \otimes k(t) \tag{2.21}$$

is surjective, where $\underline{\mathrm{Lie}}$ denotes the sheaf of tangent vectors.

Let $A/\operatorname{Spec} k(t)$ be the abelian variety classified by t. Put $T = \operatorname{Spec} k(t)[\varepsilon]/(\varepsilon^2)$. The lifts A_ϕ of A (with other PEL structures) to T correspond to homomorphisms

$$\phi\colon t^*\,{}'\underline{\omega}^\bullet \to (\underline{\mathrm{Lie}}(\mathcal{A}^\vee/{}'\mathcal{X})_1^{2,1})^{O_\mathfrak{p}} \otimes k(t).$$

Since both sides are $k(t)$-vector spaces of dimension 1, we may choose a ϕ that is surjective. Let $t_\phi\colon T \to {}'\mathcal{X}$ be the morphism that classifies A_ϕ/T. Compose the isomorphism $t_\phi^*\,{}'\underline{\omega}^\bullet \otimes k(t) \to t^*\,{}'\underline{\omega}^\bullet$ and the surjective map ϕ. By the isomorphism

$$(\underline{\mathrm{Lie}}(\mathcal{A}^\vee/{}'\mathcal{X})_1^{2,1})^{O_\mathfrak{p}} \otimes k(t) \simeq t_\phi^*(\underline{\mathrm{Lie}}(\mathcal{A}^\vee/{}'\mathcal{X})_1^{2,1})^{O_\mathfrak{p}} \otimes \Omega^1_{T/k(t)} \otimes k(t),$$

we obtain a surjective map

$$t_\phi^*\,{}'\underline{\omega}^\bullet \otimes k(t) \to t_\phi^*(\underline{\mathrm{Lie}}(\mathcal{A}^\vee/{}'\mathcal{X})_1^{2,1})^{O_\mathfrak{p}} \otimes \Omega^1_{T/k(t)} \otimes k(t),$$

which is the pullback of (2.21) under t_ϕ. Therefore, (2.21) is surjective. □

For $n \in \mathbb{N}$, denote by ${}'\mathfrak{X}(n)$ the formal completion along its special fiber, which is equipped with an $O_\mathfrak{p}$-divisible group ${}'\mathfrak{G}$ induced from ${}'\mathcal{G}$. Let ${}'\mathfrak{G}_{\mathrm{can}} \subset {}'\mathfrak{G}$ be the canonical subgroup. We have a morphism ${}'\Phi\colon {}'\mathfrak{X} \to {}'\mathfrak{X}$ defined by "dividing ${}'\mathfrak{G}_{\mathrm{can}}[\mathfrak{p}]$" which lifts the relative Frobenius on the special fiber. In fact, the induced map on the coordinate ring is simply the operator Frob defined in [Kas04, Definition 11.1].

Proof of Lemma 2.10. We only need to prove the same statement for ${}'\mathcal{X}$. Then we define ${}'\mathcal{L}^\circ$ to be the sub-sheaf of ${}'\mathcal{L}$ where ${}'\Phi$ acts by a p-adic unit. To show that it glues to a formal quasi-coherent sheaf, we may adopt the proof of [Kat73, Theorem 4.1] in the case where $\mathbb{Z}_p$ is replaced by $O_\mathfrak{p}$ and p is replaced by a uniformizer ϖ of F. The assumptions are satisfied because the Newton polygon of the underlying p-divisible group of ${}'\mathfrak{G}|_x$ for any $x \in {}'\mathfrak{X}(\kappa)$ is the one starting with $(0,0)$, ending with $(2h,1)$ and having the unique breaking point at $(h,0)$. The similar proof also confirms (1) and (2). For (3), we use the local calculation in Lemma B.9. □

Proof of Lemma 2.12 and Lemma 2.21. We only need to prove the similar statements for ${}'\mathfrak{X}$, which follow from the moduli interpretation of ${}'\mathfrak{X}$ and the Serre–Tate theorem for Lemma 2.12, and the existence of the universal abelian variety $\mathcal{A}$ for Lemma 2.21, respectively. □

Proof of Proposition 2.13. We may similarly define $'\mathfrak{X}(\infty)$ over $\operatorname{Spf} O_{\mathfrak{p}}^{\mathrm{nr}}$ and only need to construct the morphism $'\beta\colon \mathcal{LT}\times_{\operatorname{Spf} O_{\mathfrak{p}}^{\mathrm{nr}}}{}'\mathfrak{X}(\infty)\to{}'\mathfrak{X}(\infty)$ with similar properties, since the action of $\mathcal{LT}$ is supposed to preserve the special fiber. We use moduli interpretation. For a scheme S over $\operatorname{Spec} O_{\mathfrak{p}}^{\mathrm{nr}}$ where p is locally nilpotent, $'\mathfrak{X}(\infty)(S)$ is the set of isomorphism classes of quintuples $(A,\iota,\theta,k^{\mathfrak{p}},\kappa_{\mathfrak{p}})$, where $(A,\iota,\theta,k^{\mathfrak{p}})$ is the same data in [Car86, §5.2] but $k^{\mathfrak{p}}$ is an isomorphism instead of a class, and $\kappa_{\mathfrak{p}}$ is an exact sequence

$$0\longrightarrow \mathcal{LT}\longrightarrow (A_{p^\infty})_1^{2,1}\longrightarrow F_{\mathfrak{p}}/O_{\mathfrak{p}}\longrightarrow 0.$$

On the other hand, $\mathcal{LT}(S)$ is the set of isomorphism classes of (G,k_G) where k_G is an exact sequence

$$0\longrightarrow \mathcal{LT}\longrightarrow G\longrightarrow F_{\mathfrak{p}}/O_{\mathfrak{p}}\longrightarrow 0.$$

Using the group structure on $\mathcal{LT}$, we may add the above two exact sequences to a new one $\alpha(k_{\mathfrak{p}},k_G)$ as

$$0\longrightarrow \mathcal{LT}\longrightarrow \alpha((A_{p^\infty})_1^{2,1},G)\longrightarrow F_{\mathfrak{p}}/O_{\mathfrak{p}}\longrightarrow 0.$$

By the theorem of Serre–Tate and the fact that étale level structures are determined on the special fiber, we associate canonically a quintuple $(A',\iota',\theta',k'^{\mathfrak{p}},\kappa'_{\mathfrak{p}})$ with $\kappa'_{\mathfrak{p}}=\alpha(k_{\mathfrak{p}},k_G)$. This defines $'\beta$. All these properties follow from the above construction and Theorem B.1. □

3. Heegner cycles on abelian varieties

In this chapter, we reformulate our main theorems about p-adic L-functions and p-adic Waldspurger formula in terms of Heegner cycles on abelian varieties. We start from recalling some background about representations of incoherent algebras and abelian varieties of GL(2)-type in §3.1. In §3.2, we state the main theorem about p-adic L-functions in terms of Heegner cycles and show that it implies Theorem 1.6. In §3.3, we state the main theorem about p-adic Waldspurger formula in terms of Heegner cycles and show that it implies Theorem 1.8.

3.1. **Representations of incoherent algebras.** We recall some materials from [YZZ13, §3.2]. Let $\iota_1,\dots,\iota_g$ be all archimedean places of F. Let $\mathbb{B}$ be a totally definite incoherent quaternion algebra over $\mathbb{A}$, to which there is an associated projective system of Shimura curves $\{X_U\}_U$. Put $X=\varprojlim_U X_U$. We recall the following definition in [YZZ13, §3.2.2].

Definition 3.1. Let L be a field admitting embeddings into $\mathbb{C}$. Denote by $\mathcal{A}(\mathbb{B}^\times,L)$ the set of isomorphism classes of irreducible representations Π of $\mathbb{B}^{\infty\times}$ over L such that for some and hence all embeddings $L\hookrightarrow\mathbb{C}$, the Jacquet–Langlands transfer of $\Pi\otimes_L\mathbb{C}$ to $\mathrm{GL}_2(\mathbb{A}^\infty)$ is a finite direct sum of (finite components of) irreducible cuspidal automorphic representations $\mathrm{GL}_2(\mathbb{A})$ of parallel weight 2.

Remark 3.2. When L is algebraically closed, we have for every finite place v of F the local L-function $L(s,\Pi_v)$, local ϵ-factor $\epsilon(1/2,\psi,\Pi_v)$, local adjoint L-function $L(s,\Pi_v,\mathrm{Ad})$, local Rankin–Selberg L-function $L(s,\Pi_v,\chi_v)$ and ϵ-factor $\epsilon(1/2,\Pi_v,\chi_v)$ for a locally constant character $\chi_v\colon E_v^\times\to L^\times$. When $L=\mathbb{C}$, we have the global versions, which are products of local ones over all *finite* places of F.

We say an abelian variety A can be *parameterized by* $\mathbb{B}$ if there is a non-constant morphism from $X = X(\mathbb{B})$ to A. Denote by $\mathrm{AV}^0(\mathbb{B})$ the set of simple abelian varieties over F that can be parameterized by $\mathbb{B}$ up to isogeny, which is stable under duality. From $A \in \mathrm{AV}^0(\mathbb{B})$, we obtain a rational representation Π_A of $\mathbb{B}^{\infty\times}$ which is an element in $\mathcal{A}(\mathbb{B}^\times, \mathbb{Q})$. The assignment $A \mapsto \Pi_A$ induces a bijection between $\mathrm{AV}^0(\mathbb{B})$ and $\Pi \in \mathcal{A}(\mathbb{B}^\times, \mathbb{Q})$.

Notation 3.3. Recall from [YZZ13, §3.2.3] the following notation

$$\Pi_A = \varinjlim_U \mathrm{Hom}_{\xi_U}(X_U^*, A),$$

where

- the colimit is taken over all compact open subgroups U of $\mathbb{B}^{\infty\times}$;
- X_U^* is simply X_U (resp. X_U plus cusps) if X_U is proper (resp. not proper which happens exactly when it is the classical modular curve);
- ξ_U is the normalized Hodge class on X_U^* [YZZ13, §3.1.3]; and
- $\mathrm{Hom}_{\xi_U}(X_U^*, A)$ denotes the $\mathbb{Q}$-vector space of *modular parameterizations*, that is, (quasi-)morphisms from X_U^* to A that send ξ_U to torsion.

If we denote by J_U the Jacobian of X_U^*, then $\mathrm{Hom}_{\xi_U}(X_U^*, A)$ is canonically identified with $\mathrm{Hom}^0(J_U, A)$. Moreover, $M_A := \mathrm{End}^0(A)$ is a field of degree equal to the dimension of A and M_A acts on the representation Π. Denote by $A^\vee$ the dual abelian variety (up to isogeny) of A and we have $\Pi_{A^\vee}$ similarly. Then the rosati involution induces a canonical isomorphism $M_{A^\vee} \simeq M_A$.

Definition 3.4 (Canonical pairing, [YZZ13, §3.2.4]). We have a canonical pairing

$$(\ ,\ )_A \colon \Pi_A \times \Pi_{A^\vee} \to M_A$$

induced by maps

$$(\ ,\ )_U \colon\ \mathrm{Hom}^0(J_U, A) \times \mathrm{Hom}^0(J_U, A^\vee) \to M_A$$

defined by $(f_+, f_-) \mapsto \mathrm{vol}(X_U)^{-1} \circ f_+ \circ f_-^\vee \in \mathrm{End}^0(A) = M_A$ for all levels U.

Now we take a simple abelian variety A over F of GL(2)-type. For a finite place v of F, choose a rational prime ℓ that does not divide v. We have a Galois representation $\rho_{A,v}$ of D_v, the decomposition group at v, on the ℓ-adic Tate module $\mathrm{V}_\ell(A)$ of A, which is a free module over $M_{A,\ell} := M_A \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$ of rank 2. It is well-known that the characteristic polynomial

$$P_v(T) = \det{}_{M_{A,\ell}}(1 - \mathrm{Frob}_v \,|\mathrm{V}_\ell(A)^{\mathrm{I}_v})$$

belongs to $M_A[T]$ and is independent of ℓ, where $\mathrm{I}_v \subset \mathrm{D}_v$ is the inertia subgroup and $\mathrm{Frob}_v \in \mathrm{D}_v/\mathrm{I}_v$ is the geometric Frobenius.

Definition 3.5 (L-function and ϵ-factor). Let K be a field containing M_A.

(1) Define $L(s, \rho_{A,v}) = P_v(N_v^{-s-1/2})^{-1}$ to be the local L-function. In a similar manner, we define the local adjoint L-function $L(s, \rho_{A,v}, \mathrm{Ad})$; in particular, $L(1, \rho_{A,v}, \mathrm{Ad}) \in M_A$.

(2) For a locally constant character $\chi_v \colon F_v^\times \to K^\times$, we have the twisted local L-function $L(s, \rho_{A,v} \otimes \chi_v)$ and the ϵ-factor $\epsilon(1/2, \psi, \rho_{A,v} \otimes \chi_v)$.

(3) For a locally constant character $\chi_v\colon E_v^\times \to K^\times$, we have the local Rankin–Selberg L-function $L(s,\rho_{A,v},\chi_v)$ and the ϵ-factor $\epsilon(1/2,\rho_{A,v},\chi_v)$.

(4) Let $\iota\colon K \hookrightarrow \mathbb{C}$ be an embedding.

We define the global L-function

$$L(s,\rho_A^{(\iota)}) = \prod_{v<\infty} \iota L(s,\rho_{A,v}),$$

which is absolutely convergent for $\operatorname{Re} s > 1$. We say that A is *automorphic* if $L(s,\rho_A^{(\iota)})$, for some and hence all ι, is (the finite component of) the L-function of an irreducible cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A})$. We have global versions for the other L-functions and ϵ-factors.

Remark 3.6. It is conjectured that every abelian variety of GL(2)-type is automorphic. In particular, when $F=\mathbb{Q}$, every abelian variety of GL(2)-type is parameterized by modular curves. This follows from Serre's modularity conjecture (for $\mathbb{Q}$) [Rib92, Theorem 4.4], where the latter has been proved by Khare and Wintenberger [KW09].

3.2. **p-adic L-function in terms of abelian varieties.** Let A be the simple abelian variety up to isogeny (of GL(2)-type) over F that gives rise to the classical representation π in §1.2. In particular, it is automorphic. We put $M=M_A=M_{A^\vee}$ which is regarded as a subfield of $\mathbb{C}_p$. Denote by $\omega_A\colon F^\times\backslash\mathbb{A}^{\infty\times}\to M^\times$ the central character associated to A. For simplicity, we also put $F^M = F\otimes_{\mathbb{Q}} M$ equipped with a natural map to $\mathbb{C}_p$.

Definition 3.7 (Distribution algebra). Denote by Γ_E the set of compact open subgroups $V^{\mathfrak{p}}$ of $\mathbb{A}_E^{\infty\mathfrak{p}\times}$, which is a filtered partially ordered set under inclusion. Let $K/F_{\mathfrak{p}}$ be a complete field extension.

(1) A (K-valued) character

$$\chi\colon E^\times\backslash\mathbb{A}_E^{\infty\times}\to K^\times$$

is a *character of weight $w\in\mathbb{Z}$ and tame level $V^{\mathfrak{p}}\in\Gamma_E$* if
- χ is invariant under some $V^{\mathfrak{p}}\in\Gamma_E$;
- there is a compact open subgroup $V_{\mathfrak{p}}$ of $E_{\mathfrak{p}}^\times$ and $w\in\mathbb{Z}$ such that $\chi(t) = (t_{\mathfrak{P}}/t_{\mathfrak{P}^c})^w$ for $t\in V_{\mathfrak{p}}$.

We suppress the word *tame level* if $V^{\mathfrak{p}}$ is not specified.

(2) For a K-valued character χ of weight w as above, we define two characters $\check\chi_{\mathfrak{P}}$ and $\check\chi_{\mathfrak{P}^c}$ of $F_{\mathfrak{p}}^\times$ by the formula $\check\chi_{\mathfrak{P}}(t)=t^{-w}\chi_{\mathfrak{P}}(t)$ and $\check\chi_{\mathfrak{P}^c}(t)=t^{w}\chi_{\mathfrak{P}^c}(t)$.

(3) Suppose K is contained in $\mathbb{C}_p$. Let χ be a locally algebraic character of weight w. Given an isomorphism $\iota\colon\mathbb{C}_p\xrightarrow{\sim}\mathbb{C}$, we define the following local characters
- $\chi_v^{(\iota)}=1$ if $v|\infty$ but not equal to $\iota|_F$;
- $\chi_v^{(\iota)}(z)=(z/z^c)^w$ for $v=\iota|_F$, where $z\in E\otimes_{F,\iota}\mathbb{R}\xrightarrow{\iota|_E}\mathbb{C}$;
- $\chi_v^{(\iota)}=\iota\chi_v$ for $v<\infty$ but $v\neq\mathfrak{p}$;
- $\chi_{\mathfrak{p}}^{(\iota)}(t)=\iota\left(\check\chi_{\mathfrak{P}}(t_\circ)\check\chi_{\mathfrak{P}^c}(t_\bullet)\right)$ for $t\in E_{\mathfrak{p}}^\times$.

In particular, $\chi^{(\iota)}:=\otimes_v\chi_v^{(\iota)}\colon\mathbb{A}_E^\times\to\mathbb{C}^\times$ is an automorphic character, which is called the *ι-avatar* of χ.

(4) Suppose K contains M. A K-valued characters χ of weight w is *A-related* if
- $\omega_A\cdot\chi|_{\mathbb{A}^{\infty\times}}=1$;
- $\#\{v<\infty, v\neq\mathfrak{p}\mid\epsilon(1/2,\rho_{A,v},\chi_v)=-1\}\equiv g+1 \mod 2$.

Denote by $\Xi(A,K)_w$ the set of all A-related K-valued characters of weight w. Put $\Xi(A,K)=\bigcup_{\mathbb{Z}}\Xi(A,K)_w$. For $\chi\in\Xi(A,K)$, there is a unique up to isomorphism totally definite incoherent quaternion algebra $\mathbb{B}_\chi$ over $\mathbb{A}$, unramified at $\mathfrak{p}$, such that $\epsilon(1/2,\rho_{A,v},\chi_v)=\chi_v(-1)\eta_v(-1)\epsilon(\mathbb{B}_{\chi,v})$ for every finite place $v\neq\mathfrak{p}$ of F. The algebra $\mathbb{B}_\chi$ is E-embeddable and by which A can be parameterized.

(5) Suppose K contains M. For a locally constant character $\omega\colon F^\times\backslash\mathbb{A}^{\infty\times}\to M^\times$, denote by $\mathscr{C}(\omega,K)$ the set of locally analytic K-valued functions f on the locally $F_\mathfrak{p}$-analytic group $E^\times\backslash\mathbb{A}_E^{\infty\times}$ satisfying
   - f is invariant under translation by some $V^\mathfrak{p}\in\Gamma_E$;
   - $f(xt)=\omega(t)^{-1}f(x)$ for all $x\in E^\times\backslash\mathbb{A}_E^{\infty\times}$ and $t\in F^\times\backslash\mathbb{A}^{\infty\times}$.

   Then $\mathscr{C}(\omega,K)$ is a locally convex K-vector space. Let $\mathscr{D}(\omega,K)$ be the strong dual of $\mathscr{C}(\omega,K)$, which we call the *K-valued ω-related distribution algebra*. It is a commutative K-algebra by convolution.

   For a complete field extension K'/K, we have $\mathscr{D}(\omega,K)\widehat{\otimes}_K K'\simeq\mathscr{D}(\omega,K')$. In fact, if K is discretely valued, $\mathscr{D}(\omega,K)$ may be written as a projective limit, indexed by tame levels $V^\mathfrak{p}\in\Gamma_E$, of nuclear Fréchet–Stein K-algebras with finite étale transition homomorphisms (see Remark 4.16), and thus complete. We have a continuous homomorphism
$$[\ ]\colon E^\times\backslash\mathbb{A}_E^{\infty\times}\to\mathscr{D}(\omega,K)^\times$$
   given by Dirac distributions.

(6) Suppose K contains M. Define $\mathscr{D}(A,K)$ to be the quotient K-algebra of $\mathscr{D}(\omega_A,K)$ by the closed ideal generated by elements that vanish on $\Xi(A,K)\subset\mathscr{C}(\omega_A,K)$, which we call the *$K$-valued A-related distribution algebra*. For a complete field extension K'/K, we have $\mathscr{D}(A,K)\widehat{\otimes}_K K'\simeq\mathscr{D}(A,K')$. Similar to $\mathscr{D}(\omega_A,K)$, if K is discretely valued, $\mathscr{D}(A,K)$ may be written as a projective limit of nuclear Fréchet–Stein K-algebras.

(7) For $\pi\in\mathscr{A}_{\mathbb{C}_p}(\mathbb{B}^\times)$ as in §1.2, we define $\mathscr{D}(\pi)$ to be the quotient of $\mathscr{D}(\omega_A,\mathbb{C}_p)$ by the closed ideal generated by elements that vanish on $\chi\in\Xi(A,\mathbb{C}_p)$ with $\mathbb{B}_\chi\simeq\mathbb{B}$, which we call the *$\pi$-related distribution algebra*. In particular, we have a quotient map $\varsigma\colon\mathscr{D}(A,\mathbb{C}_p)\to\mathscr{D}(\pi)$.

Let K be a complete field containing $MF_\mathfrak{p}^{\mathrm{lt}}$. Consider a character $\chi\in\Xi(A,K)_k$. Take $\mathbb{B}=\mathbb{B}_\chi$, and choose an E-embedding. Then we have the F-scheme X and its closed subscheme $Y=Y^+\coprod Y^-$. Put $A^+=A$ and $A^-=A^\vee$, and $\Pi^\pm=\Pi_{A^\pm}\in\mathcal{A}(\mathbb{B}^\times,\mathbb{Q})$. We have the canonical pairing $(\ ,\ )_A\colon\Pi^+\times\Pi^-\to M$.

Define $\sigma_\chi^\pm$ to be the K-subspace of $\mathrm{H}^0(Y^\pm,\Omega_{X,Y^\pm}^{\otimes-k})\otimes_F K$ consisting of φ such that $t^*\varphi=\chi(t)^{\pm1}\varphi$, where $\Omega_{X,Y^\pm}=\Omega_X^1|_{Y^\pm}$. The abstract conjugation $\mathbf{c}$ induces a $\mathbb{A}_E^{\infty\times}$-invariant pairing
$$(\ ,\ )_\chi\colon\sigma_\chi^+\times\sigma_\chi^-\to K$$
by the formula $(\varphi_+,\varphi_-)_\chi=(\varphi_+\otimes\omega_{\psi^+}^k)\cdot\mathbf{c}^*(\varphi_-\otimes\omega_{\psi^-}^k)$, where the right-hand side is a K-valued constant function on Y^+, hence regarded as an element in K. Here,
$$\omega_{\psi^\pm}=\Upsilon_\pm^*\omega_\nu|_{Y^\pm}\tag{3.1}$$
is the a section of $\Omega_{X,Y^\pm}$, where ω_ν is the global Lubin–Tate differential in Definition 2.14. It is determined the additive character ψ.

Assume K is contained in $\mathbb{C}_p$ and $k \geq 1$. For every $\iota\colon \mathbb{C}_p \xrightarrow{\sim} \mathbb{C}$, we have a $\mathbb{B}^{\infty\times} \times \mathbb{A}_E^{\infty\times}$-invariant pairing

$$(\ , \ )^{(\iota)}_{A,\chi}\colon (\Pi^+ \otimes_{F^M} \sigma^+_\chi) \times (\Pi^- \otimes_{F^M} \sigma^-_\chi) \to (\operatorname{Lie} A^+ \otimes_{F^M} \operatorname{Lie} A^-) \otimes_{F^M,\iota} \mathbb{C},$$

such that for $f_\pm \in \Pi^\pm$, $\varphi_\pm \in \sigma^\pm_\chi$ and $\omega_\pm \in \mathrm{H}^0(A^\pm, \Omega^1_{A^\pm})$,

$$\langle \omega_+ \otimes \omega_-, (f_+ \otimes \varphi_+, f_- \otimes \varphi_-)^{(\iota)}_{A,\chi} \rangle = \\ \times \left(\iota\varphi_+ \otimes \mathfrak{c}^*_\iota \iota\varphi_- \otimes \mu^k \right) \times \int_{X_\iota(\mathbb{C})} \frac{\Theta_\iota^{k-1} f^*_+ \omega_+ \otimes \mathfrak{c}^*_\iota \Theta_\iota^{k-1} f^*_- \omega_-}{\mu^k} \,\mathrm{d}x,$$

where

- $\langle \ , \ \rangle$ is the canonical pairing between $\mathrm{H}^0(A^\pm, \Omega^1_{A^\pm})$ and $\operatorname{Lie} A^\pm$;
- μ is an arbitrary Hecke invariant hyperbolic metric on $X_\iota(\mathbb{C})$;
- $\iota\varphi_+ \otimes \mathfrak{c}^*_\iota \iota\varphi_- \otimes \mu^k$ is a constant function on $Y^+_\iota(\mathbb{C})$, hence viewed as a complex number; and
- $\mathrm{d}x$ is the Tamagawa measure on $X_\iota(\mathbb{C})$.

Define $\mathbf{P}_\iota(\chi)$ to be the unique element in $\mathbb{C}^\times$ such that

$$(\ , \ )^{(\iota)}_{A,\chi} = \mathbf{P}_\iota(\chi) \cdot \iota(\ , \ )_A \otimes \iota(\ , \ )_\chi,$$

which we call the *period ratio (at ι)*, as a function on $\bigcup_{k\geq 1} \Xi(A,K)_k$.

Theorem 3.8. *There is a unique element*

$$\mathscr{L}(A) \in (\operatorname{Lie} A^+ \otimes_{F^M} \operatorname{Lie} A^-) \otimes_{F^M} \mathscr{D}(A, MF^{\mathrm{lt}}_{\mathfrak{p}})$$

such that for every character $\chi \in \Xi(A,K)_k$ with $k \geq 1$ and $MF^{\mathrm{lt}}_{\mathfrak{p}} \subset K \subset \mathbb{C}_p$ a complete intermediate field, and every $\iota\colon \mathbb{C}_p \xrightarrow{\sim} \mathbb{C}$,

$$\iota\mathscr{L}(A)(\chi) = L(1/2, \rho^{(\iota)}_A, \chi^{(\iota)}) \cdot \frac{2^{g-3}\delta_E^{1/2}\zeta_F(2)\mathbf{P}_\iota(\chi)}{L(1,\eta)^2 L(1, \rho^{(\iota)}_A, \mathrm{Ad})} \frac{\iota\epsilon(1/2, \psi, \rho_{A,\mathfrak{p}} \otimes \check{\chi}_{\mathfrak{P}^c})}{\iota L(1/2, \rho_{A,\mathfrak{p}} \otimes \check{\chi}_{\mathfrak{P}^c})^2}. \tag{3.2}$$

Lemma 3.9. *Theorem 3.8 implies Theorem 1.6.*

Proof. Apparently, we only need to prove Theorem 1.6 for one p-adic Petersson inner product $(\ , \ )_\pi$. Choose a basis element $\omega_\pm$ of the rank-1 free F^M-module $\operatorname{Lie} A^\pm$. They together defines a pairing $(\ , \ )_\pi$ such that

$$(f^*_+ \log_{\omega^+}, f^*_- \log_{\omega^-})_\pi = (f_+, f_-)_A$$

for every $f_\pm \in \Pi^\pm$. Define $\mathscr{L}(\pi)$ to be $\langle \omega_+ \otimes \omega_-, \varsigma\mathscr{L}(A) \rangle$, where ς is introduced in 3.7 (7). Then by Lemma 2.26, $\mathscr{L}(\pi)$ satisfies the requirement in Theorem 1.6 for the above p-adic Petersson inner product. □

3.3. **Heegner cycles and p-adic Waldspurger formula.** Let K be a complete field containing M. Consider an element $\chi \in \Xi(A,K)_0$, regarded as a locally constant character of $\mathrm{Gal}(E^{\mathrm{ab}}/E)$ via global class field theory. Take $\mathbb{B} = \mathbb{B}_\chi$, and choose an E-embedding. We choose a CM point $P^+ \in Y^+(E^{\mathrm{ab}}) = Y^+(\mathbb{C}_p)$ and put $P^- = \mathbf{c}P^+$. For each $f_\pm \in \Pi^\pm$, we have a Heegner cycle $P^\pm_\chi(f_\pm)$ on $A^\pm$ defined by the formula

$$P^\pm_\chi(f_\pm) = \int_{\mathrm{Gal}(E^{\mathrm{ab}}/E)} f_\pm(\tau P^\pm) \otimes_M \chi(\tau)^{\pm 1} \,\mathrm{d}\tau, \tag{3.3}$$

where $\mathrm{d}\tau$ is the Haar measure on $\mathrm{Gal}(E^{\mathrm{ab}}/E)$ of total volume 1.

Suppose K contains $MF_{\mathfrak{p}}^{\mathrm{ab}}$. We have a K-linear map

$$\log_{A^\pm}\colon A^\pm(K)_{\mathbb{Q}} \otimes_M K \to \mathrm{Lie}\, A^\pm \otimes_{F^M} K.$$

As a functional on $\Pi^+ \times \Pi^-$, the product $\log_{A^+} P_\chi^+(f_+) \cdot \log_{A^-} P_\chi^-(f_-)$ defines an element in the following one dimensional K-vector space

$$\mathrm{Hom}_{\mathbb{A}_E^{\infty\times}}(\Pi^+ \otimes \chi, K) \otimes_K \mathrm{Hom}_{\mathbb{A}_E^{\infty\times}}(\Pi^- \otimes \chi^{-1}, K) \otimes_{F^M} (\mathrm{Lie}\, A^+ \otimes_{F^M} \mathrm{Lie}\, A^-).$$

On the other hand, using matrix coefficient integral, we construct a basis $\alpha_\chi(\ ,\ )$ of the space $\mathrm{Hom}_{\mathbb{A}_E^{\infty\times}}(\Pi^+ \otimes \chi, K) \otimes_K \mathrm{Hom}_{\mathbb{A}_E^{\infty\times}}(\Pi^- \otimes \chi^{-1}, K)$. It satisfies that for every $\iota\colon \mathbb{C}_p \xrightarrow{\sim} \mathbb{C}$,

$$\iota\alpha_\chi(f_+, f_-) = \alpha^\natural(f_+, f_-; \chi^{(\iota)}),$$

where the last term is introduced in Definition 4.4.

Theorem 3.10 (p-adic Waldspurger formula)**.** *Let the notation be as above. For $\chi \in \Xi(A, K)_0$, we have*

$$\log_{A^+} P_\chi^+(f_+) \cdot \log_{A^-} P_\chi^-(f_-) = \mathscr{L}(A)(\chi) \cdot \frac{L(1/2, \rho_{A,\mathfrak{p}} \otimes \check{\chi}_{\mathfrak{P}^c})^2}{\epsilon(1/2, \psi, \rho_{A,\mathfrak{p}} \otimes \check{\chi}_{\mathfrak{P}^c})} \cdot \alpha_\chi(f_+, f_-).$$

Lemma 3.11. *Theorem 3.10 implies Theorem 1.8.*

Proof. We only need to prove Theorem 1.8 for one $(\ ,\ )_\pi$ and one nonzero element $\varphi_\pm \in \sigma_\chi^\pm$. We use the p-adic Petersson inner product in the proof of Lemma 3.9, and choose $\varphi_\pm$ such that $\varphi_\pm(P^\pm) = 1$. Then the lemma follows by definition. □

4. Construction of p-adic L-function

This chapter is dedicated to the proofs of Theorems 3.8 and 3.10. In §4.1, we construct the distribution interpolating matrix coefficient integrals appearing in the classical Waldspurger formula. We construct the universal torus period in §4.2, which is a crucial construction toward the p-adic L-function. In §4.3, we study the relation between universal torus periods and classical torus periods, based on which we accomplish the proofs of our main theorems in §4.4.

4.1. **Distribution of matrix coefficient integrals.** Let $K/MF_{\mathfrak{p}}$ be a complete field extension.

Definition 4.1. For $w \in \mathbb{Z}$, $n \in \mathbb{N}$, and a locally constant character $\omega\colon F^\times\backslash\mathbb{A}^{\infty\times} \to M^\times$, we say a K-valued character $\chi\colon E^\times\backslash\mathbb{A}_E^{\infty\times} \to K^\times$ of weight w is *of central type ω and depth n* if

- $\omega \cdot \chi|_{\mathbb{A}^{\infty\times}} = 1$, and
- $\chi_{\mathfrak{P}^c}(t) = t^{-w}$ for all $t \in (1 + \mathfrak{p}^n)^\times$.

We denote by $\Xi(\omega, K)_w^n$ the set of all K-valued characters of weight w, central type ω and depth n. Moreover, put $\Xi(\omega, K)^n = \bigcup_{\mathbb{Z}} \Xi(\omega, K)_w^n$. We have a natural pairing $\mathscr{D}(\omega, K) \times \Xi(\omega, K)^n \to K$.

Choose a $\mathbb{B}$ by which A can be parameterized, together with an E-embedding. Recall that we put $\Pi^\pm = \Pi(\mathbb{B})_{A^\pm}$.

Definition 4.2 (Stable vector). An element $f_\pm$ in $\Pi^\pm \otimes_M K$ (resp. $\Pi^\pm_\mathfrak{p} \otimes_M K$) is a *stable vector* if

(1) $f_\pm$ is fixed by $\mathrm{N}^\pm(O_\mathfrak{p})$;
(2) $f_\pm$ satisfies the equation

$$\sum_{\mathrm{N}^\pm(\mathfrak{p}^{-1})/\mathrm{N}^\pm(O_\mathfrak{p})} \Pi^\pm_\mathfrak{p}(g) f_\pm = 0.$$

We denote by $(\Pi^\pm)^\heartsuit_K$ (resp. $(\Pi^\pm_\mathfrak{p})^\heartsuit_K$) the subset of $\Pi^\pm \otimes_M K$ (resp. $\Pi^\pm_\mathfrak{p} \otimes_M K$) of stable vectors.

For $n \in \mathbb{N}$, we say a stable vector $f_\pm \in (\Pi^\pm)^\heartsuit_K$ (or $(\Pi^\pm_\mathfrak{p})^\heartsuit_K$) is *$n$-admissible* if

$$\Pi^\pm_\mathfrak{p}(\mathrm{n}^\pm(x)) f_\pm = \psi^\pm(x) f_\pm$$

for every $x \in \mathfrak{p}^{-n}/O_\mathfrak{p}$, where $\mathrm{n}^\pm(x)$ is same as in Proposition 2.13.

Remark 4.3. If we realize $\Pi^\pm_\mathfrak{p}$ in the Kirillov model with respect to the pair $(\mathrm{N}^+, \psi^\pm)$, then $f_{\pm\mathfrak{p}}$ belongs to $(\Pi^\pm_\mathfrak{p})^\heartsuit_K$ if and only if $f_{+\mathfrak{p}}$ (resp. $\Pi^-_\mathfrak{p}(\mathrm{J}) f_{-\mathfrak{p}}$) is supported on $O^\times_\mathfrak{p}$, and is n-admissible if and only if $f_{+\mathfrak{p}}$ (resp. $\Pi^-_\mathfrak{p}(\mathrm{J}) f_{-\mathfrak{p}}$) is supported on $(1+\mathfrak{p}^n)^\times$.

We recall the definition of the classical (normalized) matrix coefficient integral. Suppose K is contained in $\mathbb{C}_p$. We take a character $\chi \in \Xi(A, K)$.

Definition 4.4 (Regularized matrix coefficient integral). Let $\iota\colon \mathbb{C}_p \xrightarrow{\sim} \mathbb{C}$ be an isomorphism. We recall the regularization of the following matrix coefficient integral

$$\alpha^\natural(f_+, f_-; \chi^{(\iota)}) \text{ “=” } \int_{\mathbb{A}^{\infty\times}\backslash \mathbb{A}_E^{\infty\times}} \iota(\Pi(t) f_+, f_-)_A \cdot \chi^{(\iota)}(t)\, \mathrm{d}t.$$

To do this, we take any decomposition $\iota(\ ,\ )_A = \prod_{v<\infty} (\ ,\ )_{\iota,v}$ where $(\ ,\ )_{\iota,v}\colon \Pi^+_v \times \Pi^-_v \to \mathbb{C}$ is a $\mathbb{B}^\times_v$-invariant bilinear pairing. For $f_\pm = \otimes_{v<\infty} f_{\pm v}$ such that $(f_{+v}, f_{-v})_{\iota,v} = 1$ for all but finitely many v, we put

$$\alpha(f_{+v}, f_{-v}; \chi^{(\iota)}_v) = \int_{F^\times_v \backslash E^\times_v} (\Pi_v(t) f_{+v}, f_{-v})_{\iota,v} \chi^{(\iota)}_v(t)\, \mathrm{d}t;$$

$$\alpha^\natural(f_{+v}, f_{-v}; \chi^{(\iota)}_v) = \left(\frac{\zeta_{F_v}(2) L(1/2, \rho^{(\iota)}_{A,v}, \chi^{(\iota)}_v)}{L(1, \eta_v) L(1, \rho^{(\iota)}_{A,v}, \mathrm{Ad})} \right)^{-1} \alpha(f_{+v}, f_{-v}; \chi^{(\iota)}_v).$$

Here, $\mathrm{d}t$ is the measure on $F^\times_v \backslash E^\times_v$ given determined in §1.4, and $\rho^{(\iota)}_{A,v}$ is the corresponding admissible complex representation of $\mathbb{B}^\times_v$ via ι. Then we have $\alpha^\natural(f_{+v}, f_{-v}; \chi^{(\iota)}_v) = 1$ for all but finitely many v, and the product

$$\alpha^\natural(f_+, f_-; \chi^{(\iota)}) := \prod_{v<\infty} \alpha^\natural(f_{+v}, f_{-v}; \chi^{(\iota)}_v),$$

which is well-defined, does not depend on the choice of the decomposition of $\iota(\ ,\ )_A$. We extend the functional $\alpha^\natural(\ ,\ ; \chi^{(\iota)})$ to all f_+, f_- by linearity.

The regularization of the matrix coefficient integral for $\iota\alpha^\natural(\phi_+, \phi_-; \varphi_+, \varphi_-)$ in §1.3 is defined similarly as above.

The following proposition is our main result.

Proposition 4.5. *Let $MF_p \subset K \subset \mathbb{C}_p$ be a complete intermediate field. Let $f_\pm \in (\Pi^\pm)_K^\heartsuit$ be an n-admissible stable vector for some $n \in \mathbb{N}$. Then there is a unique element $\mathscr{Q}(f_+, f_-) \in \mathscr{D}(\omega_A, K)$ such that for all K-valued characters $\chi \in \Xi(\omega_A, K)^n$ of central type ω_A and depth n, and $\iota\colon \mathbb{C}_p \xrightarrow{\sim} \mathbb{C}$,*

$$\iota\mathscr{Q}(f_+, f_-)(\chi) = \iota\left(\frac{L(1/2, \rho_{A,\mathfrak{p}} \otimes \check{\chi}_{\mathfrak{P}^c})^2}{\epsilon(1/2, \psi, \rho_{A,\mathfrak{p}} \otimes \check{\chi}_{\mathfrak{P}^c})}\right) \cdot \alpha^\natural(f_+, f_-; \chi^{(\iota)}).$$

The element $\mathscr{Q}(f_+, f_-)$ is called the *(K-valued) local period distribution.*

Before giving the proof, we make a convenient choice of a decomposition of $(\ ,\ )_A$. Realize the representation $\Pi_\mathfrak{p}^\pm$ in the Kirillov model as in Remark 4.3. We may assume that $f_\pm = \otimes f_{\pm v}$, with $f_{\pm v} \in \Pi_v^\pm \otimes_M K$, is decomposable and is fixed by some $V^\mathfrak{p} \in \Gamma_E$ sufficiently small. Choose a decomposition $(\ ,\ )_A = \prod_{v<\infty} (\ ,\ )_v$ such that

1. $(f_{+v}, f_{-v})_v = 1$ for all but finitely many v;
2. $(f'_{+v}, f'_{-v})_v \in K$ for all $f'_{\pm v} \in \Pi_v^\pm \otimes_M K$;
3. for $f'_{\pm\mathfrak{p}} \in \Pi_\mathfrak{p}^\pm \otimes_M K$ that is compactly supported on $F_\mathfrak{p}^\times$,

$$(f'_{+\mathfrak{p}}, f'_{-\mathfrak{p}})_\mathfrak{p} = \int_{F_\mathfrak{p}^\times} f'_{+\mathfrak{p}}(a) f'_{-\mathfrak{p}}(a)\, \mathrm{d}a,$$

where $\mathrm{d}a$ is the Haar measure on $F_\mathfrak{p}^\times$ such that the volume of $O_\mathfrak{p}^\times$ is 1.

We need two lemmas for the proof of the proposition, where for simplicity we write $\omega = \omega_A$. For each finite place $v \neq \mathfrak{p}$, let $\mathscr{D}(\omega_v, K, V_v^\mathfrak{p})$ be the quotient of $D(E_v^\times/V_v^\mathfrak{p}, K)$ by the closed ideal generated by $\{\omega_v(t)[t] - 1 \mid t \in F_v^\times\}$. Put

$$\mathscr{D}(\omega_v, K) = \varprojlim_{V_v^\mathfrak{p}} \mathscr{D}(\omega_v, K, V_v^\mathfrak{p}),$$

where the limit runs over all compact open subgroups $V_v^\mathfrak{p}$ of $E_v^\times$. Let $\mathscr{D}(\omega_\mathfrak{p}, K)$ be the quotient of $D(E_\mathfrak{p}^\times, K)$ by the closed ideal generated by $\{\omega_\mathfrak{p}(t)[t] - 1 \mid t \in F_\mathfrak{p}^\times\}$. We have natural homomorphisms $\mathscr{D}(\omega_v, K) \to \mathscr{D}(\omega, K)$ for all finite places v.

Lemma 4.6. *Let $v \neq \mathfrak{p}$ be a finite place of F.*

1. *There exists a unique element*

$$\mathcal{L}^{-1}(\rho_{A,v}) \in \mathscr{D}(\omega_v, MF_\mathfrak{p})$$

such that for every locally constant character $\chi_v\colon E_v^\times \to K^\times$ satisfying $\omega_v \cdot \chi_v|_{F_v^\times} = 1$,

$$\mathcal{L}^{-1}(\rho_{A,v})(\chi_v) = L(1/2, \rho_{A,v}, \chi_v)^{-1}.$$

2. *For $f_{\pm v} \in \Pi_v^\pm \otimes_M K$, there exists a unique element*

$$\mathscr{Q}(f_{+v}, f_{-v}) \in \mathscr{D}(\omega_v, K)$$

such that for every locally constant character $\chi_v\colon E_v^\times \to K^\times$ satisfying $\omega_v \cdot \chi_v|_{F_v^\times} = 1$, and $\iota\colon \mathbb{C}_p \xrightarrow{\sim} \mathbb{C}$,

$$\iota\mathscr{Q}(f_{+v}, f_{-v})(\chi_v) = \alpha^\natural(f_{+v}, f_{-v}; \chi_v^{(\iota)}).$$

Proof. The uniqueness is clear. In the following proof, we suppress v from the notation and we will use the subscript ι for all changing of coefficients of representations via ι.

To prove (1), we first consider the following situation. Let $\tilde{F}$ be either F or E, and $\tilde{\Pi}$ be an irreducible admissible M-representation of $\mathrm{GL}_2(\tilde{F})$. We claim that there is a (unique) element $\mathcal{L}_{\tilde{F}}^{-1}(\tilde{\Pi}) \in D_\flat(\tilde{F}, MF_\mathfrak{p})$, where $D_\flat(\tilde{F}, K) = \varprojlim_V D(\tilde{F}^\times/V, K)$ with V running over all compact open subgroups of $\tilde{F}^\times$, such that for every locally constant character $\chi\colon \tilde{F}^\times \to K^\times$ and $\iota\colon \mathbb{C}_p \xrightarrow{\sim} \mathbb{C}$,

$$\iota\mathcal{L}_{\tilde{F}}^{-1}(\tilde{\Pi})(\chi) = L(1/2, \tilde{\Pi}_\iota \otimes \chi_\iota)^{-1}.$$

In fact, for a locally constant character $\mu\colon \tilde{F}^\times \to M^\times$, define $\mathcal{L}_{\tilde{F}}^{-1}(\mu) \in D_\flat(\tilde{F}^\times, MF_\mathfrak{p})$ by the formula

$$\mathcal{L}_{\tilde{F}}^{-1}(\mu)(h) = 1 - \int_{O_{\tilde{F}}^\times} \mu(\tilde{\varpi}a)h(\tilde{\varpi}a)\,\mathrm{d}a$$

for $h \in \varinjlim_V C(\tilde{F}^\times/V, MF_\mathfrak{p})$. Here, $\tilde{\varpi}$ is an arbitrary uniformizer of $\tilde{F}$, and $\mathrm{d}a$ is the Haar measure on $O_{\tilde{F}}^\times$ with total volume 1. Then we have three cases:

- If $\tilde{\Pi}$ is supercuspidal, put $\mathcal{L}_{\tilde{F}}^{-1}(\tilde{\Pi}) = 1$.
- If $\tilde{\Pi}$ is the unique irreducible subrepresentation of the non-normalized parabolic induction of $(\mu, \mu|\ |^{-2})$ for a character $\mu\colon \tilde{F}^\times \to M^\times$, put $\mathcal{L}_{\tilde{F}}^{-1}(\tilde{\Pi}) = \mathcal{L}_{\tilde{F}}^{-1}(\mu)$.
- If $\tilde{\Pi}$ is the irreducible non-normalized parabolic induction of $(\mu^1, \mu^2|\ |^{-1})$ for a pair of characters $\mu^i\colon \tilde{F}^\times \to M^\times$ $(i = 1, 2)$, put $\mathcal{L}_{\tilde{F}}^{-1}(\tilde{\Pi}) = \mathcal{L}_{\tilde{F}}^{-1}(\mu^1) \cdot \mathcal{L}_{\tilde{F}}^{-1}(\mu^2)$.

Go back to (1). First, assume E/F is non-split. Then we define $\mathcal{L}^{-1}(\rho_A)$ to be the image of $\mathcal{L}_E^{-1}(\Pi_E)$ in $\mathscr{D}(\omega, MF_\mathfrak{p})$ where Π_E is the base change of Π to $\mathrm{GL}_2(E)$, which depends only on ρ_A. Second, assume $E = F_\bullet \times F_\circ$ is split where $F_\bullet = F_\circ = F$. Then we define $\mathcal{L}^{-1}(\rho_A)$ to be the image of $\mathcal{L}_{F_\bullet}^{-1}(\Pi) \otimes \mathcal{L}_{F_\circ}^{-1}(\Pi)$ in $\mathscr{D}(\omega, MF_\mathfrak{p})$.

Now we consider (2). First, assume E/F is non-split. Then the torus $F^\times\backslash E^\times$ is compact and hence the matrix coefficient $\Phi_{f_+,f_-}(g) := (\Pi^+(g)f_+, f_-)$ is finite under $E^\times$-translation. We may assume the restriction $\Phi_{f_+,f_-}|_{E^\times} = \sum_i a_i\chi_i$ is a finite K-linear combination of K-valued (locally constant) characters χ_i of $E^\times$ such that $\omega \cdot \chi_i|_{F^\times} = 1$. To every locally constant function h on $E^\times$ satisfying $\omega(t)h(at) = h(a)$ for all $a \in E^\times$ and $t \in F^\times$, assigning the integral

$$\sum_i a_i \int_{F^\times\backslash E^\times} \chi_i(t)h(t)\,\mathrm{d}t,$$

which is a finite sum, defines an element $\alpha(f_+, f_-)$ in $\mathscr{D}(\omega, K)$. Put

$$\mathscr{Q}(f_+, f_-) = \left(\frac{\zeta_F(2)}{L(1, \rho_A, \mathrm{Ad})L(1, \eta)}\right)^{-1} \mathcal{L}^{-1}(\rho_A)\alpha(f_+, f_-).$$

Second, assume $E = F_\bullet \times F_\circ$ is split. We assume the embedding $E \to \mathrm{Mat}_2(F)$ is given by

$$(t_\bullet, t_\circ) \mapsto \begin{pmatrix} t_\bullet & \\ & t_\circ \end{pmatrix}$$

for $t_\bullet, t_\circ \in F$. Moreover, a character χ of $E^\times$ is given by a pair $(\chi_\bullet, \chi_\circ)$ of characters of $F^\times$ such that $\chi((t_\bullet, t_\circ)) = \chi_\bullet(t_\bullet)\chi_\circ(t_\circ)$.

We realize $\Pi^{\pm}$ in the Kirillov model with respect to a (nontrivial) additive character $\psi^{\pm}\colon F\to\mathbb{C}^{\times}$ of conductor 0 where $\psi^{-}=(\psi^{+})^{-1}$. Moreover, we may assume for $f_{\pm}\in\Pi^{\pm}\otimes_M K$ that is compactly supported on $F^{\times}$,

$$(f_+,f_-)=\int_{F^\times}f_+(a)f_-(a)\,\mathrm{d}a,$$

where $\mathrm{d}a$ is the Haar measure on $F^{\times}$ such that the volume of $O_F^{\times}$ is c for some $c\in M$. We have the following formula

$$\begin{aligned}(4.1)\quad &\alpha^{\natural}(f_+,f_-;\chi_\iota)\\ &=\left(\frac{\zeta_F(2)L(1/2,\rho_A^{(\iota)},\chi_\iota)}{L(1,\eta)L(1,\rho_A^{(\iota)},\mathrm{Ad})}\right)^{-1}\int_{F^\times}\iota f_+(a)\cdot\chi_{\bullet\iota}(a)\,\mathrm{d}a\int_{F^\times}\iota f_-(b)\cdot\chi_{\bullet\iota}(b^{-1})\,\mathrm{d}b\\ &=\left(\frac{\zeta_F(2)}{L(1,\eta)}L(1,\rho_A^{(\iota)},\mathrm{Ad})\right)^{-1}Z(\iota f_+,\chi_{\bullet\iota})Z(\iota f_-,\chi_{\bullet\iota}^{-1}),\end{aligned}$$

where

$$Z(\iota f_\pm,\chi_{\bullet\iota}^{\pm1})=L(1/2,\Pi_\iota^{\pm}\otimes\chi_{\bullet\iota}^{\pm1})^{-1}\int_{F^\times}\iota f_\pm(a)\cdot\chi_{\bullet\iota}^{\pm1}(a)\,\mathrm{d}a.$$

To conclude, we only need to show that there exists an element $\mathcal{Z}(f_\pm)\in D_\flat(F^\times,K)$ such that for every locally constant character $\chi\colon F^\times\to K^\times$ and $\iota\colon\mathbb{C}_p\xrightarrow{\sim}\mathbb{C}$, $\iota\mathcal{Z}(f_\pm)(\chi)=Z(\iota f_\pm,\chi_\iota^{\pm1})$. Without loss of generality, we only construct $\mathcal{Z}(f_+)$.

Enlarging M if necessary to include $l^{1/2}$ where l is the cardinality of the residue field of F, there is a subspace $\Pi^{+,c}$ of Π^+ such that $\Pi^{+,c}\otimes_M K$ is the subspace of $\Pi^+\otimes_M K$ of functions that are compactly supported on $F^\times$. For $f_+\in\Pi^{+,c}\otimes_M K$, we may define $\mathcal{Z}(f_+)$ such that for every locally constant function h on $F^\times$

$$\mathcal{Z}(f_+)(h)=\mathcal{L}_F^{-1}(\Pi^+)(h)\times\int_{F^\times}f_+(a)h(a)\,\mathrm{d}a.$$

Therefore, we may conclude if $\dim\Pi^+/\Pi^{+,c}=0$. There are two cases remaining.

First, Π^+ is a special representation, that is, $\dim\Pi^+/\Pi^{+,c}=1$. We may choose a representative $f_+=\mu(a)\cdot\mathrm{ch}_{O_F\backslash\{0\}}(a)$ for some character $\mu\colon F^\times\to M^\times$. Then $Z(\iota f_\pm,\chi_\iota^{\pm1})=c$ (resp. 0) if $\mu\cdot\chi$ is unramified (resp. otherwise). Therefore, we may define $\mathcal{Z}(f_+)$ such that

$$\mathcal{Z}(f_+)(h)=\int_{O_F^\times}\mu(a)h(a)\,\mathrm{d}a$$

for every locally constant function h on $F^\times$.

Second, Π^+ is a principal series, that is, $\dim\Pi^+/\Pi^{+,c}=2$. There are two possibilities. In the first case, we may choose representatives $f_+^i=\mu^i(a)\cdot\mathrm{ch}_{O_F\backslash\{0\}}(a)$ for two *different* characters $\mu^1,\mu^2\colon F^\times\to M^\times$. Without loss of generality, we consider f_+^1. Then $Z(\iota f_+^1,\chi_\iota)=L(1/2,\mu_\iota^2\cdot\chi_\iota)^{-1}$ (resp. 0) if $\mu^1\cdot\chi$ is unramified (resp. otherwise). Therefore, we may define $\mathcal{Z}(f_+^1)$ such that

$$\mathcal{Z}(f_+^1)(h)=\mathcal{L}_F^{-1}(\mu^1)(h)\times\int_{O_F^\times}\mu(a)h(a)\,\mathrm{d}a$$

for every locally constant function h on $F^\times$. In the second case, we may choose representatives $f_+^1=\mu(a)\cdot\mathrm{ch}_{O_F\backslash\{0\}}(a)$ and $f_+^2=(1-\log_l|a|)\mu(a)\cdot\mathrm{ch}_{O_F\backslash\{0\}}(a)$ for some character $\mu\colon F^\times\to M^\times$. The function f_+^1 has been treated above. For f_+^2, we have $Z(\iota f_+^2,\chi_\iota)=c$

(resp. 0) if $\mu \cdot \chi$ is unramified (resp. otherwise). Therefore, we may define $\mathcal{Z}(f_+)$ such that

$$\mathcal{Z}(f_+)(h) = \int_{O_F^\times} \mu(a)h(a)\,\mathrm{d}a$$

for every locally constant function h on $F^\times$. □

Lemma 4.7. *Let $f_{\pm\mathfrak{p}} \in (\Pi_\mathfrak{p}^\pm)_K^\heartsuit$ be n-admissible stable vectors. Then there exists a unique element*

$$\mathscr{Q}(f_{+\mathfrak{p}}, f_{-\mathfrak{p}}) \in \mathscr{D}(\omega_\mathfrak{p}, K)$$

with the following property: for every character $\chi_\mathfrak{p}\colon E_\mathfrak{p}^\times \to K^\times$ satisfying $\omega_\mathfrak{p} \cdot \chi_\mathfrak{p}|_{F_\mathfrak{p}^\times} = 1$ and $\chi_{\mathfrak{P}^c}(t) = t^{-w}$ for $t \in (1+\mathfrak{p}^n)^\times$ and some $w \in \mathbb{Z}$, and $\iota\colon \mathbb{C}_p \xrightarrow{\sim} \mathbb{C}$, we have

$$\iota\mathscr{Q}(f_{+\mathfrak{p}}, f_{-\mathfrak{p}})(\chi_\mathfrak{p}) = \iota\left(\frac{L(1/2, \rho_{A,\mathfrak{p}} \otimes \check{\chi}_{\mathfrak{P}^c})^2}{\epsilon(1/2, \psi, \rho_{A,\mathfrak{p}} \otimes \check{\chi}_{\mathfrak{P}^c})}\right) \alpha^\natural(f_{+\mathfrak{p}}, f_{-\mathfrak{p}}; \chi_\mathfrak{p}^{(\iota)}).$$

Here, $\check{\chi}$ is defined similarly as in Definition 3.7 (2).

Proof. The uniqueness is clear. Note that the formula (4.1) also works at $\mathfrak{p}$. Moreover, we have the functional equation

$$Z(\iota f_{-\mathfrak{p}}, \chi_{\mathfrak{P}^c}^{(\iota)-1}) = \iota\epsilon(1/2, \psi, \rho_{A,\mathfrak{p}} \otimes \check{\chi}_{\mathfrak{P}^c}) \cdot Z(\iota(\Pi_\mathfrak{p}^-(\mathrm{J})f_{-\mathfrak{p}}), \chi_{\mathfrak{P}^c}^{(\iota)}).$$

By Remark 4.3, we only need to show that for $f \in \Pi_\mathfrak{p}^\pm \otimes_M K$ that is supported on $(1+\mathfrak{p}^n)^\times$, there exists $\mathscr{Q}'(f) \in \mathscr{D}(\omega_\mathfrak{p}, K)$ such that for $\chi_\mathfrak{p}$ as in the statement and ι,

$$\iota\mathscr{Q}'(f)(\chi_\mathfrak{p}) = \int_{O_\mathfrak{p}^\times} \iota f(a) \cdot \chi_{\mathfrak{P}^c}^{(\iota)}(a)\,\mathrm{d}a.$$

But in fact since $\chi_{\mathfrak{P}^c}^{(\iota)}$ restricts to the trivial character on $(1+\mathfrak{p}^n)^\times$,

$$\int_{O_\mathfrak{p}^\times} \iota f(a) \cdot \chi_{\mathfrak{P}^c}^{(\iota)}(a)\,\mathrm{d}a = \iota\int_{O_\mathfrak{p}^\times} f(a)\,\mathrm{d}a.$$

We may put

$$\mathscr{Q}'(f) = \int_{O_\mathfrak{p}^\times} f(a)\,\mathrm{d}a \in K$$

which is a constant (depending only on f). □

Proof of Proposition 4.5. Let $f_\pm \in (\Pi^\pm)_K^\heartsuit$ be n-admissible stable vectors. It is clear that $\mathscr{Q}(f_{+v}, f_{-v})$ constructed in Lemma 4.6 is 1 for almost all v. Therefore, we may simply define $\mathscr{Q}(f_+, f_-)$ to be the image of

$$\mathscr{Q}(f_{+\mathfrak{p}}, f_{-\mathfrak{p}}) \otimes \bigotimes_{v \neq \mathfrak{p}} \mathscr{Q}(f_{+v}, f_{-v})$$

in $\mathscr{D}(\omega_\mathbb{A}, K)$. □

4.2. **Universal torus periods.** Let $\mathbb{B}$ be as in the previous section and we choose a CM point $P^+ \in Y^+(E^{\mathrm{ab}})$. Recall that for $m \in \mathbb{N} \cup \{\infty\}$, we have the closed formal subscheme $\mathfrak{Y}^\pm(m)$ of $\mathfrak{X}(m)$ as in §2.4. For a complete field extension $K/F_\mathfrak{p}^{\mathrm{nr}}$, put

$$\mathscr{N}^\pm(m, K) = \mathrm{H}^0(\mathfrak{Y}^\pm(m), \mathscr{O}_{\mathfrak{Y}^\pm(m)}) \widehat{\otimes}_{O_\mathfrak{p}^{\mathrm{nr}}} K.$$

The point $P^\pm$ identifies $\mathscr{N}^\pm(\infty, K)$ with $\mathrm{Map}_{\mathrm{cont}}(E_{\mathrm{cl}}^\times \backslash \mathbb{A}^{\infty\times}, K)$, the K-algebra of continuous K-valued functions on $E_{\mathrm{cl}}^\times \backslash \mathbb{A}^{\infty\times}$, under which for every $f \in \mathscr{N}^\pm(\infty, K)$ and $t \in E_{\mathrm{cl}}^\times \backslash \mathbb{A}^{\infty\times}$, $f(\mathrm{T}_t P^\pm) = f(t^{\pm 1})$. Moreover, the Galois group $O_{E_\mathfrak{p}}^\times$ of $\mathfrak{X}(\infty)/\mathfrak{X}(0)$ preserves $\mathfrak{Y}^\pm(\infty)$, whose induced action on $\mathscr{N}^\pm(\infty, K)$ is given by

$$t^* f(x) = f(xt^{\pm 1}), \quad x \in E_{\mathrm{cl}}^\times \backslash \mathbb{A}^{\infty\times}.$$

The quotient of $\mathfrak{Y}^\pm(\infty)$ by the subgroup $O_{E_\mathfrak{p},m}^\times$ is simply $\mathfrak{Y}^\pm(m)$.

Definition 4.8. Consider a locally constant character

$$\omega \colon F^\times \backslash \mathbb{A}^{\infty\times} \to M^\times.$$

Let $K/MF_\mathfrak{p}$ be a complete field extension. For every $V^\mathfrak{p} \in \Gamma_E$ on which ω is trivial, denote by $\mathscr{D}(\omega, K, V^\mathfrak{p})$ the quotient of $D(E_{\mathrm{cl}}^\times \backslash \mathbb{A}^{\infty\times}/V^\mathfrak{p}, K)$ by the closed ideal generated by $\{\omega(t)[t] - 1 \mid t \in \mathbb{A}^{\infty\times}\}$. Then we have the canonical isomorphism

$$\mathscr{D}(\omega, K) \simeq \varprojlim_{\Gamma_E} \mathscr{D}(\omega, K, V^\mathfrak{p}),$$

where the former one is introduced in Definition 3.7 (5). The (unique) continuous homomorphism $D(O_{E_\mathfrak{p}}^\times, K) \to D(E_{\mathrm{cl}}^\times \backslash \mathbb{A}^{\infty\times}/V^\mathfrak{p}, K)$ sending $[t]$ to $\omega_\mathfrak{p}(t_\circ)[t]$ for $t = (t_\bullet, t_\circ) \in O_{E_\mathfrak{p}}^\times$ descends to a continuous homomorphism $\mathbf{w} \colon D(O_\mathfrak{p}^{\mathrm{anti}}, K) \to \mathscr{D}(\omega, K, V^\mathfrak{p})$ of K-algebras, which is compatible when varying K and $V^\mathfrak{p}$. In other words, we have a homomorphism

$$\mathbf{w} \colon D(O_\mathfrak{p}^{\mathrm{anti}}, K) \to \mathscr{D}(\omega, K). \tag{4.2}$$

Definition 4.9 (Universal character). We define the $\pm$-*universal character* to be

$$\chi_{\mathrm{univ}}^\pm \colon E_{\mathrm{cl}}^\times \backslash \mathbb{A}^{\infty\times} \xrightarrow{[\,]^{\pm 1}} \mathscr{D}(\omega, MF_\mathfrak{p})^\times.$$

Then $\chi_{\mathrm{univ}}^\pm$ is an element in $\mathscr{N}^\pm(\infty, F_\mathfrak{p}^{\mathrm{nr}}) \widehat{\otimes}_{F_\mathfrak{p}} \mathscr{D}(\omega, MF_\mathfrak{p})$ satisfying

$$t^* \chi_{\mathrm{univ}}^\pm = [t] \cdot \chi_{\mathrm{univ}}^\pm \tag{4.3}$$

for $t \in O_{E_\mathfrak{p},m}^\times$, the group on which $\omega_\mathfrak{p}$ is trivial.

Definition 4.10 (Universal torus period). Suppose K contains $MF_\mathfrak{p}^{\mathrm{lt}}$. Given a stable convergent modular form $f \in \mathscr{M}_\flat^w(m, K)^\heartsuit$ for some $w, m \in \mathbb{N}$, we have the global Mellin transform $\mathbf{M}(f)$ by Theorem 2.20. By restriction, we obtain elements

$$\mathbf{M}(f)|_{\mathfrak{Y}^\pm(\infty)} \in \mathscr{N}^\pm(\infty, K) \widehat{\otimes}_{F_\mathfrak{p}} D(O_\mathfrak{p}^{\mathrm{anti}}, F_\mathfrak{p}),$$

and hence by (4.2),

$$\mathbf{w}(\mathbf{M}(f)|_{\mathfrak{Y}^\pm(\infty)}) \in \mathscr{N}^\pm(\infty, K) \widehat{\otimes}_K \mathscr{D}(\omega, K).$$

By Theorem 2.20 (2) and (4.3), the product $\mathbf{w}(\mathbf{M}(f)|_{\mathfrak{Y}^\pm(\infty)}) \cdot \chi_{\mathrm{univ}}^\pm$ descends to an element in $\mathscr{N}^\pm(m, K) \widehat{\otimes}_K \mathscr{D}(\omega, K)$ for some (different) $m \in \mathbb{N}$ depending on f (and $\omega_\mathfrak{p}$). For any

$V^{\mathfrak{p}} \in \Gamma$ under which f is invariant, we regard $\mathbf{w}(\mathbf{M}(f)|_{\mathfrak{Y}^{\pm}(\infty)}) \cdot \chi^{\pm}_{\mathrm{univ}}$ as an element in $\mathscr{N}^{\pm}(m,K)\widehat{\otimes}_K \mathscr{D}(\omega, K, V^{\mathfrak{p}})$, which is then invariant under $V^{\mathfrak{p}}$. In particular,

$$\mathscr{P}^{\pm}_{\omega}(f) := \frac{1}{|E^\times\backslash \mathbb{A}_E^{\infty\times}/V^{\mathfrak{p}}O^\times_{E_{\mathfrak{p}},m}|} \sum_{E^\times\backslash \mathbb{A}_E^{\infty\times}/V^{\mathfrak{p}}O^\times_{E_{\mathfrak{p}},m}} (\mathbf{w}(\mathbf{M}(f)|_{\mathfrak{Y}^{\pm}(\infty)}) \cdot \chi^{\pm}_{\mathrm{univ}})(t) \tag{4.4}$$

is an element in $\mathscr{D}(\omega, K, V^{\mathfrak{p}})$ independent of m, and is compatible with respect to $V^{\mathfrak{p}}$. In other words, $\mathscr{P}^{\pm}_{\omega}(f)$ is an element in $\mathscr{D}(\omega, K)$, which we call the *universal torus period*.

4.3. **Interpolation of universal torus periods.** We remain the setting in the previous section. Let $MF^{\mathrm{lt}}_{\mathfrak{p}}F^{\mathrm{ab}}_{\mathfrak{p}} \subset K \subset \mathbb{C}_p$ be a complete intermediate field.

By Definition 4.2, an element $f^{\pm} \in \Pi^{\pm}\otimes_F K$ can be realized as K-linear combination of morphisms from $X_{U^{\mathfrak{p}}U_{\mathfrak{p},\pm m}}$ to A for some $U^{\mathfrak{p}} \in \Gamma_E$ and $m \in \mathbb{N}$, which we will now assume. Take $\omega_{\pm} \in \mathrm{H}^0(A^{\pm}, \Omega^1_{A^{\pm}})$. Using the notation in (2.17) and by Proposition 2.13 (3), we have $(f^*_{\pm}\omega_{\pm})_{\mathrm{ord}} \in \mathscr{M}^0_{\flat}(\infty, K)$. It is stable (resp. n-admissible (in the sense of Definition 2.15)) if and only if $f^{\pm}$ is stable (resp. n-admissible (in the sense of Definition 4.2)).

For stable vectors $f^{\pm} \in (\Pi^{\pm})^{\heartsuit}_K$, define the element

$$\mathscr{P}^{\pm}_{\mathrm{univ}}(f_{\pm}) \in \mathrm{Lie}\, A^{\pm} \otimes_{F^M} \mathscr{D}(\omega_A, K)$$

by the formula

$$\langle \omega_{\pm}, \mathscr{P}^{\pm}_{\mathrm{univ}}(f_{\pm})\rangle = \mathscr{P}^{\pm}_{\omega_A}((f^*_{\pm}\omega_{\pm})_{\mathrm{ord}}).$$

In this section, we study the relation of

$$\iota\mathscr{P}^{+}_{\mathrm{univ}}(f_+)(\chi)\cdot \iota\mathscr{P}^{-}_{\mathrm{univ}}(f_-)(\chi) \in (\mathrm{Lie}\, A^+ \otimes_{F^M} \mathrm{Lie}\, A^-)\otimes_{F^M,\iota}\mathbb{C}$$

for a given $\iota\colon \mathbb{C}_p \xrightarrow{\sim} \mathbb{C}$, with classical torus periods, for n-admissible stable vectors $f^{\pm} \in (\Pi^{\pm})^{\heartsuit}_K$ and a character $\chi \in \Xi(\omega_A, K)^n_k$ of weight $k \geq 1$ and depth n (see Definition 4.1). For this purpose, we choose an ι-nearby data for $\mathbb{B}$ (Definition 2.24). In particular, we have

$$Y^{\pm}_{\iota}(\mathbb{C}) = E^\times_{\mathrm{cl}}\backslash\{\pm i\}\times \mathbb{A}_E^{\infty\times} \subset X_{\iota}(\mathbb{C}).$$

Choose an element $t_{\pm} \in \mathbb{A}_E^{\infty\times}$ such that $\iota P^{\pm}$ is represented by $[\pm i, t_{\pm}]$. Define $\zeta^{\pm}_{\iota} \in \mathbb{C}^\times$ such that

$$\mathrm{d}z([\pm i, t_{\pm}]) = \zeta^{\pm}_{\iota}\cdot \iota\omega_{\psi^{\pm}}|_{P^{\pm}}, \tag{4.5}$$

where $\omega_{\psi^{\pm}}$ is defined in (3.1). We also introduce matrices

$$j^+_{\iota} = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}, \quad j^-_{\iota} = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$$

in $\mathrm{Mat}_2(\mathbb{R}) = B(\iota)\otimes_{F,\iota} R$.

Lemma 4.11. *Let the notation be as above. We have*

$$\begin{aligned}\iota\langle\omega_+, \mathscr{P}^{+}_{\mathrm{univ}}(f_+)(\chi)\rangle\cdot\iota\langle\omega_-, \mathscr{P}^{-}_{\mathrm{univ}}(f_-)(\chi)\rangle &= \frac{(\zeta^+_{\iota}\zeta^-_{\iota})^k\cdot \chi^{(\iota)}(t_+^{-1}t_-)}{4}\\ &\times \mathscr{P}_{\mathrm{Wa}}(\Delta^{k-1}_{+,\iota}(\mathrm{R}(j^+_{\iota})\phi_{\iota}(f^*_+\omega_+)), \chi^{(\iota)+1})\mathscr{P}_{\mathrm{Wa}}(\Delta^{k-1}_{-,\iota}(\mathrm{R}(j^-_{\iota})\phi_{\iota}(f^*_-\omega_-)), \chi^{(\iota)-1}),\end{aligned}$$

where ϕ_{ι} is defined in (2.19)*, and*

$$\mathscr{P}_{\mathrm{Wa}}(\Phi, \chi^{(\iota)\pm1}) = \int_{E^\times\mathbb{A}^\times\backslash\mathbb{A}_E^\times} \Phi(t)\chi^{(\iota)}(t)^{\pm1}\,\mathrm{d}t$$

is the classical torus period, that is, the automorphic period appearing in the classical Waldspurger formula.

Proof. Take $V^{\mathfrak{p}} \in \Gamma_E$ under which $f_\pm$ and χ are invariant. By Theorem 2.20 and Definition (4.4), we have

$$\langle \omega_\pm, \mathscr{P}^\pm_{\mathrm{univ}}(f_\pm)(\chi)\rangle = \frac{\Omega_{\mathrm{lt}}^{-k}}{|E^\times\backslash \mathbb{A}_E^{\infty\times}/V^{\mathfrak{p}}O^\times_{E_{\mathfrak{p}},m}|} \times \sum_{E^\times\backslash \mathbb{A}_E^{\infty\times}/V^{\mathfrak{p}}O^\times_{E_{\mathfrak{p}},m}} \Theta_{\mathrm{ord}}^{k-1}(f_\pm^*\omega_\pm)_{\mathrm{ord}}(t)\cdot(\chi^{\pm1}\omega_{\psi\pm}^{-k})(t) \tag{4.6}$$

for some sufficiently large $m \geq n$. By (4.5) and Lemma 2.23, we have

$$\begin{aligned}\iota\langle \omega_\pm, \mathscr{P}^\pm_{\mathrm{univ}}(f_\pm)(\chi)\rangle &= \frac{(\zeta_\iota^\pm)^k}{|E^\times\backslash \mathbb{A}_E^{\infty\times}/V^{\mathfrak{p}}O^\times_{E_{\mathfrak{p}},m}|} \\ &\quad\times \sum_{E^\times\backslash \mathbb{A}_E^{\infty\times}/V^{\mathfrak{p}}O^\times_{E_{\mathfrak{p}},m}} \Theta_\iota^{k-1} f_\pm^*\omega_\pm(t)\,\mathrm{d}z([\pm i, t_\pm t])^{-k}\cdot \chi^{(\iota)}(t)^{\pm1} \\ &= \frac{(\zeta_\iota^\pm)^k}{2}\int_{E^\times\mathbb{A}^\times\backslash\mathbb{A}_E^\times} \mathrm{R}(j_\iota^\pm)\phi_\iota(\Theta_\iota^{k-1}f_\pm^*\omega_\pm)(t)\cdot\chi^{(\iota)}(t)^{\pm1}\,\mathrm{d}t,\end{aligned}$$

which by Lemma 2.26 equals

$$= \frac{(\zeta_\iota^\pm)^k\cdot\chi^{(\iota)}(t_\pm^{\mp1})}{2}\int_{E^\times\mathbb{A}^\times\backslash\mathbb{A}_E^\times}\Delta_{\pm,\iota}^{k-1}(\mathrm{R}(j_\iota^\pm)\phi_\iota(f_\pm^*\omega_\pm))(t)\cdot\chi^{(\iota)}(t)^{\pm1}\,\mathrm{d}t.$$

This completes the proof. ∎

Proposition 4.12. *Given $\omega_\pm \in \mathrm{H}^0(A^\pm, \Omega^1_{A^\pm})$, n-admissible stable vectors $f_\pm \in (\Pi^\pm)_K^\heartsuit$, and a character $\chi \in \Xi(\omega, K)^n_k$ of weight $k \geq 1$ and depth n, we have*

$$\begin{aligned}\iota\langle\omega_+, \mathscr{P}^+_{\mathrm{univ}}(f_+)(\chi)\rangle\cdot\iota\langle\omega_-, \mathscr{P}^-_{\mathrm{univ}}(f_-)(\chi)\rangle &= \iota\mathscr{Q}(f_+, f_-)(\chi) \\ &\quad\times L(1/2, \rho_A^{(\iota)}, \chi^{(\iota)})\cdot\frac{2^{g-3}\delta_E^{1/2}\zeta_F(2)\mathbf{P}_\iota(\chi)}{L(1,\eta)^2L(1,\rho_A^{(\iota)},\mathrm{Ad})}\frac{\iota\epsilon(1/2,\psi,\rho_{A,\mathfrak{p}}\otimes\check{\chi}_{\mathfrak{P}^c})}{\iota L(1/2,\rho_{A,\mathfrak{p}}\otimes\check{\chi}_{\mathfrak{P}^c})^2}.\end{aligned}$$

Proof. By the classical Waldspurger formula [Wal85] (or see [YZZ13, Theorem 1.4.2]) and Proposition 4.5, we have that

$$\begin{aligned}&\mathscr{P}_{\mathrm{Wa}}(\Delta_{+,\iota}^{k-1}(\mathrm{R}(j_\iota^+)\phi_\iota(f_+^*\omega_+), \chi^{(\iota)+1})\mathscr{P}_{\mathrm{Wa}}(\Delta_{-,\iota}^{k-1}(\mathrm{R}(j_\iota^-)\phi_\iota(f_-^*\omega_-), \chi^{(\iota)-1}) \\ &\quad= C_\iota\frac{1}{4}\frac{\zeta_F(2)L(1/2,\rho_A^{(\iota)},\chi^{(\iota)})}{L(1,\rho_A^{(\iota)},\mathrm{Ad})L(1,\eta)}\frac{2}{2^{-g}\delta_E^{-1/2}L(1,\eta)}\frac{\iota\epsilon(1/2,\psi,\rho_{A,\mathfrak{p}}\otimes\check{\chi}_{\mathfrak{P}^c})}{\iota L(1/2,\rho_{A,\mathfrak{p}}\otimes\check{\chi}_{\mathfrak{P}^c})^2}\cdot\iota\mathscr{Q}(f_+,f_-)(\chi),\end{aligned}$$

where C_ι is the complex constant such that

$$(\Delta_{+,\iota}^{k-1}(\mathrm{R}(j_\iota^+)\phi_\iota(f_+^*\omega_+), \Delta_{-,\iota}^{k-1}(\mathrm{R}(j_\iota^-)\phi_\iota(f_-^*\omega_-))_{\mathrm{Pet}} = C_\iota\cdot\iota(f_+, f_-)_A$$

holds for all f_+ and f_-. The proposition follows from the above formula, Lemma 4.11, and the formula

$$\mathbf{P}_\iota(\chi) = C_\iota\cdot(\zeta_\iota^+\zeta_\iota^-)^k\cdot\chi^{(\iota)}(t_+^{-1}t_-).$$

∎

Corollary 4.13. *For $\chi \in \Xi(\omega_A, K)^n_k$ with $k \geq 1$, the ratio*

$$\frac{\mathscr{P}^+_{\mathrm{univ}}(f_+)(\chi)\mathscr{P}^-_{\mathrm{univ}}(f_-)(\chi)}{\mathscr{Q}(f_+, f_-)(\chi)} \in (\mathrm{Lie}\, A^+ \otimes_{F^M} \mathrm{Lie}\, A^-) \otimes_{F^M} K,$$

if the denominator is nonzero, is independent of the choice of $f_\pm \in (\Pi^\pm)^\heartsuit_K$ that is n-admissible. Moreover, for $\iota\colon \mathbb{C}_p \xrightarrow{\sim} \mathbb{C}$,

$$\iota\left(\frac{\mathscr{P}^+_{\mathrm{univ}}(f_+)(\chi)\mathscr{P}^-_{\mathrm{univ}}(f_-)(\chi)}{\mathscr{Q}(f_+, f_-)(\chi)}\right) = L(1/2, \rho^{(\iota)}_A, \chi^{(\iota)}) \cdot \frac{2^{g-3}\delta_E^{1/2}\zeta_F(2)\mathbf{P}_\iota(\chi)}{L(1,\eta)^2 L(1, \rho^{(\iota)}_A, \mathrm{Ad})} \frac{\iota\epsilon(1/2, \psi, \rho_{A,\mathfrak{p}} \otimes \check{\chi}_{\mathfrak{P}^c})}{\iota L(1/2, \rho_{A,\mathfrak{p}} \otimes \check{\chi}_{\mathfrak{P}^c})^2}.$$

Proposition 4.14. *For n-admissible stable vectors $f_\pm \in (\Pi^\pm)^\heartsuit_K$ and a character $\chi \in \Xi(\omega_A, K)^n_0$ of weight 0 and depth n, we have*

$$\mathscr{P}^\pm_{\mathrm{univ}}(f_\pm)(\chi) = \log_{A^\pm} P^\pm_\chi(f_\pm).$$

Proof. We may choose a tame level $U^\mathfrak{p} \in \Gamma$ that fixes $f_\pm$, and such that χ is fixed by $U^\mathfrak{p} \cap \mathbb{A}_E^{\infty\times}$. We may realize $f_\pm$ as a K-linear combination of morphisms from $X_{U^\mathfrak{p}U_{\mathfrak{p},\pm m}}$ to $A^\pm$ for some sufficiently large integer $m \geq n$. By linearity, we may assume $f_\pm$ is such a morphism.

For $\omega_\pm \in \mathrm{H}^0(A^\pm, \Omega^1_{A^\pm})$, we have by Theorem 2.20 (3,4) that

$$\mathrm{d}\mathbf{M}((f^*_\pm\omega_\pm)_{\mathrm{ord}})(\chi|_{O^\times_{E_{\mathfrak{P}^c}}}) = \Theta_{\mathrm{ord}}\mathbf{M}((f^*_\pm\omega_\pm)_{\mathrm{ord}})(\chi|_{O^\times_{E_{\mathfrak{P}^c}}}) = (f^*_\pm\omega_\pm)_{\mathrm{ord}},$$

which by definition is the restriction of $f^*_\pm\omega_\pm$ to $\mathfrak{X}(m, U^\mathfrak{p})$. On the other hand, by Proposition A.1, we know that $f^*_\pm \log_{\omega_\pm}$ is a Coleman integral of $f^*_\pm\omega_\pm$ on (the generic fiber of) $\mathfrak{X}(m, U^\mathfrak{p})$. Therefore,

$$\mathbf{M}((f^*_\pm\omega_\pm)_{\mathrm{ord}})(\chi|_{O^\times_{E_{\mathfrak{P}^c}}}) = f^*_\pm \log_{\omega_\pm}$$

on $\mathfrak{X}(m, U^\mathfrak{p})$ since both of them are Coleman integrals of $f^*_\pm\omega_\pm$ on $\mathfrak{X}(m, U^\mathfrak{p})$ that belong to $\mathscr{M}^0_\flat(m, K)^\heartsuit$. The proposition follows by (3.3) and (4.4). ☐

4.4. **Proof of main theorems.** Let the situation be as in Definition 4.8. Denote by $\mathscr{C}(\omega, K, V^\mathfrak{p})$ the (closed) subspace of $\mathscr{C}(\omega, K)$ of functions that are invariant under the right translation of $V^\mathfrak{p}$, which is also a closed subspace of $C(E^\times\backslash\mathbb{A}_E^{\infty\times}/V^\mathfrak{p}, K)$. By duality, the strong dual of $\mathscr{C}(\omega, K, V^\mathfrak{p})$ is canonically isomorphic to $\mathscr{D}(\omega, K, V^\mathfrak{p})$.

We consider totally definite (not necessarily incoherent) quaternion algebras $\mathbb{B}$ over $\mathbb{A}$ such that for a finite place v of F, $\epsilon(\mathbb{B}_v) = 1$ if v is split in E *or* the Galois representation $\rho_{A,v}$ corresponds to a principal series.

For such a $\mathbb{B}$, we choose an embedding as (1.1), which is possible. We may define a representation

$$\Pi(\mathbb{B})^{\mathrm{tame}}_{A^\pm} = {\bigotimes}'_M \Pi_{v,A^\pm},$$

where the restricted tensor product (over M) is taken over all finite places $v \neq \mathfrak{p}$, and $\Pi_{v,A^\pm}$ is an M-representation of $\mathbb{B}_v^\times$ determined by $\rho_{A^\pm,v}$ which is unique up to isomorphism. In particular, if $\mathbb{B}$ is incoherent, then $\Pi(\mathbb{B})^{\mathrm{tame}}_{A^{\pm}}$ is isomorphic to the away-from-$\mathfrak{p}$ component of $\Pi(\mathbb{B})_{A^{\pm}}$. Let $\mathscr{I}_{\pm}(\omega_A, K, V^{\mathfrak{p}})$ be the closed ideal of $\mathscr{D}(\omega_A, K, V^{\mathfrak{p}})$ generated by

$$\{\mathscr{Q}(f_+, f_-) \mid f_{\pm} \in (\Pi(\mathbb{B})^{\mathrm{tame}}_{A^{\pm}})^{V^{\mathfrak{p}}} \otimes_M K, \epsilon(\mathbb{B}) = \pm 1\},$$

where $\mathscr{Q}(f_+, f_-)$ is similarly defined (as the product) in Lemma 4.6. It is topologically finitely generated. Let $\mathscr{C}_{\pm}(\omega_A, K, V^{\mathfrak{p}})$ be the subspace of $\mathscr{C}(\omega_A, K, V^{\mathfrak{p}})$ of functions lying in the kernel of every element in $\mathscr{I}_{\mp}(\omega_A, K, V^{\mathfrak{p}})$, which is closed. Put $\Xi(A, K, V^{\mathfrak{p}}) = \Xi(A, K) \cap \mathscr{C}(\omega_A, K, V^{\mathfrak{p}})$ and $\Xi(\omega_A, K, V^{\mathfrak{p}}) = \Xi(\omega_A, K) \cap \mathscr{C}(\omega_A, K, V^{\mathfrak{p}})$, where $\Xi(A, K)$ and $\Xi(\omega_A, K)$ are introduced in Definition 3.7.

Lemma 4.15. *Assume $V^{\mathfrak{p}} \in \Gamma_E$ is sufficiently small. We have*

(1) $\mathscr{I}_+(\omega_A, K, V^{\mathfrak{p}}) \cap \mathscr{I}_-(\omega_A, K, V^{\mathfrak{p}}) = 0$*;*
(2) $\mathscr{I}_+(\omega_A, K, V^{\mathfrak{p}}) + \mathscr{I}_-(\omega_A, K, V^{\mathfrak{p}}) = \mathscr{D}(\omega_A, K, V^{\mathfrak{p}})$*;*
(3) $\mathscr{C}(\omega_A, K, V^{\mathfrak{p}}) = \mathscr{C}_+(\omega_A, K, V^{\mathfrak{p}}) \oplus \mathscr{C}_-(\omega_A, K, V^{\mathfrak{p}})$*;*
(4) *the subset $\Xi(A, K, V^{\mathfrak{p}})$ is contained in and generates a dense subspace of $\mathscr{C}_-(\omega_A, K, V^{\mathfrak{p}})$;*
(5) $\mathscr{I}_+(\omega_A, K, V^{\mathfrak{p}})$ *is the closed ideal generated by elements that vanish on* $\Xi(A, K, V^{\mathfrak{p}})$*.*

If we put $\mathscr{D}(A, K, V^{\mathfrak{p}}) = \mathscr{D}(\omega_A, K, V^{\mathfrak{p}})/\mathscr{I}_+(\omega_A, K, V^{\mathfrak{p}})$, then

$$\mathscr{D}(A, K) = \varprojlim_{V^{\mathfrak{p}} \in \Gamma_E} \mathscr{D}(A, K, V^{\mathfrak{p}}).$$

We have $\mathscr{D}(A, K, V^{\mathfrak{p}})\widehat{\otimes}_K K' \simeq \mathscr{D}(A, K', V^{\mathfrak{p}})$ and $\mathscr{D}(A, K)\widehat{\otimes}_K K' \simeq \mathscr{D}(A, K')$ for a complete field extension K'/K.

Proof. We first realize that $\Xi(\omega_A, K, V^{\mathfrak{p}})$ generates a dense subspace of $\mathscr{C}(\omega_A, K, V^{\mathfrak{p}})$. Thus (1) follows from the dichotomy theorem of Saito–Tunnell [Tun83, Sai93]. For (2), assume the converse and suppose that $\mathscr{I}_+(\omega_A, K, V^{\mathfrak{p}}) + \mathscr{I}_-(\omega_A, K, V^{\mathfrak{p}})$ is contained in a (closed) maximal ideal $\mathfrak{m}$ with the residue field K'. Then all local period distributions $\mathscr{Q}(f_+, f_-)$ will vanish on the character

$$E^{\times}\backslash \mathbb{A}_E^{\infty\times}/V^{\mathfrak{p}} \xrightarrow{[\,]} \mathscr{D}(\omega_A, K, V^{\mathfrak{p}}) \to K',$$

which contradicts the theorem of Saito–Tunnell. Part (3) is a direct consequence of (1) and (2). It is clear that $\Xi(A, K, V^{\mathfrak{p}})$ is contained in $\mathscr{C}_-(\omega_A, K, V^{\mathfrak{p}})$ and by Saito–Tunnell, $\Xi(\omega_A, K, V^{\mathfrak{p}})\backslash\Xi(A, K, V^{\mathfrak{p}}) \subset \mathscr{C}_+(\omega_A, K, V^{\mathfrak{p}})$, which together imply (4). Finally, (5) follows from (4). ☐

Remark 4.16. In fact, for sufficiently small $V^{\mathfrak{p}} \in \Gamma_E$, the morphism $\mathbf{w}\colon D(O_{\mathfrak{p}}^{\mathrm{anti}}, K) \to \mathscr{D}(\omega_A, K, V^{\mathfrak{p}})$ (4.2) is injective with the quotient that is a finite étale K-algebra. We also have $D(O_{\mathfrak{p}}^{\mathrm{anti}}, K) \cap \mathscr{I}_+(\omega_A, K, V^{\mathfrak{p}}) = \{0\}$. Thus if K is discretely valued, $\mathscr{D}(A, K, V^{\mathfrak{p}})$ is a (commutative) nuclear Fréchet–Stein K-algebras (defined for example in [Eme, Definition 1.2.10]). Moreover, it is not hard to see that the transition homomorphism $\mathscr{D}(A, K, V'^{\mathfrak{p}}) \to \mathscr{D}(A, K, V^{\mathfrak{p}})$ is finite étale for $V'^{\mathfrak{p}} \subset V^{\mathfrak{p}}$. The rigid analytic $MF_{\mathfrak{p}}$-variety associated to $\mathscr{D}(A, MF_{\mathfrak{p}}, V^{\mathfrak{p}})$ is a smooth curve, which may be regarded as an eigencurve for the group $\mathrm{U}(1)_{E/F}$ of tame level $V^{\mathfrak{p}}$, twisted by (the cyclotomic character) ω_A and cut off by the condition that $\epsilon(1/2, \rho_A, \ ) = -1$.

Definition 4.17 (p-adic L-function). Note that the union $\bigcup_{k\geq 1}\Xi(\omega_A,K)^0_k$ is already dense in $\mathscr{C}(\omega_A,K)$. Now let $MF^{\mathrm{lt}}_{\mathfrak{p}}F^{\mathrm{ab}}_{\mathfrak{p}}\subset K\subset\mathbb{C}_p$ be a complete intermediate field. By Corollary 4.13, the ratios

$$\frac{\mathscr{P}^+_{\mathrm{univ}}(f_+)\mathscr{P}^-_{\mathrm{univ}}(f_-)}{\mathscr{Q}(f_+,f_-)}$$

for $f_\pm$ running over $(\Pi(\mathbb{B})_{A^\pm})^\heartsuit_K$ with $\epsilon(\mathbb{B})=-1$, together define an element

$$\mathscr{L}(A)\in(\mathrm{Lie}\,A^+\otimes_{F^M}\mathrm{Lie}\,A^-)\otimes_{F^M}\mathscr{D}(A,K)$$

by Lemma 4.15, which is the *anti-cyclotomic p-adic L-function* attached to A. It actually belongs to $(\mathrm{Lie}\,A^+\otimes_{F^M}\mathrm{Lie}\,A^-)\otimes_{F^M}\mathscr{D}(A,MF^{\mathrm{lt}}_{\mathfrak{p}})$ by the lemma below.

Lemma 4.18. *The element $\mathscr{L}(A)$ belongs to* $(\mathrm{Lie}\,A^+\otimes_{F^M}\mathrm{Lie}\,A^-)\otimes_{F^M}\mathscr{D}(A,MF^{\mathrm{lt}}_{\mathfrak{p}})$.

Proof. In the definition of $\mathscr{L}(A)$, we only need to consider $f_\pm\in(\Pi(\mathbb{B})_{A^\pm})^\heartsuit_{MF^{\mathrm{lt}}_{\mathfrak{p}}}$ such that f_+ (resp. $\Pi(\mathbb{B})_{A^-}(\mathrm{J})f_-$) is invariant under $O^\times_{\mathfrak{P}^c}$. Then for $\chi\in\bigcup_{k\geq 1}\Xi(A,MF^{\mathrm{lt}}_{\mathfrak{p}})^0_k$, the value $\mathscr{L}(A)(\chi)$ belongs to $(\mathrm{Lie}\,A^+\otimes_{F^M}\mathrm{Lie}\,A^-)\otimes_{F^M}MF^{\mathrm{lt}}_{\mathfrak{p}}$ by the formula (4.6). ☐

Proof of Theorem 3.8. We only need to show that the element

$$\mathscr{L}(A)\in(\mathrm{Lie}\,A^+\otimes_{F^M}\mathrm{Lie}\,A^-)\otimes_{F^M}\mathscr{D}(A,MF^{\mathrm{lt}}_{\mathfrak{p}})$$

introduced in Definition 4.17 satisfies (3.2), which follows from Corollary 4.13. ☐

Proof of Theorem 3.10. It follows from Proposition 4.14 and Proposition 4.5. ☐

5. Remarks on cases of higher weights

In this chapter, we discuss how to generalize our formulation of p-adic Waldspurger formula to p-adic Maass forms of other weights. We keep the setup in §1.1 and assume $\mathfrak{p}$ is split in E. Let $\iota_1,\dots,\iota_g\colon F\hookrightarrow\mathbb{C}_p$ be g different embeddings. For each ι_i, there are two embeddings $\iota_i^\circ,\iota_i^\bullet\colon E\hookrightarrow\mathbb{C}_p$ extending it, such that ι_i° induces $\mathfrak{P}$ if ι_i induces $\mathfrak{p}$.

Take a totally definite incoherent quaternion algebra $\mathbb{B}$ over $\mathbb{A}$, and put $\mathbb{B}_p=\mathbb{B}\otimes_{\mathbb{Q}}\mathbb{Q}_p$. To simplify notation, we regard X as defined over $\mathbb{C}_p$. Moreover, we assume $\mathbb{B}^\infty\not\simeq\mathrm{Mat}_2(\mathbb{A}^\infty)$ when $F=\mathbb{Q}$ to avoid extra care for cusps. We also ignore all arguments like doing things on X_U first then taking limits to X.

Let $\underline{w}=(w;w_1,\dots,w_g)$ be $g+1$ integers with the same parity such that $w_i\geq 2$. The embedding ι_i gives rise to the two-dimensional standard representation std_i of $\mathbb{B}_p^\times$ with $\mathbb{C}_p$-coefficient. Put

$$V_{\underline{w}}=\bigotimes_{i=1}^g\mathrm{Sym}^{w_i-2}(\mathrm{std}_i)\otimes(\iota_i\,\mathrm{Nm}_{\mathbb{B}_p/F_p})^{\frac{w-w_i+2}{2}},$$

which is a $\mathbb{C}_p$-representation of $\mathbb{B}_p^\times$ of dimension $\prod_{i=1}^g(w_i-1)$. Let U_p be a compact open subgroup of $\mathbb{B}_p^\times$. We choose an $O_{\mathbb{C}_p}$-lattice $V^\infty_{\underline{w}}$ of $V_{\underline{w}}$ that is stable under the restricted action of U_p. For $n\geq 1$, let $U_{p,n}$ be the kernel of the induced U_p-representation $V^\infty_{\underline{w}}/p^nV^\infty_{\underline{w}}$, which is a subgroup of finite index. Then the quotient

$$\left(V^\infty_{\underline{w}}/p^nV^\infty_{\underline{w}}\times X_{U_{p,n}}\right)/(U_p/U_{p,n})$$

defines a locally constant étale $O_{\mathbb{C}_p}/p^n$-module $V^n_{\underline{w}}$ on X_{U_p}. Put

$$\mathscr{V}_{\underline{w}} = (\varprojlim_{n\geq 1} V^n_{\underline{w}}) \otimes \mathbb{Q},$$

which is a $\mathbb{B}^{\infty\times}$-equivariant $\mathbb{C}_p$-local system of rank $\prod_{i=1}^g (w_i - 1)$ over X_{U_p} and hence X by restriction. It is easy to see that up to isomorphism, $\mathscr{V}_{\underline{w}}$ is independent of the choice of U_p and the lattice $V^\infty_{\underline{w}}$.

Definition 5.1 (p-adic Maass form). Denote by X^{ord} the ordinary locus in the rigid analytification of X. We choose an exhausting family of basic wide open neighborhoods $\{X^{\mathrm{ord}}_r\}_r$ of X^{ord} parameterized by real numbers $0 < r < 1$ (see, for example, [Kas04, §9]).

(1) The space of *p-adic Maass forms of weight* $\underline{w}$ is defined to be

$$\mathscr{A}^{\underline{w}}_{\mathbb{C}_p}(\mathbb{B}^\times) = \varinjlim_r \Gamma(X^{\mathrm{ord}}_r, \mathscr{V}_{\underline{w}}),$$

where $\Gamma(X^{\mathrm{ord}}_r, \mathscr{V}_{\underline{w}})$ denotes the space of locally analytic sections of $\mathscr{V}_{\underline{w}}$ over X^{ord}_r, and the transition maps in the colimit are restriction maps.

(2) A p-adic Maass form is *classical* if it comes from some section $\phi \in \Gamma(X^{\mathrm{ord}}_r, \mathscr{V}_{\underline{w}})$ that is *the* Coleman primitive ([Col94, §10]) of some element in $\mathrm{H}^0(X, \mathscr{V}_{\underline{w}} \otimes_{\mathbb{C}_p} \Omega^1_X)$. (Strictly speaking, when $\underline{w} = (w; 2, \dots, 2)$, Coleman primitives of a global differential form are unique up to an element in $\mathrm{H}^0(X, \mathscr{O}_X)$.)

In fact, the space $\mathscr{A}^{\underline{w}}_{\mathbb{C}_p}(\mathbb{B}^\times)$ up to isomorphism, and the notion of being classical do not depend on the choice of the family of basic wide open neighborhoods. Moreover, it is not hard to see that Hecke actions of $\mathbb{B}^{\infty\times}$ preserve $\mathscr{A}^{\underline{w}}_{\mathbb{C}_p}(\mathbb{B}^\times)$. An irreducible $\mathbb{B}^{\infty\times}$-subrepresentation π of $\mathscr{A}^{\underline{w}}_{\mathbb{C}_p}(\mathbb{B}^\times)$ is *classical* if one (and hence all) of its nonzero members are classical. For a classical π, we may define its *dual* $\pi^\vee$, which is contained in $\mathscr{A}^{\underline{w}^\vee}_{\mathbb{C}_p}(\mathbb{B}^\times)$ where $\underline{w}^\vee := (-w; w_1, \dots, w_g)$.

Let $\underline{l} = (l, l_1, \dots, l_g)$ be g integers with the same parity. The embedding ι°_i can be viewed as a 1-dimensional $\mathbb{C}_p$-representation of $E^\times_p$. Put

$$W_{\underline{l}} = \bigotimes_{i=1}^g (\iota^\circ_i)^{l_i} \otimes (\iota_i \circ \mathrm{Nm}_{E_p/F_p})^{\frac{-l-l_i}{2}},$$

which is a $\mathbb{C}_p$-representation of $E^\times_p$ of dimension 1. In the same manner, we have an $\mathbb{A}^{\infty\times}_E$-equivariant $\mathbb{C}_p$-local system $\mathscr{W}_{\underline{l}}$ of rank 1 over Y. Similar to the space $\mathscr{A}^{\underline{w}}_{\mathbb{C}_p}(\mathbb{B}^\times)$, we may also define the notion of a classical irreducible $\mathbb{A}^{\infty\times}_E$-subrepresentation $\sigma \subset \Gamma(Y^+, \mathscr{W}_{\underline{l}})$, and its dual $\sigma^\vee \subset \Gamma(Y^-, \mathscr{W}_{\underline{l}^\vee})$. Recall that Y is contained in X^{ord}. The following lemma is immediate.

Lemma 5.2. *Suppose $l = w$ and $|l_i| < w_i$ for $1 \leq i \leq g$. Then the $\mathbb{A}^{\infty\times}_E$-coinvariant space of $\Gamma(Y^\pm, \mathscr{V}_{\underline{w}} \otimes \mathscr{W}_{\underline{l}})$ is of dimension* 1*.*

From now on, we assume $l = w$ and $|l_i| < w_i$ for $1 \leq i \leq g$. Denote by $\Gamma(Y^\pm, \mathscr{V}_{\underline{w}} \otimes \mathscr{W}_{\underline{l}})_\flat$ the $\mathbb{A}^{\infty\times}_E$-coinvariant (quotient) space of $\Gamma(Y^\pm, \mathscr{V}_{\underline{w}} \otimes \mathscr{W}_{\underline{l}})$, with the quotient map formally denoted by the integration $\int_{Y^\pm} \mathrm{d}y$. For $\phi \in \mathscr{A}^{\underline{w}}_{\mathbb{C}_p}(\mathbb{B}^\times)$ and $\varphi_\pm \in \Gamma(Y^\pm, \mathscr{W}_{\underline{l}})$, consider

$$\mathscr{P}_{Y^\pm}(\phi, \varphi_\pm) = \int_{Y^\pm} \phi(y)\varphi_\pm(y)\,\mathrm{d}y.$$

Now we adopt the $\pm$ convention for notation as in §1.2 and below. In particular, we have classical representations $\pi^\pm \subset \mathscr{A}^{w^\pm}_{\mathbb{C}_p}(\mathbb{B}^\times)$ and $\sigma^\pm \subset \Gamma(Y^+, \mathscr{W}_{\underline{l}^\pm})$. We make the following choices:

- an *abstract conjugation*, that is, an $\mathbb{A}_E^{\infty\times}$-equivariant automorphism $\mathbf{c}$ of Y switching Y^+ and Y^-;
- a duality pairing $(\mathscr{V}_{\underline{w}^+} \otimes \mathscr{W}_{\underline{l}^+}) \times \mathbf{c}^*(\mathscr{V}_{\underline{w}^-} \otimes \mathscr{W}_{\underline{l}^-}) \to \underline{\mathbb{C}_p}$ of local systems over Y^+;

They induce a pairing

$$(\ , \ )_\flat \colon \Gamma(Y^+, \mathscr{V}_{\underline{w}^+} \otimes \mathscr{W}_{\underline{l}^+})_\flat \times \Gamma(Y^-, \mathscr{V}_{\underline{w}^-} \otimes \mathscr{W}_{\underline{l}^-})_\flat \to \mathbb{C}_p.$$

The p-adic Waldspurger formula in this generality would be a formal seeking for the value of $(\mathscr{P}_{Y^+}(\phi_+, \varphi_+), \mathscr{P}_{Y^-}(\phi_-, \varphi_-))_\flat$ for $\phi_\pm \in \pi^\pm$ and $\varphi_\pm \in \sigma^\pm$, in terms of certain p-adic L-function.

Appendix A. Compatibility of logarithm and Coleman integral

In this appendix, we generalize a result of Coleman in [Col85] about the compatibility of p-adic logarithm and Coleman integral. Such result will only be used in the proof of Proposition 4.14.

Let F be a local field contained in $\mathbb{C}_p$ with the ring of integers O_F and the residue field k. Let X be a quasi-projective scheme over F and $U \subset X^{\mathrm{rig}}$ an affinoid domain with good reduction. We say a closed rigid analytic 1-form ω on U is *Frobenius proper* if there exits a Frobenius endomorphism ϕ of U and a polynomial $P(X)$ over $\mathbb{C}_p$ such that $P(\phi^*)\omega$ is the differential of a rigid analytic function on U and such that no root of $P(X)$ is a root of unity. Therefore, by [Col85, Theorem 2.1], there exits a locally analytic function f_ω on $U(\mathbb{C}_p)$, unique up to an additive constant on each geometric connected component, such that

- $\mathrm{d}f_\omega = \omega$;
- $P(\phi^*)f_\omega$ is rigid analytic.

Such f_ω is known as a *Coleman integral* of ω on U, which is independent of the choice of P [Col85, Corollary 2.1b].

Proposition A.1. *Let X and U be as above. Let A be an abelian variety over F which has either totally degenerate reduction or potentially good reduction. For a morphism $f\colon X \to A$ and a differential form $\omega \in \Omega^1(A/F)$, $f^*\omega|_U$ is Frobenius proper which admits $f^*\log_\omega|_U$ as a Coleman integral, where $\log_\omega \colon A(\mathbb{C}_p) \to \mathbb{C}_p$ is the p-adic logarithm associated to ω.*

Proof. We may assume X is projective, and replace F by a finite extension. Therefore, we may assume A has good reduction, or split totally degenerate reduction, that is, the connected neutral component $\mathcal{A}_s^\circ$ of the special fiber $\mathcal{A}_s$ of the Néron model $\mathcal{A}$ of A is isomorphic to $\mathbf{G}_{m,k}^d$, where d is the dimension of A. The first case follows from [Col85, Theorem 2.8, Proposition 2.2].

Now we consider the second case. Denote by $\mathcal{A}_\eta^\circ$ the analytic domain of A^{rig} of points whose reduction is in $\mathcal{A}_s^\circ$. By the well-known uniformization, we have $A^{\mathrm{rig}} \simeq (\mathbf{G}_{m,F}^{\mathrm{rig}})^d/\Lambda$ for a lattice $\Lambda \subset \mathbf{G}_{m,F}^d(F)$. Moreover, $\mathcal{A}_\eta^\circ$ is isomorphic to $\mathrm{Sp}\, F\langle T_1, \dots, T_d, T_1^{-1}, \dots, T_d^{-1}\rangle$, the rigid analytic multi-torus of multi-radius 1.

Choose an admissible covering $\mathscr{U}$ of X^{rig} containing U, which determines a formal model $X_{\mathscr{U}}$ of X over O_F. Since X is projective, we may assume $X_{\mathscr{U}}$ is algebraic. Let Z be the non-smooth locus of $X_{\mathscr{U}}$ over O_F. The set of closed points of X whose reduction is not in Z forms an analytic domain W of X^{rig}. Since U has good reduction, we have $U \subset W$. By Néron mapping property, the morphism f extends unique to a morphism $X_{\mathscr{U}} - Z \to \mathcal{A}$, which induces a morphism $f' \colon U \to A^{\mathrm{rig}}$. Without loss of generality, we assume $f'(U)$ is contained in $\mathcal{A}_\eta^\circ$. By [Col85, Proposition 2.2], we only need to show that $\omega|_{\mathcal{A}_\eta^\circ}$ is Frobenius proper and $\log_\omega|_{\mathcal{A}_\eta^\circ}$ is a Coleman integral of it.

In fact, we have

$$\left\{ \omega|_{\mathcal{A}_\eta^\circ} \middle| \omega \in \Omega^1(A/F) \right\} = \mathrm{Span}_F \left\{ \frac{\mathrm{d}T_1}{T_1}, \cdots, \frac{\mathrm{d}T_d}{T_d} \right\}.$$

By linearity, we may assume $\omega^\circ := \omega|_{\mathcal{A}_\eta^\circ} = \mathrm{d}T_1/T_1$. We choose the Frobenius endomorphism on $\mathcal{A}_\eta^\circ$ to be given by $\phi((T_1, \dots, T_d)) = (T_1^q, \dots, T_d^q)$ where $q = |k|$. We have that $P(\phi^*)\omega^\circ = 0$ for $P(X) = X - q$. On the other hand, the p-adic logarithm log on $\mathrm{Sp}\, F\langle T_1, T_1^{-1}\rangle$ is also killed by $P(\phi^*)$. Therefore, the function $(\log, 1, \dots, 1)$ on $\mathrm{Sp}\, F\langle T_1, T_1^{-1}\rangle \times \cdots \times \mathrm{Sp}\, F\langle T_d, T_d^{-1}\rangle \simeq \mathcal{A}_\eta^\circ$ is a Coleman integral of ω°, which coincides with the restriction of $\log_\omega$. □

Appendix B. Serre–Tate local moduli for $\mathcal{O}$-divisible groups (d'après N. Katz)

In this appendix, we generalize a classical theorem of Katz [Kat81] describing the Kodaira–Spencer isomorphism in terms of the Serre–Tate coordinate for ordinary p-divisible groups to ordinary $\mathcal{O}$-divisible groups. Only Theorem B.1 and Theorem B.5 will be used in the main part of the article. Some notation in this appendix may be different from those in §1.4.

B.1. **$\mathcal{O}$-divisible groups and Serre–Tate coordinates.** Let F be a finite field extension of $\mathbb{Q}_p$ where p is a rational prime. Denote by $\breve{F}$ the completion of a maximal ramified extension of F. The ring of integers of F (resp. $\breve{F}$) is denoted by $\mathcal{O}$ (resp. $\breve{\mathcal{O}}$). Let k be the residue field of $\breve{\mathcal{O}}$, which is isomorphic to $\mathbb{F}_p^{\mathrm{ac}}$. For a p-divisible group G over $\mathrm{Spec}\, R$, we denote by $\Omega(G/R)$ the R-modules of invariant differentials of G over R, which is the dual R-module of the tangent space $\mathrm{Lie}(G/R)$ at the identity.

Let S be an $\breve{\mathcal{O}}$-scheme. Recall that an $\mathcal{O}$-divisible group over S is a p-divisible group G over S with an action by $\mathcal{O}$ such that the induced action of $\mathcal{O}$ on the sheaf $\underline{\mathrm{Lie}}(G/S)$ coincides with the natural action as an $\mathscr{O}_S$-module (hence an $\mathcal{O}$-module). Denote by $\mathsf{BT}_S^{\mathcal{O}}$ the category of $\mathcal{O}$-divisible groups over S, which is an abelian category. We omit the superscript $\mathcal{O}$ if it is $\mathbb{Z}_p$. The height h of G, as a p-divisible group, must be divisible by $[F : \mathbb{Q}_p]$. We define the *$\mathcal{O}$-height* of G to be $[F : \mathbb{Q}_p]^{-1}h$. An $\mathcal{O}$-divisible group G is connected (resp. étale) if its underlying p-divisible group is. We denote by $\mathcal{LT}$ the Lubin–Tate $\mathcal{O}$-group over $\mathrm{Spec}\, \breve{\mathcal{O}}$, which is unique up to isomorphism. We use the same notation for its base change to S.

For an $\mathcal{O}$-divisible group G over S, define its *$\mathcal{O}$-Cartier dual* to be $G^D := \varinjlim_n \underline{\mathrm{Hom}}_{\mathcal{O}}(G[p^n], \mathcal{LT}[p^n])$ [Fal02]. An $\mathcal{O}$-divisible group G is *ordinary* if $(G^0)^D$ is étale, where G^0 is the connected part of G. Denote by $\mathrm{T}_pG = \varprojlim_n G[p^n]$ the Tate module functor. Denote by $\mathsf{Nilp}_{\breve{\mathcal{O}}}$ the category of $\breve{\mathcal{O}}$-schemes on which p is locally nilpotent.

Theorem B.1 (Serre–Tate coordinate)**.** *Let $\mathbf{G}$ be an ordinary $\mathcal{O}$-divisible group over k. Consider the moduli functor $\mathfrak{M}_{\mathbf{G}}$ on $\mathsf{Nilp}_{\breve{\mathcal{O}}}$ such that for every $\breve{\mathcal{O}}$-scheme S on which p is locally nilpotent, $\mathfrak{M}_{\mathbf{G}}(S)$ is the set of isomorphism classes of pairs (G,φ) where G is an object in $\mathsf{BT}_S^{\mathcal{O}}$ and $\varphi\colon G\times_S(S\otimes_{\breve{\mathcal{O}}}k)\to\mathbf{G}\times_{\mathrm{Spec}\,k}(S\otimes_{\breve{\mathcal{O}}}k)$ is an isomorphism. Then $\mathfrak{M}_{\mathbf{G}}$ is canonically pro-represented by the $\breve{\mathcal{O}}$-formal scheme $\mathrm{Hom}_{\mathcal{O}}(\mathrm{T}_p\mathbf{G}(k)\otimes_{\mathcal{O}}\mathrm{T}_p\mathbf{G}^D(k),\mathcal{LT})$.*

In particular, for every artinian local $\breve{\mathcal{O}}$-algebra R with the maximal ideal $\mathfrak{m}_R$ and G/R a deformation of $\mathbf{G}$, we have a pairing

$$q(G/R;\ ,\ )\colon \mathrm{T}_p\mathbf{G}(k)\otimes_{\mathcal{O}}\mathrm{T}_p\mathbf{G}^D(k)\to\mathcal{LT}(R)=1+\mathfrak{m}_R.$$

It satisfies:

(1) *For every $\alpha\in\mathrm{T}_p\mathbf{G}(k)$ and $\alpha_D\in\mathrm{T}_p\mathbf{G}^D(k)$,*

$$q(G/R;\alpha,\alpha_D)=q(G^D/R;\alpha_D,\alpha).$$

(2) *Suppose we have another ordinary $\mathcal{O}$-divisible groups $\mathbf{H}$ over k, and its deformation H over R. Let $\mathbf{f}\colon\mathbf{G}\to\mathbf{H}$ be a homomorphism and $\mathbf{f}^D$ be its dual. Then $\mathbf{f}$ lifts to a (unique) homomorphism $f\colon G\to H$ if and only if*

$$q(G/R;\alpha,\mathbf{f}^D\beta_D)=q(H/R;\mathbf{f}\alpha,\beta_D)$$

for every $\alpha\in\mathrm{T}_p\mathbf{G}(k)$ and $\beta_D\in\mathrm{T}_p\mathbf{H}^D(k)$.

By abuse of notation, we will use $\mathfrak{M}_{\mathbf{G}}$ to denote the formal scheme $\mathrm{Hom}_{\mathcal{O}}(\mathrm{T}_p\mathbf{G}(k)\otimes_{\mathcal{O}}\mathrm{T}_p\mathbf{G}^D(k),\mathcal{LT})$. The proof of the theorem follows exactly in the way of [Kat81, Theorem 2.1].

Proof. The fact that $\mathfrak{M}_G$ is pro-presentable is well-known. Now we determine the representing formal scheme.

Since $\mathbf{G}$ is ordinary, we have a canonical isomorphism

$$\mathbf{G}\simeq\mathbf{G}^0\times\mathrm{T}_p\mathbf{G}(k)\otimes_{\mathcal{O}}F/\mathcal{O}.$$

By the definition of $\mathcal{O}$-Cartier duality, we have a morphism

$$e_{p^n}\colon\mathbf{G}[p^n]\times\mathbf{G}^D[p^n]\to\mathcal{LT}[p^n].$$

The restriction of the first factor to $\mathbf{G}^0[p^n]$ gives rise an isomorphism

$$\mathbf{G}^0[p^n]\xrightarrow{\sim}\mathrm{Hom}_{\mathcal{O}}(\mathbf{G}^D[p^n](k),\mathcal{LT}[p^n])$$

of group schemes over k preserving $\mathcal{O}$-actions. Passing to limit, we obtain an isomorphism of $\mathcal{O}$-formal groups over k

$$\mathbf{G}^0\xrightarrow{\sim}\mathrm{Hom}_{\mathcal{O}}(\mathrm{T}_p\mathbf{G}^D(k),\mathcal{LT}),$$

which induces a pairing

$$E_{\mathbf{G}}\colon\mathbf{G}^0\times\mathrm{T}_p\mathbf{G}^D(k)\to\mathcal{LT}.$$

Let G/R be a deformation of $\mathbf{G}$, then we have an extension

$$0\longrightarrow G^0\longrightarrow G\longrightarrow\mathrm{T}_p\mathbf{G}(k)\otimes_{\mathcal{O}}F/\mathcal{O}\longrightarrow 0\tag{B.1}$$

of $\mathcal{O}$-divisible groups. We have pairings

$$E_{G,p^n}\colon G^0[p^n]\times \mathbf{G}^D[p^n]\to \mathcal{LT}[p^n],$$
$$E_G\colon G^0\times \mathrm{T}_p\mathbf{G}^D(k)\to \mathcal{LT},$$

which lift e_{p^n} and $E_{\mathbf{G}}$, respectively.

Similar to the p-divisible group case, the extension (B.1) is obtained from the extension

$$0\longrightarrow \mathrm{T}_p\mathbf{G}(k)\longrightarrow \mathrm{T}_p\mathbf{G}(k)\otimes_{\mathcal{O}} F\longrightarrow \mathrm{T}_p\mathbf{G}(k)\otimes_{\mathcal{O}} F/\mathcal{O}\longrightarrow 0$$

by pushing out along a unique $\mathcal{O}$-linear homomorphism

$$\varphi_{G/R}\colon \mathrm{T}_p\mathbf{G}(k)\to G^0(R).$$

The homomorphism $\varphi_{G/R}$ may be recovered from (B.1) in the way described in [Kat81, page 151]. It is the composite

$$\mathrm{T}_p\mathbf{G}(k)\to \mathrm{T}_p\mathbf{G}[p^n](k)\xrightarrow{\langle p^n\rangle} G^0(R)$$

for any $n\geq 1$ such that $\mathfrak{m}_R^{n+1}=0$. Therefore, from G/R, we obtain a pairing

$$q(G/R;\ ,\ )=E_G(R)\circ(\varphi_{G/R},\mathrm{id})\colon \mathrm{T}_p\mathbf{G}(k)\otimes_{\mathcal{O}}\mathrm{T}_p\mathbf{G}^D(k)\to\mathcal{LT}(R)=1+\mathfrak{m}_R.$$

This shows that the functor $\mathfrak{M}_{\mathbf{G}}$ is canonically pro-represented by the $\breve{\mathcal{O}}$-formal scheme $\mathrm{Hom}_{\mathcal{O}}(\mathrm{T}_p\mathbf{G}(k)\otimes_{\mathcal{O}}\mathrm{T}_p\mathbf{G}^D(k),\mathcal{LT})$.

For (2), if the given homomorphism $\mathbf{f}\colon\mathbf{G}\to\mathbf{H}$ can be lifted to $f\colon G\to H$, then we must have the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_{\mathcal{O}}(\mathrm{T}_p\mathbf{G}^D(k),\mathcal{LT}) \arrow[r] \arrow[d, "\circ\mathrm{T}_p\mathbf{f}^D(k)"'] & G \arrow[r] \arrow[d, "f"] & \mathrm{T}_p\mathbf{G}(k)\otimes_{\mathcal{O}} F/\mathcal{O} \arrow[r] \arrow[d, "\mathrm{T}_p\mathbf{f}(k)\otimes_{\mathcal{O}}F/\mathcal{O}"] & 0\\
0 \arrow[r] & \mathrm{Hom}_{\mathcal{O}}(\mathrm{T}_p\mathbf{H}^D(k),\mathcal{LT}) \arrow[r] & H \arrow[r] & \mathrm{T}_p\mathbf{H}(k)\otimes_{\mathcal{O}} F/\mathcal{O} \arrow[r] & 0.
\end{tikzcd}$$

Conversely, if we may fill f in the above diagram, then $\mathbf{f}$ lifts.

The existence of the middle arrow is equivalent to that the push-out of the top extension by the left arrow is isomorphic to the pull-back of the lower extension by the right arrow. The above mentioned push-out is an element of

$$\mathrm{Ext}_{\mathsf{BT}_R^{\mathcal{O}}}(\mathrm{T}_p\mathbf{G}(k)\otimes_{\mathcal{O}}F/\mathcal{O},\mathrm{Hom}_{\mathcal{O}}(\mathrm{T}_p\mathbf{H}^D(k),\mathcal{LT}))$$

which is isomorphic to

$$\mathrm{Hom}_{\mathcal{O}}(\mathrm{T}_p\mathbf{G}(k)\otimes_{\mathcal{O}}\mathrm{T}_p\mathbf{H}^D(k),\mathcal{LT}(R))$$

by the bilinear pairing

$$(\alpha,\beta_D)\mapsto q(G/R;\alpha,\mathbf{f}^D\beta_D).$$

Similarly, the above mentioned pull-back is an element in

$$\mathrm{Hom}_{\mathcal{O}}(\mathrm{T}_p\mathbf{G}(k)\otimes_{\mathcal{O}}\mathrm{T}_p\mathbf{H}^D(k),\mathcal{LT}(R))$$

defined by the bilinear pairing

$$(\alpha,\beta_D)\mapsto q(H/R;\mathbf{f}\alpha,\beta_D).$$

It remains to prove (1). Choose n such that $\mathfrak{m}_R^{n+1}=0$. Then both $G^0(R)$ and $(G^D)^0(R)$ are annihilated by p^n. Denote by $\alpha(n)$ the image of α under the canonical projection

$\mathrm{T}_p\mathbf{G}(k) \to G[\varpi^n](k)$ and similarly for $\alpha_D(n)$. By construction, we have $\varphi_{G/R}(\alpha) = \langle p^n\rangle\alpha(n) \in G^0(R)$ and $\varphi_{G^D/R}(\alpha_D) = \langle p^n\rangle\alpha_D(n) \in (G^D)^0(R)$. Therefore, we have

$$q(G/R;\alpha,\alpha_D) = E_{G,p^n}(\langle p^n\rangle\alpha(n),\alpha_D(n)).$$

Similarly, we have $q(G^D/R;\alpha_D,\alpha) = E_{G^D,p^n}(\langle p^n\rangle\alpha_D(n),\alpha(n))$.

The remaining argument is formal and one only needs to replace $\widehat{\mathbf{G}}_m$ (resp. abelian varieties) by $\mathcal{LT}$ (resp. $\mathcal{O}$-divisible groups) in the proof of [Kat81, Theorem 2.1]. In particular, we have the following lemma. □

Lemma B.2. *Given any $n \geq 1$, $x \in G^0[p^n](R)$ and $y \in \mathbf{G}^D[p^n](k)$, there exist an artinian local ring R' that is finite and flat over R, and a point $Y \in G^D[p^n](R')$ lifting y. For every such R' and Y, we have the equality $E_{G,p^n}(x,y) = e_{p^n}(x,Y)$ inside $\mathcal{LT}(R')$.*

B.2. **Main theorem.** We fix an ordinary $\mathcal{O}$-divisible group $\mathbf{G}$ over k. Denote by $\mathfrak{R}$ the coordinate ring of $\mathfrak{M}_{\mathbf{G}}$, which is a complete $\breve{\mathcal{O}}$-algebra. We have the universal pairing

$$q\colon \mathrm{T}_p\mathbf{G}(k)\otimes_{\mathcal{O}}\mathrm{T}_p\mathbf{G}^D(k) \to \mathcal{LT}(\mathfrak{R}) \subset \mathfrak{R}^\times.$$

Therefore, we may regard $q(\alpha,\alpha_D)$ as a regular function on $\mathfrak{M}_{\mathbf{G}}$. For each $\mathcal{O}$-linear form $\ell \in \mathrm{Hom}_{\mathcal{O}}(\mathrm{T}_p\mathbf{G}(k)\otimes_{\mathcal{O}}\mathrm{T}_p\mathbf{G}^D(k),\mathcal{O})$, denote by $D(\ell)$ the translation-invariant continuous derivation of $\mathfrak{R}$ given by

$$\mathbf{D}(\ell)q(\alpha,\alpha_D) = \ell(\alpha\otimes\alpha_D)\cdot q(\alpha,\alpha_D).$$

By abuse of notation, we also denote by $\mathbf{D}(\ell)$ the corresponding map $\Omega_{\mathfrak{R}/\breve{\mathcal{O}}} \to \mathfrak{R}$. Denote by $\mathfrak{G}$ the universal $\mathcal{O}$-divisible group over $\mathfrak{M}_{\mathbf{G}}$.

We choose a normalized logarithm $\log\colon \mathcal{LT} \to \widehat{\mathbf{G}}_a$ over $\breve{\mathcal{O}}\otimes\mathbb{Q}$, and put $\omega_0 = \log^*\mathrm{d}T$, which is a generator of the free $\breve{\mathcal{O}}$-module $\Omega(\mathcal{LT}/\breve{\mathcal{O}})$ of rank 1.

Let R be as in Theorem B.1 and G/R be a deformation of $\mathbf{G}$. We have the canonical isomorphism of $\mathcal{O}$-modules

$$\lambda_G\colon \mathrm{T}_p\mathbf{G}^D(k) \xrightarrow{\sim} \mathrm{Hom}_{\mathsf{BT}_R^{\mathcal{O}}}(G^0,\mathcal{LT}).$$

Define the $\mathcal{O}$-linear map $\omega_G\colon \mathrm{T}_p\mathbf{G}^D(k) \to \Omega(G/R)$ by the formula

$$\omega_G(\alpha_D) = \lambda_G(\alpha_D)^*\omega_0 \in \Omega(G^0/R) = \Omega(G/R).$$

Let $L_G\colon \mathrm{Hom}_{\mathcal{O}}(\mathrm{T}_p\mathbf{G}^D(k),\mathcal{O}) \to \mathrm{Lie}(G/R)$ be the unique $\mathcal{O}$-linear map such that

$$\omega_G(\alpha_D)\cdot L_G(\alpha_D^\vee) = \alpha_D\cdot\alpha_D^\vee \in \mathcal{O}.$$

In fact, the R-linear extensions

$$\omega_G\colon \mathrm{T}_p\mathbf{G}^D(k)\otimes_{\mathcal{O}} R \to \Omega(G/R)$$

and

$$L_G\colon \mathrm{Hom}_{\mathcal{O}}(\mathrm{T}_p\mathbf{G}^D(k),R) \to \mathrm{Lie}(G/R)$$

are isomorphisms. Similarly, we have an isomorphism

$$\lambda_{G^\vee}\colon \mathrm{T}_p\mathbf{G}(k) = \mathrm{T}_p\mathbf{G}^{\text{ét}}(k) = \mathrm{T}_pG^{\text{ét}}(R) \xrightarrow{\sim} \mathrm{Hom}_{\mathsf{BT}_R}((G^{\text{ét}})^\vee,\widehat{\mathbf{G}}_m),$$

which induces an isomorphism

$$\mathrm{T}_p\mathbf{G}(k)\otimes_{\mathbb{Z}_p} R \xrightarrow{\sim} \Omega((G^{\text{ét}})^\vee/R)$$

by pulling back $\mathrm{d}T/T$. It further induces an isomorphism

$$\omega_{G^\vee}\colon \mathrm{T}_p\mathbf{G}(k)\otimes_{\mathcal{O}} R = (\mathrm{T}_p\mathbf{G}(k)\otimes_{\mathbb{Z}_p} R)_{\mathcal{O}} \xrightarrow{\sim} \Omega((G^{\text{ét}})^\vee/R)_{\mathcal{O}}.$$

Here, the subscript $\mathcal{O}$ denotes the maximal flat quotient on which $\mathcal{O}$ acts via the structure map. By construction, we have the following functoriality.

Lemma B.3. *Let $\mathbf{f}\colon \mathbf{G}\to\mathbf{H}$ be as in Theorem B.1 and $f\colon G\to H$ lifts $\mathbf{f}$. Then*

(1) $((f^{\text{ét}})^\vee)^*(\omega_{G^\vee}(\alpha)) = \omega_{H^\vee}(\mathbf{f}\alpha)$ *for every $\alpha\in\mathrm{T}_p\mathbf{G}(k)$, where $f^{\text{ét}}\colon G^{\text{ét}}\to H^{\text{ét}}$ is the induced homomorphism on the étale quotient.*

(2) $f_*(L_H(\alpha_D^\vee)) = L_G(\alpha_D^\vee\circ\mathbf{f}^D)$ *for every $\alpha_D^\vee\in\mathrm{Hom}_{\mathcal{O}}(\mathrm{T}_p\mathbf{G}^D(k),\mathcal{O})$.*

Denote by $\mathrm{D}(\mathbf{G})$ the (contra-variant) Dieudonné crystal of $\mathbf{G}$. We have the following exact sequence

$$0\longrightarrow\Omega(\mathfrak{G}^\vee/\mathfrak{R})\longrightarrow\mathrm{D}(\mathbf{G}^\vee)_{\mathfrak{R}}\longrightarrow\mathrm{Lie}(\mathfrak{G}/\mathfrak{R})\longrightarrow 0$$

and the *Gauss–Manin connection*

$$\nabla\colon \mathrm{D}(\mathbf{G}^\vee)_{\mathfrak{R}}\to\mathrm{D}(\mathbf{G}^\vee)_{\mathfrak{R}}\otimes_{\mathfrak{R}}\Omega_{\mathfrak{R}/\breve{\mathcal{O}}}.$$

They together define the following *(universal) Kodaira–Spencer map*

$$\mathrm{KS}\colon \Omega(\mathfrak{G}^\vee/\mathfrak{R})\to\mathrm{Lie}(\mathfrak{G}/\mathfrak{R})\otimes_{\mathfrak{R}}\Omega_{\mathfrak{R}/\breve{\mathcal{O}}},$$

which factors through the quotient $\Omega(\mathfrak{G}^\vee/\mathfrak{R})\to\Omega(\mathfrak{G}^\vee/\mathfrak{R})_{\mathcal{O}}$. The following lemma is immediate.

Lemma B.4. *The natural map $\Omega(\mathfrak{G}^\vee/\mathfrak{R})_{\mathcal{O}}\to\Omega((\mathfrak{G}^{\text{ét}})^\vee/\mathfrak{R})_{\mathcal{O}}$ is an isomorphism.*

In particular, we may regard $\omega_{\mathfrak{G}^\vee}$ as a map from $\mathrm{T}_p\mathbf{G}(k)$ to $\Omega(\mathfrak{G}^\vee/\mathfrak{R})_{\mathcal{O}}$. The following result on the compatibility of the Kodaira–Spencer map and the Serre–Tate coordinate is the main theorem of this appendix.

Theorem B.5. *We have the following equality in $\Omega_{\mathfrak{R}/\breve{\mathcal{O}}}$*

$$\omega_{\mathfrak{G}}(\alpha_D)\cdot\mathrm{KS}(\omega_{\mathfrak{G}^\vee}(\alpha)) = \mathrm{d}\log(q(\alpha,\alpha_D))$$

for every $\alpha\in\mathrm{T}_p\mathbf{G}(k)$ and $\alpha_D\in\mathrm{T}_p\mathbf{G}^D(k)$.

Note that the definition of $\omega_{\mathfrak{G}}$, but not $\omega_{\mathfrak{G}^\vee}$, depends on the choice of log, which is compatible with the right-hand side.

B.3. **Frobenius.** Denote by σ the Frobenius automorphism of $\breve{\mathcal{O}}$ that fixes every element in $\mathcal{O}$. Put $X^\sigma = X\otimes_{\breve{\mathcal{O}},\sigma}\breve{\mathcal{O}}$ for every $\breve{\mathcal{O}}$-(formal) scheme X, $\Sigma_X\colon X^\sigma\to X$ the natural projection, and $\mathsf{F}_X\colon X\to X^\sigma$ the relative Frobenius morphism which is $\breve{\mathcal{O}}$-linear. We omit the subscript X when it is $\mathfrak{M}_{\mathbf{G}}$.

Lemma B.6. *We have*

(1) *There is a natural isomorphism*

$$\mathfrak{M}_{\mathbf{G}}^\sigma\xrightarrow{\sim}\mathfrak{M}_{\mathbf{G}^\sigma}$$

under which the regular function $q(\sigma(\alpha),\sigma(\alpha_D))$ is mapped to $\Sigma^*q(\alpha,\alpha_D)$.

(2) *Under* $\Sigma_{(G^{\text{ét}})^\vee}\colon (G^{\sigma\text{ét}})^\vee \simeq ((G^{\text{ét}})^\vee)^\sigma \to (G^{\text{ét}})^\vee$, *we have*

$$\Sigma^*_{(G^{\text{ét}})^\vee}\omega_{G^\vee}(\alpha) = \omega_{(G^\sigma)^\vee}(\sigma\alpha)$$

for every $\alpha \in \mathrm{T}_p\mathbf{G}(k)$.

(3) *Under* $\mathsf{F}_G\colon G \to G^\sigma$, *we have*

$$\mathsf{F}_{G*}L_G(\alpha_D^\vee) = L_{G^\sigma}(\alpha_D^\vee \circ \sigma^{-1})$$

for every $\alpha_D^\vee \in \mathrm{Hom}_{\mathcal{O}}(\mathrm{T}_p\mathbf{G}^D(k), \mathcal{O})$.

Proof. The proof is same as [Kat81, Lemma 4.1.1 & 4.1.1.1]. □

From now on, we choose a uniformizer ϖ of F, which gives rise to an isomorphism $\mathcal{LT}^\sigma \simeq \mathcal{LT}$. In particular, we may identify $(G^D)^\sigma$ and $(G^\sigma)^D$.

For a deformation G/R of $\mathbf{G}$, we denote by G'/R the quotient of G by subgroup $G^0[\varpi]$. The induced projection map

$$\mathcal{F}_G\colon G \to G'$$

lifts the relative Frobenius morphism

$$\mathsf{F}_{\mathbf{G}}\colon \mathbf{G} \to \mathbf{G}^\sigma.$$

Define the *Verschiebung* to be

$$\mathsf{V}_{\mathbf{G}} = (\mathsf{F}_{\mathbf{G}^D})^D\colon \mathbf{G}^\sigma \simeq \mathbf{G}^{D\sigma D} \to \mathbf{G}.$$

Note that the isomorphism depends on ϖ.

Lemma B.7. *For* $\alpha \in \mathrm{T}_p\mathbf{G}(k)$ *and* $\alpha_D \in \mathrm{T}_p\mathbf{G}^D(k)$, *we have formulas*

(1) $\mathsf{F}_{\mathbf{G}}(\alpha) = \sigma\alpha$ *and* $\mathsf{V}_{\mathbf{G}}(\sigma\alpha) = \varpi\alpha_D$;

(2) $q(G'/R; \sigma\alpha, \sigma\alpha_D) = \varpi.q(G/R; \alpha, \alpha_D)$.

Proof. The proof is the same as [Kat81, Lemma 4.1.2], with $\mathsf{V}_{\mathbf{G}} \circ \mathsf{F}_{\mathbf{G}} = \varpi$. □

Lemma B.8. *For* $\alpha \in \mathrm{T}_p\mathbf{G}(k)$ *and* $\alpha_D^\vee \in \mathrm{Hom}_{\mathcal{O}}(\mathrm{T}_p\mathbf{G}^D(k), \mathcal{O})$, *we have*

(1) $((\mathcal{F}_G^{\text{ét}})^\vee)^*\omega_{G^\vee}(\alpha) = \omega_{G'^\vee}(\sigma\alpha)$;

(2) $\mathcal{F}_{G*}L_G(\alpha_D^\vee) = \varpi L_{G'}(\alpha_D^\vee \circ \sigma^{-1})$.

Proof. It follows from Lemmas B.3 and B.7. □

If we apply the construction to the universal object $\mathfrak{G}$, we obtain a formal deformation $\mathfrak{G}'/\mathfrak{R}$ of G^σ. Its classifying map is the unique morphism

$$\Phi\colon \mathfrak{M}_{\mathbf{G}} \to \mathfrak{M}_{\mathbf{G}^\sigma} \xrightarrow{\sim} \mathfrak{M}_{\mathbf{G}}^\sigma$$

such that $\Phi^*\mathfrak{G}^\sigma \simeq \mathfrak{G}'$. Therefore, we may regard $\mathcal{F}_{\mathfrak{G}}$ as a morphism

$$\mathcal{F}_{\mathfrak{G}}\colon \mathfrak{G} \to \Phi^*\mathfrak{G}^\sigma$$

of $\mathcal{O}$-divisible groups over $\mathfrak{M}_{\mathbf{G}}$. Taking dual, we have

$$\mathcal{F}_{\mathfrak{G}}^\vee\colon \Phi^*\mathfrak{G}^{\sigma\vee} \simeq (\Phi^*\mathfrak{G}^\sigma)^\vee \to \mathfrak{G}^\vee.$$

Lemma B.9. *We have*

(1) *The map* $\omega_{\mathfrak{G}^\vee}\colon \mathrm{T}_p\mathbf{G}(k) \otimes_{\mathcal{O}} \mathfrak{R} \to \Omega(\mathfrak{G}^\vee/\mathfrak{R})_{\mathcal{O}}$ *induces an isomorphism*

$$\mathrm{T}_p\mathbf{G}(k) \xrightarrow{\sim} \Omega(\mathfrak{G}^\vee/\mathfrak{R})^1_{\mathcal{O}} := \{\omega \in \Omega(\mathfrak{G}^\vee/\mathfrak{R})_{\mathcal{O}} \mid (\mathcal{F}_{\mathfrak{G}}^\vee)^*\omega = \Phi^*\Sigma^*_{\mathfrak{G}^\vee}\omega\}$$

of $\mathcal{O}$*-modules.*

(2) *The map* $L_{\mathfrak{G}}\colon \mathrm{Hom}_{\mathcal{O}}(\mathrm{T}_p\mathbf{G}^D(k),\mathfrak{R}) \to \mathrm{Lie}(\mathfrak{G}/\mathfrak{R})$ *induces an isomorphism*
$$\mathrm{Hom}_{\mathcal{O}}(\mathrm{T}_p\mathbf{G}^D(k),\mathcal{O}) \xrightarrow{\sim} \mathrm{Lie}(\mathfrak{G}/\mathfrak{R})^0 := \{\delta \in \mathrm{Lie}(\mathfrak{G}/\mathfrak{R}) \mid \mathcal{F}_{\mathfrak{G}*}\delta = \varpi\Phi^*\mathsf{F}_{\mathfrak{G}*}\delta\}$$
of $\mathcal{O}$*-modules.*

Proof. It can be proved by the same way as [Kat81, Corollary 4.1.5] by using Lemmas B.6 and B.8. ☐

Consider the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & \Omega(\mathfrak{G}^\vee/\mathfrak{R})_{\mathcal{O}} \arrow[r, "a"] \arrow[d, "(\mathcal{F}^\vee_{\mathfrak{G}})^*"'] & (\mathrm{D}(\mathbf{G}^\vee)_{\mathfrak{R}})_{\mathcal{O}} \arrow[r, "b"] \arrow[d, "\mathrm{D}(\mathsf{F}^\vee_{\mathbf{G}})"] & \mathrm{Lie}(\mathfrak{G}/\mathfrak{R}) \arrow[r] \arrow[d, "\mathcal{F}_{\mathfrak{G}*}"] & 0 \\
0 \arrow[r] & \Omega(\mathfrak{G}'^\vee/\mathfrak{R})_{\mathcal{O}} \arrow[r] & (\mathrm{D}(\mathbf{G}^{\sigma\vee})_{\mathfrak{R}})_{\mathcal{O}} \arrow[r] & \mathrm{Lie}(\mathfrak{G}'/\mathfrak{R}) \arrow[r] & 0 \\
0 \arrow[r] & \Omega(\mathfrak{G}^\vee/\mathfrak{R})_{\mathcal{O}} \arrow[r] \arrow[u, "\Phi^*\circ\Sigma^*_{\mathfrak{G}^\vee}"] & (\mathrm{D}(\mathbf{G}^\vee)_{\mathfrak{R}})_{\mathcal{O}} \arrow[r] \arrow[u, "\mathrm{D}(\Sigma_{\mathbf{G}^\vee})"'] & \mathrm{Lie}(\mathfrak{G}/\mathfrak{R}) \arrow[r] \arrow[u, "\Phi^*\circ\mathsf{F}_{\mathfrak{G}*}"'] & 0.
\end{tikzcd} \tag{B.2}$$

For $k \in \mathbb{Z}$, we define $\mathcal{O}$-modules
$$\begin{aligned}\mathrm{D}(\mathbf{G}^\vee)^k_{\mathfrak{R}} &= \{\xi \in (\mathrm{D}(\mathbf{G}^\vee)_{\mathfrak{R}})_{\mathcal{O}} \mid \mathrm{D}(\mathsf{F}^\vee_{\mathbf{G}})\xi = \varpi^{1-k}\mathrm{D}(\Sigma_{\mathbf{G}^\vee})\xi\} \\ &= \{\xi \in (\mathrm{D}(\mathbf{G}^\vee)_{\mathfrak{R}})_{\mathcal{O}} \mid \mathrm{D}(\mathsf{V}^\vee_{\mathbf{G}})\mathrm{D}(\Sigma_{\mathbf{G}^\vee})\xi = \varpi^{k}\xi\}.\end{aligned}$$

Lemma B.10. *The maps* $\omega_{\mathfrak{G}^\vee}$ *and* a *in* (B.2) *induce an isomorphism*
$$a_1\colon \mathrm{T}_p\mathbf{G}(k) \xrightarrow{\sim} \mathrm{D}(\mathbf{G}^\vee)^1_{\mathfrak{R}}$$
of $\mathcal{O}$*-modules. The maps* $L_{\mathfrak{G}}$ *and* b *in* (B.2) *induce an isomorphism*
$$b_0\colon \mathrm{D}(\mathbf{G}^\vee)^0_{\mathfrak{R}} \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{O}}(\mathrm{T}_p\mathbf{G}^D(k),\mathcal{O})$$
of $\mathcal{O}$*-modules.*

Proof. For the first part, by a similar argument in [Kat81, Lemma 4.2.1], we know that $b(\xi) = 0$ for $\xi \in \mathrm{D}(\mathbf{G}^\vee)^1_{\mathfrak{R}}$, that is, ξ is in the image of a. The conclusion then follows from Lemma B.9 (1).

For the second part, it is easy to see that $\mathrm{Im}(a) \cap \mathrm{D}(\mathbf{G}^\vee)^0_{\mathfrak{G}} = \{0\}$ by choosing an $\mathcal{O}$-basis of $\mathrm{T}_p\mathbf{G}(k)$. Therefore, b restricts to an injective map $\mathrm{D}(\mathbf{G}^\vee)^0_{\mathfrak{R}} \to \mathrm{Lie}(\mathfrak{G}/\mathfrak{R})^0$. We only need to show that this map is also surjective. For every $\delta \in \mathrm{Lie}(\mathfrak{G}/\mathfrak{R})^0$, choose an element $\xi_0 \in (\mathrm{D}(\mathbf{G}^\vee)_{\mathfrak{R}})_{\mathcal{O}}$. Put $\xi_{n+1} = \mathrm{D}(\mathsf{V}^\vee_{\mathbf{G}})\mathrm{D}(\Sigma_{\mathbf{G}^\vee})\xi_n$ for $n \geq 0$. Then $b(\xi_n) = \delta$ and $\{\xi_n\}$ converge to an element $\xi \in \mathrm{D}(\mathbf{G}^\vee)^0_{\mathfrak{R}}$. ☐

Lemma B.11. *For every* $\ell \in \mathrm{Hom}_{\mathcal{O}}(\mathrm{T}_p\mathbf{G}(k) \otimes_{\mathcal{O}} \mathrm{T}_p\mathbf{G}^D(k),\mathcal{O})$*, the action of* $D(\ell)$ *under the Gauss–Manin connection on* $(\mathrm{D}(G^\vee)_{\mathfrak{R}})_{\mathcal{O}}$ *satisfies the formula*
$$D(\ell)(\nabla(\mathrm{D}(\mathsf{V}^\vee_{\mathbf{G}})\mathrm{D}(\Sigma_{G^\vee})\xi)) = \varpi\mathrm{D}(\mathsf{V}^\vee_{\mathbf{G}})\mathrm{D}(\Sigma_{\mathbf{G}^\vee})(D(\ell)(\nabla\xi))$$
for every $\xi \in (\mathrm{D}(\mathbf{G}^\vee)_{\mathfrak{R}})_{\mathcal{O}}$*.*

Proof. It is proved in the same way as [Kat81, Lemma 4.3.3]. ☐

Lemma B.12. *If* $\xi \in (\mathrm{D}(\mathbf{G}^\vee)_{\mathfrak{R}})_{\mathcal{O}}$ *satisfies* $\mathrm{D}(\mathsf{V}^\vee_{\mathbf{G}})\mathrm{D}(\Sigma_{\mathbf{G}^\vee})\xi = \lambda\xi$ *for some* $\lambda \in \breve{\mathcal{O}}$*, then for every* $\ell \in \mathrm{Hom}_{\mathcal{O}}(\mathrm{T}_p\mathbf{G}(k) \otimes_{\mathcal{O}} \mathrm{T}_p\mathbf{G}^D(k),\mathcal{O})$*, the element* $D(\ell)(\nabla\xi) \in (\mathrm{D}(\mathbf{G}^\vee)_{\mathfrak{R}})_{\mathcal{O}}$ *satisfies*
$$\varpi\mathrm{D}(\mathsf{V}^\vee_{\mathbf{G}})\mathrm{D}(\Sigma_{\mathbf{G}^\vee})(D(\ell)(\nabla\xi)) = \lambda D(\ell)(\nabla\xi).$$

Proof. It follows immediately from Lemma B.11. □

Proposition B.13. *For $\alpha \in \mathrm{T}_p\mathbf{G}(k)$ and $\alpha_D \in \mathrm{T}_p\mathbf{G}^D(k)$, there exists a unique character $Q(\alpha, \alpha_D)$ of $\mathfrak{M}_{\mathbf{G}}$ such that*

$$\omega_{\mathfrak{G}}(\alpha_D) \cdot \mathrm{KS}(\omega_{\mathfrak{G}^\vee}(\alpha)) = \mathrm{d}\log Q(\alpha, \alpha_D).$$

Proof. Let $\{\alpha_i\}$ (resp. $\{\alpha_{D,j}\}$) be an $\mathcal{O}$-basis of $\mathrm{T}_p\mathbf{G}(k)$ (resp. $\mathrm{T}_p\mathbf{G}^D(k)$). Let $\{\ell_{i,j}\}$ be the basis of $\mathrm{Hom}_{\mathcal{O}}(\mathrm{T}_p\mathbf{G}(k) \otimes_{\mathcal{O}} \mathrm{T}_p\mathbf{G}^D(k), \mathcal{O})$ dual to $\{\alpha_i \otimes \alpha_{D,j}\}$. Then for every element $\xi \in (\mathrm{D}(\mathbf{G}^\vee)_{\mathfrak{R}})_{\mathcal{O}}$, we have

$$\nabla\xi = \sum_{i,j} D(\ell_{i,j})(\nabla\xi) \otimes \mathrm{d}\log q(\alpha_i, \alpha_{D,j}).$$

In particular, for $\xi = \omega_{\mathfrak{G}^\vee}(\alpha)$, we have

$$\nabla\omega_{\mathfrak{G}^\vee}(\alpha) = \sum_{i,j} D(\ell_{i,j})(\nabla\omega_{\mathfrak{G}^\vee}(\alpha)) \otimes \mathrm{d}\log q(\alpha_i, \alpha_{D,j}).$$

By Lemmas B.9 and B.12, $\nabla\omega_{\mathfrak{G}^\vee}(\alpha) \in \mathrm{D}(\mathbf{G}^\vee)^0_{\mathfrak{R}}$. Therefore, there exist unique elements $\alpha^\vee_{D,i,j} \in \mathrm{Hom}_{\mathcal{O}}(\mathrm{T}_p\mathbf{G}^D(k), \mathcal{O})$ such that

$$\nabla\omega_{\mathfrak{G}^\vee}(\alpha) = b_0^{-1}(\alpha^\vee_{D,i,j})$$

for every i, j. By definition,

$$\mathrm{KS}(\omega_{\mathfrak{G}^\vee}(\alpha)) = \sum_{i,j} L_{\mathfrak{G}}(\alpha^\vee_{D,i,j}) \otimes \mathrm{d}\log q(\alpha_i, \alpha_{D,j}),$$

and

$$\omega_{\mathfrak{G}}(\alpha_D) \cdot \mathrm{KS}(\omega_{\mathfrak{G}^\vee}(\alpha)) = \mathrm{d}\log\left(\prod_{i,j} q(\alpha_i, \alpha_{D,j})^{\alpha_D \cdot \alpha^\vee_{D,i,j}}\right).$$

□

Corollary B.14. *For elements $\alpha \in \mathrm{T}_p\mathbf{G}(k)$, $\alpha_D \in \mathrm{T}_p\mathbf{G}^D(k)$ and $\ell \in \mathrm{Hom}_{\mathcal{O}}(\mathrm{T}_p\mathbf{G}(k) \otimes_{\mathcal{O}} \mathrm{T}_p\mathbf{G}^D(k), \mathcal{O})$, we have that $D(\ell)(\omega_{\mathfrak{G}}(\alpha_D) \cdot \mathrm{KS}(\omega_{\mathfrak{G}^\vee}(\alpha)))$ is a constant in $\mathcal{O}$.*

Corollary B.15. *Suppose for every integer $n \geq 1$, we can find a homomorphism $f_n \colon \mathfrak{R} \to \breve{\mathcal{O}}/p^n$ such that*

$$f_n(D(\ell)(\omega_{\mathfrak{G}}(\alpha_D) \cdot \mathrm{KS}(\omega_{\mathfrak{G}^\vee}(\alpha)))) = \ell(\alpha \otimes \alpha_D)$$

holds W_n. Then $Q = q$ and Theorem B.5 follows.

The condition of this corollary is fulfilled by Theorem B.17. Therefore, we have reduced Theorem B.5 to Theorem B.17.

B.4. **Infinitesimal computation.** Let R be an (artinian) local $\breve{\mathcal{O}}$-algebra with the maximal ideal $\mathfrak{m}_R$ satisfying $\mathfrak{m}_R^{n+1} = 0$. We suppose G/R is canonical deformation of $\mathbf{G}$. Let $\tilde{G}$ be a deformation of G to $\tilde{R} := R[\varepsilon]/(\varepsilon^2)$, which gives rise to a map $\partial\colon \Omega(G^\vee/R) \to \mathrm{Lie}(G/R)$. Note that the target may be identified with $\mathrm{Ker}(G^0(\tilde{R}) \to G^0(R))$.

Lemma B.16. *The reduction map $\mathrm{T}_pG(R) \to \mathrm{T}_p\mathbf{G}(k)$ is an isomorphism.*

Proof. It follows from the same argument in [Kat81, Lemma 6.1]. □

In particular, we may define $\lambda_{G^\vee}\colon \mathrm{T}_p\mathbf{G}(k) \to \mathrm{Hom}_{\mathsf{BT}_R}(G^\vee, \widehat{\mathbf{G}}_m)$ and

$$\omega_{G^\vee}\colon \mathrm{T}_p\mathbf{G}(k) \to \Omega(G^\vee/R). \tag{B.3}$$

Theorem B.17. *The Serre–Tate coordinate for $\tilde{G}/\tilde{R}$ satisfies*

$$q(\tilde{G}/\tilde{R}; \alpha, \alpha_D) = 1 + \varepsilon\omega_G(\alpha_D)\cdot\partial(\omega_{G^\vee}(\alpha)).$$

Lemma B.18. *For $\alpha_D \in \mathrm{T}_p\mathbf{G}^D(k)$ and $\alpha \in \mathrm{Ker}(G^0(\tilde{R}) \to G^0(R)) = \mathrm{Lie}(G/R)$, we have*

$$E_G(\alpha, \alpha_D) = 1 + \varepsilon\omega_G(\alpha_D)\alpha.$$

Proof. By functoriality, we only need to prove the lemma for the universal object $\mathfrak{G}/\mathfrak{R}$. By definition,

$$1 + \varepsilon\omega_{\mathfrak{G}}(\alpha_D)\alpha = 1 + \varepsilon\left(\lambda_{\mathfrak{G}}(\alpha_D)_*\alpha\cdot\omega_0\right) \in \mathcal{LT}(\tilde{R}).$$

We also have

$$\lambda_{\mathfrak{G}}(\alpha_D)_*\alpha\cdot\omega_{\mathcal{LT}} = (\log\circ\lambda_{\mathfrak{G}}(\alpha_D))_*\alpha\cdot\mathrm{d}T$$

in $\mathfrak{R}[p^{-1}]$. Therefore, we have the equality

$$E_{\mathfrak{G}}(\alpha, \alpha_D) = 1 + \varepsilon\omega_{\mathfrak{G}}(\alpha_D)\alpha$$

in $\mathrm{Ker}(\mathcal{LT}(\tilde{\mathfrak{R}}[p^{-1}]) \to \mathcal{LT}(\mathfrak{R}[p^{-1}]))$. ◻

For an integer $N > n$, denote by α_N the image of α in $G[p^N](R)$. Let $\tilde{\alpha}_N \in \tilde{G}(\tilde{R})$ be an arbitrary lifting of α_N. Then

$$p^N\tilde{\alpha}_N \in \mathrm{Ker}(\tilde{G}(\tilde{R}) \to G(R)) = \mathrm{Ker}(\tilde{G}^0(\tilde{R}) \to G^0(R)) \simeq \mathrm{Lie}(G/R).$$

Such process defines a map $\varphi_G\colon \mathrm{T}_pG(R) \to \mathrm{Lie}(G/R)$.

Proposition B.19. *We have $\partial\omega_{G^\vee}(\alpha) = \varphi_G(\alpha)$ for every $\alpha \in \mathrm{T}_pG(R)$.*

Assuming the above proposition, we prove Theorem B.17.

Proof of Theorem B.17. It is clear that $G^0 \otimes_R \tilde{R}$ is the unique, up to isomorphism, deformation of G^0 to $\tilde{R}$. Then the deformation $\tilde{G}$ corresponds to the extension

$$0 \longrightarrow G^0 \otimes_R \tilde{R} \longrightarrow \tilde{G} \longrightarrow \mathrm{T}_p\mathbf{G}(k) \otimes_{\mathcal{O}} F/\mathcal{O} \longrightarrow 0.$$

In particular, we may identify $\tilde{G}^0$ with $G^0 \otimes_R \tilde{R}$. We have

$$\mathrm{Ker}(\tilde{G}(\tilde{R}) \to G(R)) = \mathrm{Ker}(\tilde{G}^0(\tilde{R}) \to G^0(R)) = \mathrm{Ker}(G^0(\tilde{R}) \to G^0(R)) = \mathrm{Lie}(G^0/R).$$

For $D \in \mathrm{Lie}(G^0/R)$, we have

$$E_{\tilde{G}}(D, -) = E_G(D, -)\colon \mathrm{T}_p\mathbf{G}^D(k) \simeq \mathrm{T}_p(\mathbf{G}^0)^D(k) \to \mathcal{LT}(\tilde{R}),$$

where in the pairing $E_{\tilde{G}}$ (resp. E_G), we view D as an element of $\tilde{G}^0(\tilde{R})$ (resp. $G^0(\tilde{R})$). For $\alpha_D \in \mathrm{T}_p\mathbf{G}^D(k)$, we have

$$E_G(D, \alpha_D) = 1 + \varepsilon\omega_G(\omega_\alpha)D$$

by definition. Therefore, Theorem B.17 follows from Proposition B.19 and the construction of q. ◻

The rest of the appendix is devoted to the proof of Proposition B.19. We will reduce it to certain statements in [Kat81] about abelian varieties. It is interesting to find a proof purely using $\mathcal{O}$-divisible groups.

Recall that ordinary $\mathcal{O}$-divisible groups over k are classified by its dimension and $\mathcal{O}$-height. Let $\mathbf{G}_{r,s}$ be an $\mathcal{O}$-divisible group of dimension r and $\mathcal{O}$-height $r+s$ with $r \geq 0, r+s>0$.

Choose a totally real number field E^+ such that $F \simeq E^+ \otimes_{\mathbb{Q}} \mathbb{Q}_p \simeq$, and an imaginary quadratic field K in which $p = \mathfrak{p}^+\mathfrak{p}^-$ splits. Put $E = E^+ \otimes_{\mathbb{Q}} K$. Suppose $\tau_1, \tau_2, \dots, \tau_h$ are all complex embeddings of E^+. Consider the data $(\mathbf{A}_{r,s}, \theta, i)$ where

- $\mathbf{A}_{r,s}$ is an abelian variety over k;
- $\theta\colon \mathbf{A}_{r,s} \to \mathbf{A}_{r,s}^\vee$ is a prime-to-p polarization;
- $i\colon O_E \to \operatorname{End}_k \mathbf{A}_{r,s}$ is an O_E-action which sends the complex conjugation on O_E to the Rosati involution and such that, in the induced decomposition
$$\mathbf{A}_{r,s}[p^\infty] = \mathbf{A}_{r,s}[p^\infty]^+ \oplus \mathbf{A}_{r,s}[p^\infty]^-$$
of the $O_E \otimes \mathbb{Z}_p$-module $\mathbf{A}_{r,s}[p^\infty]$, $\mathbf{A}_{r,s}[p^\infty]^+$ is isomorphic to $\mathbf{G}_{r,s}$ as an $\mathcal{O}$-divisible group.

It is clear that the polarization θ induces an isomorphism $\mathbf{A}_{r,s}[p^\infty]^+ \xrightarrow{\sim} (\mathbf{A}_{r,s}[p^\infty]^-)^\vee$. By Serre–Tate theorem, $\mathfrak{M}_{\mathbf{G}_{r,s}}$ also parameterizes deformation of the triple $(\mathbf{A}_{r,s}, \theta, i)$. In what follows, we fix r, s and suppress them from notation. Let R be as in Theorem B.1, A/R be the canonical deformation of $\mathbf{A}/k$, and $\tilde{A}$ be a deformation of A to $\tilde{R}$ such that $\tilde{G} \simeq \tilde{A}[p^\infty]^+$.

There is a similar map (B.3) for A and we have $\omega_{G^\vee}(\alpha) = \omega_{A^\vee}(\alpha)$ for $\alpha \in \mathrm{T}_p G(R) \subset \mathrm{T}_p A(R)$, where we view $\Omega(G^\vee/R)$ as a submodule of $\mathrm{H}^0(A^\vee, \Omega^1_{A^\vee/R})$. Moreover, the map $\varphi_G\colon \mathrm{T}_p G(R) \to \mathrm{Lie}(G/R)$ can be extended in a same way to a map $\varphi_A\colon \mathrm{T}_p A(R) \to \mathrm{Lie}(A/R)$. Then Proposition B.19 follows from [Kat81, Lemma 5.4 & §6.5], where the argument uses normalized cocycles and does *not* require A to be ordinary in the usual sense.

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